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# Reduced Words and Plane Partitions

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**Abstract.** Let $w_0$ be the element of maximal length in the symmetric group $S_n$, and let $Red(w_0)$ be the set of all reduced words for $w_0$. We prove the identity

$$\sum_{(a_1,a_2,\ldots)\in Red(w_0)} (x+a_1)(x+a_2)\cdots = \binom{n}{2}! \prod_{1\le i<j\le n} \frac{2x+i+j-1}{i+j-1}, \qquad (*)$$

which generalizes Stanley's [20] formula for the cardinality of $Red(w_0)$, and Macdonald's [11] formula $\sum a_1 a_2 \cdots = \binom{n}{2}!$.

Our approach uses an observation, based on a result by Wachs [21], that evaluation of certain specializations of Schubert polynomials is essentially equivalent to enumeration of plane partitions whose parts are bounded from above. Thus, enumerative results for reduced words can be obtained from the corresponding statements about plane partitions, and vice versa. In particular, identity $(*)$ follows from Proctor's [14] formula for the number of plane partitions of a staircase shape, with bounded largest part.

Similar results are obtained for other permutations and shapes; $q$-analogues are also given.

**Keywords:** reduced word, plane partition, Schubert polynomial

## 1. Main result

We study enumerative problems related to reduced words (or reduced decompositions) in the symmetric group $S_n$. Recall that a reduced word for a permutation $w \in S_n$ is a sequence of indices $\mathbf{a} = (a_1, \ldots, a_l)$ such that $l = l(w)$ is the *length* of $w$ (the number of inversions), and $s_{a_1} \cdots s_{a_l} = w$, where $s_a$ denotes a simple transposition $(a\ a+1)$. The set of all reduced words for $w$ is denoted by $Red(w)$.

Let

$$w_0 = n\ n-1\ \cdots\ 2\ 1$$

be the permutation of maximal length in $S_n$; obviously, $l(w_0) = \binom{n}{2}$. Stanley [20] proved that the number of reduced words for $w_0$ is equal to the number $f^{\lambda_0}$ of standard Young

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tableaux of the staircase shape $\lambda_0 = (n-1, n-2, \ldots, 1)$, and therefore can be computed via the hooklength formula (see, e.g., [12]):

$$|Red(w_0)| = f^{\lambda_0} = \frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}}. \tag{1.1}$$

Edelman and Greene [2] then found an explicit bijection between reduced words and standard tableaux that underlies Stanley's formula. For example, if $n = 3$, then $w_0 = 321$, and there are two reduced words in $Red(w_0) = \{121, 212\}$, as well as two standard Young tableaux of shape $\lambda_0 = (2, 1)$.

Macdonald [11] discovered that, amazingly,

$$\sum_{\mathbf{a}\in Red(w_0)} a_1 a_2 \cdots a_{\binom{n}{2}} = \binom{n}{2}!. \tag{1.2}$$

For instance, in the example above,

$$1\cdot 2\cdot 1 + 2\cdot 1\cdot 2 = \binom{3}{2}!.$$

We found the following common generalization of these formulas of Stanley and Macdonald.

**Theorem 1.1** *We have*

$$\begin{aligned}\sum_{\mathbf{a}=(a_1,a_2,\ldots)\in Red(w_0)} (x+a_1)\cdots\left(x+a_{\binom{n}{2}}\right) &= \binom{n}{2}! \prod_{1\le i<j\le n} \frac{2x+i+j-1}{i+j-1}\\ &= f^{\lambda_0} \prod_{1\le i<j\le n} \frac{2x+i+j-1}{2}.\end{aligned} \tag{1.3}$$

The last two expressions are equal by virtue of the hooklength formula (1.1). Specializing $x = 0$, we obtain (1.2); equating the leading coefficients yields (1.1).

Identity (1.3) can also be rewritten as

$$\sum_{\mathbf{a}\in Red(w_0)} (x+a_1)\cdots\left(x+a_{\binom{n}{2}}\right) = \binom{n}{2}! \prod_{t\in\lambda_0} \frac{x+\frac{c(t)+n}{2}}{h(t)},$$

where $t$ runs over all boxes of the staircase shape $\lambda_0$, and $c(t)$ and $h(t)$ denote the content and hooklength of $t$, respectively (see, e.g., [19]). To obtain the last formula, it suffices to observe that

$$\prod_{1\le i<j\le n} \frac{i+j-1}{2} = \prod_{t\in\lambda_0} \frac{c(t)+n}{2} = \prod_{t\in\lambda_0} h(t) = \prod_{i=1}^{n-1}(2i-1)^{n-i}.$$

To illustrate (1.3), take $n = 3$. Then it becomes

$$(x+1)(x+2)(x+1) + (x+2)(x+1)(x+2)$$
$$= \binom{3}{2}! \cdot \frac{2x+2}{2} \cdot \frac{2x+3}{3} \cdot \frac{2x+4}{4}.$$

Let us note that if $x$ is a nonnegative integer, then, by a theorem of Proctor [14] (see also [8, 15, 16]), the first product appearing in (1.3) is exactly the number $\mathrm{pp}^{\lambda_0}(x)$ of (weak) plane partitions of shape $\lambda_0$ whose parts do not exceed $x$:

$$\mathrm{pp}^{\lambda_0}(x) = \prod_{1\le i<j\le n} \frac{2x+i+j-1}{i+j-1}. \tag{1.4}$$

For example, if $n = 3$, then $\mathrm{pp}^{\lambda_0}(x)$ is the number of plane partitions of the form

$$\begin{array}{|c|c|}\hline k & a \\ \hline b & \\ \cline{1-1}\end{array}\ ,$$

with $0 \le a, b \le k \le x$. This number obviously is $\sum_{k=0}^{x}(k+1)^2 = \frac{(x+1)(x+2)(2x+3)}{6}$, agreeing with our previous computations.

Appearance of plane partitions in this context is not accidental. We will show that there is a close connection between counting plane partitions with bounded parts and enumerative problems concerning reduced words. It is this connection that will allow us to prove Theorem 1.1.

In what follows, we will need some results from the theory of *Schubert polynomials* of Lascoux and Schützenberger (see, e.g., [10, 11]). For a permutation $w = (w(1), \ldots, w(n)) \in S_n$, we will denote by $\mathfrak{S}_w(x_1, \ldots, x_{n-1})$ the corresponding Schubert polynomial. We will also use the notation

$$1_x \times w = (1, 2, \ldots, x, x + w(1), \ldots, x + w(n)) \in S_{n+m}, \tag{1.5}$$

provided $x$ is a nonnegative integer.

**Lemma 1.2** *Let $x$ be a nonnegative integer. Then*

$$\sum_{\mathbf{a}\in Red(w_0)} (x+a_1)\cdots\left(x+a_{\binom{n}{2}}\right) = \binom{n}{2}!\,\mathfrak{S}_{1_x\times w_0}(1,\ldots,1). \tag{1.6}$$

(Note that $1_x \times w_0 = (1, 2, \ldots, x, x+n, x+n-1, \ldots, x+1)$.)

**Proof:** Since $(x+a_1, \ldots, x+a_{\binom{n}{2}})$ is a general form of a reduced word for $1_x \times w_0$, the lemma follows from [11, (6.11)]. □

**Proof of Theorem 1.1:** By a theorem of Wachs [21] (cf. also [11, 17]), the Schubert polynomial of any vexillary permutation (see [11, p. 11]) can be expressed as a certain *flagged Schur function.* In the case under consideration, Wachs' theorem gives

$$\mathfrak{S}_{1_x \times w_0}(x_1, \ldots, x_{n-1}) = \sum_T x^T, \tag{1.7}$$

where the sum is over all semi-standard Young tableaux of a staircase shape $\lambda_0$ such that every entry in row $i$ is $\le i + x$, and $x^T$ denotes the monomial associated with such a tableau, in the usual way. These tableaux are in obvious one-to-one correspondence with reverse plane partitions of shape $\lambda_0$ and parts $\le x$; namely, subtract $i$ from all entries in the $i$th row. Hence substituting $x_1 = x_2 = \cdots = 1$ in (1.7) yields

$$\mathfrak{S}_{1_x \times w_0}(1, \ldots, 1) = \mathrm{pp}^{\lambda_0}(x). \tag{1.8}$$

Combining this with (1.6) and (1.4), we obtain (1.3). □

We remark that both sides of the identity (1.8) have several interpretations:

(i) purely combinatorial;
(ii) representation-theoretic;
(iii) as certain determinants.

Let us explain what we mean by (i), (ii) and (iii).

Combinatorially, the left-hand side of (1.8) can be described by means of Stanley's formula for a Schubert polynomial [1, 3]. In the case under consideration, $\mathfrak{S}_{1_x \times w_0}(1, \ldots, 1)$ is equal to the number of subwords of

$$(1\ 2\ \cdots\ n-1)^x (1)(2\ 1)(3\ 2\ 1) \cdots (n-1\ \cdots\ 2\ 1)$$

which are reduced words for $w_0$. Other equivalent descriptions can be given in terms of resolutions of pseudo-line arrangements (see [4]), or in terms of balanced flagged labellings (see [5]). None of these can be trivially bijected to the plane partitions enumerated by $\mathrm{pp}^{\lambda_0}(x)$.

The number $\mathrm{pp}^{\lambda_0}(x)$ is the dimension of a certain indecomposable representation of a symplectic group. In fact, the multiplicative formula (1.4) for this number can be obtained (see [8, 14]) by combining the classical product formula for this dimension (see, e.g., [18, Corollary VII.8.1]) with an explicit combinatorial description of the corresponding Gelfand-Tsetlin basis given by Zhelobenko [22] and King [7]. (It was also realized (see [6, 14, 16]) that (1.4) can be derived in a purely combinatorial way, by factoring MacMahon's determinantal expression for $\mathrm{pp}^{\lambda_0}(x)$.) On the other hand, the specialization at $x_1 = x_2 = \cdots = 1$ of any Schubert polynomial, and in particular $\mathfrak{S}_{1_x \times w_0}(1, \ldots, 1)$, is the dimension of a certain naturally defined representation of the Borel subgroup of upper-triangular matrices, namely, the *Schubert module* of Kraśkiewicz and Pragacz [9] (see also, e.g., [5, Section 7]). It would be interesting to find a direct connection between these two representation-theoretic constructions.

## 2. Other shapes and permutations

We will now replace $\lambda_0$ by an arbitrary Ferrers shape $\lambda$. The role of $w_0$ will then be played by a *dominant permutation* (see, e.g., [11, p. 12]) whose Rothe diagram is $\lambda$. This is a permutation $w_\lambda$ such that

$$\lambda = \left\{(i, j) \,:\, w_\lambda(i) > j \text{ and } w_\lambda^{-1}(j) > i\right\}.$$

The arguments of Section 1 can be repeated, with obvious changes, and Theorem 1.1 generalizes as follows.

**Theorem 2.1** *Let $\lambda$ be a Ferrers shape of size $l$, and let $w_\lambda$ be the corresponding dominant permutation. Then*

$$\sum_{\mathbf{a}\in Red(w_\lambda)} (x + a_1)\cdots(x + a_l) \;=\; l!\,\mathrm{pp}^\lambda(x)\,, \tag{2.1}$$

*where* $\mathrm{pp}^\lambda(x)$ *denotes the number of plane partitions of shape $\lambda$ with parts $\le x$.*

This theorem can be used to compute the polynomials $\mathrm{pp}^\lambda(x)$. For example, if $\lambda$ is a rectangular shape $[n_1] \times [n_2]$, then

$$w_\lambda = (n_2 + 1, n_2 + 2, \ldots, n_2 + n_1, 1, 2, \ldots, n_2) \in S_{n_1+n_2}.$$

This permutation is 321-avoiding (see [1]), which means that all its reduced words are permutations of one another. Hence all summands in the left-hand side of (2.1) are equal, and we easily arrive at the famous MacMahon's formula [13, Section 495] for the number of plane partitions whose 3-dimensional shape is contained in a box.

In the other direction, Theorem 2.1 provides a product formula for the expression in the left-hand side of (2.1) whenever such a formula exists for $\mathrm{pp}^\lambda(x)$. The most general result of the latter kind that we know is due to Proctor [14] who gave product formulas in the case when the rows (equivalently, columns) of $\lambda$ form an arithmetic progression.

For a general $\lambda$, let us compute the greatest common divisor of the summands in (2.1). To this end, we employ the following observation.

**Lemma 2.2** *For any permutation $w$, the number of occurrences of an entry $k$ in any reduced word for $w$ is at least*

$$m_k \;=\; \#\{i : i \le k \text{ and } w(i) > k\}. \tag{2.2}$$

**Proof:** Let us interpret a reduced word as a process of converting the identity permutation into $w$ by means of adjacent transpositions. Since $m_k$ numbers have to be moved from some of the first $k$ positions to some of the remaining ones, it follows that the transposition $s_k$ has to be applied at least $m_k$ times. □

**Corollary 2.3** *Let $\lambda$ be a Ferrers shape of size $l$. For $k = 1, 2, \ldots$, let $m_k$ be the maximal number of boxes $(i, j)$ in an intersection of the diagonal $i + j = k + 1$ with some rectangular shape $\mu$ contained in $\lambda$. Then*

$$\mathrm{pp}^{\lambda}(x) = \frac{1}{l!} T_{\lambda}(x)(x+1)^{m_1}(x+2)^{m_2}\cdots, \tag{2.3}$$

*where $T_{\lambda}(x)$ is a polynomial in $x$ with nonnegative integer coefficients.*

**Proof:** In view of Lemma 2.2, each product $(x + a_1)\cdots(x + a_l)$ in (2.1) is divisible by $\prod (x + a_k)^{m_k}$, where the $m_k$ are computed according to (2.2), for $w = w_{\lambda}$. It remains to check that these $m_k$ coincide with those defined in Corollary 2.3. □

Corollary 2.3 enables us to compute polynomials $\mathrm{pp}^{\lambda}(x)$ for shapes $\lambda$ which are "almost rectangular," so that we can calculate $T_{\lambda}(x)$ for small values of $x$ by brute force. Note that the degree of $T_{\lambda}$ is $|\lambda| - \sum m_k$, and the leading coefficient is $f^{\lambda}$, the number of standard Young tableaux of shape $\lambda$.

**Example 2.4** Let $\lambda = (3, 3, 3, 2, 2)$. Then, by Corollary 2.3,

$$\mathrm{pp}^{\lambda}(x) = \frac{1}{13!}(x+1)(x+2)^2(x+3)^3(x+4)^2(x+5)^2(x+6)(ax^2 + bx + c),$$

where $a = f^{\lambda} = 3432$. To find $b$ and $c$, note that $\mathrm{pp}^{\lambda}(0) = 1$, and $\mathrm{pp}^{\lambda}(1)$ is the number of Ferrers shapes contained in $\lambda$, which in this case is equal to 52. Straightforward computations result in

$$\mathrm{pp}^{\lambda}(x) = \frac{2}{10!}(x+1)(x+2)^2(x+3)^3(x+4)^2(x+5)^2(x+6)(x^2 + 5x + 7).$$

**Corollary 2.5** *For any Ferrers shape $\lambda$ and any rectangular shape $\mu$ contained in $\lambda$, the polynomial* $\mathrm{pp}^{\mu}(x)$ *divides* $\mathrm{pp}^{\lambda}(x)$, *and the quotient has nonnegative rational coefficients.*

**Proof:** Follows from Corollary 2.3 and MacMahon's product formula [13, Section 495] for $\mathrm{pp}^{\mu}(x)$. □

## 3. *q*-analogues

Most results stated above have natural $q$-analogues. Instead of simply counting plane partitions, we can $q$-enumerate them by the sum of their parts; this will translate into computing a principal specialization of the corresponding flagged Schur function or, equivalently, the corresponding Schubert polynomial.

Our next result generalizes Theorem 2.1. To state it, we will need to recall some conventional notation. The *comajor index* of a finite sequence $\mathbf{a} = (a_1, a_2, \ldots)$ is defined to be the number

$$\mathrm{comaj}(\mathbf{a}) = \sum_{a_i < a_{i+1}} i.$$

The $q$-analogue of a nonnegative integer is defined by $[k] = (1-q^k)/(1-q)$. We will denote by $[\mathrm{rpp}^\lambda(x)]_q$ the generating function for (weak) reverse plane partitions of shape $\lambda$ and parts $\leq x$, which are $q$-enumerated with respect to the sum of their parts.

**Theorem 3.1** *Let $\lambda$ be a Ferrers shape of size $l$, and let $w_\lambda$ be the corresponding dominant permutation. Then*

$$\sum_{\mathbf{a}=(a_1,a_2,\ldots)\in Red(w_\lambda)} q^{\mathrm{comaj}(\mathbf{a})}[x+a_1]\cdots[x+a_l] = [l!]q^{b(\lambda)}[\mathrm{rpp}^\lambda(x)]_q, \tag{3.1}$$

*where $b(\lambda) = \sum_i (i-1)\lambda_i$.*

**Proof:** We first rewrite the left-hand side as

$$\sum_{\mathbf{a}\in Red(1_x\times w_\lambda)} q^{\mathrm{comaj}(\mathbf{a})}[a_1]\cdots[a_l]. \tag{3.2}$$

According to the formula for the principal specialization of a Schubert polynomial, conjectured in [11] and proved in [3], the expression (3.2) is equal to

$$[l!]\mathfrak{S}_{1_x\times w_\lambda}(1,q,q^2,\ldots).$$

By Wachs' theorem, $\mathfrak{S}_{1_x\times w_\lambda}$ is a certain flagged Schur function for the shape $\lambda$, whose principal specialization can easily be seen to coincide, up to an appropriate power of $q$, with the generating function for reverse plane partitions of shape $\lambda$ with bounded part size. This yields (3.1). □

## 4. Open problems and comments

**1.** It would be very nice to have a bijective proof of our main identity

$$\sum_{\mathbf{a}\in Red(w_0)} (x+a_1)\cdots\left(x+a_{\binom{n}{2}}\right) = \binom{n}{2}!\ \mathrm{pp}^{\lambda_0}(x) \tag{4.1}$$

(cf. (1.3)–(1.4)) or even its $q$-analogue (3.1). This seems to be quite tricky even in the case of $x=0$ (that is, in the case of Macdonald's identity (1.2)), where a fairly complicated bijection has been constructed by B. Sagan and the first author (unpublished).

We have already mentioned that there may also exist a representation-theoretic proof of (4.1).

**2.** It is natural to look for results similar to Theorem 1.1 for other finite Coxeter groups, for example, for the hyperoctahedral group. And even in the case of the symmetric group, it is not clear what should be the analogue of this theorem for other classes of permutations (not necessarily dominant). In particular, for which permutations $w$ is the polynomial

$$\sum_{\mathbf{a}\in Red(w)} (x+a_1)\cdots(x+a_{l(w)})$$

a product of linear factors?

**3.** The product expression for $\mathrm{pp}^{\lambda_0}(x)$ (see (1.4)) has yet another combinatorial interpretation. It is straightforward to show that

$$\prod_{1\le i<j\le n} \frac{2x+i+j-1}{i+j-1} = 2^{-\binom{n}{2}} s_{\lambda_0}(\underbrace{1,\ldots,1}_{n+2x}),$$

where $s_{\lambda_0}$ denotes the corresponding Schur function. Since $s_{\lambda_0}(\underbrace{1,\ldots,1}_{n}) = 2^{\binom{n}{2}}$, we obtain the identity

$$\mathrm{pp}^{\lambda_0}(x) s_{\lambda_0}(\underbrace{1,\ldots,1}_{n}) = s_{\lambda_0}(\underbrace{1,\ldots,1}_{n+2x}),$$

which suggests that there exists an explicit bijection between

(i) pairs (plane partition of shape $\lambda_0$ with parts $\le x$, semi-standard Young tableaux of shape $\lambda_0$ and entries $\le n$), and

(ii) semi-standard Young tableaux of shape $\lambda_0$ and entries $\le n+2x$.

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## References

1. S.C. Billey, W. Jockusch, and R.P. Stanley, "Some combinatorial properties of Schubert polynomials," *J. Alg. Combin.* **2** (1993), 345–374.
2. P. Edelman and C. Greene, "Balanced tableaux," *Advances in Math.* **63** (1987), 42–99.
3. S. Fomin and R.P. Stanley, "Schubert polynomials and the nilCoxeter algebra," *Advances in Math.* **103** (1994), 196–207.
4. S. Fomin and A.N. Kirillov, "The Yang-Baxter equation, symmetric functions, and Schubert polynomials," *Discrete Math.* **153** (1996), 123–143.
5. S. Fomin, C. Greene, V. Reiner, and M. Shimozono, "Balanced labellings and Schubert polynomials," *European J. Combin.* **18** (1997), 373–389.
6. I.M. Gessel and G.X. Viennot, "Determinants, paths, and plane partitions," (preprint).

7. R.C. King, “Weight multiplicities for the classical groups,” Springer Lecture Notes in Physics **50** (1976).
8. K. Koike and I. Terada, “Young-diagrammatic methods for the representation theory of the classical groups of type $B_n$, $C_n$, $D_n$,” *J. Algebra* **107** (1987), 466–511.
9. W. Kraśkiewicz and P. Pragacz, “Schubert functors and Schubert polynomials,” 1986 (preprint).
10. A. Lascoux, “Polynômes de Schubert. Une approche historique,” *Séries formelles et combinatoire algébrique*, P. Leroux and C. Reutenauer (Eds.), Université du Québec à Montréal, LACIM, pp. 283–296, 1992.
11. I.G. Macdonald, *Notes on Schubert polynomials*, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, Montréal, 1991.
12. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford Univ. Press, Oxford, 1995.
13. P.A. MacMahon, *Combinatory Analysis*, Vols. 1–2, Cambridge University Press, 1915, 1916; reprinted by Chelsea, New York, 1960.
14. R.A. Proctor, unpublished research announcement, 1984.
15. R.A. Proctor, “Odd symplectic groups,” *Invent. Math.* **92** (1988), 307–332.
16. R.A. Proctor, “New symmetric plane partition identities from invariant theory work of De Concini and Procesi,” *European J. Combin.* **11** (1990), 289–300.
17. V. Reiner and M. Shimozono, “Key polynomials and a flagged Littlewood-Richardson rule,” *J. Combin. Theory, Ser. A* **70** (1995), 107–143.
18. J.-P. Serre, *Algebres de Lie Semi-Simples Complexes*, W.A. Benjamin, New York, 1966.
19. R.P. Stanley, “Theory and applications of plane partitions,” *Studies in Appl. Math.* **50** (1971), 167–188, 259–279.
20. R.P. Stanley, “On the number of reduced decompositions of elements of Coxeter groups,” *European J. Combin.* **5** (1984), 359–372.
21. M.L. Wachs, “Flagged Schur functions, Schubert polynomials, and symmetrizing operators,” *J. Combin. Theory, Ser. A* **40** (1985), 276–289.
22. D.P. Zhelobenko, “The classical groups. Spectral analysis of their finite dimensional representations,” *Russ. Math. Surv.* **17** (1962), 1–94.