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Elementary Proof of MacMahon's Conjecture

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Abstract. Major Percy A. MacMahon's first paper on plane partitions [4] included a conjectured generating function for symmetric plane partitions. This conjecture was proven almost simultaneously by George Andrews and Ian Macdonald, Andrews using the machinery of basic hypergeometric series [1] and Macdonald employing his knowledge of symmetric functions [3]. The purpose of this paper is to simplify Macdonald's proof by providing a direct, inductive proof of his formula which expresses the sum of Schur functions whose partitions fit inside a rectangular box as a ratio of determinants.

Keywords: plane partition, symmetric plane partition, Schur function

By a **plane partition**, we mean a finite set, $\mathcal{P}$, of lattice points with positive integer coefficients, $\{(i, j, k)\} \subseteq \mathbb{N}^3$, with the property that if $(r, s, t) \in \mathcal{P}$ and $1 \leq i \leq r,\ 1 \leq j \leq s,\ 1 \leq k \leq t$, then (i, j, k) must also be in $\mathcal{P}$. A plane partition is **symmetric** if $(i, j, k) \in \mathcal{P}$ if and only if $(j, i, k) \in \mathcal{P}$. MacMahon's conjecture states that the generating function for symmetric plane partitions whose x and y coordinates are less than or equal to n and whose z coordinate is less than or equal to m is given by

$$\prod_{i=1}^{n} \frac{1 - q^{m+2i-1}}{1 - q^{2i-1}} \prod_{1 \leq i < j \leq n} \frac{1 - q^{2(m+i+j-1)}}{1 - q^{2(i+j-1)}}.$$

Our proof parallels that of Ian Macdonald [3] which divides into three distinct pieces. We shall concentrate on the middle piece which is the most difficult and the heart of his argument. Macdonald derived it as a corollary of a formula for Hall-Littlewood polynomials. Details of the proof of Macdonald's formula as well as a generalization may be found in [2]. We shall prove the middle piece directly by induction on the number of variables.

The first piece of Macdonald's proof is the observation, known before Macdonald, that there is a one-to-one correspondence, preserving the number of lattice points, between bounded symmetric plane partitions and **column-strict plane partitions** with y coordinates bounded by m, z coordinates bounded by $2n - 1$, and in which and non-empty columns have odd height. The column at position (i, j) is the set of $(i, j, k) \in \mathcal{P}$, and the column height is the cardinality of this set. To say that the partition is column-strict means that if $1 \leq h < i$ and the column at (h, j) is non-empty, then the column height at (h, j) must be strictly greater than the column height at (i, j).

From this observation and the definition of the Schur function, s_λ, as a sum over semistandard tableaux of shape λ, it follows that the generating function for bounded symmetric plane partitions is given by

$$\sum_{\lambda\subseteq\{m^n\}} s_\lambda(q^{2n-1}, q^{2n-3}, \ldots, q),$$

where the sum is over all partitions, λ, into at most n parts each of which is less than or equal to m.

The second piece of Macdonald's proof is the following theorem which is the result that we shall prove in this paper.

Theorem *For arbitrary positive integers m and n,*

$$\sum_{\lambda\subseteq\{m^n\}} s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{j-1} - x_i^{m+2n-j}\right)}{\det\left(x_i^{j-1} - x_i^{2n-j}\right)}. \tag{1}$$

The final piece of Macdonald's proof is to rewrite the right side of Eq. (1) when $x_i = q^{2(n-i)+1}$, $1 \le i \le n$, as a ratio of products by employing the Weyl denominator formula for the root system B_n:

$$\det\left(x_i^{j-1} - x_i^{2n-j}\right) = \prod_{i=1}^{n}(1-x_i) \prod_{1\le i<j\le n}(x_i - x_j)(x_i x_j - 1). \tag{2}$$

There is a very simple inductive proof of this case of the Weyl denominator formula. Let $D_n(x_1, \ldots, x_n) = \det(x_j^{i-1} - x_j^{2n-i})$. This is a polynomial of degree $2n-1$ in x_1 with roots at $1, x_2, \ldots, x_n, x_2^{-1}, \ldots, x_n^{-1}$. The coefficient of x_1^{2n-1} is $-x_2 \cdots x_n D_{n-1}(x_2, \ldots, x_n)$.

Before we begin the proof of the theorem, we note that it similarly implies Gordon's identity ([3], p. 86):

$$\sum_{\lambda\subseteq\{m^n\}} s_\lambda(q^n, q^{n-1}, \ldots, q) = \prod_{1\le i\le j\le n} \frac{1-q^{m+i+j-1}}{1-q^{i+j-1}}.$$

Proof of the Theorem

We shall need the following lemma.

Lemma

$$x_1\cdots x_n \sum_{k=1}^{n}(-1)^{k-1}(1-x_k)x_k^{-1}\prod_{i\ne k}(1-x_i x_k) \prod_{\substack{1\le i<j\le n\\ i,j\ne k}}(x_j - x_i)$$

$$= (1 - x_1\cdots x_n)\prod_{1\le i<j\le n}(x_j - x_i). \tag{3}$$

Proof: We verify that this lemma is correct for $n = 2$ or 3 and proceed by induction. The left side of Eq. (3) is an anti-symmetric polynomial. If we divide it by $\prod_{1\le i<j\le n}(x_j - x_i)$, we obtain a symmetric polynomial. Let us denote this ratio by

$$F(x_1,\ldots,x_n) = x_1\cdots x_n \sum_{k=1}^{n}(1-x_k)x_k^{-1}\prod_{i\neq k}\frac{1-x_i\,x_k}{x_i - x_k}.$$

As a function of x_1, F is a polynomial of degree at most n divided by a polynomial of degree $n-1$, and is therefore a linear polynomial in x_1. It is easily verified that

$$\begin{aligned} F(0,x_2,\ldots,x_n) &= 1,\\ F(1,x_2,\ldots,x_n) &= F(x_2,\ldots,x_n)\\ &= 1 - x_2x_3\cdots x_n. \end{aligned}$$ □

We use Eq. (2) to rewrite the right-hand side of the theorem as

$$\frac{\det\left(x_i^{j-1} - x_i^{m+2n-j}\right)}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_i - x_j)(x_i\,x_j - 1)}.$$

We shall also use the representation of the Schur function as a ratio of determinants:

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-i}\right)}{\prod_{1\le i<j\le n}(x_i - x_j)}.$$

Combining these, our theorem can be restated as

$$\det\left(x_i^{j-1} - x_i^{m+2n-j}\right) = \sum_{\lambda\subseteq\{m^n\}}\det\left(x_i^{\lambda_j+n-j}\right)\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_i\,x_j - 1). \tag{4}$$

When we expand these determinants, we see that the theorem to be proved is equivalent to

$$\begin{aligned} &\sum_{\sigma,S}(-1)^{\mathcal{I}(\sigma)+|S|}\prod_{i\in S}x_i^{m+2n-\sigma(i)}\prod_{i\notin S}x_i^{\sigma(i)-1}\\ &\quad= \sum_{\lambda,\sigma}(-1)^{\mathcal{I}(\sigma)}\prod_{i=1}^{n}x_i^{\lambda_{\sigma(i)}+n-\sigma(i)}\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_i\,x_j - 1), \end{aligned} \tag{5}$$

where $\mathcal{I}(\sigma)$ is the inversion number. The first sum is over all permutations, σ, and subsets, S, of $\{1,\ldots,n\}$. The second sum is over partitions $\lambda\subseteq\{m^n\}$ and permutations.

Our proof will be by induction on n. It is easy to check that this equation is correct for $n=1$ or 2. Let RHS denote the right-hand side of Eq. (5). We shall sum over all possible values of λ_n and $k = \sigma^{-1}(n)$. Given λ_n and k, we subtract λ_n from each part in λ to get

$\lambda' \subseteq \{(m-\lambda_n)^{n-1}\}$. The permutation σ is uniquely determined by k and a one-to-one mapping $\sigma' : \{1, \dots, n\} \backslash \{k\} \to \{1, \dots, n-1\}$. We can express the right-hand side of Eq. (5) as:

$$\begin{aligned}\text{RHS} = &\sum_{\lambda_n=0}^{m} \sum_{k=1}^{n} (-1)^{n+k}(1-x_k)x_k^{-1}(x_1 \cdots x_n)^{\lambda_n+1} \prod_{i \neq k}(x_i\, x_k - 1) \\ &\times \sum_{\lambda',\sigma'} (-1)^{\mathcal{I}(\sigma')} \prod_{i\neq k} x_i^{\lambda'_{\sigma'(i)}+(n-1)-\sigma'(i)} \prod_{\substack{i=1\\ i\neq k}}^{n} (1-x_i) \prod_{\substack{1\leq i<j\leq n\\ i,j\neq k}} (x_i\, x_j - 1).\end{aligned}$$

We apply the induction hypothesis to the inner sum and then sum over λ_n:

$$\begin{aligned}\text{RHS} = &\sum_{k=1}^{n} (-1)^{n+k}(1-x_k) \prod_{i\neq k}(x_i\, x_k - 1) \\ &\times \sum_{\sigma,S} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i\in S} x_i^{m+1+2n-2-\sigma(i)} \prod_{i\in \bar{S}} x_i^{\sigma(i)} \frac{1 - x_k^{m+1}\prod_{i\in\bar{S}} x_i^{m+1}}{1 - x_k \prod_{i\in\bar{S}} x_i},\end{aligned}$$

where the inner sum is over all one-to-one mappings σ from $\{1, \dots, n\}\backslash\{k\} \to \{1, \dots, n-1\}$ and subsets S of $\{1, \dots, n\}\backslash\{k\}$. We use $\bar{S}$ to denote the complement of S in $\{1, \dots, n\}\backslash\{k\}$.

It is convenient at this point to replace x_i^{m+1} by $t_i\, x_i^{2-2n}$ on each side of the equation to be proved. Our theorem is now seen to be equivalent to

$$\begin{aligned}&\sum_{\sigma,S} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i\in S} t_i x_i^{1-\sigma(i)} \prod_{i\notin S} x_i^{\sigma(i)-1} \\ &\quad = \sum_{k=1}^{n} (-1)^{n+k}(1-x_k) \prod_{i\neq k}(x_i\, x_k - 1) \\ &\quad\quad \times \sum_{\sigma,S} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i\in S} t_i\, x_i^{-\sigma(i)} \prod_{i\in\bar{S}} x_i^{\sigma(i)} \frac{1 - \prod_{i\notin S} t_i\, x_i^{2-2n}}{1 - \prod_{i\notin S} x_i}.\end{aligned} \tag{6}$$

The sum on σ on the right-hand side is a Vandermonde determinant in $n-1$ variables. We replace it with the appropriate product and then interchange the summation on S, which must be a proper subset of $\{1, \dots, n\}$, and k, which cannot be an element of S:

$$\begin{aligned}\text{RHS} = &\sum_{S\subset\{1,\dots,n\}} (-1)^{|S|} \prod_{i\in S} t_i x_i^{-1} \prod_{i\notin S} x_i \left(\frac{1 - \prod_{i\notin S} t_i\, x_i^{2-2n}}{1 - \prod_{i\notin S} x_i} \right) \\ &\times \sum_{k\notin S} (-1)^{n+k}(1-x_k)x_k^{-1} \prod_{i\neq k}(x_i x_k - 1) \prod_{\substack{i<j\\ i,j\neq k}} \left(x_j^{\epsilon_j} - x_i^{\epsilon_i}\right),\end{aligned}$$

where $\epsilon_i = -1$ if $i \in S$, $= +1$ if $i \notin S$. We rewrite

$$(-1)^{n+k} \prod_{i \neq k} (x_i\, x_k - 1) = \prod_{i<k} (x_i\, x_k - 1) \prod_{i>k} (1 - x_i\, x_k)$$
$$= \prod_{\substack{i<k \\ i \notin S}} (x_i\, x_k - 1) \prod_{\substack{i>k \\ i \notin S}} (1 - x_i\, x_k)$$
$$\times \prod_{i \in S} x_i \prod_{\substack{i<k \\ i \in S}} \left(x_k - x_i^{-1}\right) \prod_{\substack{i>k \\ i \in S}} \left(x_i^{-1} - x_k\right),$$

and then factor all terms that involve x_i, $i \in S$, out of the sum on k. The sum on $k \notin S$ can now be evaluated using the lemma:

$$\text{RHS} = \sum_{S \subset \{1,\ldots,n\}} (-1)^{|S|} \prod_{i \in S} t_i \left(1 - \prod_{i \notin S} t_i\, x_i^{2-2n}\right) \prod_{1 \le i < j \le n} \left(x_j^{\epsilon_j} - x_i^{\epsilon_i}\right)$$
$$= \sum_{S,\sigma} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i \in S} t_i x_i^{1-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1}$$
$$- t_1 \cdots t_n \sum_{S,\sigma} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i \in S} x_i^{1-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)+1-2n},$$

where both sums are over all *proper* subsets S of $\{1, \ldots, n\}$. Equation (6)—which we have seen is equivalent to the theorem—now follows from the observation that when we sum over all subsets S of $\{1, \ldots, n\}$,

$$\sum_{S,\sigma} (-1)^{\mathcal{I}(\sigma)+|S|} \prod_{i \in S} x_i^{1-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)+1-2n} = \det\left(x_i^{j+1-2n} - x_i^{1-j}\right) = 0.$$

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