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Association Schemes with Multiple Q-polynomial Structures*

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Abstract. It is well known that an association scheme $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ with $k_1 > 2$ has at most two P-polynomial structures. The parametrical condition for an association scheme to have two P-polynomial structures is also known. In this paper, we give a similar result for Q-polynomial association schemes. In fact, if $d > 5$, then we obtain exactly the same parametrical conditions for the dual intersection numbers or Krein parameters.

Keywords: Q-polynomial, association scheme, multiple Q-polynomial structure, Krein parameter, distance-regular graph, integrality of eigenvalue

1. Introduction

A *d-class symmetric association scheme* is a pair $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$, where X is a finite set, each R_i is a nonempty subset of $X \times X$ for $i = 0, 1, \ldots, d$ satisfying the following.

(i) $R_0 = \{(x, x)|x \in X\}$.

(ii) $\{R_i\}_{0\leq i\leq d}$ is a partition of $X \times X$, i.e.,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d,\ R_i \cap R_j = \emptyset \text{ if } i \neq j.$$

(iii) ${}^tR_i = R_i$ for $i = 0, 1, \ldots, d$, where ${}^tR_i = \{(y, x)|(x, y) \in R_i\}$.

(iv) There exist integers $p^h_{i,j}$ such that for all $x, y \in X$ with $(x, y) \in R_h$,

$$p^h_{i,j} = |\{z \in X|(x, z) \in R_i,\ (z, y) \in R_j\}|.$$

We refer to X as the *vertex set* of $\mathcal{X}$, and to the integers $p^h_{i,j}$ as the *intersection numbers* of $\mathcal{X}$.

Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a symmetric association scheme. Let $\mathrm{Mat}_X(\boldsymbol{R})$ denote the algebra of matrices over the reals $\boldsymbol{R}$ with rows and columns indexed by X. The *i-th adjacency matrix* $A_i \in \mathrm{Mat}_X(\boldsymbol{R})$ of $\mathcal{X}$ is defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

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From $(i) - (iv)$ above, it is easy to see the following.

$(i)'$ $A_0 = I$.

$(ii)'$ $A_0 + A_1 + \cdots + A_d = J$, where J is the all-1s matrix, and $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq d$, where $\circ$ denotes the entry-wise matrix product.

$(iii)'$ ${}^t A_i = A_i$ for $0 \leq i \leq d$.

$(iv)'$ $A_i A_j = \displaystyle\sum_{h=0}^{d} p_{i,j}^h A_h$ for $0 \leq i, j \leq d$.

By the *Bose-Mesner algebra* of $\mathcal{X}$ we mean the subalgebra $\mathcal{M}$ of $\mathrm{Mat}_X(\boldsymbol{R})$ generated by the adjacency matrices $A_0, A_1, \ldots, A_d$. Observe by $(iv)'$ above that the adjacency matrices form a basis for $\mathcal{M}$. Moreover, $\mathcal{M}$ consists of symmetric matrices and it is closed under $\circ$. In particular, $\mathcal{M}$ is commutative in both multiplications.

Since the algebra $\mathcal{M}$ consists of commutative symmetric matrices, there is a second basis $E_0, E_1, \ldots, E_d$ satisfying the following.

$(i)''$ $E_0 = \dfrac{1}{|X|} J$.

$(ii)''$ $E_0 + E_1 + \cdots + E_d = I$, and $E_i E_j = \delta_{i,j} E_i$ for $0 \leq i, j \leq d$.

$(iii)''$ ${}^t E_i = E_i$ for $0 \leq i \leq d$.

$(iv)''$ $E_i \circ E_j = \dfrac{1}{|X|} \displaystyle\sum_{h=0}^{d} q_{i,j}^h E_h$, $(0 \leq i, j \leq d)$ for some real numbers $q_{i,j}^h$.

$E_0, E_1, \ldots, E_d$ are the primitive idempotents of the Bose-Mesner algebra. The parameters $q_{i,j}^h$ are called *Krein parameters* or *dual intersection numbers*.

Conventionally, we assume $p_{i,j}^h$ and $q_{i,j}^h$ are zero if one of the indices h, i, j is out of range $\{0, 1, \ldots, d\}$ otherwise mentioned clearly.

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ with respect to the ordering $R_0, R_1, \ldots, R_d$ of the relations is called a *P-polynomial association scheme* if the following conditions are satisfied.

$(P1)$ $p_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two.

$(P2)$ $p_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two for $0 \leq h, i, j \leq d$.

In this case we write $c_i = p_{i-1,1}^i$, $a_i = p_{i,1}^i$, $b_i = p_{i+1,1}^i$ and $k_i = p_{i,i}^0$ for $i = 0, 1, \ldots, d$.

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of the Bose-Mesner algebra is called a *Q-polynomial association scheme* if the following conditions are satisfied.

$(Q1)$ $q_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two.

$(Q2)$ $q_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two for $0 \leq h, i, j \leq d$.

In this case we write $c_i^* = q_{i-1,1}^i$, $a_i^* = q_{i,1}^i$, $b_i^* = q_{i+1,1}^i$ and $k_i^* = q_{i,i}^0$ for $i = 0, 1, \ldots, d$.

If $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is a P-polynomial association scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, then the graph $\Gamma = (X, R_1)$ with vertex set X, edge set defined by R_1 becomes a distance-regular graph. In this case,

$$R_i = \{(x, y) \in X \times X | \partial(x, y) = i\},$$

where $\partial(x, y)$ denotes the distance between x and y. Conversely, every distance-regular graph is obtained in this way.

Q-polynomial association schemes appear in design theory in connection with tight conditions, but it is not much studied compared with P-polynomial association schemes, though there are extensive studies of P- and Q-polynomial association schemes.

Recently the author studied imprimitive Q-polynomial association schemes and showed in [9] that if $d > 6$ and $k_1^* > 2$, then imprimitive Q-polynomial association schemes are either dual bipartite or dual antipodal, i.e., dual intersection numbers satisfy either $a_i^* = 0$ for all i, or $b_i^* = c_{d-i}^*$ for all $i \neq [d/2]$. This is a continuation of the study of Q-polynomial association schemes.

As is well known, the Bose-Mesner algebra of a symmetric association scheme becomes a so-called C-algebra and satisfies Kawada-Delsarte duality, and by this duality Q-polynomial association schemes correspond to P-polynomial association schemes in ‘algebraic level’. On the other hand, the combinatorial properties of association schemes can be easily seen as those of distance-regular graphs for P-polynomial association schemes but the Q-polynomial property is not well understood. See [10, 11].

Some of the properties of P-polynomial association schemes are expected to hold in Q-polynomial association schemes as dual. But it is also true that some of the properties such as the unimodal property of ith valencies k_i’s do not hold for k_i^*’s in Q-polynomial association schemes. Until recently, there was no break through to replace the parametrical conditions obtained by combinatorial argument in distance-regular graphs by something in Q-polynomial association schemes.

Recently, Garth A. Dickie proved the following:

> Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme. Then for each i with $0 < i < d$, $q_{1,i}^i = 0$ implies that $q_{1,1}^1 = 0$.

The corresponding result for P-polynomial association scheme is easily shown by a simple combinatorial argument. Dickie substituted that part by matrix identities in [6], which is a part of [5, Chapter 4]. In [9], the author generalized Dickie’s result and obtained Proposition 2 and Corollary 1, which played the key roles in the proof of the main theorem in it.

In this paper, we prepare another identity using matrix identities to treat the problem to determine association schemes with multiple Q-polynomial structures.

The following is our main result in this paper.

Theorem 1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ with $k_1^* > 2$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents.*

(1) *Suppose $\mathcal{X}$ is Q-polynomial with respect to another ordering. Then the new ordering is one of the following:*

(I) $E_0, E_2, E_4, E_6, \ldots, E_5, E_3, E_1$,

(II) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \ldots$,

(III) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \ldots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$,

(IV) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \ldots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$, *or*

(V) $d = 5$ *and* $E_0, E_5, E_3, E_2, E_4, E_1$.

(2) *$\mathcal{X}$ has at most two Q-polynomial structures.*

It is well known that Q-polynomial association schemes with $k_1^* = 2$ are the association schemes attached to ordinary n-gons as distance-regular graphs. We also give parametrical conditions in each of the cases in the theorem above. See Theorem 2. Association schemes with multiple P-polynomial structures were studied by Eiichi Bannai and Etsuko Bannai in [1], see also [2, 3, 7]. On the other hand, the corresponding problem for Q-polynomial association schemes was raised by Eiichi Bannai and Tatsuro Ito in [2, Sections III.4, III.7] in connection with the integrality condition of the eigenvalues of P- and Q-polynomial association schemes. In his thesis [5], Garth A. Dickie classified P-polynomial association schemes with multiple Q-polynomial structures. Our result in this paper is a generalization of a part of his result and actually it can substitute a part of his proof. It is worth noting that Dickie's proof uses the additional condition P-polynomial property fully. He proves first that the association schemes in question is thin in Terwilliger's terminology. This part can be seen without difficulty as a corollary of our result.

The author believes that P-polynomial association schemes and Q-polynomial association schemes share many more properties which cannot be seen at C-algebra level. That means we may be able to expect higher duality between these types of schemes. On the other hand, each of these classes of association schemes should be studied separately to understand their peculiarity. Just as the graph theoretical arguments developed in distance-regular graphs have successfully applied in the study of association schemes, the representation theory in Q-polynomial association schemes should shed light from different direction.

2. Basic Properties of Q-polynomial Schemes

In this section, we collect the properties of Krein parameters $q_{i,j}^h$ which are derived algebraically from the conditions of P-polynomial C-algebra with nonnegative structure constants. See the definitions and the proofs in [9].

Lemma 1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Let $k_i^* = q_{i,i}^0$. Then the following hold.*

(1) $q_{i+1,j}^h c_{i+1}^* = q_{i,j-1}^h b_{j-1}^* + q_{i,j}^h (a_j^* - a_i^*) + q_{i,j+1}^h c_{j+1}^* - q_{i-1,j}^h b_{i-1}^*$.

(2) $k_h^* q_{i,j}^h = k_i^* q_{j,h}^i$ *and* $k_i^* > 0$ *for* $i = 0, 1, \ldots, d$. *In particular,* $q_{i,j}^h \neq 0$ *if and only if* $q_{j,h}^i \neq 0$.

(3) $q^{i+h}_{i,h+1}c^*_{h+1} = q^{i+h}_{i,h}(a^*_i + \cdots + a^*_{i+h} - a^*_1 - \cdots - a^*_h)$.

Well known Krein condition asserts that Krein parameters $q^h_{i,j}$ are all nonnegative, and we can derive some more properties of them using this condition.

Lemma 2 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Then $q^h_{i,j} \ge 0$ for all $0 \le h, i, j \le d$ and the following hold.*

(1) *If $q^h_{i+1,j-1} = q^h_{i+1,j} = q^h_{i+1,j+1} = 0$ for $0 \le i < d$, then $q^h_{i,j} = q^h_{i+2,j} = 0$.*

(2) *If $q^h_{l,j-l+i} = q^h_{l,j-l+i+1} = \cdots = q^h_{l,j+l-i} = 0$ for $i \le l$ and $0 \le i < d$, then $q^h_{i,j} = q^h_{2l-i,j} = 0$.*

(3) *For all i, j with $0 \le i, h, i+h \le d$, $a^*_i = a^*_{i+1} = \cdots = a^*_{i+h} = 0$ implies $a^*_1 = \cdots = a^*_h = 0$.*

(4) *For all h and i with $0 \le h, i, i+h \le d$, the following hold.*

   (i) *If $q^h_{i,i+h-1} = 0$, then $a^*_i \le a^*_{i+h}$. Moreover if $a^*_i = a^*_{i+h}$, then $q^h_{i+1,i+h} = 0$.*

   (ii) *If $q^h_{i+1,i+h} = 0$, then $a^*_i \ge a^*_{i+h}$. Moreover if $a^*_i = a^*_{i+h}$, then $q^h_{i,i+h-1} = 0$.*

   (iii) *If $q^h_{i,i+h-1} = q^h_{i+1,i+h} = 0$, then $a^*_i = a^*_{i+h}$.*

(5) *For all h and i with $0 \le i \le h \le d$, the following hold.*

   (i) *If $q^h_{i,h-i+1} = 0$, then $a^*_i \le a^*_{h-i}$. Moreover if $a^*_i = a^*_{h-i}$, then $q^h_{i+1,h-i} = 0$.*

   (ii) *If $q^h_{i+1,h-i} = 0$, then $a^*_i \ge a^*_{h-i}$. Moreover if $a^*_i = a^*_{h-i}$, then $q^h_{i,h-i+1} = 0$.*

   (iii) *If $q^h_{i,h-i+1} = q^h_{i+1,h-i} = 0$, then $a^*_i = a^*_{h-i}$.*

3. New Conditions on Krein Parameters

Only a few restrictions of the Krein parameters $q^h_{i,j}$ of symmetric association schemes are known except those derived algebraically using Lemma 1 or Krein conditions in Lemma 2. We list other restrictions on Krein parameters. The first one is shown in [4]. See also [2, Theorem 2.3.8, Proposition 2.8.3]. This is the key to connect the conditions on Krein parameters with representations or matrix identities. Actually, all the rest follow from this identity.

Proposition 1 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. Let $E_0, E_1, \ldots, E_d$ be primitive idempotents and let $q^h_{i,j}$ be the Krein parameters. Then for $0 \le h, i, j \le d$, we have*

$$q^h_{i,j} = 0 \Leftrightarrow \sum_{u\in X} (E_h)_{ux}(E_i)_{uy}(E_j)_{uz} = 0 \text{ for all } x, y, z \in X.$$

The following three results are proved in [9]. Proposition 2 is shown by Lemma 3 and Corollary 1 is a direct consequence of Proposition 2.

Lemma 3 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a symmetric association scheme. Let $E_0, E_1, \ldots, E_d$ be primitive idempotents and let $q^h_{i,j}$ be the Krein parameters. Suppose $\{i \mid q^i_{j,k} q^i_{l,m} \neq 0\} \subset \{h\}$. Then for all integers $0 \le h, i, j, k, l, m \le d$ and the vertices a, a', b, b', the following hold.*

(1) $$\sum_{e \in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} = \frac{q^h_{l,m}}{|X|} \sum_{e \in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}.$$

(2) $$\sum_{e \in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} = \sum_{e, e' \in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}.$$

Proposition 2 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Suppose that*

$$\{l \mid q^l_{j,h+i} q^l_{i-j,h+j} \neq 0\} \subset \{h + i - j\}.$$

Then for $h \ge 0$, $i \ge j \ge 1$ with $h + i + j \le d$, $q^{h+i}_{i,h+j} = 0$ implies that $q^{h+j}_{j,h+j} = 0$.

Corollary 1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents.*

(1) *For $h \ge 0$, $i \ge 1$ with $h + i + 1 \le d$,*

$$q^{h+i}_{i,h+1} = q^{h+i}_{1,h+i} = 0 \text{ implies that } q^{h+1}_{1,h+1} = 0.$$

(2) *For $h \ge 0$, $i \ge 2$ with $h + i + 2 \le d$,*

$$q^{h+i}_{i,h+2} = q^{h+i}_{2,h+i} = q^{h+i}_{2,h+i-1} = 0 \text{ implies that } q^{h+2}_{2,h+2} = 0.$$

By setting $h = 0$ in Corollary 1, we have the main result in [6], i.e., $a^*_i = 0$ implies $a^*_1 = 0$ for all $1 \le i \le d - 1$.

We give another application of the matrix identities, which gives a basic tool to handle Krein parameters of association schemes.

Proposition 3 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a symmetric association scheme. Suppose the following.*

$$(1)\ \{t \mid q^t_{i,k} q^t_{h,m} \neq 0\} \subset \{l\}, \quad (2)\ \{t \mid q^t_{j,k} q^t_{h,l} \neq 0\} \subset \{m\}.$$

Then $q^h_{i,j} \neq 0$ implies that $q^i_{k,l} = q^j_{k,m}$.

Proof: Let $X(h, i, j, k, l, m)$ be the following sum.

$$\sum_{w,x,y,z \in X} (E_h)_{w,x}(E_i)_{w,y}(E_j)_{w,z}(E_k)_{y,z}(E_l)_{x,y}(E_m)_{x,z}.$$

We evaluate $X(h,i,j,k,l,m)$ using Lemma 3 under our assumption.

Rearranging first the order of the product and apply Lemma 3 (2) by our assumption (1), we have the following.

$$\begin{aligned}
&X(h,i,j,k,l,m)\\
&= \sum_{w,z\in X}(E_j)_{w,z}\sum_{x,y\in X}(E_i)_{y,w}(E_k)_{y,z}(E_l)_{y,x}(E_h)_{x,w}(E_m)_{x,z}\\
&= \sum_{w,z\in X}(E_j)_{w,z}\sum_{x\in X}(E_i)_{x,w}(E_k)_{x,z}(E_h)_{x,w}(E_m)_{x,z}\\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}\sum_{z\in X}(E_k)_{x,z}(E_m)_{x,z}(E_j)_{w,z}\\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}\sum_{z\in X}(E_k\circ E_m)_{x,z}(E_j)_{z,w}\\
&= \sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}((E_k\circ E_m)E_j)_{x,w}\\
&= \frac{q^j_{k,m}}{|X|}\sum_{w,x\in X}(E_i)_{x,w}(E_h)_{x,w}(E_j)_{x,w}\\
&= \frac{q^j_{k,m}}{|X|}\sum_{w\in X}\sum_{x\in X}(E_i\circ E_j)_{w,x}(E_h)_{x,w}\\
&= \frac{q^j_{k,m}}{|X|}\sum_{w\in X}((E_i\circ E_j)E_h)_{w,w}\\
&= \frac{q^j_{k,m}}{|X|}\frac{q^h_{i,j}}{|X|}\sum_{w\in X}(E_h)_{w,w}\\
&= \frac{q^j_{k,m}q^h_{i,j}}{|X|^2}k^*_h.
\end{aligned}$$

Now by symmetry we swap i with j, and l with m to obtain the following.

$$X(h,i,j,k,l,m)=\frac{q^i_{k,l}q^h_{i,j}}{|X|^2}k^*_h.$$

Therefore if $q^h_{i,j}\neq 0$, we have $q^i_{k,l}=q^j_{k,m}$ as desired. $\square$

Corollary 2 *Let $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0,E_1,\ldots,E_d$ of primitive idempotents. If $q^h_{i,j}\neq 0$, then the following hold.*

(1) *If $q^h_{i-1,j}=q^h_{i-1,j-1}=q^h_{i,j+1}=q^h_{i+1,j+1}=0$, then $c^*_i=b^*_j$.*

(2) *If $q^h_{i-1,j}=q^h_{i-1,j+1}=q^h_{i,j-1}=q^h_{i+1,j-1}=0$, then $c^*_i=c^*_j$.*

(3) *If* $q^h_{i-1,j+1} = q^h_{i,j+1} = q^h_{i+1,j-1} = q^h_{i+1,j} = 0$, *then* $b^*_i = b^*_j$.

Proof: It is easy to check each of the following.

(1) $\{t \mid q^t_{i,1}q^t_{h,j+1} \neq 0\} \subset \{i-1\}$, and $\{t \mid q^t_{j,1}q^t_{h,i-1} \neq 0\} \subset \{j+1\}$.

(2) $\{t \mid q^t_{i,1}q^t_{h,j-1} \neq 0\} \subset \{i-1\}$, and $\{t \mid q^t_{j,1}q^t_{h,i-1} \neq 0\} \subset \{j-1\}$.

(3) $\{t \mid q^t_{i,1}q^t_{h,j+1} \neq 0\} \subset \{i+1\}$, and $\{t \mid q^t_{j,1}q^t_{h,i+1} \neq 0\} \subset \{j+1\}$.

Hence we have the assertions as direct consequences of the previous proposition. □

Corollary 3 *Let* $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ *be a* Q*-polynomial association scheme with respect to the ordering* $E_0, E_1, \ldots, E_d$ *of primitive idempotents. If* $k^*_1 > 2$, *then the following hold.*

(1) *If* $q^h_{2,h} = q^h_{1,h} = q^h_{2,h-1} = 0$ *with* $2 \le h \le d$, *then* $h = d$.

(2) *If* $q^h_{2,h} = q^h_{1,h} = q^h_{2,h+1} = 0$ *with* $2 \le h \le d$, *then* $h = d$.

Proof: (1) By Corollary 2, $b^*_1 = c^*_{h-1}$, $c^*_1 = c^*_{h+1}$. Suppose $h < d$. Then $q^2_{h+1,h-1} \neq 0$ and $q^2_{h,h-1} = q^2_{h,h} = 0$ by our assumption. Hence by Corollary 2, $c^*_{h+1} = c^*_{h-1}$. Thus $b^*_1 = c^*_1 = 1$. This implies $k^*_1 = 2$, because $a^*_h = q^h_{1,h} = 0$ implies $a^*_1 = 0$ by Corollary 1. This is not the case. Therefore $h = d$.

(2) In this case, we have $b^*_1 = b^*_{h+1}$ and $c^*_1 = b^*_{h-1}$. Similarly, if $h < d$, then we have $b^*_{h+1} = b^*_{h-1}$ and $k^*_1 = 2$. This is a contradiction. □

4. Multiple Q-polynomial Structures

In this section and the next, we prove the following result. It is obvious that Theorem 1 is a direct consequence of it.

Theorem 2 *Let* $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ *with* $k^*_1 > 2$ *be a* Q*-polynomial association scheme with respect to the ordering* $E_0, E_1, \ldots, E_d$ *of the primitive idempotents.*

(1) *Suppose* $\mathcal{X}$ *is* Q*-polynomial with respect to another ordering. Then the new ordering is one of the following:*

  (I) $E_0, E_2, E_4, E_6, \ldots, E_5, E_3, E_1$,

  (II) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \ldots$,

  (III) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \ldots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$,

  (IV) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \ldots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$, *or*

  (V) $d = 5$ *and* $E_0, E_5, E_3, E_2, E_4, E_1$.

(2) *Let* $q^h_{i,j}$ *be the Krein parameters with respect to the original ordering. Suppose* $d \ge 3$. *Then,*

(I) *holds if and only if* $q^1_{1,1} = \cdots = q^{d-1}_{1,d-1} = 0 \neq q^d_{1,d}$.

(II) *holds if and only if* $q^d_{1,d} \neq 0 = q^d_{2,d} = \cdots = q^d_{d,d}$.

(III) *holds if and only if one of the following holds:*

(i) $d = 3$, *and* $q^3_{1,3} = 0 \neq q^3_{2,3}$,

(ii) $d = 4$, $q^4_{1,4} = q^4_{3,4} = 0$, *and* $q^4_{2,4} \neq 0 \neq q^4_{2,3}$, *or*

(iii) $d \geq 5$, $q^d_{2,d} \neq 0 = q^d_{1,d} = q^d_{3,d} = \cdots = q^d_{d,d}$. *Moreover if* $d = 2e - 1$, *then* $q^j_{1,j} \neq 0$ *implies* $j = e$ *and if* $d = 2e$, *then* $q^j_{1,j} \neq 0$ *if and only if* $j = e,\ e+1$.

(IV) *holds if and only if one of the following holds:*

(i) $d = 3$, $q^2_{1,2} \neq 0 = q^2_{3,2}$, *or*

(ii) $d \geq 4$, $q^{d-1}_{2,d} = \cdots = q^{d-1}_{d,d} = 0$. *Moreover, if* $d = 2e$, *then* $q^j_{1,j} \neq 0$ *implies* $j = e$ *and if* $d = 2e + 1$, *then* $q^j_{1,j} \neq 0$ *if and only if* $j = e,\ e+1$.

(V) *holds if and only if* $q^5_{1,5} = q^5_{2,5} = q^5_{4,5} = q^5_{5,5} = 0 \neq q^5_{3,5}$ *and* $q^5_{3,4} = 0$.

(3) *$\mathcal{X}$ has at most two Q-polynomial structures.*

Before we start the proof, we prepare some lemmas to illustrate the structures of Q-polynomial association schemes appeared in the theorem above.

Lemma 4 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. For $3 \leq m \leq d$, the following are equivalent.*

(1) $a^*_1 = a^*_2 = \cdots = a^*_{m-1} = 0 \neq a^*_m$.

(2) $q^2_{1,2} = q^2_{2,3} = \cdots = q^2_{m-2,m-1} = 0 \neq q^2_{m-1,m}$.

Proof: Suppose (1) holds. Since $m \geq 3$, $a^*_2 = q^2_{1,2} = 0$. Now by induction, we have $q^2_{i,i+1} = 0$ for $i = 1, 2, \ldots, m-2$ from Lemma 2 $(4)(i)$. Moreover $q^2_{m-1,m} \neq 0$ by Lemma 2 $(4)(iii)$.

Conversely, suppose (2). Then by Corollary 1 (1), $a^*_1 = 0$ as $d \geq 3$. Now we have (1) by Lemma 2 $(4)(i), (iii)$. □

Lemma 5 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Suppose*

$$q^d_{1,d} \neq 0 = q^d_{2,d} = \cdots = q^d_{d,d}.$$

Then the following hold.

(1) *For $0 \leq i, j \leq d$, $q^d_{i,j} \neq 0$ if and only if $0 \leq i + j - d \leq 1$.*

(2) *If, in addition, $q^1_{1,1} = \cdots = q^{d-1}_{1,d-1} = 0 \neq q^d_{1,d}$, then $k^*_1 = 2$.*

Proof: (1) This is a direct consequence of Lemma 2 (2).

(2) We first observe the following:

1. $q^d_{i,d-i} \neq 0$ for $i = 1, \ldots, d-1$, $\{t \mid q^t_{i,1} q^t_{d,d-i+1} \neq 0\} \subset \{i-1\}$ and $\{t \mid q^t_{d-i,1} q^t_{d,i-1} \neq 0\} \subset \{d-i+1\}$.
2. $q^d_{i+1,d-i} \neq 0$ for $i = 1, \ldots, d-1$, $\{t \mid q^t_{i+1,1} q^t_{d,d-i+1} \neq 0\} \subset \{i\}$ and $\{t \mid q^t_{d-i,1} q^t_{d,i} \neq 0\} \subset \{d-i+1\}$.

Now we apply Proposition 3. We obtain $c^*_i = b^*_{d-i}$ for $i = 1, \ldots, d-1$ from the first set of conditions, and $c^*_{i+1} = b^*_{d-i}$ for $i = 1, \ldots, d-1$ from the second set of conditions. Thus

$$1 = c^*_1 = b^*_{d-1} = c^*_2 = b^*_{d-2} = \cdots = c^*_{d-1} = b^*_1 = c^*_d.$$

Since $a^*_1 = 0$, we have $k^*_1 = c^*_1 + a^*_1 + b^*_1 = 2$ as desired. □

Lemma 6 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Suppose*

$$q^{d-1}_{2,d} = \cdots = q^{d-1}_{d,d} = 0.$$

Then the following hold.

(1) *For $0 \le i, j \le d$, $q^{d-1}_{i,j} \neq 0$ only if $0 \le i + j - (d-1) \le 2$. Moreover, if $i + j - (d-1) \in \{0, 2\}$, $q^{d-1}_{i,j} \neq 0$.*

(2) *Suppose $d \ge 3$. Then, for $0 \le i, j \le d$, $q^d_{i,j} \neq 0$ if and only if $i + j = d$.*

(3) *If $d \ge 3$, then $b^*_i = c^*_{d-i}$ for $i = 0, 1, \ldots, d$, i.e., $\mathcal{X}$ is dual antipodal 2-cover.*

Proof: (1) This is a direct consequence of Lemma 2 (2).

(2) By the assumption, $q^d_{2,d-1} = \cdots = q^d_{d,d-1} = 0$. Since $d \ge 3$, $q^d_{d-1,d-1} = q^d_{d-1,d} = 0$. Hence $q^d_{d,d} = 0$, by Lemma 2 (1). Now by induction on i in reverse order, $q^d_{i,d} = 0$ for $1 \le i \le d$, as $q^d_{i+1,d} = q^d_{i+1,d-1} = 0$.

(3) By (2), we can apply Corollary 2 (1) by setting $h = d$, $j = d - i$ for $i = 0, 1, \ldots, d$. □

From now on assume the following:

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents. Let $q^h_{i,j}$'s (and c^*_i's, a^*_i's, b^*_i's) be Krein parameters or dual intersection numbers with respect to this ordering of the primitive idempotents. Suppose $k^*_1 > 2$.

Let $\Delta = \Delta^{(h)}$ be a graph on the vertex set $V\Delta = \{0, 1, \ldots, d\}$ such that i is adjacent to j, or $i \sim j$, if and only if $q^h_{i,j} \neq 0$. Hence for this particular graph, we allow loops. Let $\partial(i,j)$ denote the distance between i and j in this graph and $\Delta_l(i) = \{j \mid \partial(i,j) = l\}$. Let $\Delta(i) = \Delta_1(i)$, and $\Delta^*(i) = \Delta(i) - \{i\}$.

It is easy to see that $\Delta^{(h)}$ is a path graph, if and only if $\mathcal{X}$ is Q-polynomial with respect to the ordering $E_0, E_h, \ldots$ of the primitive idempotents. Here by *a path graph* we mean a path which may include some loops.

We prove Theorem 2 in a series of lemmas. In Lemma 7, we check that if the parameters satisfy one of the conditions in (2) then $\mathcal{X}$ is Q-polynomial with respect to the corresponding ordering. In Lemma 8, we show (3) assuming that (1) and (2) are valid. In the next section we show the main part, i.e., if $\mathcal{X}$ has different Q-polynomial ordering of primitive idempotents then the ordering is the one listed in (1) and the parameters satisfy the conditions in (2).

Lemma 7 *If Krein parameters $q^h_{i,j}$'s satisfy the conditions of one of the cases in* Theorem 2 (2)*, then $\mathcal{X}$ is Q-polynomial with respect to the corresponding ordering.*

Proof: It suffices to show that the graph $\Delta = \Delta^{(h)}$ is a path graph, where $h = 2$ in (I), $h = d$ in (II), (III), $h = d-1$ in (IV), and $h = 5$ in (V).

Suppose the condition in (I) holds, i.e., $q^1_{1,1} = \cdots = q^{d-1}_{1,d-1} = 0 \neq q^d_{1,d}$. Then by Lemma 4, we have that

$$q^2_{1,2} = q^2_{2,3} = \cdots = q^2_{d-2,d-1} = 0 \neq q^2_{d-1,d}.$$

Since $q^2_{i,i+2} \neq 0$ for $i = 0, 1, \ldots, d-2$, we have $\Delta^*(i) = \{i-2, i+2\}$ for $i = 2, \ldots, d-2$, $\Delta^*(d-1) = \{d-3, d\}$ and $\Delta^*(d) = \{d-2, d-1\}$. Therefore, it is easy to check that $\Delta = \Delta^{(2)}$ is a path graph

$$0 \sim 2 \sim 4 \sim 6 \sim \cdots \sim 5 \sim 3 \sim 1.$$

Suppose the condition in (II) holds. Then by Lemma 5 (1) we have that $q^d_{i,j} \neq 0$ if and only if $0 \leq i + j - d \leq 1$. It is easy to check that Δ is a path graph

$$0 \sim d \sim 1 \sim d-1 \sim 2 \sim d-2 \sim 3 \sim d-3 \sim \cdots.$$

Suppose the condition in (III) holds. The assertions are easily checked for the cases $d < 5$ from the conditions in $(i), (ii)$. Suppose $d \geq 5$. Then by Lemma 2 (2), $q^d_{i,j} = 0$ if $i + j - d > 2$ and $q^d_{i,j} \neq 0$ if $i + j - d = 2$ for $0 \leq i, j \leq d$. The additional conditions on $q^j_{1,j}$'s imply the following by Lemma 2 $(5)(i)$ and (iii).

1. If $d = 2e - 1$, then $q^d_{i,d-i+1} \neq 0$ implies that $i = e$.

2. If $d = 2e$, then $q^d_{i,d-i+1} \neq 0$ if and only if $i \in \{e, e+1\}$.

Now it is easy to check that Δ is a path graph

$$0 \sim d \sim 2 \sim d-2 \sim 4 \sim d-4 \sim \cdots \sim d-5 \sim 5 \sim d-3 \sim 3 \sim d-1 \sim 1.$$

Suppose the condition in (IV) holds. The assertion is trivial if $d = 3$. Suppose $d \geq 4$. Then by Lemma 6 (1), $q^{d-1}_{i,j} = 0$ if $i + j > d + 1$ and $q^{d-1}_{i,j} \neq 0$ if $i + j = d + 1$ for $0 \leq i, j \leq d$. The additional conditions on $q^j_{1,j}$'s imply the following by Lemma 2 $(5)(i)$ and (iii).

1. If $d = 2e$, then $q^{d-1}_{i,d-i} \neq 0$ implies that $i = e$.
2. If $d = 2e+1$, then $q^{d-1}_{i,d-i} \neq 0$ if and only if $i \in \{e, e+1\}$.

Now it is easy to check that Δ is a path graph

$$0 \sim d-1 \sim 2 \sim d-3 \sim 4 \sim d-5 \sim \cdots \sim 5 \sim d-4 \sim 3 \sim d-2 \sim 1 \sim d.$$

Finally if the condition in (V) holds, then Δ is a path graph

$$0 \sim 5 \sim 3 \sim 2 \sim 4 \sim 1.$$

Note that $q^5_{2,4} \neq 0$ follows from Lemma 2 (1) as $q^5_{3,5} \neq 0$ while $q^5_{2,6} = q^5_{2,5} = 0$. □

Lemma 8 *$\mathcal{X}$ has at most two Q-polynomial structures.*

Proof: It suffices to show that no two of the sets of parametrical conditions of Theorem 2 (2) hold simultaneously. We may assume $d \geq 3$.

Suppose (I) holds. Then $a^*_1 = \cdots = a^*_{d-1} = 0 \neq a^*_d$. Since $a^*_d \neq 0$ and $a^*_2 = 0$, the only possible case is (II). Now by Lemma 5 (2), we have $k^*_1 = 2$, which contradicts our assumption.

Suppose (II) holds. Then $a^*_d \neq 0$ and $q^3_{2,2} \neq 0$ if $d = 3$ by Lemma 5 (1). Hence none of the other cases can occur.

Suppose (III) holds. If $d = 3$, then $q^3_{1,3} = 0 \neq q^3_{2,3}$. Hence if (IV) holds as well, $q^3_{2,2} = 0$. By Lemma 1 (1) with $h = 3, i = 3, j = 2$, we have $q^3_{3,2}a^*_2 + q^3_{3,3}c^*_3 = 0$. Thus $a^*_2 = q^2_{1,2} = 0$, which is not the case in (IV). If $d \geq 4$, then $q^d_{2,d} \neq 0$. Hence (V) cannot occur. (IV) does not occur either, as $\mathcal{X}$ is dual antipodal 2-cover by Lemma 6.

(IV) and (V) cannot occur simultaneously, as $q^5_{3,5} \neq 0$ in the case (V), while $q^5_{3,5} = 0$ in the case (IV) as it is dual antipodal 2-cover.

Therefore $\mathcal{X}$ has at most two Q-polynomial structures. □

5. Proof of Main Theorem

Suppose $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ has another Q-polynomial structure, i.e., $\mathcal{X}$ is Q-polynomial with respect to another ordering $E_0, E_{i_1}, E_{i_2}, \ldots, E_{i_d}$ of primitive idempotents. Just to simplify the notation, let $i_1 = h$, and $i_2 = i$. We may assume that $d > 2$ and $h > 1$. We determine the order $0, i_1, i_2, \ldots$ and the conditions of $q^h_{l,m}$'s in the following. Let $\Delta = \Delta^{(h)}$. By our assumption, Δ is a path graph. One of the keys is that for each l with $1 \leq l \leq d$, $|\Delta(l)| \leq 3$ or $|\Delta^*(l)| \leq 2$.

In Lemma 9, we show that the first member in the new ordering $h = 2$, $d-1$ or d. In Lemma 10, we treat the case when $h = 2$ and show that we have (I) or $(IV)(i)$ in the theorem. Lemma 11 is for the case when $h = d-1$ and we show that (IV) occurs. The case when $h = d$ requires a little more work. In Lemma 12, we determine the second member i in the new ordering to show $i = 1$ or 2 with an exception when we have (V). In Lemmas 13 and 14, we determine the case when $i = 1$ and 2 respectively by showing that we have either the case (II) or (III) respectively.

Lemma 9 *The first member $h = i_1$ of the new ordering is either 2, $d-1$ or d.*

Proof: By way of contradiction, assume that $3 \le h \le d-2$. In particular, $d \ge 5$.

We first claim that $q^h_{1,h} = q^h_{2,h} = 0$. Suppose $q^h_{l,h} \ne 0$ with $l = 1$ or 2. Since $3 \le h \le d-2$, $q^h_{l,h-l} \ne 0 \ne q^h_{l,h+l}$ by $(Q2)$. Hence $\Delta(l) \ni h, h-l, h+l$. Since $l \ne h$, the only possibility is that $h = 2l = 4$ or $h = 4$, $i = 2$, $i_3 = 6$ and $\Delta(4) \subset \{0, 2, 4\}$. Observe that $q^4_{4,8} = 0$ since $q^4_{4,8} \ne 0$ implies $8 \in \Delta(4)$, so either $d = 6$ or $d = 7$.

If $d = 7$, then we have $q^4_{3,7} \ne 0$ while $q^4_{4,6} = q^4_{4,7} = q^4_{4,8} = 0$, which contradicts Lemma 2 (1).

Suppose $d = 6$. Then $q^4_{2,6} \ne 0$. Hence by Lemma 2 (1), we have either $q^4_{3,5} \ne 0$ or $q^4_{3,6} \ne 0$. Since there is a path $3 \sim 1 \sim 5$ in Δ, which is guaranteed to exist by $(Q2)$, $3 \not\sim 5$ and $q^4_{3,6} \ne 0$. Hence the new ordering has to be $0 \sim 4 \sim 2 \sim 6 \sim 3 \sim 1 \sim 5$. This contradicts Lemma 2 (1), as $q^4_{3,6} \ne 0$, while $q^4_{4,5} = q^4_{4,6} = q^4_{4,7} = 0$. Thus we have the claim.

As $q^h_{1,h} = q^h_{2,h} = 0$, we have $q^h_{2,h-1} \ne 0 \ne q^h_{2,h+1}$ by Corollary 3. Therefore, we have $h-2, h-1, h+1, h+2 \in \Delta(2)$. This is impossible. □

Lemma 10 *Suppose $d \ge 3$ and $h = i_1 = 2$. Then one of the following holds.*

(i) *$d = 3$, the new ordering is $0, 2, 1, 3$ and $q^2_{1,2} \ne 0 = q^2_{2,3}$, i.e., the case $(IV)(i)$ holds.*

(ii) *The new ordering is $0, 2, 4, 6, \ldots, 5, 3, 1$ and $q^1_{1,1} = \cdots = q^{d-1}_{1,d-1} = 0 \ne q^d_{1,d}$, i.e., the case (I) holds.*

Proof: Suppose $q^2_{1,2} \ne 0$. Then $h = 2$, $i = i_2 = 1$, and $q^2_{2,3} = q^2_{2,4} = 0$ as $\Delta^*(2) = \{0, 1\}$. We have (i) in this case.

Suppose $q^2_{1,2} = 0$. By Corollary 1, $q^1_{1,1} = 0$ as $d \ge 3$. There exists an m such that

$$a^*_1 = a^*_2 = \cdots = a^*_{m-1} = 0 \ne a^*_m$$

since otherwise Δ is not connected. Then by Lemma 4, we have

$$q^2_{1,2} = q^2_{2,3} = \cdots = q^2_{m-2,m-1} = 0 \ne q^2_{m-1,m}.$$

If $m < d$, then $q^2_{m-1,m+1} \ne 0$. Hence $m-3, m, m+1 \in \Delta^*(m-1)$, which is a contradiction. Thus $m = d$ and we have (I). □

Lemma 11 *Suppose $3 \le h = i_1 = d-1$. Then $q^{d-1}_{2,d} = \cdots = q^{d-1}_{d,d} = 0$. Moreover, if $d = 2e$, then $q^j_{1,j} \ne 0$ implies $j = e$ and if $d = 2e+1$, then $q^j_{1,j} \ne 0$ if and only if $j = e,\ e+1$. In particular, the case (IV) holds.*

Proof: First we claim that $q^{d-1}_{1,d-1} = 0 \ne q^{d-1}_{2,d-1}$ and $i = i_2 = 2$. If $q^{d-1}_{1,d-1} \ne 0$, then $d-2, d-1, d \in \Delta(1)$. This is impossible as $d \ge 4$. Suppose $q^{d-1}_{2,d-1} = 0$. By Corollary 3, $q^{d-1}_{2,d-2} \ne 0 \ne q^{d-1}_{2,d}$. Hence $d-3, d-2, d \in \Delta(2)$. Thus $d = 4$ or 5. $d \ne 5$, because $3 \sim 2 \sim 5 \sim 1 \sim 3$ in $\Delta = \Delta^{(4)}$. $d \ne 4$, because $1 \sim 2 \sim 4 \sim 1$ in $\Delta = \Delta^{(3)}$. Thus we have the claim. In particular, $0 \sim d-1 \sim 2$ in this case.

Next we claim that $q^{d-1}_{d,2} = 0$. As otherwise, $d-3, d-1, d \in \Delta(2)$. Hence $d-3 = 2$ or $d = 5$. We have $0 \sim 4 \sim 2 \sim 5 \sim 1 \sim 3$ in $\Delta = \Delta^{(4)}$. This is impossible as $q^4_{2,5} \neq 0 = q^4_{3,5} = q^4_{3,4}$ contradicts Lemma 2 (1).

We claim $q^{d-1}_{d,3} = 0$. Suppose not. Then $1 \sim d \sim 3 \sim d-4$. Since $0 \sim d-1 \sim 2$, $3 \neq d-1$ and that $q^{d-1}_{d,4} = q^{d-1}_{d-1,3} = 0$. Hence in order to have $q^{d-1}_{d,3} \neq 0$, $q^{d-1}_{d-1,4} \neq 0$ by Lemma 2 (1). Since $\Delta(d-1) \subset \{0, 2, d-1\}$, we have $d = 5$. This is impossible as $1 \sim 5 \sim 3 \sim 1$ in Δ. Thus $q^{d-1}_{d,3} = 0$.

We now claim that $q^{d-1}_{d,j} \neq 0$ if and only if $j = 1$. Since $q^{d-1}_{2,d} = q^{d-1}_{3,d} = 0$, in order to show $q^{d-1}_{d,j} = 0$ for $j > 1$ by induction, it suffices to show that $q^{d-1}_{d,j-1} = q^{d-1}_{d-1,j-1} = 0$ for each j with $4 \leq j \leq d$. Since $j \geq 4$, we need only to check $q^{d-1}_{d,d} = 0$. Note that $q^{d-1}_{d-1,j-1} \neq 0$ only if $j = 1, 3, d$. Suppose $q^{d-1}_{d,d} \neq 0$. Then $q^{d-1}_{d-1,d-1} \neq 0$ as $q^{d-1}_{d-1,d} = 0$. By our observation above, $\Delta^*(d) = \{1\}$ and d comes as the last member in the new ordering as well. Let $\tilde{a}^*_j$ denote the dual intersection number with respect to the new ordering. Since $d \geq 4$, $q^{d-1}_{1,1} = 0$. In our case $q^{d-1}_{d-1,d-1} \neq 0$ or $\tilde{a}^*_1 \neq 0$. In view of Corollary 1 (1), this is impossible as $q^{d-1}_{1,1} = 0$ and 1 is not the last member in the new ordering.

Therefore, $q^{d-1}_{d,j} \neq 0$ if and only if $j = 1$ and that $q^{d-1}_{l,m} = 0$ if $l + m \geq d + 2$. Moreover we have $q^{d-1}_{l,m} \neq 0$ if $l + m = d - 1$ or $d + 1$. In particular, we have the ordering of (IV). Now the rest follows easily. See also the proof of Lemma 7. $\square$

Lemma 12 *Suppose* $3 \leq h = i_1 = d$. *If* $i = i_2 \geq 3$, *then* $d = 5$, *and the new ordering is* $0, 5, 3, 2, 4, 1$ *and* $q^5_{1,5} = q^5_{2,5} = q^5_{4,5} = q^5_{5,5} = 0 \neq q^5_{3,5}$ *and* $q^5_{3,4} = 0$, *i.e., the case* (V) *holds.*

Proof: Suppose $i = i_2 \geq 3$. Then we have

$$q^d_{1,d} = q^d_{2,d} = \cdots = q^d_{i-2,d} = q^d_{i-1,d} = 0 \neq q^d_{i,d}.$$

Thus $q^d_{d-1,1} \neq 0$ and $q^d_{d-1,i-1} \neq 0$. We claim that $q^d_{d-1,i+1} \neq 0$. As $b^*_i \neq 0$ and $q^d_{i,d} \neq 0$, $q^d_{d-1,i+1} = 0$ implies that $q^d_{d,i+1} \neq 0$. Hence $i + 1 = d$ or $i = d - 1$. Then $q^d_{i,d} = q^d_{d-1,i+1} = 0$, which is absurd. Thus $q^d_{d-1,i+1} \neq 0$ and $1, i-1, i+1 \in \Delta(d-1)$. Hence we must have $d - 1 = i + 1$ or $i = d - 2$, i.e., $0 \sim d \sim d-2$.

Since $\Delta^*(d-1) = \{1, d-3\}$, we have $2, d-4, d \in \Delta(d-2)$ if $d \geq 6$. Note that $d \geq 6$ with $q^d_{d-2,d} \neq 0$ implies that $q^d_{d-3,d-1} \neq 0$ and $q^d_{d-4,d-2} \neq 0$ by Lemma 2 (2). Hence $d = 5$ or 6.

Suppose $d = 6$. Then $0 \sim 6 \sim 4 \sim 2$, $q^6_{3,4} = q^6_{5,4} = 0$ and $q^6_{1,6} = q^6_{2,6} = q^6_{3,6} = q^6_{5,6} = 0$. Hence $q^6_{2,5} = 0$ as well and 2 becomes the end vertex in Δ by $(Q1)$, which is absurd.

Suppose $d = 5$. Then the new ordering is $0 \sim 5 \sim 3 \sim 2 \sim 4 \sim 1$ and we have desired conditions. $\square$

Lemma 13 *Suppose* $3 \leq h = i_1 = d$ *and* $i = i_2 = 1$. *Then the new ordering is*

$$0, d, 1, d-1, 2, d-2, 3, d-3, \ldots$$

and we have $q^d_{1,d} \neq 0 = q^d_{2,d} = \cdots = q^d_{d,d}$, *i.e.,* (II) *holds.*

Proof: Since $q^d_{1,d} \neq 0 = q^d_{2,d} = \cdots = q^d_{d-1,d}$, we need to show that $q^d_{d,d} = 0$. We use $\tilde{a}^*_j$ to denote the dual intersection numbers with respect to the new ordering. Suppose $q^d_{d,d} \neq 0$. Then $\tilde{a}^*_1 \neq 0$. Since $d \geq 3$, $q^d_{1,1} = 0$. This contradicts Corollary 1. Thus $q^d_{d,d} = 0$. By Lemma 5 (1), $q^d_{l,m} \neq 0$ if and only if $l+m \in \{d, d+1\}$. Hence $\Delta(l) = \{d-l, d-l+1\}$ for $l = 1, 2, \ldots, d$ and we have the assertion. □

Lemma 14 *Suppose* $3 \leq h = i_1 = d$ *and* $i = i_2 = 2$*. Then the new ordering is*

$$0, d, 2, d-2, 4, d-4, \ldots, d-5, 5, d-3, 3, d-1, 1.$$

Moreover we have the conditions in (III)*, i.e., one of the following holds.*

(i) $d = 3$*, and* $q^3_{1,3} = 0 \neq q^3_{2,3}$*,*

(ii) $d = 4$*,* $q^4_{1,4} = q^4_{3,4} = 0$*, and* $q^4_{2,4} \neq 0 \neq q^4_{2,3}$*, or*

(iii) $d \geq 5$*,* $q^d_{2,d} \neq 0 = q^d_{1,d} = q^d_{3,d} = \cdots = q^d_{d,d}$*. Moreover if* $d = 2e-1$*, then* $q^j_{1,j} \neq 0$ *implies* $j = e$ *and if* $d = 2e$*, then* $q^j_{1,j} \neq 0$ *if and only if* $j = e,\ e+1$*.*

Proof: Suppose $d = 3$. Then the new ordering is $0, 3, 2, 1$ and the condition in (i) is easy to check.

Suppose $d = 4$. Then the new ordering is $0, 4, 2, 3, 1$ as $q^4_{2,1} = 0$. Now the condition in (ii) is easy to check.

Hence assume $d \geq 5$. By our assumption, we have that

$$q^d_{2,d} \neq 0 = q^d_{1,d} = q^d_{3,d} = \cdots = q^d_{d-1,d}.$$

We claim that $q^d_{d,d} = 0$. As otherwise, $q^d_{2,2} \neq 0$ by Corollary 1 (1) as before. This is absurd as $d \geq 5$.

Therefore by Lemma 6, we have $q^d_{l,m} = 0$ if $l+m > d+2$ and $q^d_{l,m} \neq 0$ if $l+m = d+2$ for all $0 \leq l, m \leq d$. Therefore $d-l, d-l+2 \in \Delta(l)$ and the conditions are easily checked. □

This completes the proof of Theorem 2.

6. Concluding Remarks

1. One of the keys to our proof is Lemma 9, and for P-polynomial scheme case the corresponding result is easily obtained by the unimodal property of k_i's and the equality condition for $k_1 = k_i$. Here we used vanishing conditions of the structure constants. In this sense our proof follows [8].

2. Our proof can be applied to association schemes with multiple P-polynomial structures simply by replacing Krein parameters $q^h_{i,j}$ with $p^h_{i,j}$ and hence c^*_i, a^*_i, b^*_i by c_i, a_i, b_i. Actually, the theorem is proved for the following class of P-polynomial C-algebras.

$\mathcal{X}$ is a P-polynomial C-algebra satisfying the following.

(A) The structure constants $p^h_{i,j}$'s are all nonnegative.

(B) For $h \geq 0$, $i \geq j \geq 1$ with $h + i + j \leq d$,

$$\{l \mid p^l_{j,h+i} p^l_{i-j,h+j} \neq 0\} \subset \{h + i - j\}$$

and $p^{h+i}_{i,h+j} = 0$ implies that $p^{h+j}_{j,h+j} = 0$.

(C) If $p^h_{i,j} \neq 0$, $\{t \mid p^t_{i,k} p^t_{h,m} \neq 0\} \subset \{l\}$, and $\{t \mid p^t_{j,k} p^t_{h,l} \neq 0\} \subset \{m\}$, then $p^i_{k,l} = p^j_{k,m}$.

Since P-polynomial C-algebras associated with P-polynomial association schemes satisfy these conditions, our proof in this paper is applicable to association schemes with multiple P-polynomial structures. In that case (V) can be eliminated easily by a result of A. Gardiner in [3, Proposition 5.5.7].

3. The author does not know any primitive Q-polynomial association schemes which is not P-polynomial. Are there such examples?

4. Is it possible to eliminate or classify the cases in Theorem 2? See [5, 7].

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