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Partial Flocks of Quadratic Cones with a Point Vertex in PG(n, q), n Odd

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Abstract. We generalise the definition and many properties of flocks of quadratic cones in PG$(3, q)$ to partial flocks of quadratic cones with vertex a point in PG(n, q), for $n \geq 3$ odd.

Keywords: Galois geometry, flock, cone, ovoid, cap

1. Introduction

For information on the properties of quadrics in projective spaces, see [4, Section 5.1], [5, Chapter 16] and especially [8, Chapter 22]. In the following, we always assume that $n \geq 3$ is odd.

In PG(n, q), $n \geq 3$ odd, let $\mathcal{K} = v\mathcal{Q}$ be a cone with vertex the point v and base $\mathcal{Q}$, where $\mathcal{Q}$ is a non-singular (parabolic) quadric in a hyperplane PG$(n-1, q)$ not on v.

A *partial flock* of $\mathcal{K}$ of size k is a set of hyperplanes $\pi_1, \ldots, \pi_k$ of PG(n, q), each not on v, such that for each $i, j \in \{1, \ldots, k\}$ with $i \neq j$ the $(n-2)$-dimensional space $\pi_i \cap \pi_j$ meets $\mathcal{K}$ in a non-singular elliptic quadric. The set of (non-singular, parabolic) quadrics $\pi_i \cap \mathcal{K}$ for $i = 1, \ldots, k$ is also called a *partial flock* of $\mathcal{K}$.

In the case $n = 3$, since an elliptic quadric in PG$(1, q)$ has no points, the above definition coincides with the existing definition of a partial flock of a quadratic cone in PG$(3, q)$.

2. The size of a partial flock, q even

It is easy to see that a partial flock of a quadratic cone in PG$(3, q)$, q odd or even, has size at most q, since the conics in the flock are disjoint. In this section we use Lemma 1 (a generalisation of [12, 1.5.2]) to show that this bound also holds for odd $n \geq 5$ and q even. Our proof is also valid in the case $n = 3$.

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Lemma 1 *In* $PG(n, q)$, *where* $n \geq 3$ *is odd and* q *is even, let* $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ *be a partial flock of the cone* $\mathcal{K} = v\mathcal{Q}$. *Let* u *be the nucleus of* $\mathcal{Q}$ *in the subspace* $PG(n-1, q)$ *of* $PG(n, q)$. *Then each space* $\pi_i \cap \pi_j$, $i \neq j$, *is disjoint from the line* vu.

Proof: Suppose, to the contrary, that there exist $i \neq j$ such that $\pi_i \cap \pi_j \cap vu = u'$, say. Then u' is the nucleus of the (parabolic) quadric $\mathcal{K} \cap \pi_i$, so $\pi_i \cap \pi_j \cap \mathcal{K}$ is parabolic, a contradiction. □

Theorem 2 *In* $PG(n, q)$, *where* $n \geq 3$ *is odd and* q *is even, a partial flock of a quadratic cone has size at most* q.

Proof: Let $\mathcal{F}$ be a partial flock of the cone $\mathcal{K} = v\mathcal{Q}$. Let u be the nucleus of $\mathcal{Q}$ in the subspace PG$(n-1, q)$ of PG(n, q). By Lemma 1, no two elements of $\mathcal{F}$ can meet on the line vu. Since each element of $\mathcal{F}$ must meet $vu \backslash \{v\}$, we have $k \leq q$. □

3. Generalising known results

In this section we generalise some results which are well-known for flocks of quadratic cones in PG$(3, q)$. In particular, the dual setting for q even generalises [12, 1.5.3], the algebraic condition generalises [12, 1.5.5], the existence of the partial ovoid of $\mathcal{Q}^+(n+2, q)$ generalises [12, 1.3], the process of derivation for q odd generalises [1] and the construction of herds of caps for q even generalises [2, Theorem 1] (see also [11, Theorem 2.1]).

4. The dual setting

Case (1) q odd: First suppose that q is odd. In PG(n, q), let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ be a partial flock of the cone $\mathcal{K} = v\mathcal{Q}$. We apply a duality to PG(n, q). The point v is mapped to a hyperplane V of PG(n, q) and the set of lines of $\mathcal{K}$ on v is mapped to the set of all tangent hyperplanes to a non-singular quadric $\mathcal{Q}'$ of V. The hyperplanes $\pi_1, \ldots, \pi_k$ of $\mathcal{F}$ are mapped to points $p_1, \ldots, p_k$ of PG$(n, q) \backslash V$. For $i \neq j$ the $(n-2)$-dimensional space $\pi_i \cap \pi_j$ meets $\mathcal{K}$ in the points of a non-singular elliptic quadric $\mathcal{Q}^-(n-2, q)$; so the hyperplane $\langle \pi_i \cap \pi_j, v \rangle$, generated by $\pi_i \cap \pi_j$ and v, contains exactly the lines of $v\mathcal{Q}$ on the cone $v\mathcal{Q}^-(n-2, q)$. It follows that the line $p_i p_j$ meets V in a point p_{ij} on exactly the tangent hyperplanes of $\mathcal{Q}'$ which correspond under the duality to the lines of $v\mathcal{Q}^-(n-2, q)$; so the tangent points of these hyperplanes are the points of a non-singular elliptic quadric $\hat{\mathcal{Q}}^-(n-2, q)$ on $\mathcal{Q}'$. Hence p_{ij} is an interior point of $\mathcal{Q}'$.

Thus, for n and q odd, a *dual partial flock* of a non-singular quadric $\mathcal{Q}'$ of a hyperplane PG$(n-1, q)$ of PG(n, q) is a set of points of PG$(n, q) \backslash$PG$(n-1, q)$ such that the line joining any two of them meets PG$(n-1, q)$ in a point interior to $\mathcal{Q}'$. It is clear that a partial flock gives rise to a dual partial flock and conversely.

Case (2) q even: Now suppose that q is even.

We use the following notation, introduced in [7]. Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q)$, let $\mathrm{PG}(n-1, q)$ be a hyperplane and let Q be a point of $\mathrm{PG}(n, q)\backslash\mathrm{PG}(n-1, q)$ not lying on $\mathcal{Q}$ and distinct from its nucleus. The projection of $\mathcal{Q}$ from Q onto $\mathrm{PG}(n-1, q)$ is the set $\mathcal{R} = \{PQ \cap \mathrm{PG}(n-1, q) : P \in \mathcal{Q}\}$. If n is odd and $\mathcal{Q}$ is hyperbolic then we write $\mathcal{R} = \mathcal{R}^+$ while if $\mathcal{Q}$ is elliptic then we write $\mathcal{R} = \mathcal{R}^-$. We note, see [7], that a set $\mathcal{R}$ has type $(1, q/2 + 1, q + 1)$ with respect to lines, that a set $\mathcal{R}^+$ contains a unique hyperplane $\mathrm{PG}(n-2, q)$ such that $(\mathrm{PG}(n-1, q)\backslash\mathcal{R}^+) \cup \mathrm{PG}(n-2, q)$ is a set $\mathcal{R}^-$ and that a set $\mathcal{R}^-$ contains a unique hyperplane $\mathrm{PG}(n-2, q)$ such that $(\mathrm{PG}(n-1, q)\backslash\mathcal{R}^-) \cup \mathrm{PG}(n-2, q)$ is a set $\mathcal{R}^+$.

In $\mathrm{PG}(n, q)$, for odd $n \geq 5$, let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ be a partial flock of the cone $\mathcal{K} = v\mathcal{Q}$. Again, we apply a duality to $\mathrm{PG}(n, q)$. The point v is mapped to a hyperplane $V = \mathrm{PG}(n-1, q)$ of $\mathrm{PG}(n, q)$. Let $\mathcal{G}$ be the set of generators ($((n-3)/2)$-dimensional subspaces) lying on $\mathcal{Q}$. A $((n-1)/2)$-dimensional subspace vG, $G \in \mathcal{G}$, is mapped by the duality to an $((n-1)/2)$-dimensional subspace of V, and we denote by $\mathcal{R}$ the union of the points lying on such $((n-1)/2)$-dimensional subspaces of V. The set $\mathcal{R}$ contains the subspace $\mathrm{PG}(n-2, q)$ of V which is the dual of the line uv, with u the nucleus of $\mathcal{Q}$. It can be shown that $\mathcal{R}$ has type $(1, q/2+1, q+1)$ with respect to lines, by showing that an $(n-2)$-dimensional subspace of $\mathrm{PG}(n, q)$ on v lies in exactly 1, $q/2+1$ or $q+1$ hyperplanes containing an element vG, $G \in \mathcal{G}$. Then, since $\mathcal{R}$ contains $((n-1)/2)$-dimensional subspaces not in $\mathrm{PG}(n-2, q)$, it follows that $\mathcal{R}$ is a set $\mathcal{R}^+$ in V (this also follows from $|\mathcal{R}| = q^{n-1}/2+q^{n-2}+\cdots+q+1+q^{(n-1)/2}/2$ and [7]). The hyperplanes $\pi_1, \ldots, \pi_k$ of $\mathcal{F}$ are mapped to points $p_1, \ldots, p_k$ of $\mathrm{PG}(n, q)\backslash V$. For $i \neq j$ the $(n-2)$-dimensional space $\pi_i \cap \pi_j$ does not meet the line uv and meets $\mathcal{K}$ in exactly the points of a non-singular elliptic quadric $\mathcal{Q}^-(n-2, q)$; hence the hyperplane $\langle \pi_i \cap \pi_j, v\rangle$ does not contain any element of $\mathcal{G}$. So the line $p_i p_j$ meets V in a point of $V\backslash\mathcal{R}^+ = \mathcal{R}^-\backslash\mathrm{PG}(n-2, q)$.

For n odd and q even a *dual partial flock* of a set $\mathcal{R}^+$ of type $(1, q/2 + 1, q + 1)$ in a hyperplane $\mathrm{PG}(n-1, q)$ of $\mathrm{PG}(n, q)$ is a set of points of $\mathrm{PG}(n, q)\backslash\mathrm{PG}(n-1, q)$ such that the line joining any two of them meets $\mathrm{PG}(n-1, q)$ in a point of $\mathrm{PG}(n-1, q)\backslash\mathcal{R}^+$. It is clear that a partial flock gives rise to a dual partial flock and conversely.

We remark that the results of this last section also hold in the case $n = 3$ (see [12]); here a set $\mathcal{R}^+$ is the set of points of a dual regular hyperoval.

4.1. The algebraic conditions

For $q = 2^h$, the map trace is defined by

$$\text{trace: } \mathrm{GF}(q) \to \mathrm{GF}(2), \quad x \mapsto \sum_{i=0}^{h-1} x^{2^i}.$$

Theorem 3 *In $\mathrm{PG}(n, q)$ for $n \geq 3$ odd, let $\mathcal{K} = v\mathcal{Q}$ be a quadratic cone with vertex the point v and base $\mathcal{Q}$, where $\mathcal{Q}$ is a non-singular quadric in a hyperplane not on v, and let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ be a set of hyperplanes not on v. Without loss of generality, we can suppose that the quadratic cone $\mathcal{K} = v\mathcal{Q}$ has equation $x_0x_1+x_2x_3+\cdots+x_{n-3}x_{n-2} = x_{n-1}^2$, so that $v = (0, \ldots, 0, 1)$ and $\mathcal{Q}$ has equation $x_0x_1 + x_2x_3 + \cdots + x_{n-3}x_{n-2} = x_{n-1}^2$ in the* *hyperplane* $PG(n-1, q)$ *with equation* $x_n = 0$. *For* $i = 1, \ldots, k$ *the hyperplane* π_i *has equation* $a_0^{(i)} x_0 + \cdots + a_{n-1}^{(i)} x_{n-1} + x_n = 0$ *for some* $a_j^{(i)} \in GF(q)$. *If* q *is odd*, $\mathcal{F}$ *is a partial flock of* $\mathcal{K}$ *if and only if*

$$-4\left(a_0^{(i)} - a_0^{(j)}\right)\left(a_1^{(i)} - a_1^{(j)}\right) - \cdots \\ - 4\left(a_{n-3}^{(i)} - a_{n-3}^{(j)}\right)\left(a_{n-2}^{(i)} - a_{n-2}^{(j)}\right) + \left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)^2$$

is a non-square in $GF(q)$ *for all* $i, j \in \{1, \ldots, k\}, i \neq j$. *If* q *is even*, $\mathcal{F}$ *is a partial flock of* $\mathcal{K}$ *if and only if* $a_{n-1}^{(i)} - a_{n-1}^{(j)} \neq 0$ *and*

$$\mathit{trace}\left(\frac{\left(a_0^{(i)} - a_0^{(j)}\right)\left(a_1^{(i)} - a_1^{(j)}\right) + \cdots + \left(a_{n-3}^{(i)} - a_{n-3}^{(j)}\right)\left(a_{n-2}^{(i)} - a_{n-2}^{(j)}\right)}{\left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)^2}\right) = 1$$

for all $i, j \in \{1, \ldots, k\}, i \neq j$.

Proof: For $i, j \in \{1, \ldots, k\}, i \neq j$, the hyperplane $\langle \pi_i \cap \pi_j, v\rangle$ meets $\mathcal{K} \cap \mathrm{PG}(n-1, q) = \mathcal{Q}$ in the quadric $\mathcal{Q}'$ with equations

$$\begin{aligned} \left(a_0^{(i)} - a_0^{(j)}\right)x_0 + \cdots + \left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)x_{n-1} = 0, \\ x_0x_1 + x_2x_3 + \cdots + x_{n-3}x_{n-2} = x_{n-1}^2. \end{aligned} \tag{1}$$

At least one of $(a_0^{(i)} - a_0^{(j)}), \ldots, (a_{n-2}^{(i)} - a_{n-2}^{(j)})$ is not zero, for otherwise $\langle \pi_i \cap \pi_j, v\rangle$ meets $\mathcal{K}$ in a hyperbolic quadratic cone with vertex v, so $\pi_i \cap \pi_j$ meets $\mathcal{K}$ in a hyperbolic quadric, contrary to the definition of partial flock. Therefore, without loss of generality, we suppose that $a_0^{(i)} \neq a_0^{(j)}$. The quadric $\mathcal{Q}'$ is the intersection of the cone

$$\left(a_0^{(j)} - a_0^{(i)}\right)^{-1}\left(\left(a_1^{(i)} - a_1^{(j)}\right)x_1 + \cdots + \left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)x_{n-1}\right)x_1 \\ + x_2x_3 + \cdots + x_{n-3}x_{n-2} = x_{n-1}^2,$$

that is,

$$\begin{aligned} &\left(a_1^{(i)} - a_1^{(j)}\right)x_1^2 + \left(a_0^{(i)} - a_0^{(j)}\right)x_{n-1}^2 + \left(a_2^{(i)} - a_2^{(j)}\right)x_1x_2 + \cdots \\ &\quad + \left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)x_1x_{n-1} + \left(a_0^{(j)} - a_0^{(i)}\right)x_2x_3 + \left(a_0^{(j)} - a_0^{(i)}\right)x_4x_5 + \cdots \\ &\quad + \left(a_0^{(j)} - a_0^{(i)}\right)x_{n-3}x_{n-2} = 0, \end{aligned} \tag{2}$$

with the hyperplane (1) not through its vertex. We determine exactly when the quadric $\mathcal{Q}'$ is non-singular and elliptic. Let the matrix $A = [a_{ij}]_{i,j=1,\ldots,n-1}$, where a_{ii} is twice the coefficient of x_i^2 in (2) and for $i < j$ $a_{ij} = a_{ji}$ is the coefficient of x_ix_j in (2).

Then A is

$$\begin{pmatrix}
2\left(a_1^{(i)}-a_1^{(j)}\right) & \left(a_2^{(i)}-a_2^{(j)}\right) & \left(a_3^{(i)}-a_3^{(j)}\right) & \dots & \dots & \left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right) & \left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right) & \left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right)\\
\left(a_2^{(i)}-a_2^{(j)}\right) & 0 & \left(a_0^{(j)}-a_0^{(i)}\right) & 0 & \dots & 0 & 0 & 0\\
\left(a_3^{(i)}-a_3^{(j)}\right) & \left(a_0^{(j)}-a_0^{(i)}\right) & 0 & 0 & \dots & 0 & 0 & 0\\
\left(a_4^{(i)}-a_4^{(j)}\right) & 0 & 0 & \ddots & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots\\
\left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right) & 0 & 0 & 0 & \dots & 0 & \left(a_0^{(j)}-a_0^{(i)}\right) & 0\\
\left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right) & 0 & 0 & 0 & \dots & \left(a_0^{(j)}-a_0^{(i)}\right) & 0 & 0\\
\left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right) & 0 & 0 & 0 & \dots & 0 & 0 & 2\left(a_0^{(i)}-a_0^{(j)}\right)
\end{pmatrix}$$

with determinant (expanding by the last row; then expanding the two resulting subdeterminants by the last column and first row respectively)

$$|A| = (-1)^{(n-3)/2}\left(a_0^{(i)}-a_0^{(j)}\right)^{n-3}\Big(4\Big(\left(a_0^{(i)}-a_0^{(j)}\right)\left(a_1^{(i)}-a_1^{(j)}\right)+\left(a_2^{(i)}-a_2^{(j)}\right)$$
$$\times\left(a_3^{(i)}-a_3^{(j)}\right)+\cdots+\left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right)\left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right)\Big)-\left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right)^2\Big).$$

If q is odd, by [8, 22.2.1], the quadric $\mathcal{Q}'$ is non-singular and elliptic if and only if $(-1)^{(n-1)/2}|A|$ is a non-square in GF(q), which is if and only if

$$-4\left(a_0^{(i)}-a_0^{(j)}\right)\left(a_1^{(i)}-a_1^{(j)}\right)-\cdots-4\left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right)\left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right)+\left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right)^2$$

is a non-square in GF(q).

For q even, by [8, 22.2.1], the quadric $\mathcal{Q}'$ is non-singular if and only if $|A| \neq 0$, that is, if and only if $a_{n-1}^{(i)} - a_{n-1}^{(j)} \neq 0$. Further, the non-singular quadric $\mathcal{Q}'$ is elliptic if and only if trace$((|B| - (-1)^{(n-1)/2}|A|)/(4|B|)) = 1$, where the matrix $B = [b_{ij}]_{i,j=1,\dots,n-1}$ has $b_{ii} = 0$ and $b_{ji} = -b_{ij} = -a_{ij}$ for $i < j$. (The formula $(|B| - (-1)^{(n-1)/2}|A|)/(4|B|)$ should be interpreted as follows: the terms a_{ij} are replaced by indeterminates z_{ij}, the formula is evaluated as a rational function over the integers Z, and then z_{ij} is specialized to a_{ij} to give the result.) Thus B is

$$\begin{pmatrix}
0 & \left(a_2^{(i)}-a_2^{(j)}\right) & \left(a_3^{(i)}-a_3^{(j)}\right) & \dots & \dots & \left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right) & \left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right) & \left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right)\\
-\left(a_2^{(i)}-a_2^{(j)}\right) & 0 & \left(a_0^{(j)}-a_0^{(i)}\right) & 0 & \dots & 0 & 0 & 0\\
-\left(a_3^{(i)}-a_3^{(j)}\right) & -\left(a_0^{(j)}-a_0^{(i)}\right) & 0 & 0 & \dots & 0 & 0 & 0\\
-\left(a_4^{(i)}-a_4^{(j)}\right) & 0 & 0 & \ddots & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots\\
-\left(a_{n-3}^{(i)}-a_{n-3}^{(j)}\right) & 0 & 0 & 0 & \dots & 0 & \left(a_0^{(j)}-a_0^{(i)}\right) & 0\\
-\left(a_{n-2}^{(i)}-a_{n-2}^{(j)}\right) & 0 & 0 & 0 & \dots & -\left(a_0^{(j)}-a_0^{(i)}\right) & 0 & 0\\
-\left(a_{n-1}^{(i)}-a_{n-1}^{(j)}\right) & 0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix}$$

and $|B| = (a_0^{(i)} - a_0^{(j)})^{n-3}(a_{n-1}^{(i)} - a_{n-1}^{(j)})^2$. Thus, the non-singular quadric $\mathcal{Q}'$ is elliptic if and only if

$$\text{trace}\left(\frac{\left(a_0^{(i)} - a_0^{(j)}\right)\left(a_1^{(i)} - a_1^{(j)}\right) + \cdots + \left(a_{n-3}^{(i)} - a_{n-3}^{(j)}\right)\left(a_{n-2}^{(i)} - a_{n-2}^{(j)}\right)}{\left(a_{n-1}^{(i)} - a_{n-1}^{(j)}\right)^2}\right) = 1. \qquad \Box$$

4.2. The corresponding partial ovoid of $\mathcal{Q}^+(n+2, q)$

Theorem 4 *In PG(n, q), $n \geq 3$ odd, let $\mathcal{F}$ be a partial flock of size k of the quadratic cone $\mathcal{K} = v\mathcal{Q}$. Then there exists a partial ovoid of the non-singular hyperbolic quadric $\mathcal{Q}^+(n+2, q)$ of size $kq+1$ comprising k conics mutually tangent at a common point. Conversely, given any such partial ovoid there exists a partial flock $\mathcal{F}$ of $\mathcal{K}$.*

Proof: Embed $\mathcal{K}$ in a non-singular hyperbolic quadric $\mathcal{Q}^+$ in PG$(n+2, q)$ and let $\perp$ denote the polarity determined by $\mathcal{Q}^+$. Let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$. First, since PG$(n, q) \cap \mathcal{Q}^+ = v\mathcal{Q}$, the line $L = \text{PG}(n, q)^\perp$ meets $\mathcal{Q}^+$ in the single point v. For $i = 1, \ldots, k$, $\pi_i^\perp$ is a plane on L meeting $\mathcal{Q}^+$ in a (non-singular) conic $\mathcal{C}_i$ on v. Since, for $i, j \in \{1, \ldots, k\}$, $i \neq j$, $\pi_i \cap \pi_j$ meets $\mathcal{K}$ and hence also $\mathcal{Q}^+$ in a non-singular elliptic quadric, it follows that $\langle \pi_i^\perp, \pi_j^\perp \rangle$ also meets $\mathcal{Q}^+$ in a non-singular elliptic quadric. Hence no two points of $\mathcal{C}_i \cup \mathcal{C}_j$ are collinear on $\mathcal{Q}^+$, so $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$ is a partial ovoid of $\mathcal{Q}^+$ of size $kq+1$. The converse is immediate as the polarity is bijective and involutory. $\Box$

Corollary 5 *Let q be even. A partial ovoid of $\mathcal{Q}^+(n+2, q)$ which is a union of conics mutually tangent at a common point has size at most $q^2 + 1$.*

Proof: Theorems 2 and 4. $\Box$

The construction in Theorem 4 gives a bound on the size of a partial flock. If $n > 3$ and q is even, this is not as good as the bound in Theorem 2.

Theorem 6 *In PG(n, q), $n \geq 3$ odd, let $\mathcal{F}$ be a partial flock of size k of the quadratic cone $\mathcal{K} = v\mathcal{Q}$ in PG(n, q). Then $k \leq q^{(n-1)/2}$.*

Proof: Given $\mathcal{F}$, by Theorem 4 there exists a partial ovoid $\mathcal{O}$ of size $kq+1$ of $\mathcal{Q}^+(n+2, q)$. Thus $\mathcal{O} \leq q^{(n+1)/2} + 1$ ([8, A VI]) and the result follows. $\Box$

We remark that in the case $n = 3$, the bound is best possible as there exist partial flocks of size q of a quadratic cone in PG$(3, q)$, called *flocks,* associated with certain ovoids of $\mathcal{Q}^+(5, q)$.

Let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ be a partial flock of $\mathcal{K} = v\mathcal{Q}$ in PG(n, q), n odd. If the elements of the partial flock contain a common m-dimensional subspace ξ, then the corresponding partial ovoid of $\mathcal{Q}^+(n+2, q)$ is contained in an $(n+1-m)$-dimensional subspace. In particular, if $m = n-3$ and if $\xi \cap \mathcal{K}$ is non-singular then the corresponding partial ovoid is contained in a quadric $\mathcal{Q}(4, q)$. If, further, q is odd then there corresponds a partial spread of size $kq+1$ of the generalized quadrangle $W(q)$. If $k = q$ then this is a spread and there arises a translation plane.

4.3. Derivation of a partial flock of $\mathcal{K}$, q odd

Let $\mathcal{Q}(n+1, q)$ be the non-singular quadric of $\mathrm{PG}(n+1, q)$ defined by the equation $x_0x_1+x_2x_3+\cdots+x_{n-3}x_{n-2}-x_{n-1}^2+x_nx_{n+1} = 0$ and let $\perp$ denote the polarity determined by $\mathcal{Q}(n+1, q)$. The tangent hyperplane H_0 of $\mathcal{Q}(n+1, q)$ at the point $p_0 = (0, \ldots, 0, 1, 0)$ has equation $x_{n+1} = 0$ and intersects $\mathcal{Q}(n+1, q)$ in the quadratic cone $\mathcal{K}_0$ with equation $x_0x_1 + x_2x_3 + \cdots + x_{n-3}x_{n-2} - x_{n-1}^2 = x_{n+1} = 0$ and vertex p_0.

Let $\mathcal{F}_0$ be a partial flock of size k of $\mathcal{K}_0$, where for $i = 1, \ldots, k$ the element π_i of $\mathcal{F}_0$ has equations $a_0^{(i)}x_0 + \cdots + a_{n-1}^{(i)}x_{n-1} + x_n = x_{n+1} = 0$. For $i = 1, \ldots, k$, we define the line $L_i = \pi_i^{\perp}$, and note that L_i meets $\mathcal{Q}(n+1, q)$ in p_0 and the further point

$$p_i = \left(a_1^{(i)}, a_0^{(i)}, a_3^{(i)}, a_2^{(i)}, \ldots, a_{n-2}^{(i)}, a_{n-3}^{(i)}, \frac{-1}{2}a_{n-1}^{(i)}, \frac{1}{4}\left(a_{n-1}^{(i)}\right)^2 - a_0^{(i)}a_1^{(i)}\right.$$
$$\left. - a_2^{(i)}a_3^{(i)} - \cdots - a_{n-3}^{(i)}a_{n-2}^{(i)}, 1\right).$$

Since $p_i \in \mathcal{Q}(n+1, q)$, it follows that the hyperplane $H_i = p_i^{\perp}$ with equation

$$a_0^{(i)}x_0 + a_1^{(i)}x_1 + \cdots + a_{n-1}^{(i)}x_{n-1} + x_n + a_{n+1}^{(i)}x_{n+1} = 0,$$

where

$$a_{n+1}^{(i)} = 1/4\left(a_{n-1}^{(i)}\right)^2 - a_0^{(i)}a_1^{(i)} - a_2^{(i)}a_3^{(i)} - \cdots - a_{n-3}^{(i)}a_{n-2}^{(i)}, \tag{3}$$

meets $\mathcal{Q}(n+1, q)$ in a quadratic cone $\mathcal{K}_i$. For each $i, j \in \{1, \ldots, k\}$ with $i \neq j$, define the $(n-1)$-dimensional space $\pi_{ij} = H_i \cap H_j$. For each $j \in \{1, \ldots, k\}$ let π_{jj} be the $(n-1)$-dimensional space π_j.

Theorem 7 *With the notation introduced above, for any $j \in \{1, \ldots, k\}$, the set $\mathcal{F}_j = \{\pi_{ij} : i = 1, \ldots, k\}$ is a partial flock of the quadratic cone $\mathcal{K}_j$ in H_j.*

Proof: We use the notation and definitions made in this subsection. Let the collineation σ of $\mathrm{PG}(n+1, q)$ be defined by

$$\sigma : (x_0, x_1, \ldots, x_{n+1}) \mapsto \left(x_0 - a_1^{(j)}x_{n+1}, x_1 - a_0^{(j)}x_{n+1}, \ldots, x_{n-3} - a_{n-2}^{(j)}x_{n+1},\right.$$
$$x_{n-2} - a_{n-3}^{(j)}x_{n+1}, x_{n-1} + \frac{1}{2}a_{n-1}^{(j)}x_{n+1}, x_n + a_0^{(j)}x_0$$
$$\left. + a_1^{(j)}x_1 + \cdots + a_{n-1}^{(j)}x_{n-1} + a_{n+1}^{(j)}x_{n+1}, x_{n+1}\right).$$

Then σ fixes $\mathcal{Q}(n+1,q)$ setwise and fixes the point p_0 and the hyperplane H_0, hence also fixes $\mathcal{K}_0$. For $i=1,\ldots,k$ the $(n-1)$-dimensional space π_i is mapped to the space with equations

$$A_0^{(i)}x_0+\cdots+A_{n-1}^{(i)}x_{n-1}+x_n=x_{n+1}=0,$$

where $A_0^{(i)}=a_0^{(i)}-a_0^{(j)},\ldots,A_{n-1}^{(i)}=a_{n-1}^{(i)}-a_{n-1}^{(j)}$. Thus, without loss of generality we can suppose that $a_0^{(j)}=\cdots=a_{n-1}^{(j)}=0$; so $p_j=(0,\ldots,0,1)$, H_j is the hyperplane with equation $x_n=0$, $\mathcal{K}_j$ is the cone with equations $x_0x_1+\cdots+x_{n-3}x_{n-2}-x_{n-1}^2=x_n=0$ and $\mathcal{F}_j$ comprises the k $(n-1)$-dimensional spaces $x_n=x_{n+1}=0$ and $a_0^{(i)}x_0+a_1^{(i)}x_1+\cdots+a_{n-1}^{(i)}x_{n-1}+a_{n+1}^{(i)}x_{n+1}=x_n=0$, for $i=1,\ldots,j-1,j+1,\ldots,k$.

We will use Theorem 3 to show that $\mathcal{F}_j$ is a partial flock. First, let $i,\ell\in\{1,2,\ldots,k\}$, with $j\neq i\neq\ell\neq j$. We must prove that

$$-4\left(\frac{a_0^{(i)}}{a_{n+1}^{(i)}}-\frac{a_0^{(\ell)}}{a_{n+1}^{(\ell)}}\right)\left(\frac{a_1^{(i)}}{a_{n+1}^{(i)}}-\frac{a_1^{(\ell)}}{a_{n+1}^{(\ell)}}\right)-\cdots-4\left(\frac{a_{n-3}^{(i)}}{a_{n+1}^{(i)}}-\frac{a_{n-3}^{(\ell)}}{a_{n+1}^{(\ell)}}\right)\left(\frac{a_{n-2}^{(i)}}{a_{n+1}^{(i)}}-\frac{a_{n-2}^{(\ell)}}{a_{n+1}^{(\ell)}}\right)$$
$$+\left(\frac{a_{n-1}^{(i)}}{a_{n+1}^{(i)}}-\frac{a_{n-1}^{(\ell)}}{a_{n+1}^{(\ell)}}\right)^2$$

is a non-square in GF(q). Put $b_j=a_j^{(i)}$ and $c_j=a_j^{(\ell)}$. So we must prove that

$$-4\left(\frac{b_0}{b_{n+1}}-\frac{c_0}{c_{n+1}}\right)\left(\frac{b_1}{b_{n+1}}-\frac{c_1}{c_{n+1}}\right)-\cdots-4\left(\frac{b_{n-3}}{b_{n+1}}-\frac{c_{n-3}}{c_{n+1}}\right)\left(\frac{b_{n-2}}{b_{n+1}}-\frac{c_{n-2}}{c_{n+1}}\right)$$
$$+\left(\frac{b_{n-1}}{b_{n+1}}-\frac{c_{n-1}}{c_{n+1}}\right)^2$$

is a non-square in GF(q). Multiplying by $(b_{n+1})^2(c_{n+1})^2$, we see that this is equivalent to showing that

$$\begin{aligned}F(i,\ell)=&-4b_0b_1(c_{n+1})^2-4c_0c_1(b_{n+1})^2+4b_0c_1b_{n+1}c_{n+1}\\&+4b_1c_0b_{n+1}c_{n+1}-\cdots-4b_{n-3}b_{n-2}(c_{n+1})^2-4c_{n-3}c_{n-2}(b_{n+1})^2\\&+4b_{n-3}c_{n-2}b_{n+1}c_{n+1}+4b_{n-2}c_{n-3}b_{n+1}c_{n+1}+(b_{n-1})^2(c_{n+1})^2\\&+(c_{n-1})^2(b_{n+1})^2-2b_{n-1}c_{n-1}b_{n+1}c_{n+1}\end{aligned}$$

is a non-square. On rearranging this expression, we find that

$$\begin{aligned}F(i,\ell)=&(c_{n+1})^2((b_{n-1})^2-4b_0b_1-\cdots-4b_{n-3}b_{n-2})\\&+(b_{n+1})^2((c_{n-1})^2-4c_0c_1-\cdots-4c_{n-3}c_{n-2})+b_{n+1}c_{n+1}\\&\times(-2b_{n-1}c_{n-1}+4b_0c_1+4b_1c_0+\cdots+4b_{n-3}c_{n-2}+4b_{n-2}c_{n-3})\end{aligned}$$

and hence, taking account of (3), that

$$\begin{aligned}F(i,\ell) = {}& 4(c_{n+1})^2 b_{n+1} + 4(b_{n+1})^2 c_{n+1} \\ & + b_{n+1}c_{n+1}(-2b_{n-1}c_{n-1} + 4b_0c_1 + 4b_1c_0 + \cdots + 4b_{n-3}c_{n-2} + 4b_{n-2}c_{n-3}) \\ = {}& b_{n+1}c_{n+1}(4c_{n+1} + 4b_{n+1} - 2b_{n-1}c_{n-1} + 4b_0c_1 + 4b_1c_0 + \cdots \\ & + 4b_{n-3}c_{n-2} + 4b_{n-2}c_{n-3}) \\ = {}& b_{n+1}c_{n+1}((c_{n-1})^2 - 4c_0c_1 - \cdots - 4c_{n-3}c_{n-2} + (b_{n-1})^2 - 4b_0b_1 - \cdots \\ & - 4b_{n-3}b_{n-2} - 2b_{n-1}c_{n-1} + 4b_0c_1 + 4b_1c_0 + \cdots \\ & + 4b_{n-3}c_{n-2} + 4b_{n-2}c_{n-3}).\end{aligned}$$

Simplifying, we find that

$$\begin{aligned}F(i,\ell) = {}& c_{n+1}b_{n+1}((c_{n-1} - b_{n-1})^2 - 4(c_0 - b_0)(c_1 - b_1) - \cdots \\ & - 4(c_{n-3} - b_{n-3})(c_{n-2} - b_{n-2})).\end{aligned}$$

Applying Theorem 3 to the pairs π_i, π_j and π_ℓ, π_j of hyperplanes in the partial flock $\mathcal{F}_0$ of $\mathcal{K}_0$ shows that each of b_{n+1} and c_{n+1} is a non-square in $\mathrm{GF}(q)$. Similarly, applying Theorem 3 to the planes π_i and π_ℓ of the partial flock $\mathcal{F}_0$ of $\mathcal{K}_0$ shows that the third factor is a non-square in $\mathrm{GF}(q)$. Thus $F(i,\ell)$ is a non-square in $\mathrm{GF}(q)$.

Finally, let $i \in \{1, \ldots, k\}$ with $i \neq j$. We must prove that

$$\left(\frac{a_{n-1}^{(i)}}{a_{n+1}^{(i)}}\right)^2 - 4\left(\frac{a_0^{(i)}}{a_{n+1}^{(i)}}\right)\left(\frac{a_1^{(i)}}{a_{n+1}^{(i)}}\right) - \cdots - 4\left(\frac{a_{n-3}^{(i)}}{a_{n+1}^{(i)}}\right)\left(\frac{a_{n-2}^{(i)}}{a_{n+1}^{(i)}}\right)$$

is a non-square in $\mathrm{GF}(q)$. But this expression is $4(a_{n+1}^{(i)})^{-1}$ and the result follows, since $a_{n+1}^{(i)}$ is a non-square in $\mathrm{GF}(q)$ as above. □

We say that the partial flocks $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are *derived* from the partial flock $\mathcal{F}_0$.

For n and q odd, let $p_0, p_1, \ldots, p_k$ be $k+1$ points of the non-singular quadric $\mathcal{Q}(n+1, q)$ and let $H_0, H_1, \ldots, H_k$ be the tangent hyperplanes to $\mathcal{Q}(n+1, q)$ at these points, respectively. The k $(n-1)$-dimensional spaces $H_0 \cap H_i$ for $i = 1, \ldots, k$ determine a partial flock of the cone $\mathcal{K}_0 = H_0 \cap \mathcal{Q}(n+1, q)$ if and only if the space $H_0 \cap H_i \cap H_j$ meets $\mathcal{Q}(n+1, q)$ in a non-singular elliptic quadric for each $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

Let $\mathcal{F}_0$ be a partial flock of $\mathcal{K}_0 = H_0 \cap \mathcal{Q}(n+1, q)$ and let $p_0, p_1, \ldots, p_k$ be the $k+1$ points associated with $\mathcal{F}_0$ as above. For any $j \in \{1, \ldots, k\}$ the $(n-1)$-dimensional spaces $H_0 \cap H_j$ and $H_i \cap H_j$, for $i = 1, \ldots, k$ with $i \neq j$, determine a partial flock of the cone $\mathcal{K}_j = H_j \cap \mathcal{Q}(n+1, q)$ by Theorem 7. Thus, any three distinct elements H_i, H_j, H_ℓ of $\{H_0, \ldots, H_k\}$ intersect in an $(n-2)$-dimensional space which meets $\mathcal{Q}(n+1, q)$ in a non-singular elliptic quadric, that is, the polar space $(p_i p_j p_\ell)^\perp$ meets $\mathcal{Q}(n+1, q)$ in a non-singular elliptic quadric.

Following the convention established in the case $n = 3$, we refer to a set of points $p_0, \ldots, p_k$ with the above properties as a *partial BLT-set.*

Let $\{p_0, p_1, \ldots, p_k\}$ be a partial BLT-set of the quadric $\mathcal{Q}(n+1, q)$. From $p_i, i \in \{0, 1, \ldots, k\}$, we project $\mathcal{Q}(n+1, q)$ onto a hyperplane $\mathrm{PG}(n, q)$ not containing p_i, thereby obtaining a well-known representation of $\mathcal{Q}(n+1, q)$ in $\mathrm{PG}(n, q)$ (see [10, 3.2.2, 3.2.4]). If H_i is the tangent hyperplane of $\mathcal{Q}(n+1, q)$ at p_i, then $H_i \cap \mathcal{Q}(n+1, q) \cap \mathrm{PG}(n, q)$ is a non-singular quadric $\mathcal{Q}(n-1, q)$ in the $(n-1)$-dimensional space $H_i \cap \mathrm{PG}(n, q) = \mathrm{PG}(n-1, q)$. If $p_i p_j \cap \mathrm{PG}(n, q) = p'_j$ for $j \in \{0, 1, \ldots, k\}$ and $j \neq i$, then it is easy to see that $\{p'_0, p'_1, \ldots, p'_{i-1}, p'_{i+1}, \ldots, p'_k\}$ is a dual partial flock $\mathcal{F}'_i$ of $\mathcal{Q}(n-1, q)$; it is also clear that $\mathcal{F}'_i$ is the dual of the flock $\mathcal{F}_i$. Conversely, if $\mathcal{F}'$ is any dual partial flock of $\mathcal{Q}(n-1, q)$ then p_i together with the points of $\mathcal{Q}(n+1, q)$ which correspond to the points of $\mathcal{F}'$ form a partial BLT-set of $\mathcal{Q}(n+1, q)$.

Further, we can construct a partial ovoid of size $kq+1$ of $\mathcal{Q}^+(n+2, q)$ directly from a partial BLT-set of $\mathcal{Q}(n+1, q)$ of size $k+1$, without going via the associated partial flock as in Section 4.2. Let $\{p_0, p_1, \ldots, p_k\}$ be a partial BLT-set of the quadric $\mathcal{Q}(n+1, q)$ in $\mathrm{PG}(n+1, q)$. Now embed $\mathrm{PG}(n+1, q)$ as a hyperplane in $\mathrm{PG}(n+2, q)$ so that $\mathcal{Q}(n+1, q)$ is embedded in a quadric $\mathcal{Q}^+(n+2, q)$ in $\mathrm{PG}(n+2, q)$. Let p be the pole of $\mathrm{PG}(n+1, q)$ under the polarity determined by $\mathcal{Q}^+(n+2, q)$. Each of the planes $\langle p, p_0, p_i\rangle$ for $i = 1, \ldots, k$ meets $\mathcal{Q}^+(n+2, q)$ in a conic, and the union of these conics is a partial ovoid of size $kq+1$ of $\mathcal{Q}^+(n+2, q)$.

4.4. *Herds of caps, q even*

Theorem 8 *In $PG(n, q)$, for n odd and q even, for $i = 1, \ldots, k$ and for $c \in GF(q)$, let*

$$\begin{aligned}
&\pi_i \;:\; a_0^{(i)}x_0 + \cdots + a_{n-1}^{(i)}x_{n-1} + x_n = 0,\\
&\mathcal{C}_\infty = \left\{\left(1, a_1^{(i)}, a_3^{(i)}, \ldots, a_{n-2}^{(i)}, \left(a_{n-1}^{(i)}\right)^2\right) \;:\; i = 1, \ldots, k\right\} \cup \{(0, \ldots, 0, 1)\} \text{ and}\\
&\mathcal{C}_c = \left\{\left(1, a_0^{(i)} + ca_1^{(i)} + c^{1/2}a_{n-1}^{(i)}, a_2^{(i)} + ca_3^{(i)} + c^{1/2}a_{n-1}^{(i)}, \ldots, a_{n-3}^{(i)} + ca_{n-2}^{(i)}\right.\right.\\
&\qquad\left.\left. + c^{1/2}a_{n-1}^{(i)}, \left(a_{n-1}^{(i)}\right)^2\right) \;:\; i = 1, \ldots, k\right\} \cup \{(0, \ldots, 0, 1)\},
\end{aligned}$$

for some $a_j^{(i)} \in GF(q)$. If the set $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ of k hyperplanes is a partial flock of the quadratic cone $\mathcal{K} : x_0x_1 + x_2x_3 + \cdots + x_{n-3}x_{n-2} = x_{n-1}^2$ then each of $\mathcal{C}_\infty$ and $\mathcal{C}_c$, for all $c \in GF(q)$, is a $(k+1)$-cap in $PG((n+1)/2, q)$ for $n > 3$ and a $(k+1)$-arc in $PG(2, q)$ for $n = 3$.

Proof: Suppose $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ is a partial flock of the quadratic cone $\mathcal{K}$. We first show that no three points of $\mathcal{C}_\infty \backslash \{(0, \ldots, 0, 1)\}$ are collinear. Suppose to the contrary that for some $i, j, \ell \in \{1, \ldots, k\}$ the matrix

$$\begin{pmatrix}
1 & a_1^{(i)} & a_3^{(i)} & \ldots & a_{n-2}^{(i)} & \left(a_{n-1}^{(i)}\right)^2\\
1 & a_1^{(j)} & a_3^{(j)} & \ldots & a_{n-2}^{(j)} & \left(a_{n-1}^{(j)}\right)^2\\
1 & a_1^{(\ell)} & a_3^{(\ell)} & \ldots & a_{n-2}^{(\ell)} & \left(a_{n-1}^{(\ell)}\right)^2
\end{pmatrix}$$

has rank 2. It follows easily that there exist elements $\alpha_1, \alpha_3, \ldots, \alpha_{n-2} \in \mathrm{GF}(q)$ such that

$$\frac{a_1^{(i)} + a_1^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2} = \frac{a_1^{(j)} + a_1^{(\ell)}}{\left(a_{n-1}^{(j)}\right)^2 + \left(a_{n-1}^{(\ell)}\right)^2} = \frac{a_1^{(\ell)} + a_1^{(i)}}{\left(a_{n-1}^{(\ell)}\right)^2 + \left(a_{n-1}^{(i)}\right)^2} = \alpha_1,$$

$$\vdots$$

$$\frac{a_{n-2}^{(i)} + a_{n-2}^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2} = \frac{a_{n-2}^{(j)} + a_{n-2}^{(\ell)}}{\left(a_{n-1}^{(j)}\right)^2 + \left(a_{n-1}^{(\ell)}\right)^2} = \frac{a_{n-2}^{(\ell)} + a_{n-2}^{(i)}}{\left(a_{n-1}^{(\ell)}\right)^2 + \left(a_{n-1}^{(i)}\right)^2} = \alpha_{n-2}.$$

Using the algebraic condition in Theorem 3 we obtain:

$$\mathrm{trace}\Big(\alpha_1\big(a_0^{(i)} + a_0^{(j)}\big) + \alpha_3\big(a_2^{(i)} + a_2^{(j)}\big) + \cdots + \alpha_{n-2}\big(a_{n-3}^{(i)} + a_{n-3}^{(j)}\big)\Big) = 1, \tag{4}$$

$$\mathrm{trace}\Big(\alpha_1\big(a_0^{(j)} + a_0^{(\ell)}\big) + \alpha_3\big(a_2^{(j)} + a_2^{(\ell)}\big) + \cdots + \alpha_{n-2}\big(a_{n-3}^{(j)} + a_{n-3}^{(\ell)}\big)\Big) = 1, \tag{5}$$

$$\mathrm{trace}\Big(\alpha_1\big(a_0^{(\ell)} + a_0^{(i)}\big) + \alpha_3\big(a_2^{(\ell)} + a_2^{(i)}\big) + \cdots + \alpha_{n-2}\big(a_{n-3}^{(\ell)} + a_{n-3}^{(i)}\big)\Big) = 1. \tag{6}$$

Adding Eqs. (4), (5) and (6) implies that $\mathrm{trace}(0) = 1$, a contradiction. Thus $\mathcal{C}_\infty \backslash \{(0, \ldots, 0, 1)\}$ is a k-cap of $\mathrm{PG}((n+1)/2, q)$. Finally, suppose that two points of $\mathcal{C}_\infty$ are collinear with $(0, \ldots, 0, 1)$. Then there exist $i, j \in \{1, \ldots, k\}$ such that $a_1^{(i)} + a_1^{(j)}, a_3^{(i)} + a_3^{(j)}, \ldots, a_{n-2}^{(i)} + a_{n-2}^{(j)}$ are all zero. But this contradicts the condition in Theorem 3. Thus $\mathcal{C}_\infty$ is a $(k+1)$-cap of $\mathrm{PG}((n+1)/2, q)$.

Next, for $c \in \mathrm{GF}(q)$, we consider $\mathcal{C}_c$. For $r = 0, 2, \ldots, n-3$ and for $i, j \in \{1, \ldots, k\}$ let

$$\alpha_r^{ij} = \frac{a_r^{(i)} + a_r^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2} + c\frac{a_{r+1}^{(i)} + a_{r+1}^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2} + c^{1/2}\frac{a_{n-1}^{(i)} + a_{n-1}^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2}.$$

Suppose that some three points of $\mathcal{C}_c \backslash \{(0, \ldots, 0, 1)\}$ are collinear; so for some $i, j, \ell \in \{1, \ldots, k\}$ there exist $\alpha_0, \alpha_2, \ldots, \alpha_{n-3}$ such that

$$\alpha_0^{ij} = \alpha_0^{j\ell} = \alpha_0^{\ell i} = \alpha_0,$$
$$\alpha_2^{ij} = \alpha_2^{j\ell} = \alpha_2^{\ell i} = \alpha_2,$$
$$\vdots$$
$$\alpha_{n-3}^{ij} = \alpha_{n-3}^{j\ell} = \alpha_{n-3}^{\ell i} = \alpha_{n-3}.$$

Consider

$$\begin{aligned}\alpha_0\big(a_1^{(i)} + a_1^{(j)}\big) &= \frac{a_0^{(i)} + a_0^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2}\big(a_1^{(i)} + a_1^{(j)}\big) + c\frac{a_1^{(i)} + a_1^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2}\big(a_1^{(i)} + a_1^{(j)}\big) \\ &\quad + c^{1/2}\frac{a_{n-1}^{(i)} + a_{n-1}^{(j)}}{\left(a_{n-1}^{(i)}\right)^2 + \left(a_{n-1}^{(j)}\right)^2}\big(a_1^{(i)} + a_1^{(j)}\big) \\ &= c_0^{ij} + b_0^{ij},\end{aligned}$$

where $c_0^{ij} = (a_0^{(i)} + a_0^{(j)})(a_1^{(i)} + a_1^{(j)})/((a_{n-1}^{(i)})^2 + (a_{n-1}^{(j)})^2)$ and $\mathrm{trace}(b_0^{ij}) = 0$, as b_0^{ij} is of the form $t + t^2$ for some $t \in \mathrm{GF}(q)$. Analogously, we write

$$\alpha_0\big(a_1^{(j)} + a_1^{(\ell)}\big) = c_0^{j\ell} + b_0^{j\ell},$$
$$\alpha_0\big(a_1^{(\ell)} + a_1^{(i)}\big) = c_0^{\ell i} + b_0^{\ell i},$$

where $\mathrm{trace}(b_0^{j\ell}) = \mathrm{trace}(b_0^{\ell i}) = 0$. On adding these three equations, we obtain $0 = c_0^{ij} + c_0^{j\ell} + c_0^{\ell i} + b_0$, where $b_0 = b_0^{ij} + b_0^{j\ell} + b_0^{\ell i}$ satisfies $\mathrm{trace}(b_0) = 0$. Repeating these calculations with 0 replaced by r for $r = 2, 4, \ldots, n-3$, we obtain:

$$0 = b_2 + c_2^{ij} + c_2^{j\ell} + c_2^{\ell i},$$
$$\vdots$$
$$0 = b_{n-3} + c_{n-3}^{ij} + c_{n-3}^{j\ell} + c_{m-3}^{\ell i},$$

for analogous expressions $b_r, c_r^{ij}, c_r^{j\ell}, c_r^{\ell i} \in \mathrm{GF}(q)$ satisfying $\mathrm{trace}(b_2) = \cdots = \mathrm{trace}(b_{n-3}) = 0$. Adding these $(n-1)/2$ equations shows that $0 = b + c_{ij} + c_{j\ell} + c_{\ell i}$, where $b = b_0 + b_2 + \cdots + b_{n-3}$, $c_{ij} = c_0^{ij} + c_2^{ij} + \cdots + c_{n-3}^{ij}$ and $c_{j\ell}, c_{\ell i}$ are analogous. Further, $\mathrm{trace}(b) = 0$, and by Theorem 3, we have $\mathrm{trace}(c_{ij}) = \mathrm{trace}(c_{j\ell}) = \mathrm{trace}(c_{\ell i}) = 1$, implying that $\mathrm{trace}(0) = 1$, a contradiction. Thus we have shown that $\mathcal{C}_c \backslash \{(0, \ldots, 0, 1)\}$ is a k-cap. Finally, suppose that two points of $\mathcal{C}_c$ are collinear with $(0, \ldots, 0, 1)$. Then there exist $i, j \in \{1, \ldots, k\}$ such that $a_0^{(i)} + a_0^{(j)} + c(a_1^{(i)} + a_1^{(j)}) + c^{1/2}(a_{n-1}^{(i)} + a_{n-1}^{(j)}), \ldots, a_{n-3}^{(i)} + a_{n-3}^{(j)} + c(a_{n-2}^{(i)} + a_{n-2}^{(j)}) + c^{1/2}(a_{n-1}^{(i)} + a_{n-1}^{(j)})$ are all zero. Multiplying the first expression by $(a_1^{(i)} + a_1^{(j)})/((a_{n-1}^{(i)})^2 + (a_{n-1}^{(j)})^2)$, we see that $0 = c_0^{ij} + d$, where $\mathrm{trace}(d) = 0$. Thus $\mathrm{trace}(c_0^{ij}) = 0$, and analogously (multiplying the remaining expressions by $(a_3^{(i)} + a_3^{(j)})/((a_{n-1}^{(i)})^2 + (a_{n-1}^{(j)})^2), \ldots, (a_{n-2}^{(i)} + a_{n-2}^{(j)})/((a_{n-1}^{(i)})^2 + (a_{n-1}^{(j)})^2)$ respectively), we find that $\mathrm{trace}(c_2^{ij}) = \cdots = \mathrm{trace}(c_{n-3}^{ij}) = 0$. Thus $\mathrm{trace}(c_{ij}) = \mathrm{trace}(c_0^{ij} + c_2^{ij} + \cdots + c_{n-3}^{ij}) = 0$, contradicting Theorem 3. Hence, for $c \in \mathrm{GF}(q)$, $\mathcal{C}_c$ is a $(k+1)$-cap of $\mathrm{PG}((n+1)/2, q)$. □

Such a set of $(k+1)$-caps, of which there are $q+1$, is called a *herd* of $(k+1)$-caps. By Theorem 2, the caps have maximum size $q+1$.

Remarks:

(1) For $n = 3$ we refer to [2] and [11]. In this case the $(k+1)$-arcs of Theorem 8 extend to $(k+2)$-arcs by adjoining the point $(0, 1, 0)$. Further, the converse of Theorem 8 holds.

(2) There are $2^{(n-1)/2}$ herds of caps projectively equivalent to those arising in Theorem 8 and obtained by interchanging in turn each subset of the pairs of coordinates $(x_0, x_1), (x_2, x_3), \ldots, (x_{n-3}, x_{n-2})$.

5. Examples and characterisations of partial flocks of $\mathcal{K}$

5.1. *The linear partial flocks*

Let $\mathcal{K} = v\mathcal{Q}$ be a quadratic cone in PG(n, q), where n is odd. Let PG$(n-2, q)$ be an $(n-2)$-dimensional subspace of PG(n, q) such that PG$(n-2, q) \cap \mathcal{K}$ is a non-singular elliptic quadric. Then k hyperplanes on PG$(n-2, q)$ not containing v are a partial flock of $\mathcal{K}$ of size k, called a *linear* partial flock; clearly $k \leq q$.

A partial flock is linear if and only if the corresponding dual partial flock is k points of a line.

Theorem 9 *Let $\mathcal{F} = \{\pi_1, \ldots, \pi_k\}$ be a partial flock of size k of the quadratic cone $\mathcal{K} = v\mathcal{Q}$ in PG(n, q), $n > 3$ odd. Suppose that for some $i, j \in \{1, \ldots, k\}$ with $i \neq j$ the elements of $\mathcal{F}$ cover the points of $\mathcal{K}\backslash v\mathcal{E}_{ij}$, where $\mathcal{E}_{ij} = \pi_i \cap \pi_j \cap \mathcal{K}$. Then $k \geq q$ and if $k = q$ then $\mathcal{F}$ is linear.*

Proof: Let $\mathcal{S} = \mathcal{K}\backslash v\mathcal{E}_{ij}$ and suppose the elements of $\mathcal{F}$ cover the points of $\mathcal{S}$.

For $P \in \mathcal{S}$, let N_P denote the number of elements of $\mathcal{F}$ on P. By hypothesis, $N_P \geq 1$ for $P \in \mathcal{S}$. Now count the ordered pairs (P, π_ℓ) where $P \in \mathcal{S}, \pi_\ell \in \mathcal{F}$ and $P \in \pi_\ell$. We obtain:

$$q(|\mathcal{Q}| - |\mathcal{E}_{ij}|) = |\mathcal{S}| \leq \sum_{P \in \mathcal{S}} N_P = k(|\mathcal{Q}| - |\mathcal{E}_{ij}|).$$

Thus $k \geq q$ and if $k = q$ then equality must hold throughout the expression, so $N_P = 1$ for all $P \in \mathcal{S}$ and $\mathcal{F}$ partitions $\mathcal{K}\backslash v\mathcal{E}_{ij}$. We note that $\pi_i \cap v\mathcal{E}_{ij} = \pi_j \cap v\mathcal{E}_{ij} = \mathcal{E}_{ij}$. Let $\ell, m \in \{1, \ldots, q\}, \ell \neq m$, and let $\mathcal{E}_{\ell m} = \pi_\ell \cap \pi_m \cap \mathcal{K}$. We have shown that $\mathcal{E}_{\ell m} \subseteq v\mathcal{E}_{ij}$; so $\pi_\ell \cap v\mathcal{E}_{ij} = \pi_m \cap v\mathcal{E}_{ij} = \mathcal{E}_{\ell m}$. We may assume that $i \neq \ell$. Then $\pi_i \cap \pi_\ell \cap \mathcal{K} = \pi_i \cap \pi_\ell \cap v\mathcal{E}_{ij} = \mathcal{E}_{ij} \cap \mathcal{E}_{\ell m}$ is a non-singular elliptic quadric in some $(n-2)$-dimensional subspace of PG(n, q). Thus $\mathcal{E}_{ij} = \mathcal{E}_{\ell m}$, hence $\mathcal{F}$ is linear. □

The elements of a linear partial flock of size k have a common $(n-2)$-dimensional subspace; so the corresponding partial ovoid of size $kq + 1$ lies in a 3-dimensional space. In fact this partial ovoid lies in an elliptic quadric.

5.2. *Partial flocks with partial BLT-set a normal rational curve*, q *odd*

These examples generalise the Fisher-Thas-Walker flocks in PG$(3, q)$ q odd, [3, 13], since by [1] such a flock in PG$(3, q)$ has BLT-set a normal rational curve on $\mathcal{Q}(4, q)$.

Theorem 10 *In PG(n, q) for $n \geq 3$ odd and q odd, let $\mathcal{K}$ be the quadratic cone with equation $x_0x_1 + \cdots + x_{n-3}x_{n-2} = x_{n-1}^2$. For $t \in GF(q)$, let π_t be the hyperplane with equation* $a_n t^n x_0 + a_1 t x_1 + a_{n-1} t^{n-1} x_2 + a_2 t^2 x_3 + \cdots + a_{(n+3)/2} t^{(n+3)/2} x_{n-3} + a_{(n-1)/2} t^{(n-1)/2} x_{n-2} +$

$a_{(n+1)/2}t^{(n+1)/2}x_{n-1} + x_n = 0$ *where for* $i = 1, 2, \ldots, (n-1)/2$ *and for some element* α *a non-square in* $GF(q)$*, we have*

$$4a_{n+1-i}a_i = (-1)^i \binom{n+1}{i}\alpha \qquad \text{and} \qquad a^2_{(n+1)/2} = \frac{\alpha}{2}(-1)^{(n+3)/2}\binom{n+1}{\frac{n+1}{2}}.$$

Then the set $\mathcal{F} = \{\pi_t : t \in GF(q)\}$ *is a partial flock of size* q *of* $\mathcal{K}$*, with BLT-set a normal rational curve of* $PG(n+1, q)$ *if and only if* $a_1a_2\cdots a_n \neq 0$. (*For a given non-square* $\alpha \in GF(q)$, q *odd, there exists such a partial flock if and only if* $(1/2)(-1)^{(n+3)/2}\binom{n+1}{\frac{n+1}{2}}$ *is either zero or a non-square.*)

Proof: We use Theorem 3. For $s, t \in \mathrm{GF}(q)$, $s \neq t$, we have

$$\begin{aligned}
&-4(a_nt^n - a_ns^n)(a_1t - a_1s) - 4(a_{n-1}t^{n-1} - a_{n-1}s^{n-1})(a_2t^2 - a_2s^2) - \cdots \\
&\quad - 4\big(a_{(n+3)/2}t^{(n+3)/2} - a_{(n+3)/2}s^{(n+3)/2}\big)\big(a_{(n-1)/2}t^{(n-1)/2} - a_{(n-1)/2}s^{(n-1)/2}\big) \\
&\quad + \big(a_{(n+1)/2}t^{(n+1)/2} - a_{(n+1)/2}s^{(n+1)/2}\big)^2 \\
&= (t^{n+1} + s^{n+1})\big(-4a_na_1 - 4a_{n-1}a_2 - \cdots - 4a_{(n+3)/2}a_{(n-1)/2} + a^2_{(n+1)/2}\big) \\
&\quad + (t^ns + ts^n)(4a_na_1) + (t^{n-1}s^2 + t^2s^{n-1})(4a_{n-1}a_2) + \cdots + \big(t^{(n+3)/2}s^{(n-1)/2} \\
&\quad + t^{(n-1)/2}s^{(n+3)/2}\big)\big(4a_{(n+3)/2}a_{(n-1)/2}\big) + t^{(n+1)/2}s^{(n+1)/2}\big(-2a^2_{(n+1)/2}\big) \\
&= \alpha(t-s)^{n+1},
\end{aligned}$$

by the definition of $a_1, \ldots, a_n$ and noting that the coefficient of $(t^{n+1} + s^{n+1})$ in the expression is

$$\alpha\sum_{i=1}^{(n-1)/2}(-1)^{i+1}\binom{n+1}{i} + \frac{\alpha}{2}(-1)^{(n+3)/2}\binom{n+1}{\frac{n+1}{2}} = \frac{\alpha}{2}\sum_{i=1}^{n}(-1)^{i+1}\binom{n+1}{i} = \alpha.$$

By Theorem 3, $\mathcal{F}$ is a partial flock of $\mathcal{K}$ of size q. The associated BLT-set is the normal rational curve $\{(a_1t, a_nt^n, a_2t^2, a_{n-1}t^{n-1}, \ldots, a_{(n-1)/2}t^{(n-1)/2}, a_{(n+3)/2}t^{(n+3)/2}, (-1/2)a_{(n+1)/2}t^{(n+1)/2}, (\alpha/4)t^{n+1}, 1) : t \in \mathrm{GF}(q)\} \cup \{(0, \ldots, 0, 1, 0)\}$. □

5.3. *Other non-linear partial flocks*

The first examples generalise the Kantor flocks in $\mathrm{PG}(3, q)$, for q odd [9], see also [12, 1.5.6].

Theorem 11 *For* $t \in \mathcal{T} \subseteq GF(q)$, q *odd, let* π_t *have equation* $a_0^{(t)}x_0 + a_1^{(t)}x_1 + \cdots + a_{n-1}^{(t)}x_{n-1} + x_n = 0$, *where* $a_j^{(t)} \in GF(q)$. *For each* $t \in \mathcal{T}$, *let* $a_1^{(t)} + a_3^{(t)} + \cdots + a_{n-2}^{(t)} = -bt^\sigma$, *where* b *is a non-square in* $GF(q)$ *and* $\sigma \in AutGF(q)$, *let* $a_{n-1}^{(t)} = 0$ *and for* $j = 2i$, $i = 0, 1, \ldots, (n-3)/2$, *let* $a_j^{(t)} = t$. *Then* $\mathcal{F} = \{\pi_t : t \in \mathcal{T}\}$ *is a partial flock of size* $|\mathcal{T}|$ *of the cone* $\mathcal{K}$ *in* $PG(n, q)$ *with equation* $x_0x_1 + \cdots + x_{n-3}x_{n-2} = x^2_{n-1}$.

Proof: We use Theorem 3, noting that for $i, j \in \mathcal{T}, i \neq j$, we have

$$-4\big(a_0^{(i)} - a_0^{(j)}\big)\big(a_1^{(i)} - a_1^{(j)}\big) - \cdots - 4\big(a_{n-3}^{(i)} - a_{n-3}^{(j)}\big)\big(a_{n-2}^{(i)} - a_{n-2}^{(j)}\big) + \big(a_{n-1}^{(i)} - a_{n-1}^{(j)}\big)^2$$
$$= 4b(i-j)^{\sigma+1},$$

which is a non-square in $\mathrm{GF}(q)$. □

For example, let $a_1^{(t)} = -bt^\sigma$ and let all the terms $a_{2i+1}^{(t)}$ be zero, $i = 1, \ldots, (n-3)/2$. Then π_t has equation $tx_0 - bt^\sigma x_1 + tx_2 + tx_4 + \cdots + tx_{n-3} + x_n = 0$, so contains the subspace with equation $x_0 = x_1 = x_2 = x_4 = \cdots = x_{n-3} = x_n = 0$. If $\mathcal{T} = \mathrm{GF}(q)$ then the above partial flock induces a Kantor flock of the cone $x_0x_1 = x_{n-1}^2$ in the subspace with projective coordinates (x_0, x_1, x_{n-1}, x_n).

In this example, for $\sigma \neq 1$, the hyperplanes of the partial flock intersect in the $(n-3)$-dimensional space $x_1 = x_n = x_0 + x_2 + \cdots + x_{n-3} = 0$. So for $\sigma \neq 1$ we have a partial ovoid of $\mathcal{Q}^+(n+2, q)$ of size q^2+1 and lying in a 4-dimensional space $\mathrm{PG}(4, q)$. As $\mathrm{PG}(4, q)$ intersects $\mathcal{Q}^+(n+2, q)$ in a non-singular quadric $\mathcal{Q}(4, q)$, we obtain an ovoid of $\mathcal{Q}(4, q)$ (which is, in fact, a Kantor ovoid of $\mathcal{Q}(4, q)$ [9]).

Now, let $\mathcal{F}$ be a partial flock of size k of a quadratic cone $\mathcal{K}$ in $\mathrm{PG}(m, q)$, for some odd $m \geq 3$, and suppose that all the hyperplanes in $\mathcal{F}$ intersect in a common r-dimensional subspace. By Theorem 4, there is associated a partial ovoid $\mathcal{O}$ of size $kq+1$ of $\mathcal{Q}^+(m+2, q)$ such that the points of $\mathcal{O}$ generate an $(m-r+1)$-dimensional subspace. Now embed $\mathcal{Q}^+(m+2, q)$ in $\mathcal{Q}^+(n+2, q)$ where n is odd and $n \geq m$. Then $\mathcal{O}$ is a partial ovoid of size $kq+1$ of $\mathcal{Q}^+(n+2, q)$ consisting of k mutually tangent conics, so by Theorem 4 there is associated a partial flock of size k of a quadratic cone in $\mathrm{PG}(n, q)$ such that the hyperplanes of the partial flock intersect in a common $(n-m+r)$-dimensional subspace.

For example, let $m = 3$ and $k = q$ and let $\mathcal{F}$ be a linear flock of $\mathcal{K}$ (so $r = 1$). Then there exists a partial flock of size q of a quadratic cone in $\mathrm{PG}(n, q)$ for each odd $n \geq 3$ such that the hyperplanes in the partial flock intersect in a common $(n-2)$-dimensional subspace, that is, the partial flock is linear.

More generally, let $\mathcal{O}$ be a partial ovoid of size $kq+1$ of a (singular or non-singular) quadric $\mathcal{Q}$ in $\mathrm{PG}(m, q)$ (where m is odd or even), and suppose that $\mathcal{O}$ comprises k mutually tangent conics. Embed $\mathcal{Q}$ in $\mathcal{Q}^+(n+2, q)$ where $n+2 \geq m$ and n is odd (the smallest possible value for n will depend on the type of $\mathcal{Q}$). Then $\mathcal{O}$ is a partial ovoid of $\mathcal{Q}^+(n+2, q)$ comprising k mutually tangent conics, hence determines a partial flock of size k of a quadratic cone in $\mathrm{PG}(n, q)$. If the points of $\mathcal{O}$ generate an l-dimensional space then the hyperplanes in the partial flock intersect in a common $(n-\ell+1)$-dimensional subspace.

For example, let $m = 6$ and let $\mathcal{Q} = L\mathcal{Q}'$ be the singular quadric with vertex a line L and base a non-singular quadric $\mathcal{Q}'$ in $\mathrm{PG}(4, q)$. Let $\mathcal{O}$ be an ovoid of $\mathcal{Q}$ consisting of q mutually tangent conics (from an ovoid $\mathcal{O}'$ of $\mathcal{Q}'$ consisting of q mutually tangent conics many such ovoids $\mathcal{O}$ can be constructed). Embed $\mathcal{Q}$ in a $\mathcal{Q}^+(n+2, q)$, n odd and $n \geq 7$. Then there arises a partial flock of size q of a quadratic cone in $\mathrm{PG}(n, q)$, the hyperplanes of which intersect in at least an $(n-5)$-dimensional space (if $\mathcal{O}'$ is an elliptic quadric, then they intersect in at least an $(n-4)$-dimensional space).

6. Partial flocks for small *q*

In PG$(n, 2)$, a partial flock of a quadratic cone $\mathcal{K} = v\mathcal{Q}$ with vertex v has size at most two. Further, every partial flock of $\mathcal{K}$ of cardinality 2 is linear.

In PG$(5, 3)$, let $\mathcal{K} = v\mathcal{Q}$ be the quadratic cone with equation $x_0x_1 + x_2x_3 = x_4^2$. Using the notation $[a_0, a_1, \ldots, a_5]$ for the hyperplane $a_0x_0 + a_1x_1 + \cdots + a_5x_5 = 0$, a partial flock of $\mathcal{K}$ of size six in PG$(5, 3)$ is $\mathcal{F} = \{[0, 0, 0, 0, 0, 1], [0, 0, 1, 1, 0, 1], [0, 1, 2, 2, 0, 1], [2, 0, 2, 2, 0, 1], [2, 1, 0, 1, 1, 1], [2, 1, 1, 0, 2, 1]\}$. Thus for $n > 3$ and q odd, there exist partial flocks of size greater than q.

It is an open problem to determine the maximum size of a partial flock of a quadratic cone in PG(n, q) for q odd.

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References

1. L. Bader, G. Lunardon, and J.A. Thas, "Derivation of flocks of quadratic cones," *Forum Math.* **2** (1990), 163–174.
2. W.E. Cherowitzo, T. Penttila, I. Pinneri, and G.F. Royle, "Flocks and ovals," *Geom. Dedicata* **60** (1996), 17–37.
3. J.C. Fisher and J.A. Thas, "Flocks in PG$(3, q)$," *Math. Z.* **169** (1979), 1–11.
4. J.W.P. Hirschfeld, *Projective Geometries Over Finite Fields*, Oxford University Press, Oxford, 1979.
5. J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions,* Oxford University Press, Oxford, 1985.
6. J.W.P. Hirschfeld and J.A. Thas, "Sets of type $(1, n, q + 1)$ in PG(d, q)," *Proc. London Math. Soc. (3)* **41** (1980), 254–278.
7. J.W.P. Hirschfeld and J.A. Thas, "The characterization of projections of quadrics over finite fields of even order," *J. London Math. Soc. (2)* **22** (1980), 226–238.
8. J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries,* Oxford University Press, Oxford, 1991.
9. W.M. Kantor, "Ovoids and translation planes," *Canad. J. Math.* **34** (1982), 1195–1207.
10. S.E. Payne and J.A. Thas, *Finite Generalized Quadrangles*, Pitman, London, 1984.
11. L. Storme and J.A. Thas, "*k*-arcs and partial flocks," *Linear Algebra and Its Applications* **226–228** (1995), 33–45.
12. J.A. Thas, "Generalized quadrangles and flocks of cones," *European J. Combin.* **8** (1987), 441–452.
13. M. Walker, "A class of translation planes," *Geometriae Dedicata* **5** (1976), 135–146.