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Rational Functions and Association Scheme Parameters

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Abstract. The parameters of metric, cometric, symmetric association schemes with $q \neq \pm 1$ (the same as the parameters of the underlying orthogonal polynomials) can be given in general by evaluating a single rational function of degree (4, 4) in the complex variable q^j. But in all known examples, save the simple n-gons, these reduce to polynomials of degree at most 2 in q^j with q an integer. One reason this occurs is that the rational function can have singularities at points which would determine some of the parameters. This paper deals with the case in which not all of the singularities are removable, thus giving some reason why the n-gons might naturally be the only exceptions to schemes with parameters being polynomials of degree at most 2 in q^j, except possibly for schemes of very small diameter.

Keywords: metric, cometric, symmetric association scheme, discrete orthogonal polynomial, rational function, monic Tchebyshev polynomial

0. Introduction

Association schemes with d (non-trivial) relations R_i in general have $(d+1)^3$ connection parameters

$$p_{i,j,k} := |\{z\colon (x,z) \in R_i, (z,y) \in R_j\}| \quad \text{for } (x,y) \in R_k.$$

While these are not all independent, there are still $O(d^3)$ independent parameters. Even symmetric, metric (P-polynomial) schemes, for which there is a distance function determining these would seem to have $2d$ parameters

$$b_j := p_{1,j+1,j} \quad \text{and} \quad c_j := p_{1,j-1,j}.$$

In the early 80's it was proposed that a classification of all metric, cometric, (that is, P- and Q-polynomial) symmetric association schemes should be possible. [For a reasonable background to this material, the reader is referred to either Bannai and Ito [1] (Chapter III) or Brouwer et al. [2] (Chapter 8).] Part of this classification scheme, namely the determination of the parameters, was settled in some sense in Leonard [3], in that the parameters b_j and c_j of the underlying discrete orthogonal polynomials (and hence, the same parameters of the schemes) were given as rational functions of degree (4, 4) in the complex variable q^j.

This, in turn, means that said parameters are described by a fixed number of variables, independent of the diameter d.

However, in all known examples *except* the common n-gons, this rational function actually degenerates to a polynomial of degree 2 in q^j, with q an integer. In fact, it is conjectured that this must be the case. Proving this conjecture may depend only the condition that the parameters of the scheme must be non-negative integers (rather than just complex numbers) and *not* on any other properties of the scheme. [It should be noted that, in this paper, very little about the scheme itself is used. In fact, so little about such a scheme is used, that we choose not to even define it here. The proof is entirely in terms of a rational function having certain integer values at ceratin prescribed points. So even though the assumption in the theorems is that a metric, cometric, symmetric association scheme exists, the only use made of that assumption is that there are two sequences of parameters, b_j and c_j, which are non-negative integers, that they have a known form (given in the literature but revised immediately below), that $c_0 = 0 < c_1 = 1 < \cdots < c_d \leq b_0 > b_1 > \cdots > b_d = 0$, and that the dual eigenvalues θ_j^* are real and distinct. Given these as ground rules, it is possible to read this paper as a paper about rational functions with such integer values at ceratin prescribed points, though the results will be of little import unless applied to metric, cometric, symmetric association schemes, studied in either reference [1] or [2] mentioned above.]

We shall assume that the parameters in question satisfy the equations (in complex variables):

$$\begin{aligned}
(\sigma_0^* - \sigma_3^* q^{2j})(\sigma_0^* - \sigma_3^* q^{2j-1}) b_{j-1} &= (\sigma_0^* - \sigma_3^* q^j) \\
&\quad \times (\sigma_0^* \sigma_0 - \delta_1 q^j + \delta_2 q^{2j} - \sigma_3^* \sigma_3 q^{3j}) \\
(\sigma_0^* - \sigma_3^* q^{2j})(\sigma_0^* - \sigma_3^* q^{2j+1}) c_j &= q(1 - q^j) \\
&\quad \times \left(\sigma_0^{*2} \sigma_3 - \sigma_0^* \delta_2 q^j + \sigma_3^* \delta_1 q^{2j} - \sigma_3^{*2} \sigma_0 q^{3j}\right) \\
\theta_j^* - \theta_0^* &= q^{-j}(1 - q^j)(\sigma_0^* - \sigma_3^* q^{j+1}),
\end{aligned}$$

(with the θ_j^*'s, real and distinct) as do the same parameters of the underlying orthogonal polynomials. [The form in Bannai and Ito (Case(I), page 264) can be gotten by the change of variables $\sigma_0 := h, \delta_1 := hh^*(r_1 + r_2 + r_3), \delta_2 := hh^*(r_1 r_2 + r_1 r_3 + r_2 r_3), \sigma_3 := hs$, $\sigma_0^* := h^*, \sigma_3^* := h^* s^*$; or these can be derived directly from Leonard (Eq. (2.11)), given the form of θ_j^* (and dually θ_j). It is advisable not to divide yet to solve for b_{j-1} and c_j. This is done in Bannai and Ito (Theorem 5.1), wherein some attention is paid to separating the cases $c_0 = 0, b_0, b_d = 0, c_d$. But the special form for b_0 given there is unnecessary for their $s^* \neq q^{-1}$, and doesn't follow from what is given when $s^* = q^{-1}$. Also in this form it is more natural to replace Bannai and Ito (Case(I), $s^* \neq 0$) with the above with $\sigma_3^* \neq 0$, (Case(I), $s^* = 0$) with the above with $\sigma_3^* = 0$, and treat (IA) and (IB) as unnecessary special cases of the latter.]

For metric, cometric, symmetric association schemes, the parameters b_j and c_j have one obvious *extra* condition that, since they count something, they must be non-negative integers rather than just complex numbers.

In most known examples of such schemes, $\sigma_3^* = 0$ (that is, $s^* = 0$ in Bannai and Ito notation), in which case (with $\sigma_1 := \delta_1/\sigma_0^*$ and $\sigma_2 := \delta_2/\sigma_0^*$) the equations for the parameters reduce to $b_{j-1} = \sigma_0 - \sigma_1 q^j + \sigma_2 q^{2j}$ and $c_j = q(1-q^j)(\sigma_3 - \sigma_2 q^j)$. The only known examples for which $q \neq \pm 1$ and $\sigma_3^* \neq 0$ are the simple n-gons. In fact, the conjecture of Bannai and Ito (page 366) alluded to above is that $s^* = 0$ in Case (I), except for these n-gons.

For $q \neq \pm 1$, there is, given in Theorem 3 a common function

$$h(z) := \frac{q(1-z)\left(\sigma_3^{*2}\sigma_0 - \sigma_0^*\sigma_3^*\sigma_1 z + \sigma_0^{*2}\sigma_2 z^2 - \sigma_0^{*2}\sigma_3 z^3\right)}{(\sigma_3^* q - \sigma_0^* z^2)(\sigma_3^* - \sigma_0^* z^2)},$$

which generates (most of) both parameter sequences in a natural way, namely $b_j = h(q^{j+1}\sigma_3^*/\sigma_0^*)$ and $c_j = h(q^{-j})$ for $0 \leq j \leq d$. The cases in which $c_0 = 0$ cannot be solved for in this function (or equivalently in the equations above) are special, and will be treated in this paper (by using rational functions and monic Tchebyshev polynomials). The results, summarized in Theorems 5 and 6, are that in these cases the only possible sequences of parameters are those for the n-gons, except possibly for some schemes of very small diameter.

1. Monic Tchebyshev polynomials

Let $\omega := q + q^{-1}, q \neq \pm 1$. [When $\sigma_0^*\sigma_3^* \neq 0$, ω is a much better parameter than q in the sense that the dual eigenvalues (or the eigenvalues) being real, forces ω to be real, as opposed to forcing q to be real or to lie on the complex unit circle. [It is also like preferring $\cos\theta$ or $\cosh\theta$ instead of $e^{i\theta}$.] Also the parameters can be given in terms of either q or q^{-1}, but both reduce to the same equations in terms of ω.]

Consider the polynomials in ω defined by

$$\begin{aligned} p_{2m+1} &= p_{2m+1}(\omega) := q^{-m}(q^{2m+1} - 1)/(q-1) \quad \text{and} \\ p_{2m+2} &= p_{2m+2}(\omega) := q^{-m}(q^{2m+2} - 1)/(q^2 - 1), \end{aligned}$$

normally used for writing $\sin(\frac{1}{2}(2m+1)\theta)/\sin(\frac{1}{2}\theta)$ and $\sin((m+1)\theta)/\sin\theta$ in terms of $\omega := 2\cos\theta$. Both are monic of degree m for $m \geq 0$. The following simply deduced facts about these polynomials are useful.

Lemma 1
1. $\omega p_k(\omega) = p_{k+2}(\omega) + p_{k-2}(\omega)$,
2. $p_{2k+1}(\omega) = p_{2k+2}(\omega) + p_{2k}(\omega)$,
3. $\gcd(p_a(\omega), p_b(\omega)) = p_{\gcd(a,b)}(\omega)$.

Proof: Straightforward from the definition. □

If ω is rational, write it as α/β with $\alpha, \beta \in \mathcal{Z}$ and $\gcd(\alpha, \beta) = 1$. Then define $P_{2m+1}(\alpha, \beta) := \beta^m p_{2m+1}(\omega)$ and $P_{2m+2}(\alpha, \beta) := \beta^m p_{2m+2}(\omega)$.

2. Rational functions for association scheme parameters

The dual eigenvalues θ_j^*, $0 \le j \le d$, are assumed to be distinct and real. Assume as well that $q \neq \pm 1$ and $\sigma_0^* \sigma_3^* \neq 0$.

Lemma 2 *$q^m \neq 1$ for $1 \le m \le d$ and $q^m \neq \sigma_0^*/\sigma_3^*$ for $2 \le m \le 2d$.*

Proof: For $0 \le j < k \le d$, $\theta_k^* - \theta_j^* = q^{-k}(1 - q^{k-j})(\sigma_0^* - \sigma_3^* q^{k+j+1}) \neq 0$. □

Theorem 3 *Let*

$$h(z) := \frac{q(1-z)\big(\sigma_3^{*2}\sigma_0 - \sigma_0^*\sigma_3^*\sigma_1 z + \sigma_0^{*2}\sigma_2 z^2 - \sigma_0^{*2}\sigma_3 z^3\big)}{(\sigma_3^* q - \sigma_0^* z^2)(\sigma_3^* - \sigma_0^* z^2)}.$$

Then $b_j = h(q^{j+1}\sigma_3^*/\sigma_0^*)$ and $c_j = h(q^{-j})$ for $0 \le j \le d$, unless $h(z)$ is undefined because the denominator is zero, which happens for b_0 if $\sigma_0^*/\sigma_3^* = q$, for $c_0 = 0$ if $\sigma_0^*/\sigma_3^* = q, 1$, for $b_d = 0$ if $\sigma_0/\sigma_3^* = q^{2d+1}, q^{2d+2}$, and for c_d if $\sigma_0^*/\sigma_3^* = q^{2d+1}$.

Proof: Immediate. □

[The importance of noting the exceptions here is that the simple n-gons occur for $q = \sigma_0^*/\sigma_3^* = q^{2d+1}, q^{2d+2}$ and $h(z) \equiv 1$ for those z for which the denominator is not zero.]

Lemma 4 *If $q^n = 1$ and $\omega := q + q^{-1}$ is rational, then $n \le 6$.*

Proof: [This is undoubtedly folklore attributable to many, but the proof is short enough to give here.] If $q^n = 1$, then $p_n(\omega)$ is an algebraic integer. Since ω is rational, ω must be an integer. Since $|q| = 1$, it follows that $|\omega| \le 2$. If $\omega = -2$, then $q^2 = 1$. If $\omega = -1$, then $q^3 = 1$. If $\omega = 0$, then $q^4 = 1$. If $\omega = 1$, then $q^6 = 1$. And if $\omega = 2$, then $q^1 = 1$. □

The remainder of this paper treats the exceptional cases in which $c_0 = 0$ is not given by $h(q^{-0})$ because the latter is undefined because $\sigma_0^*/\sigma_3^* = 1, q$. Since σ_0^* may be assumed to be nonzero, let $\sigma_1 := \delta_1/\sigma_0^*$ and $\sigma_2 := \delta_2/\sigma_0^*$.

3. The case $\boldsymbol{\sigma_0^*/\sigma_3^* = 1}$

In this case, the formula for $h(z)$ (for $z \neq 1$) reduces to

$$h(z) = \frac{q(\sigma_0 - \sigma_1 z + \sigma_2 z^2 - \sigma_3 z^3)}{(q - z^2)(1+z)}.$$

Since $c_0 = 0$ is not given by $h(q^{-0})$, consider $c_1 = 1 = h(q^{-1})$ and $d \ge 2$. Then $f(z) := h(z) - 1$ is given by

$$f(z) = \frac{(qz-1)(-(1+\sigma_1)q^3 + (1+\sigma_2 q)q(qz+1) - (1+\sigma_3 q)(q^2z^2 + qz + 1))}{q^3(q - z^2)(1+z)}.$$

If $f_j := f(q^j)$, then for $d \geq 3$, use Lagrange interpolation on $f(z)(q^{-1}z^2-1)(z+1)/(qz-1)$ to get

$$f(z)\frac{(z+1)(q^{-1}z^2-1)}{(qz-1)} = f_1\frac{(z-q^2)(z-q^3)}{(q-q^2)(q-q^3)} + f_2(1+q^2)\frac{(z-q)(z-q^3)}{(q^2-q)(q^2-q^3)} + f_3\frac{(1+q^3)(q^5-1)(z-q)(z-q^2)}{(q^4-1)(q^3-q)(q^3-q^2)}$$

In terms of ω and the monic Tchebyshev polynomials of Section 1, this becomes

$$f_j\omega p_{2j}p_{2j-1} = p_{j+1}p_j(f_1\omega p_{j-2}p_{j-3} - f_2\omega^2\epsilon p_{j-1}p_{j-3} + f_3(\omega-1)p_5p_{j-1}p_{j-2}),$$

with ϵ being 1 if j is even and $\omega+2$ if j is odd.

Theorem 5 *Suppose that $q \neq \pm 1$ and $\omega := q + q^{-1}$. Let $\mathcal{X}$ be a metric, cometric, symmetric association scheme with parameters given as in Theorem* 3. *Suppose further that* $\sigma_0^*/\sigma_3^* = 1$.
1. *If ω is rational, then $d \leq 2$.*
2. *If ω is not rational, but $\sigma_0+\sigma_1+\sigma_2+\sigma_3 = 0$, then $d \leq 4$.*
3. *If ω is not rational and $\sigma_0+\sigma_1+\sigma_2+\sigma_3 \neq 0$, then $d \leq 4$.*

Proof: [This is proven as three separate cases.]

Case 1. Suppose that $d \geq 3$ and ω is rational. Then from Lemma 4, $q^{2d+1} \neq 1$ and $q^{2d+2} \neq 1$. So $f_{d+1} = b_d - 1 = -1$. From Lemma 2.1, $\gcd(p_{2j}, p_{j+1}) = 1$, $\gcd(p_{2j-1}, p_{j+1}) = p_{\gcd(3,j+1)}$, and $\gcd(\omega, p_{j+1}) = p_{\gcd(4,j+1)}$. With $e := \gcd(12, d+2)$, we have $P_e \mid P_{d+2} \mid P_{\gcd(3,d+2)}P_{\gcd(4,d+2)} \mid P_e$. But then $P_e = \pm P_{d+2}$, which forces $p_e = p_{d+2}$, so that either $q^{d+2+e} = 1$ or $q^{d+2-e} = 1$. But $d+1 \leq e \mid d+2$, so $5 \leq e = d+2 \mid 12$. Hence either $e = 6$ or $e = 12$. If $e = 6$, then $P_6(\alpha,\beta) = \pm P_3(\alpha,\beta)$, so $\alpha - \beta = \pm\beta$, $\omega = 0$, or $\omega = 2$, $q^4 = 1$ or $q = 1$. And if $e = 12$, then $P_{12}(\alpha,\beta) = \pm P_3(\alpha,\beta)\ P_4(\alpha,\beta)$, so $(\alpha^2 - 3\beta^2)(\alpha-\beta) = \pm\beta^3$, and hence $\omega = 2$, $q = 1$.

Case 2. If ω is not rational, but $\sigma_0+\sigma_1+\sigma_2+\sigma_3 = 0$, then for $z \neq \pm 1$,

$$h(z) = \frac{q(-\sigma_1 + \sigma_2(z-1) - \sigma_3(z^2-z+1))}{(q-z^2)}.$$

Since $c_1 = 1 = h(q^{-1})$,

$$f(z) := h(z) - 1 = \frac{(qz-1)((qz+1)(1-\sigma_3 q) + q^2(\sigma_2+\sigma_3))}{q^2(q-z^2)}.$$

Use Lagrange interpolation on the function $f(z)(q^{-1}z^2-1)/(qz-1)$ to get

$$f(z)\left(\frac{q^{-1}z^2-1}{qz-1}\right) = f_2\left(\frac{q^{-1}z-1}{q-1}\right) - f_1 q\left(\frac{q^{-2}z-1}{q^2-1}\right),$$

and hence

$$f_j\left(\frac{q^{2j-1}-1}{q-1}\right)=\left(\frac{q^{j+1}-1}{q-1}\right)\left(f_2\left(\frac{q^{j-1}-1}{q-1}\right)-qf_1\left(\frac{q^{j-2}-1}{q^2-1}\right)\right).$$

Again using ω and the monic Tchebyshev polynomials above,

$$f_j p_{2j-1} = p_{j+1}(f_2 p_{j-1}\epsilon - f_1 p_{j-2})$$

with $\epsilon = 1$ if j is even and $\epsilon = \omega + 2$ if j is odd. If $d \geq 4$, then

$$(f_2 - f_3)\omega^2 + (-f_1 + 2f_2 - f_3)\omega + f_3 = 0,$$

and

$$(f_2 - f_4)\omega^3 + (-f_1 + 2f_2 - f_4)\omega^2 + (2f_4 - f_1)\omega + f_4 - f_2 + f_1 = 0.$$

From these,

$$\begin{aligned}&\omega\big((f_2 - f_4)\big(f_1^2 - 3f_1f_2 + f_1f_3 + 2f_2f_3 - f_3^2\big)\\&\quad + (f_2 - f_3)\big(-f_2f_3 - f_1^2 + 2f_1f_2\big)\big) + (-(f_2 - f_4)((f_1 - f_2)f_3\\&\quad + (f_2 - f_3)^2) + (f_2 - f_3)f_2(f_1 - f_3)) = 0.\end{aligned}$$

Clearly ω is rational unless both

$$\begin{aligned}&(f_2 - f_4)\big(f_1^2 - 3f_1f_2 + f_1f_3 + 2f_2f_3 - f_3^2\big)\\&\quad + (f_2 - f_3)\big(-f_2f_3 - f_1^2 + 2f_1f_2\big) = 0\end{aligned}$$

and

$$-(f_2 - f_4)((f_1 - f_2)f_3 + (f_2 - f_3)^2) + (f_2 - f_3)f_2(f_1 - f_3) = 0.$$

If $f_2 - f_3 = 0$, then $b_1 = b_2$. So $f_2 - f_3 \neq 0$, and $(f_2 - f_3)((f_1 - f_2)^2 + 3f_2(f_1 - f_2) + f_2^2) = (f_1 - f_2)^3$. From the equation above involving $f_2 - f_3$, $((f_1 - f_2)^2\omega + f_1^2 - 4f_1f_2 + 2f_2^2)((f_1 - f_2)\omega - f_1) = 0$. So ω is rational unless $f_1 - f_2 = 0$ and $f_2 = 0$, which would mean that $b_0 = b_1 = 1$.

Case 3. Suppose that ω is not rational and that $\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 \neq 0$. Let

$$e(z) := h(qz) + h(z^{-1}) = (\sigma_0 + \sigma_3 q) - \frac{qz(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3)}{(1+z)(1+qz)},$$

and

$$\epsilon(z) := e(qz) - e(z) = \frac{qz(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3)(1-q)(1-qz)}{(1+z)(1+qz)(1+q^2z)},$$

with

$$\epsilon_j := \epsilon(q^j), 1 \le j \le d-1.$$

Then in this case, $\epsilon_j \neq 0$. If $d \geq 4$, then $\epsilon_2(\omega^2 - 2) - \epsilon_1(\omega+1) = 0$ and $\epsilon_3(\omega^3 - 2\omega - 1) - \epsilon_2\omega^2 = 0$. From these, $(\omega+1)\epsilon_1(\epsilon_2\epsilon_3 - \epsilon_2^2 + \epsilon_1\epsilon_3) = \epsilon_2(\epsilon_2\epsilon_3 + 2\epsilon_2^2 - \epsilon_1\epsilon_3)$, so ω would be rational unless both $\epsilon_2\epsilon_3 - \epsilon_2^2 + \epsilon_1\epsilon_3 = 0$ and $\epsilon_2\epsilon_3 + 2\epsilon_2^2 - \epsilon_1\epsilon_3 = 0$. This forces $\epsilon_1 = 6\epsilon_3$, $\epsilon_2 = -2\epsilon_3$, and $\omega^2 + 3\omega + 1 = 0$. But $f_2\omega = f_1\omega - f_2\omega p_3 + f_3(\omega - 1)p_3$, which means that $\omega(-2f_2 + 3f_3 + f_{-2} - f_1) - f_2 + 2f_3 = 0$. So $f_2 = 2f_3$ and $f_{-2} = f_1 + f_3$, $b_1 = 2b_2 - 1$, and $c_2 = b_0 + b_2 - 1 \geq b_0 > c_2$. □

4. The case $\sigma_0^*/\sigma_3^* = q$

In this case, for $z \neq 1$, $h(z)$ is given by

$$h(z) = \frac{(\sigma_0 - \sigma_1 qz + \sigma_2 q^2 z^2 - \sigma_3 q^2 z^3)}{(1 - qz^2)(1+z)}.$$

Again $c_0 = 0$ is not given by $h(q^{-0})$, but $c_1 = 1 = h(q^{-1})$ when $d \geq 2$. So for $z \neq 1$, $f(z) := h(z) - 1$ is given by

$$f(z) = \frac{(qz-1)(q^2(z-\sigma_1) + \sigma_2 q^2(qz+1) + (1-\sigma_3 q)(q^2z^2 + qz + 1))}{(1-qz^2)(1+z)}.$$

If $f_j := f(q^j)$, then Lagrange interpolation on the function $f(z)(z+1)(qz^2-1)/(qz-1)$, gives

$$\begin{aligned} f(z)\frac{(z+1)(qz^2-1)}{qz-1} = {} & f_1\frac{(q^3-1)(z-q^2)(z-q^3)}{(q-1)(q-q^2)(q-q^3)} \\ & + f_2\frac{(1+q^2)(q^5-1)(z-q)(z-q^3)}{(q^3-1)(q^2-q)(q^2-q^3)} \\ & + f_3\frac{(1+q^3)(q^7-1)(z-q)(z-q^2)}{(q^4-1)(q^3-q)(q^3-q^2)}. \end{aligned}$$

In terms of ω and the monic Tchebyshev polynomials, this becomes

$$f_j\omega p_3 p_{2j} p_{2j+1} = p_{j+1}p_j\big(f_1\omega p_3^2 p_{j-2}p_{j-3} - f_2\omega^2\epsilon p_5 p_{j-1}p_{j-3} + f_3 p_6 p_7 p_{j-1}p_{j-2}\big).$$

Theorem 6 *Suppose that $q \neq \pm 1$ and $\omega := q + q^{-1}$. Let $\mathcal{X}$ be a metric, cometric, symmetric association scheme with parameters as in Theorem 3. Suppose further that $\sigma_0^*/\sigma_3^* = q$.*

1. *If ω is rational, then $d \leq 3$.*
2. *If ω is not rational, but $\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3 = 0$, then $d \leq 3$, unless $b_j = 1 = c_j$ for $1 \leq j \leq d-1$ and $q^{2d+1} = 1$ or $q^{2d+2} = 1$.*
3. *If ω is not rational and $\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3 \neq 0$, then $d \leq 4$.*

Proof: [This, too, is proven in cases.]

Case 1. Suppose ω is rational and $d \geq 4$. Then $q^{2d+1} \neq 1$ and $q^{2d} \neq 1$. So $f_d + 1 = b_d - 1 = -1$. Since $\gcd(p_{2j}, p_{j+1}) = 1$, $\gcd(p_{2j+1}, p_{j+1}) = 1$, and $\gcd(\omega, p_{j+1}) = p_{\gcd(4,j+1)}$, then with $e := \gcd(12, d+1)$, $P_e \mid P_{d+1} \mid P_{\gcd(3,d+1)} P_{\gcd(4,d+1)} \mid P_e$. But $P_e = \pm P_{d+1}$ means $p_e = \pm p_{d+1}$, so $q^{d+1+e} = 1$ or $q^{d+1-e} = 1$. But $d+1 \leq e \mid d+1$, so $5 \leq d+1 = e \mid 12$. This leads to the same contradictions as before.

Case 2. If ω is not rational and $\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3 = 0$, then for $z \neq \pm 1$, $h(z)$ is given by

$$h(z) = \frac{q(-\sigma_1 + \sigma_2 q(z-1) - \sigma_3 q(z^2 - z + 1))}{1 - qz^2}.$$

Since $c_1 = 1 = h(q^{-1})$, for $z \neq \pm 1$,

$$f(z) := h(z) - 1 = \frac{(qz-1)((qz+1)(1-q\sigma_3) + q^2(\sigma_2 + \sigma_3))}{q(1-qz^2)},$$

Use Lagrange interpolation on the function $f(z)(qz^2-1)/(qz-1)$ to get

$$f(z)\left(\frac{qz^2-1}{qz-1}\right) = f_2\left(\frac{q^{-1}z-1}{q^{-1}-1}\right) + f_1\left(\frac{q^2z-1}{q^2-1}\right),$$

and hence

$$f_j\left(\frac{q^{2j+1}-1}{q-1}\right) = \left(\frac{q^{j+1}-1}{q-1}\right)\left(f_2\left(\frac{q^{j-1}-1}{q^{-1}-1}\right) + f_1\left(\frac{q^{j+2}-1}{q^2-1}\right)\right).$$

In terms of ω and the monic Tchebyshev polynomials above,

$$f_j p_{2j+1} = p_{j+1}(f_1 p_{j+2} - f_{-2} p_{j-1}\epsilon)$$

again with $\epsilon = 1$ for j even and $\epsilon = \omega + 2$ for j odd. But then

$$\begin{aligned}
&f_2(\omega^2 + \omega - 1) - (\omega+1)(f_1\omega - f_{-2}) = 0\\
&f_3(\omega^3 + \omega^2 - 2\omega - 1) - \omega(f_1(\omega^2 + \omega - 1) - f_{-2}(\omega+2)) = 0.
\end{aligned}$$

From these,

$$\begin{aligned}
&\omega\big((f_3 - f_2)\big(f_{-2}^2 - 2f_1^2 + 3f_1f_2 - f_2^2\big) + (f_1 - f_2)^2(f_1 - f_2 + f_{-2})\big)\\
&\quad + (f_3 - f_2)\big(f_{-2}^2 - f_2 f_{-2} - (f_1 - f_2)^2\big) = 0.
\end{aligned}$$

So ω is rational unless both $(f_3 - f_2)(f_{-2}^2 - 2f_1^2 + 3f_1f_2 - f_2^2) + (f_1 - f_2)^2(f_1 - f_2 + f_{-2}) = 0$ and $(f_3 - f_2)(f_{-2}^2 - f_2 f_{-2} - (f_1 - f_2)^2) = 0$. Then $(f_1^2 + f_1 f_{-2} - f_{-2}^2)$

$f_2 = (f_1 + f_{-2})^2(f_1 - f_{-2})$, so $f_{-2}^3\omega^2 + f_{-2}(f_{-2} - f_1)(2f_{-2} + f_1)\omega + f_1(f_1^2 - 2f_{-2}^2) = (f_{-2}^2\omega + 2f_{-2}^2 - f_1^2)(f_{-2}\omega - f_1) = 0$. So ω is rational unless $f_{-2} = 0$. Then $f_2 = f_1 = 0$, meaning that $f(z) \equiv 0$ where it is defined. This gives the n-gon case since it means that $b_j = 1 = c_j$ except for c_0, b_d.

Case 3. Suppose that ω is not rational and that $\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3 \neq 0$. Let

$$g(z) := h(q^{-1}z) + h(z^{-1}) = \sigma_0 + q\sigma_3 - \frac{z(\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3)}{(1+z)(q+z)}$$

and

$$\gamma(z) := g(qz) - g(z) = \frac{z(\sigma_0 + q\sigma_1 + q^2\sigma_2 + q^2\sigma_3)(1-q)(1-z)}{(1+z)(q+z)(1+qz)}.$$

Let $\gamma_j := \gamma(q^j)$. Then $\gamma_j \neq 0$. If $d \geq 5$, then $\gamma_3(\omega^2 - 2) - \gamma_2(\omega + 1) = 0$ and $\gamma_4(\omega^3 - 2\omega - 1) - \gamma_3\omega^2 = 0$. So $(\omega+1)\gamma_2(\gamma_3\gamma_4 - \gamma_3^2 + \gamma_2\gamma_4) = \gamma_3(\gamma_3\gamma_4 + 2\gamma_3^2 - \gamma_2\gamma_4)$. Since ω is not rational, this forces $\gamma_3\gamma_4 - \gamma_3^2 + \gamma_2\gamma_4 = 0$ and $\gamma_3\gamma_4 + 2\gamma_3^2 - \gamma_2\gamma_4 = 0$. This in turn forces $\gamma_2 = 6\gamma_4$, $\gamma_3 = -2\gamma_4$, and $\omega^2 + 3\omega + 1 = 0$. But then

$$\begin{aligned} &- f_{-2}\omega(\omega + 1) + f_1\omega(\omega + 1)(\omega^2 + \omega - 1) - f_2\omega(\omega^2 + \omega - 1)^2 \\ &\quad + f_3(\omega^2 - 1)(\omega^3 + \omega^2 - 2\omega - 1) = 0. \end{aligned}$$

So $2\omega(f_{-2} - 3f_1 - 6f_2 + 9f_3) + (f_{-2} - 2f_1 - 4f_2 + 7f_3) = 0$. Because ω is not rational, this means that $f_{-2} = 3(f_1 + 2f_2 - 3f_3) = 2f_1 + 4f_2 - 7f_3$. So $0 < f_1 + 2(f_2 - f_3) = 0$. Hence $f_1 = f_2 = f_3 = f_{-2} = 0$. $\square$

References

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