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On Spin Models, Modular Invariance, and Duality

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Abstract. A spin model is a triple (X, W^+, W^-), where W^+ and W^- are complex matrices with rows and columns indexed by X which satisfy certain equations (these equations allow the construction of a link invariant from (X, W^+, W^-)). We show that these equations imply the existence of a certain isomorphism Ψ between two algebras $\mathfrak{M}$ and $\mathfrak{H}$ associated with (X, W^+, W^-). When $\mathfrak{M} = \mathfrak{H} = \mathfrak{A}$, $\mathfrak{A}$ is the Bose-Mesner algebra of some association scheme, and Ψ is a duality of $\mathfrak{A}$. These results had already been obtained in [15] when W^+, W^- are symmetric, and in [5] in the general case, but the present proof is simpler and directly leads to a clear reformulation of the modular invariance property for self-dual association schemes. This reformulation establishes a correspondence between the modular invariance property and the existence of "spin models at the algebraic level". Moreover, for Abelian group schemes, spin models at the algebraic level and actual spin models coincide. We solve explicitly the modular invariance equations in this case, obtaining generalizations of the spin models of Bannai and Bannai [3]. We show that these spin models can be identified with those constructed by Kac and Wakimoto [20] using even rational lattices. Finally we give some examples of spin models at the algebraic level which are not actual spin models.

Keywords: spin model, association scheme, duality, modular invariance, Abelian group

1. Introduction

1) *A brief history of spin models for link invariants*

The concept of spin models considered here was first introduced by Jones [19] to produce invariants of links. Namely, a spin model is defined as a triple (X, W^+, W^-) of a finite set X and two complex square matrices W^+ and W^- indexed by the elements of X satisfying certain conditions. The fact that association schemes and their Bose-Mesner algebras provide a convenient and natural framework for the study of spin models was first pointed out by Jaeger [15]. For several reasons, it is natural to consider the situation when the matrices W^+ and W^- belong to the Bose-Mesner algebra of an association scheme.

First, let us recall the main results of Jaeger [15]. The Potts models for the Jones polynomial link invariant can be regarded as spin models corresponding to complete graphs (i.e., association schemes of class $d = 1$). Furthermore, the existence of spin models which give specializations of the Kauffman polynomial link invariant is equivalent to the existence of very special strongly regular graphs (i.e., symmetric triply regular self-dual association schemes of class $d = 2$). In particular, an interesting example, corresponding to the Higman-Sims graph on 100 vertices, was discovered (cf. Jaeger [15], or de la Harpe [13] for further discussions of this and related topics).

Another important part of Jaeger [15] is devoted to a general theory of spin models in connection with association schemes, and is summarized as follows. Let (X, W^+, W^-) be a symmetric spin model with loop variable D (cf. [15, 19]). Let $\mathfrak{M} = \langle W^+, J, \cdot\rangle$ (generated by W^+ and J as an algebra with respect to ordinary matrix product) and let $\mathfrak{H} = \langle W^-, I, \circ\rangle$ (generated by W^- and I as an algebra with respect to Hadamard product). Then there exists a unique algebra isomorphism Ψ from $\mathfrak{M}$ onto $\mathfrak{H}$ satisfying:

$$\Psi(W^+) = DW^- \quad \text{and} \quad \Psi(J) = |X|I.$$

Moreover, if $\mathfrak{M} = \mathfrak{H}$ (as sets), then it is the Bose-Mesner algebra of a formally self-dual symmetric association scheme. Then Ψ defines a dual map, i.e., $\Psi(E_j) = A_j$ and $\Psi(A_j) = |X|E_j$ for suitable orderings of the adjacency matrices $\{A_0, A_1, \ldots, A_d\}$ and the primitive idempotents $\{E_0, E_1, \ldots, E_d\}$. Moreover, simplified conditions for the existence of a spin model associated in this way with a self-dual Bose-Mesner algebra were described (see, e.g., Proposition 5 in [15]).

Inspired by a preprint of Jaeger [15], a number of people, particularly those in Japan including the first two authors of the present paper, started the study of spin models from the point of view of association schemes. These researches headed for several directions as mentioned below, and are related to each other:

a) generalization of the concept of spin models,
b) constructions of new spin models,
c) study of general properties of association schemes which contain spin models, and classification of spin models with certain properties.

a) The concept of generalized spin model, which drops the symmetry conditions (in the original definition of Jones [19]), was obtained and studied by Kawagoe et al. [23]. A further generalization, namely the concept of 4-weight spin models $(X, W_i (i = 1, 2, 3, 4))$, was introduced and studied by Bannai and Bannai [4].

b) Spin models on Hamming association schemes $H(d, q)$ were constructed by Bannai et al. [7], by using solutions of the modular invariance property of the Hamming schemes, which had been previously studied in Bannai and Bannai [2]. However, these spin models turned out to be nothing else than those obtained by a tensor product construction from the Potts models. (cf. [13, 23]). A family of generalized spin models was constructed on finite cyclic groups by Bannai and Bannai [3]. There, the complete solutions of the modular invariance property for cyclic group association schemes were first obtained, and then for each solution a generalized spin model was constructed. (This approach will be repeated in the present paper for finite Abelian groups.) The reason why we began to give importance to the modular invariance property is the following. In the context of fusion algebras at algebraic level (cf. [1, 2]) we had been interested in the modular invariance property for commutative association schemes. Then we noticed that some of the necessary conditions (given in [15]) for the existence of a spin model in an association scheme are closely connected to the solutions of the modular invariance property for the association scheme.

Also, we noticed a similarity of the self-duality appearing in the theory of fusion algebras at algebraic level and in the theory of association schemes admitting spin models. Recently, motivated by the construction on cyclic groups by Bannai and Bannai [3], Kac and Wakimoto constructed new examples of spin models [20]. They produced many generalized (and also 4-weight) spin models on finite Abelian groups by using rational valued bilinear forms. A construction of spin models on Hadamard graphs was obtained by Nomura [25]. The link invariants corresponding to Nomura's spin models were determined by Jaeger [16, 18].

c) There are many recent works on the connection between spin models and association schemes (for a survey on this topic, see [6, 16]). Bannai and Bannai [5] generalized most of the general theory of (symmetric) spin models in symmetric association schemes (in Jaeger [15]) to generalized spin models in non symmetric association schemes. Several properties of association schemes related to the existence of spin models in their Bose-Mesner algebra and to the effective computation of the corresponding link invariants are studied in [17]. In particular a "matrix-free" approach to this computation is introduced in the case of self-dual triply regular schemes, which covers all spin models discussed above except those of [7].

Recent work by Nomura [26] studied twisted extensions of spin models (which generalizes the tensor product construction). Other important contributions by Nomura are (i) study of spin models with small multiplicity of eigenvalues [27], and (ii) the determination of distance-regular graphs which admit certain spin models ([28], see also [34]). Yamada [31, 32] studied generalized and 4-weight spin models which are associated with Hadamard matrices and generalized Hadamard matrices, and obtained another version of twisted extensions of such spin models. Bannai et al. [9] studied spin models with small sizes, and for example gave a classification of spin models (in the sense of Jones) with up to 7 vertices. Nomiyama [24] classified all association schemes with at most 10 vertices, which is expected to be useful for the determination of various kinds of (including generalized and 4-weight) spin models with small sizes.

2) The contents of the present paper

In this subsection, we summarize what we will discuss in the present paper.

Section 2 is devoted to preliminaries.

Our first purpose is to give an alternative proof for results given in [15] (for the symmetric case) and [5] (for the nonsymmetric case). Namely, we give the following results in Section 3.1.

Let (X, W^+, W^-) be a generalized spin model in the sense of [23]. Then, defining $\mathfrak{M} = \langle W^+, {}^tW^+, J, \cdot\rangle$ and $\mathfrak{H} = \langle W^-, {}^tW^-, I, \circ\rangle$, we have a unique isomorphism Ψ from $\mathfrak{M}$ to $\mathfrak{H}$ satisfying

$$\begin{aligned}
\Psi(I) &= J,\\
\Psi(J) &= nI,\\
\Psi(W^+) &= DW^-,\\
\Psi({}^tW^+) &= D\,{}^tW^-,\\
\Psi(W^-) &= D\,{}^tW^+,\\
\Psi({}^tW^-) &= DW^+.
\end{aligned}$$

As before, we prove that if $\mathfrak{M} = \mathfrak{H}$ then it is the Bose-Mesner algebra of a formally self-dual (not necessarily symmetric) commutative association scheme, and that Ψ defines the dual map.

The second purpose of this paper is to study the modular invariance property for (the character table of) a (not necessarily symmetric) commutative association scheme. Let $\mathfrak{X}$ be a commutative formally self-dual association scheme (i.e., association scheme whose Bose-Mesner algebra is self-dual, i.e., $P = \bar{Q}$ where P and Q are the first and second eigenmatrices of $\mathfrak{X}$ (cf. [8, 12]). Then we say that $\mathfrak{X}$ satisfies the modular invariance property (with respect to P) if

$$(P\Delta)^3 = (\text{constant}) \cdot I$$

for an invertible diagonal matrix Δ. (Note that this property has many equivalent expressions, and this will be studied carefully in Section 3.2.)

The third purpose is to show that the existence of a (generalized) spin model which generates the Bose-Mesner algebra of a (commutative) association scheme implies the modular invariance property for the association scheme. This result is proved in Corollary 3.5. Therefore, if we are interested in spin models which generate the Bose-Mesner algebra of a commutative association scheme, then we shall first try to find the solutions of the modular invariance property, and then check if those solutions give spin models.

The fourth purpose of this paper is to solve the problem just mentioned in the previous paragraph for group association schemes of finite Abelian groups. Namely, let $\mathfrak{X}(G)$ be the group association scheme of a finite Abelian group G, let Ψ be any fixed dual map on $\mathfrak{X}$, and let P be the character table of $\mathfrak{X}(G)$ corresponding to the dual map Ψ. Then in Theorem 4.4, the complete explicit solutions of the modular equivalence equation $(P\Delta)^3 = t_0 D^3 I$ are given. On the other hand, in Proposition 4.1, it is shown that each solution of the modular invariance equation (for an Abelian group scheme) gives a generalized spin model. This completes the classification of spin models on Abelian groups which are associated with a modular invariance property.

We also establish connections between the above construction of generalized spin models and the Kac-Wakimoto construction of generalized spin models on finite Abelian groups [20].

In the final Section 5, some examples are also discussed. In particular we study the modular invariance property for the character tables of strongly regular graphs and also of the Sylow 2-subgroup of the Suzuki simple group $Sz(8)$ and remark that a spin model can not be constructed from the solutions of modular invariance equation of the Sylow 2-subgroup of the Suzuki simple group $Sz(8)$. As for the strongly regular graphs we remark that some do and some do not give spin models.

2. Spin models and association schemes: preliminaries

1) Association schemes

Let X be a finite non-empty set. We shall denote by $M(X)$ the vector space of matrices with rows and columns indexed by X and with complex entries. We denote by ${}^tA$ the transpose of the matrix A, by I the identity matrix, by J the matrix with all entries equal to 1, and by $A \circ B$ the Hadamard product of the matrices A, B defined by $(A \circ B)[i, j] = A[i, j]B[i, j]$.

A (commutative) *d-class association scheme on X* (see [8]) is a $(d+1)$-tuple $\mathfrak{X} = (A_i, i = 0, 1, \ldots, d)$ of non-zero matrices of $M(X)$ satisfying

(1) $A_i \circ A_j = \delta_{i,j} A_i$, where δ is the Kronecker symbol.
(2) $A_0 = I$.
(3) $\sum_{i=0}^{d} A_i = J$.
(4) For every i in $\{0, 1, \ldots, d\}$ there exists i' in $\{0, 1, \ldots, d\}$ with ${}^tA_i = A_{i'}$.
(5) There exist integers $p_{i,j}^k$ (for all i, j, k in $\{0, 1, \ldots, d\}$) such that

$$A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{i,j}^k A_k \quad \text{(for all } i, j \text{ in } \{0, 1, \ldots, d\}).$$

In view of (1) and (3) we may consider the matrices A_i, $0 \le i \le d$ as the adjacency matrices of $d+1$ binary relations on X which form a partition of $X \times X$, and give a combinatorial interpretation of properties (2), (4), (5). We shall not adopt this combinatorial point of view here. On the other hand we shall need the following algebraic concepts.

The *Bose-Mesner algebra* of the scheme $\mathfrak{X}$ is the linear span of the matrices A_i, $i = 0, 1, \ldots, d$, which we shall denote by $\mathfrak{A}$. We observe that $\mathfrak{A}$ has the following properties:

(6) $\mathfrak{A}$ contains I and J.
(7) $\mathfrak{A}$ is closed under Hadamard product.
(8) $\mathfrak{A}$ is closed under ordinary matrix product, which is commutative when restricted to $\mathfrak{A}$.
(9) $\mathfrak{A}$ is closed under transposition.

Conversely, it is easy to show that any vector subspace $\mathfrak{A}$ of $M(X)$ satisfying properties (6)–(9) is the Bose-Mesner algebra of some association scheme on X (this is a straightforward extension of [11], Theorem 2.6.1; see also [14]). Thus we shall call any such subspace a *Bose-Mesner algebra* (*on X*).

The matrices A_i, $i = 0, 1, \ldots, d$, form a basis of $\mathfrak{A}$, and property (1) means that this is a basis of orthogonal idempotents for the Hadamard product. It is well known (see for instance [8], Section 2.3) that $\mathfrak{A}$ has also a basis of orthogonal idempotents for the ordinary matrix product, which is necessarily unique. One may denote these ordinary idempotents by E_i, $i = 0, 1, \ldots, d$, in such a way that the following properties are satisfied:

(10) $E_i \neq 0$, $E_i E_j = \delta_{i,j} E_i$.
(11) $E_0 = \frac{1}{|X|} J$.
(12) $\sum_{i=0}^{d} E_i = I$.

The *eigenmatrices* P and Q of $\mathfrak{A}$ relate the two bases of idempotents as follows:

(13) $A_j = \sum_{i=0}^{d} P_{i,j} E_i$.
(14) $E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{i,j} A_i$.

Thus

(15) $PQ = |X|I.$

We shall call the Bose-Mesner algebra $\mathfrak{A}$ *self-dual* if $P = \bar{Q}$ for an appropriate choice of the indices of the idempotents (this corresponds to the property of formal self-duality for the association scheme $\mathfrak{X}$).

Let Ψ be the invertible linear map from $\mathfrak{A}$ to itself defined by

(16) $\Psi(E_i) = A_i, i = 0, 1, \ldots, d.$

In other words, in view of (13), Ψ is the linear map defined by the matrix P in the basis $\{E_i, i = 0, 1, \ldots, d\}$. Since E_i is the conjugate of a diagonal (0, 1)-matrix by a unitary matrix, E_i is hermitian (see [8], Section 2.3). Hence by (14) and (16):

$$^tE_j = \bar{E}_j = \frac{1}{|X|}\sum_{i=0}^{d} \bar{Q}_{i,j} A_i = \frac{1}{|X|}\sum_{i=0}^{d} \bar{Q}_{i,j}\Psi(E_i).$$

It follows that, denoting by τ the transposition map on $\mathfrak{A}$, $\bar{Q}$ is the matrix of $|X|\Psi^{-1}\tau$ in the basis $\{E_i, i = 0, 1, \ldots, d\}$. Thus $P = \bar{Q}$ if and only if

(17) $\Psi^2 = |X|\tau.$

Clearly the linear maps Ψ which satisfy (16) for some indexing of the idempotents are characterized by the property

(18) $\Psi(AB) = \Psi(A) \circ \Psi(B)$ for every A, B in $\mathfrak{A}$.

Hence $\mathfrak{A}$ is self-dual if and only if there exists a linear map $\Psi : \mathfrak{A} \to \mathfrak{A}$ which satisfies (17), (18). We shall call any such map a *duality of* $\mathfrak{A}$. It is easy to show that any duality Ψ of $\mathfrak{A}$ satisfies

(19) $\Psi(A \circ B) = \frac{1}{|X|}\Psi(A)\Psi(B)$ for every A, B in $\mathfrak{A}$.

2) Spin models

We shall consider here spin models as defined in [23] (see also [4]). Thus a *spin model* will be a triple (X, W^+, W^-), where X is a finite non-empty set of size n and W^+, W^- are matrices in $M(X)$ satisfying the following properties:

(20) $^tW^+ \circ W^- = J,$
(21) $W^+W^- = nI,$
(22) $\sum_{x\in X} W^+[\alpha, x]W^+[x, \beta]W^-[x, \gamma] = DW^+[\alpha, \beta]W^-[\beta, \gamma]W^-[\alpha, \gamma]$ for all α, β, γ in X, where $D^2 = n$.

The square root D of n appearing in (22) is the *loop variable* of the model. It is easy to show (see [23] Section 3, or [4] Propositions 4 and 5) that, under the hypotheses (20), (21), (22) there exists a non-zero complex number a (called the *modulus* of the model) such that the following two properties hold:

(23) $I \circ W^+ = aI,\ I \circ W^- = a^{-1}I,$
(24) $JW^+ = W^+J = Da^{-1}J,\ JW^- = W^-J = DaJ.$

The following result can be found in [23] (Proposition 2.1, cases (iii) and (vii)) and in [4] as a special case of Theorem 1.

Proposition 2.1 *Under the hypotheses* (20), (21), *the property* (22) *is equivalent to each one of the following properties*:
(25) $\sum_{x\in X} W^+[x,\alpha]W^+[x,\beta]W^-[\gamma,x] = DW^+[\alpha,\beta]W^-[\beta,\gamma]W^-[\gamma,\alpha]$ *for all* α,β,γ *in* X.
(26) $\sum_{x\in X} W^+[\alpha,x]W^+[\beta,x]W^-[x,\gamma] = DW^+[\alpha,\beta]W^-[\beta,\gamma]W^-[\gamma,\alpha]$ *for all* α,β,γ *in* X.

Finally, we shall need the following auxiliary result which gives a matrix formulation of equations such as (22), (25) or (26).

Proposition 2.2 *For* A, B, C, A', B', C' *in* $M(X)$, *the following properties are equivalent.*
(i) $\sum_{x\in X} A[\alpha,x]B[x,\beta]C[x,\gamma] = DA'[\alpha,\beta]B'[\alpha,\gamma]C'[\gamma,\beta]$ *for all* α,β,γ *in* X.
(ii) $A(B\circ(CM)) = DA'\circ(B'(C'\circ M))$ *for all* M *in* $M(X)$.

Proof: The $[\alpha,\beta]$-entry of $A(B\circ(CM))$ is

$$\sum_{x\in X} A[\alpha,x]\left(B[x,\beta]\sum_{\gamma\in X} C[x,\gamma]M[\gamma,\beta]\right)$$
$$= \sum_{\gamma\in X}\left(\sum_{x\in X} A[\alpha,x]B[x,\beta]C[x,\gamma]\right)M[\gamma,\beta].$$

The $[\alpha,\beta]$-entry of $DA'\circ(B'(C'\circ M))$ is

$$DA'[\alpha,\beta]\sum_{\gamma\in X} B'[\alpha,\gamma](C'[\gamma,\beta]M[\gamma,\beta])$$
$$= D\sum_{\gamma\in X}(A'[\alpha,\beta]B'[\alpha,\gamma]C'[\gamma,\beta])M[\gamma,\beta].$$

These two entries are equal for every M if and only if

$$\sum_{x\in X} A[\alpha,x]B[x,\beta]C[x,\gamma] = DA'[\alpha,\beta]B'[\alpha,\gamma]C'[\gamma,\beta] \text{ for all } \alpha,\beta,\gamma \text{ in } X. \qquad \square$$

This yields the following proposition.

Proposition 2.3 *Under the hypotheses* (20), (21), *the property* (22) *is equivalent to each one of the following properties*:

(27) ${}^tW^+({}^tW^- \circ (W^+M)) = D\,{}^tW^- \circ (W^+(W^- \circ M))$ *for all* M in $M(X)$.

(28) ${}^tW^+({}^tW^- \circ (W^+M)) = DW^- \circ ({}^tW^+({}^tW^- \circ M))$ *for all* M *in* $M(X)$.

(29) $W^+(W^- \circ ({}^tW^+M)) = D\,{}^tW^- \circ (W^+(W^- \circ M))$ *for all* M *in* $M(X)$.

Proof: Applying Proposition 2.2 with $A = {}^tW^+$, $B = {}^tW^-$, $C = W^+$, $A' = {}^tW^-$, $B' = W^+$, $C' = W^-$, we see that (27) is equivalent to (25) where β and γ have been exchanged. We also observe that exchanging α and β in (25) amounts to replacing the three matrices appearing in the right-hand side by their transposes, and thus (27) and (28) are equivalent. Finally (26) is obtained from (25) by transposing the three matrices appearing in the left-hand side, and thus (27) and (29) are equivalent. The result now follows from Proposition 2.1. □

3. Duality and modular invariance

1) Duality

Let (X, W^+, W^-) be a spin model with loop variable D (so that $D^2 = |X| = n$) and modulus a. Using Proposition 2.3, we may define a linear map $\Psi : M(X) \to M(X)$ by the following equivalent equations (for all M in $M(X)$):

(30) $\Psi(M) = D^{-1}a\,{}^tW^+({}^tW^- \circ (W^+M))$
(31) $\Psi(M) = D^{-1}aW^+(W^- \circ ({}^tW^+M))$
(32) $\Psi(M) = a\,{}^tW^- \circ (W^+(W^- \circ M))$
(33) $\Psi(M) = aW^- \circ ({}^tW^+({}^tW^- \circ M))$

It is clear from (20), (21) that Ψ is invertible. Choosing appropriately one of the above expressions for Ψ and using (20), (21), (23) and (24), it is easy to check the following equations.

(34) $\Psi(I) = J$.
(35) $\Psi(J) = nI$.
(36) $\Psi(W^+) = DW^-$.
(37) $\Psi({}^tW^+) = D\,{}^tW^-$.
(38) $\Psi(W^-) = D\,{}^tW^+$.
(39) $\Psi({}^tW^-) = DW^+$.

Let $\mathfrak{M} = \langle W^-, {}^tW^-, J, \cdot\rangle$ be the algebra generated by W^-, ${}^tW^-$, J with product the ordinary matrix product, and let $\mathfrak{H} = \langle W^+, {}^tW^+, I, \circ\rangle$ be the algebra generated by W^+, ${}^tW^+$, I with product the Hadamard product.

Remark It easily follows from (21) that $W^+, {}^tW^+, I$ belong to $\mathfrak{M}$, and moreover $\mathfrak{M} = \langle W^+, {}^tW^+, J, \cdot\rangle$. Similarly, by (20), $W^-, {}^tW^-, J$ belong to $\mathfrak{H}$, and $\mathfrak{H} = \langle W^-, {}^tW^-, I, \circ\rangle$.

Note that both $\mathfrak{M}$ and $\mathfrak{H}$ are closed under transposition. We shall denote by $\tau_{\mathfrak{M}}$ (respectively: $\tau_{\mathfrak{H}}$) the restriction of the transposition map to $\mathfrak{M}$ (respectively: $\mathfrak{H}$). Similarly we denote by $\Psi_{\mathfrak{M}}$ (respectively: $\Psi_{\mathfrak{H}}$) the restriction of Ψ to $\mathfrak{M}$ (respectively: $\mathfrak{H}$).

Theorem 3.1 *$\Psi_{\mathfrak{M}}$ is an algebra isomorphism from $\mathfrak{M}$ onto $\mathfrak{H}$, and $\frac{1}{n}\Psi_{\mathfrak{H}}$ is an algebra isomorphism from $\mathfrak{H}$ onto $\mathfrak{M}$. Hence $\mathfrak{M}$ is a commutative algebra. Moreover*

$$\Psi_{\mathfrak{M}}\Psi_{\mathfrak{H}} = n\tau_{\mathfrak{H}} \quad \text{and} \quad \Psi_{\mathfrak{H}}\Psi_{\mathfrak{M}} = n\tau_{\mathfrak{M}}$$

Proof: Let us show first that

(i) $\Psi_{\mathfrak{M}}$ is an algebra homomorphism from $\mathfrak{M}$ to $\mathfrak{H}$.

It will be enough to prove that, for every M in $\mathfrak{M}$,

(i1) $\Psi(W^-M) = \Psi(W^-) \circ \Psi(M)$,
(i2) $\Psi({}^tW^-M) = \Psi({}^tW^-) \circ \Psi(M)$,
(i3) $\Psi(JM) = \Psi(J) \circ \Psi(M)$.

Indeed, by iterating (i1), (i2), (i3) we shall obtain that Ψ transforms any finite product $M_1 \cdots M_r$ with $M_i \in \{W^-, {}^tW^-, J\}$ for $i = 1, 2, \ldots, r$ into $\Psi(M_1) \circ \cdots \circ \Psi(M_r)$, and this Hadamard product will belong to $\mathfrak{H}$ by (35), (38), (39).
Let us prove (i1). By (30),

$$\begin{aligned}\Psi(W^-M) &= D^{-1}a\, {}^tW^+({}^tW^- \circ (W^+W^-M)) \\ &= Da\, {}^tW^+({}^tW^- \circ M) \quad \text{(by (21))}.\end{aligned}$$

On the other hand, by (38) and (33),

$$\begin{aligned}\Psi(W^-) \circ \Psi(M) &= D\, {}^tW^+ \circ aW^- \circ ({}^tW^+({}^tW^- \circ M)) \\ &= Da\, {}^tW^+({}^tW^- \circ M) \quad \text{(by (20))} \\ &= \Psi(W^-M),\end{aligned}$$

as required.
The proof of (i2) is exactly similar and is left to the reader. Finally, to prove (i3), we note that (24) implies the existence of a linear one-dimensional representation $\theta^* : \mathfrak{M} \to \mathbf{C}$ of $\mathfrak{M}$ such that $JM = \theta^*(M)J$ for all $M \in \mathfrak{M}$. Thus, by (35), (i3) is equivalent to

(i4) $I \circ \Psi(M) = \theta^*(M)I$ for all M in $\mathfrak{M}$.

Assume that (i4) holds for some M in $\mathfrak{M}$. Then

$$\begin{aligned} I \circ \Psi(W^- M) &= I \circ \Psi(W^-) \circ \Psi(M) && \text{(by (i1))} \\ &= D\,{}^t W^+ \circ (I \circ \Psi(M)) && \text{(by (38))} \\ &= D\,{}^t W^+ \circ \theta^*(M) I \\ &= D\theta^*(M) a I && \text{(by (23))} \\ &= \theta^*(W^-)\theta^*(M) I && \text{(by (24))} \\ &= \theta^*(W^- M) I. \end{aligned}$$

Thus (i4) also holds for $W^- M$. One shows in exactly the same way that (i4) holds for ${}^t W^- M$. Now

$$\begin{aligned} I \circ \Psi(JM) &= I \circ \Psi(\theta^*(M) J) \\ &= \theta^*(M) n I \quad \text{(by (35))} \\ &= \theta^*(M)\theta^*(J) I = \theta^*(JM) I. \end{aligned}$$

Thus (i4) holds for $W^- M, {}^t W^- M$ and JM. Since it holds for $M = I$ by (34), it follows (by induction on length) that (i4) holds for any finite product of matrices in $\{W^-, {}^t W^-, J\}$ and (by linearity) for all M in $\mathfrak{M}$. This completes the proof of (i).
The proof of

(ii) $\frac{1}{n}\Psi_{\mathfrak{H}}$ is an algebra homomorphism from $\mathfrak{H}$ to $\mathfrak{M}$

is quite similar and will be omitted.

Now since Ψ is invertible, both $\Psi_{\mathfrak{M}}$ and $\Psi_{\mathfrak{H}}$ are injective. Then $\dim \mathfrak{M} \leq \dim \mathfrak{H}$ and $\dim \mathfrak{H} \leq \dim \mathfrak{M}$. Hence $\dim \mathfrak{M} = \dim \mathfrak{H}$ and both $\Psi_{\mathfrak{M}}$ and $\Psi_{\mathfrak{H}}$ are bijective. Since $\mathfrak{H}$ is commutative, $\mathfrak{M}$ is also commutative. The equality $\Psi_{\mathfrak{M}}(\frac{1}{n}\Psi_{\mathfrak{H}}) = \tau_{\mathfrak{H}}$ of algebra automorphisms of $\mathfrak{H}$ is easily checked, using (34)–(39), on the generating set $W^+, {}^t W^+, I$. The other equality $\Psi_{\mathfrak{H}}\Psi_{\mathfrak{M}} = n\tau_{\mathfrak{M}}$ is proved similarly. □

The above result was essentially proved in [15] when W^+, W^- are symmetric and in [5] for the general case. The proof given here is conceptually simpler and will lead us to a clear understanding of the modular invariance property for self-dual Bose-Mesner algebras. From now on we shall be mostly interested in the situation described by the following result.

Corollary 3.2 *The following properties are equivalent.*
(i) $\mathfrak{M}$ *is closed under Hadamard product.*
(ii) $\mathfrak{H}$ *is closed under ordinary matrix product.*
(iii) $\mathfrak{H} = \mathfrak{M}$ *is a self-dual Bose-Mesner algebra.*

Proof: Clearly (iii) implies (i) and (ii). Conversely, if (i) holds, $\mathfrak{H} \subseteq \mathfrak{M}$ since $W^+, {}^t W^+, I$ belong to $\mathfrak{M}$. Then $\mathfrak{H} = \mathfrak{M}$ because $\dim \mathfrak{M} = \dim \mathfrak{H}$. Similarly, if (ii) holds, $\mathfrak{H} = \mathfrak{M}$. It is clear that $\mathfrak{A} = \mathfrak{H} = \mathfrak{M}$ then satisfies the properties (6)–(9) which characterize Bose-Mesner algebras. Now $\Psi = \Psi_{\mathfrak{M}} = \Psi_{\mathfrak{H}}$ satisfies (17), (18) and hence this map is a duality of $\mathfrak{A}$. □

Thus it is natural to look for spin models (X, W^+, W^-) such that W^+, W^- belong to some Bose-Mesner algebra $\mathfrak{A}$ on X. As explained in [5, 15, 17] this allows a very significant simplification in the study of equations (20), (21), (23), (24), which bear on $\dim \mathfrak{A} = d + 1 \le n$ variables rather than on $\dim M(X) = n^2$ variables.

In this setting it is useful to state the following result.

Corollary 3.3 *Let $\mathfrak{A}$ be a Bose-Mesner algebra on X, and let (X, W^+, W^-) be a spin model with loop variable D and modulus a such that W^+, W^- belong to $\mathfrak{A}$. The following properties are equivalent*:

(i) *$W^-, {}^tW^-, J$ generate $\mathfrak{A}$ under ordinary matrix product.*

(ii) *$W^+, {}^tW^+, I$ generate $\mathfrak{A}$ under Hadamard product.*

Moreover if (i) *and* (ii) *hold*, *$\mathfrak{A}$ has a duality Ψ satisfying properties* (30)–(33) *for all M in $\mathfrak{A}$*, *and hence also satisfying* (34)–(39).

If (i) and (ii) hold in Corollary 3.3, we shall say that (X, W^+, W^-) *generates* $\mathfrak{A}$.

2) Modular invariance

Let $\mathfrak{A}$ be a self-dual Bose-Mesner algebra on X with eigenmatrices P, Q satisfying $P = \bar{Q}$. The following property has been considered in relation with fusion algebras of conformal field theories [1] and with the construction of spin models [3, 7]. We shall say that $\mathfrak{A}$ satisfies the *modular invariance property* with respect to P and the diagonal matrix Δ (of size $\dim \mathfrak{A} = d + 1$) if $(P\Delta)^3$ is a non-zero multiple of the identity. Let Ψ be the duality on $\mathfrak{A}$ defined by the matrix P in the basis $\{E_i, i = 0, 1, \ldots, d\}$ (or equivalently, defined by (16)). Let $W^+ = D \sum_{i=0}^{d} \Delta[i, i] E_i$ and $W^- = D^{-1}\Psi(W^+)$, with $D^2 = |X| = n$.

Theorem 3.4 *The following properties are equivalent for any non-zero complex number a.*

(i) $(P\Delta)^3 = a^{-1} D^3 I$.

(ii) $\Psi(M) = D^{-1}a\, {}^tW^+({}^tW^- \circ (W^+M))$ *for all M in $\mathfrak{A}$.*

(iii) $\Psi(M) = D^{-1}a W^+(W^- \circ ({}^tW^+M))$ *for all M in $\mathfrak{A}$.*

(iv) $\Psi(M) = a\, {}^tW^- \circ (W^+(W^- \circ M))$ *for all M in $\mathfrak{A}$.*

(v) $\Psi(M) = a W^- \circ ({}^tW^+({}^tW^- \circ M))$ *for all M in $\mathfrak{A}$.*

Proof: For every A in $\mathfrak{A}$, let us denote by μ_A (respectively: μ_A^*) the linear map from $\mathfrak{A}$ to itself defined by the equality $\mu_A(M) = AM$ (respectively $\mu_A^*(M) = A \circ M$) for every M in $\mathfrak{A}$. Clearly $D\Delta$ is the matrix of the linear map μ_{W^+} with respect to the basis $\{E_i, i = 0, 1, \ldots, d\}$. Hence (i) is equivalent to

(vi) $(\Psi\mu_{W^+})^3 = a^{-1}n^3 id$.

We may rewrite (vi) successively as follows.

$$\begin{aligned} \mu_{W^+}\Psi\mu_{W^+}\Psi\mu_{W^+} &= a^{-1}n^3\Psi^{-1} = a^{-1}n^2\tau\Psi \qquad \text{(by (17))} \\ \Psi &= an^{-2}\tau\mu_{W^+}\Psi\mu_{W^+}\Psi\mu_{W^+} \\ &= an^{-2}\mu_{\tau(W^+)}(\tau\Psi\mu_{W^+}\Psi)\mu_{W^+}. \end{aligned}$$

It is easy to check that $\tau\Psi\mu_{W^+}\Psi = n\mu^*_{\tau\Psi(W^+)} = n\mu^*_{D\,{}^tW^-}$. Thus (vi) is equivalent to

$$\Psi = aD^{-1}\mu_{{}^tW^+}\mu^*_{{}^tW^-}\mu_{W^+}$$

and this shows the equivalence of (i) and (ii). Let us now rewrite (vi) as follows.

$$\begin{aligned}\Psi\mu_{W^+}\Psi\mu_{W^+}\Psi\mu_{W^+}\Psi &= a^{-1}n^3\Psi\\ \Psi &= an^{-3}(\tau\Psi\mu_{W^+}\Psi)\mu_{W^+}(\Psi\mu_{W^+}\Psi\tau)\\ &= an^{-3}(nD\mu^*_{{}^tW^-})\mu_{W^+}(nD\mu^*_{W^-})\\ &= a\mu^*_{{}^tW^-}\mu_{W^+}\mu^*_{W^-}.\end{aligned}$$

This shows the equivalence of (i) and (iv). Finally (ii) and (iii) correspond to each other by multiplication by τ and replacement of M by $\tau(M)$, and (iv) and (v) correspond to each other in the same way. □

We may now apply the above result to spin models (cf. [10]).

Corollary 3.5 *Assume that the spin model (X, W^+, W^-) with loop variable D and modulus a generates the Bose-Mesner algebra $\mathfrak{A}$. Let Ψ be the corresponding duality on $\mathfrak{A}$ given by Corollary* 3.3 *and choose the indices of the Hadamard idempotents $\{A_i(i = 0, 1, \ldots, d)\}$ and ordinary idempotents $\{E_i(i = 0, 1, \ldots, d)\}$ in such a way that $\Psi(E_i) = A_i(i = 0, 1, \ldots, d)$. Let P be the corresponding eigenmatrix of $\mathfrak{A}$. Then $(P\Delta)^3 = a^{-1}D^3I$, where the diagonal matrix $\Delta = Diag[t_0, t_1, \ldots, t_d]$ is determined by one of the equivalent equations*
(i) $W^+ = \sum_{i=0}^{d} t_i^{-1}A_{i'}$.
(ii) $W^+ = D\sum_{i=0}^{d} t_iE_i$.

Proof: Define Δ by Eq. (ii). Thus $W^+ = D\sum_{i=0}^{d}\Delta[i, i]E_i$. We also have $W^- = D^{-1}\Psi(W^+)$ by (36). Since (30)–(33) hold for every M in $\mathfrak{A}$ Theorem 3.4 gives $(P\Delta)^3 = a^{-1}D^3I$. Equations (i) and (ii) are equivalent because applying Ψ to (ii) yields $\Psi(W^+) = D\sum_{i=0}^{d} t_iA_i$, i.e., $W^- = \sum_{i=0}^{d} t_iA_i$, which is equivalent to (i) via (20). □

Thus if a spin model generates a Bose-Mesner algebra, this Bose-Mesner algebra is self-dual, with a modular invariance property directly given by the spin model matrices. We now look for a (partial) converse to this statement.

We consider again a self-dual Bose-Mesner algebra $\mathfrak{A}$ on X with duality Ψ given by the eigenmatrix P in the basis $\{E_i\ (i = 0, 1, \ldots, d)\}$. Define the linear forms θ and θ^* on $\mathfrak{A}$ by the equalities $I \circ A = \theta(A)I$ and $JA = \theta^*(A)J$ for every A in $\mathfrak{A}$. Since $\Psi(E_i) = A_i(i = 0, 1, \ldots, d)$, $\Psi(I) = J$ by (3) and (12). Let $n = |X|$ and let A be any element of $\mathfrak{A}$. Then, by (19), $J\Psi(A) = \Psi(I)\Psi(A) = n\Psi(I \circ A) = n\theta(A)J$. Hence

(40) $\theta^*\Psi = n\theta$.

Assume now that for some diagonal matrix Δ of size $d + 1$, $(P\Delta)^3$ is a non-zero multiple of the identity. Note that Δ is invertible and hence $\Delta[0, 0] \neq 0$. Hence we may write

$(P\Delta)^3 = \Delta[0,0]\lambda I$ for some non-zero complex number λ. If D is a square root of n, we may multiply Δ by a suitable factor to obtain $\lambda = D^3$, that is, $(P\Delta)^3 = \Delta[0,0]D^3 I$.

Theorem 3.6 *Let Δ be a diagonal matrix of size $d+1$ such that $\Delta[0,0] = a^{-1} \neq 0$ and $(P\Delta)^3 = a^{-1}D^3 I$. Let $W^+ = D\sum_{i=0}^{d}\Delta[i,i]E_i$ and $W^- = D^{-1}\Psi(W^+)$. Then W^+, W^- satisfy*
(20) ${}^tW^+ \circ W^- = J.$
(21) $W^+W^- = nI.$
(23) $I \circ W^+ = aI,\ I \circ W^- = a^{-1}I.$
(24) $JW^+ = W^+J = Da^{-1}J,\ JW^- = W^-J = DaJ.$

and

(41) ${}^tW^+({}^tW^- \circ (W^+M)) = D\,{}^tW^- \circ (W^+(W^- \circ M))$ *for all M in $\mathfrak{A}$.*

Proof: We have property (i) of Theorem 3.4 and hence also properties (ii)–(v). Property (41) follows immediately. Let us apply (ii) to $M = J$. We obtain

$$\begin{aligned} nI = \Psi(J) &= D^{-1}a\,{}^tW^+({}^tW^- \circ \theta^*(W^+)J) \\ &= D^{-1}a\theta^*(W^+)\,{}^tW^+\,{}^tW^-. \end{aligned}$$

It is clear from (11) and the definition of W^+ that $\theta^*(W^+) = D\Delta[0,0] = Da^{-1}$. This yields (21). From this we have $\theta^*(W^+)\theta^*(W^-) = \theta^*(W^+W^-) = n\theta^*(I) = n$ and hence $\theta^*(W^-) = Da$. This yields (24). Applying Ψ to (21) gives $\Psi(W^+) \circ \Psi(W^-) = nJ$ and (20) follows. Finally, by (40), $\theta(W^+) = n^{-1}\theta^*\Psi(W^+) = n^{-1}\theta^*(DW^-) = a$ and this together with (20) gives (23). □

Note that if we replace in (41) the condition "for all M in $\mathfrak{A}$" by the condition "for all M in $M(X)$" we obtain (27), which together with (20), (21) gives (according to Proposition 2.3) the definition of a spin model. This leads us to define *a spin model at the algebraic level in* $\mathfrak{A}$ as a pair W^+, W^- of elements of $\mathfrak{A}$ satisfying properties (20), (21), (23), (24), (41). Thus we may interpret Theorem 3.6 as the statement that the modular invariance property for a self-dual Bose-Mesner algebra implies the existence of a spin model at the algebraic level in this algebra.

The above considerations indicate the following natural strategy for the construction of spin models.

(i) Given a self-dual Bose-Mesner algebra $\mathfrak{A}$, enumerate its dualities.
(ii) For each duality, solve the corresponding modular invariance equation.
(iii) For each solution, check whether the corresponding spin model at the algebraic level is an actual spin model or not.

The main interest of this approach is that steps (i) and (ii) are much easier than the study of the general spin model Eqs. (20)–(22), and leave only a small number of cases to be checked in step (iii). Also, any spin model which generates a Bose-Mesner algebra can be found by this method.

We now realize this program in the case of Bose-Mesner algebras of Abelian group schemes.

4. Spin models on Abelian groups

1) Abelian group schemes and Bose-Mesner algebras

Let X be an Abelian group of finite order n written additively. For each i in X, let A_i in $M(X)$ be defined by $A_i[x, y] = \delta_{i,y-x}(x, y \text{ in } X)$. Properties (1)–(5) are immediate, with $i' = -i$ and $p^k_{i,j} = \delta_{i+j,k}$ (for convenience we replace the index set $\{0, 1, \ldots, d\}$ by X). Thus we have an association scheme $(A_i, i \in X)$ and a corresponding Bose-Mesner algebra $\mathfrak{A}$ on X.

Let $\{E_i, i \in X\}$ be the ordinary idempotents of $\mathfrak{A}$, with $E_0 = \frac{1}{n}J$. Let P be the eigenmatrix defined by

$$A_j = \sum_{i \in X} P_{i,j} E_i \quad \text{for all } j \text{ in } X.$$

Then for all j, k in X

$$\sum_{i \in X} P_{i,j+k} E_i = A_{j+k} = A_j A_k = \sum_{i \in X} P_{i,j} P_{i,k} E_i$$

and hence $P_{i,j+k} = P_{i,j} P_{i,k}$ for all i in X. Hence for each i in X, the map $j \to P_{i,j}$ from X to $\mathbf{C}$ is a character of X. Since P is invertible, each character of X appears exactly once as a row of P (the row indexed by 0 corresponds to the trivial character since $JA_i = J$ for all i in X).

The first orthogonality relation for characters states that $P\,{}^t\bar{P} = nI$. Hence the self-duality relation $P = \bar{Q}$ holds if and only if P is symmetric, i.e., $P_{i,j} = \chi_i(j)(i, j \in X)$, where $\chi_i, i \in X$ are the characters of X with indices chosen so that $\chi_i(j) = \chi_j(i)$ for all i, j in X.

2) Spin models at the algebraic level are actual spin models

Let us compare conditions (41) and (27) when $\mathfrak{A}$ is the Bose-Mesner algebra of the Abelian group X. It is clear that for M in $M(X)$, the column of ${}^tW^+({}^tW^- \circ (W^+M))$ indexed by a given element x of X only depends on the column of M indexed by x, and similarly for $D\,{}^tW^- \circ (W^+(W^- \circ M))$. Since for every M in $M(X)$, there exists a matrix M' in $\mathfrak{A}$ with the same column indexed by x as M, property (41) implies property (27). This means that every spin model at the algebraic level is an actual spin model.

Putting this together with Theorem 3.6 we obtain the following result (where $P = (\chi_i(j))_{i,j \in X}$ and $D^2 = n$).

Proposition 4.1 *Let Δ be a diagonal matrix indexed by X such that $\Delta[0, 0] = a^{-1} \neq 0$ and $(P\Delta)^3 = a^{-1}D^3 I$. Let $W^+ = D\sum_{i \in X} \Delta[i, i]E_i$ and $W^- = \sum_{i \in X} \Delta[i, i]A_i$. Then (X, W^+, W^-) is a spin model.*

3) Reformulation of the modular invariance equation

Let $\Delta = \mathrm{Diag}[t_i]_{(i \in X)}$ be invertible. The following result generalizes Proposition 3 of [8].

Proposition 4.2 $(P\Delta)^3 = t_0 D^3 I$ *if and only if*
(42) $\chi_u(v) t_u t_v = t_0 t_{u+v}$ *for all* u, v *in* X,

and

(43) $\sum_{x \in X} t_x^{-1} = D t_0$.

Proof: The equation $(P\Delta)^3 = t_0 D^3 I$ is equivalent to

(i) $\Delta P \Delta = t_0 D^3 (\frac{1}{n} Q) \Delta^{-1} (\frac{1}{n} Q)$

The $[u, v]$-entry of the left-hand side of (i) is $t_u P_{u,v} t_v = \chi_u(v) t_u t_v$. The $[u, v]$-entry of the right-hand side of (i) is $t_0 D^{-1} \sum_{x \in X} Q_{u,x} t_x^{-1} Q_{x,v}$. Now $Q_{u,x} = \overline{P_{u,x}} = \overline{\chi_u(x)} = \overline{\chi_x(u)}$, $Q_{x,v} = \overline{P_{x,v}} = \overline{\chi_x(v)}$ and hence $Q_{u,x} Q_{x,v} = \overline{\chi_x(u)\chi_x(v)} = \overline{\chi_x(u+v)}$. Thus the right-hand side of (i) only depends on $u + v$ and (42) expresses the same property for the left-hand side. Then (i) restricted to the $[u, v]$-entries becomes

(ii) $t_0 t_{u+v} = t_0 D^{-1} \sum_{x \in X} \overline{\chi_x(u+v)} t_x^{-1}$

which gives (43) when $u + v = 0$. Conversely, assuming (42), (43), to prove (i) it is enough to prove (ii) for all u, v with $u + v \neq 0$, that is $\sum_{x \in X} \overline{\chi_x(u)} t_x^{-1} t_u^{-1} = D$ for all $u \neq 0$. By (42), the left-hand side is

$$\sum_{x \in X} \overline{\chi_x(u)} \chi_x(u) (t_0 t_{x+u})^{-1} = t_0^{-1} \sum_{x \in X} (t_{x+u})^{-1}$$

and the result follows from (43). □

Corollary 4.3 *Let* $W^+ = D \sum_{x \in X} t_i E_i$, $W^- = \sum_{x \in X} t_i A_i$, *where the* t_i, $i \in X$, *are non-zero complex numbers. If* (42) *and* (43) *hold, then* (X, W^+, W^-) *is a spin model.*

Remarks

(i) Corollary 4.3. was proved in a different way in [17] Proposition 23.
(ii) Any solution to (42) with $\sum_{x \in X} t_x^{-1} \neq 0$ can be normalized to give a solution of both (42) and (43).

4) Explicit solution of the modular invariance equation

All finite groups are considered as $\mathbf{Z}$-modules. Let $X = X_1 \oplus X_2 \oplus \cdots \oplus X_r$ be a decomposition of X into a direct sum of cyclic groups $X_1, X_2, \ldots, X_r$. For each $i \in \{1, 2, \ldots, r\}$ let a_i be a generator and n_i be the order of the cyclic group X_i. Hence $|X| = n = \prod_{i=1}^r n_i$. Let $x = \sum_{i=1}^r x_i a_i$, $y = \sum_{i=1}^r y_i a_i$, where $x_i, y_i \in \mathbf{Z}$, be elements in X. Then $x = y$ in X if and only if $x_i \equiv y_i \pmod{n_i}$ for all $i \in \{1, 2, \ldots, r\}$. Let $d_{i,j} (= d_{j,i})$ be the greatest common divisor of n_i and n_j for $i, j \in \{1, 2, \ldots, r\}$. Then $\chi_{a_i}(a_j)^{d_{i,j}} = \chi_{a_j}(a_i)^{d_{i,j}} = 1$ for $i, j \in \{1, 2, \ldots, r\}$. We can easily check that

(44) $\chi_x(y) = \chi_y(x) = \prod_{i=1}^{r} \prod_{j=1}^{r} \chi_{a_i}(a_j)^{x_i y_j}$

for any expressions $x = \sum_{i=1}^{r} x_i a_i$ and $y = \sum_{i=1}^{r} y_i a_i$ of x, y.

We have the following result (for the special case where X is a cyclic group, refer to [3]).

Theorem 4.4 *The Bose-Mesner algebra $\mathfrak{A}$ on X has the modular invariance property $(P\Delta)^3 = t_0 D^3 I$ with an invertible diagonal matrix $\Delta = Diag[t_x]_{(x \in X)}$ if and only if*

(45) $t_x = t_0 \{\prod_{i=1}^{r} {\eta_i}^{x_i} \chi_{a_i}(a_i)^{\frac{x_i(x_i-1)}{2}}\}\{\prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{x_l x_k}\}$,

where $x = \sum_{i=1}^{r} x_i a_i$, $x_i \in \mathbf{Z}$, η_i is a complex number satisfying ${\eta_i}^{n_i} = \chi_{a_i}(a_i)^{-\frac{n_i(n_i-1)}{2}}$ for $i \in \{1, 2, \ldots, r\}$, and

(46) $t_0^2 = D^{-1} \sum_{x = \sum_i x_i a_i \in X} \{\prod_{j=1}^{r} {\eta_j}^{-x_j} \chi_{a_j}(a_j)^{-\frac{x_j(x_j-1)}{2}}\}\{\prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{-x_l x_k}\}$.

Proof: By Proposition 4.2, it is enough to give the complete solutions for (42) and (43). First assume that Δ satisfies (42) and (43). Let $t_{a_i} = \eta_i t_0$ for $i = 1, 2, \ldots, r$. Then by (42) and induction on j we have

(47) $t_{ja_i} = {\eta_i}^{j} \chi_{a_i}(a_i)^{\frac{j(j-1)}{2}} t_0$

for $j \ge 0$ and $i = 1, 2, \ldots, r$. Let $j = n_i$ in (47). We obtain

(48) $\eta_i^{n_i} = \chi_{a_i}(a_i)^{-\frac{n_i(n_i-1)}{2}}$

for $i = 1, 2, \ldots, r$. Using (42) and (47) and induction on r we get (45). Then (46) follows from (43). Conversely assume that Δ satisfies (45) and (46). Then clearly (43) holds. Let $x = \sum_{i=1}^{r} x_i a_i$ and $y = \sum_{i=1}^{r} y_i a_i$. Then by (45) we obtain

$$t_{x+y} t_0 = t_0^2 \left\{ \prod_{i=1}^{r} \eta_i^{x_i+y_i} \chi_{a_i}(a_i)^{\frac{(x_i+y_i)(x_i+y_i-1)}{2}} \right\} \left\{ \prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{(x_l+y_l)(x_k+y_k)} \right\}$$

and

$$\begin{aligned} \chi_x(y) t_x t_y &= t_0^2 \left\{ \prod_{l=1}^{r} \prod_{k=1}^{r} \chi_{a_l}(a_k)^{x_l y_k} \right\} \left\{ \prod_{i=1}^{r} \eta_i^{x_i} \chi_{a_i}(a_i)^{\frac{x_i(x_i-1)}{2}} \right\} \left\{ \prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{x_l x_k} \right\} \\ &\quad \cdot \left\{ \prod_{i=1}^{r} \eta_i^{y_i} \chi_{a_i}(a_i)^{\frac{y_i(y_i-1)}{2}} \right\} \left\{ \prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{y_l y_k} \right\} \\ &= t_0^2 \left\{ \prod_{i=1}^{r} \eta_i^{x_i+y_i} \chi_{a_i}(a_i)^{\frac{(x_i+y_i)(x_i+y_i-1)}{2}} \right\} \left\{ \prod_{1 \le l < k \le r} \chi_{a_l}(a_k)^{(x_l+y_l)(x_k+y_k)} \right\}. \end{aligned}$$

This implies (42). □

Remark If the eigenmatrix P is a tensor product of the eigenmatrices $P_1, P_2, \ldots, P_r$ of the cyclic factors, then $\chi_{a_l}(a_k) = 1$ whenever $l \neq k$ and it follows from (45) that every solution Δ to the modular invariance equation with respect to P is a tensor product of the solutions $\Delta_1, \Delta_2, \ldots, \Delta_r$ to the modular invariance equations with respect to $P_1, P_2, \ldots, P_r$ respectively. However there are some eigenmatrices P which are not expressed as tensor products of eigenmatrices of cyclic factors. We can easily check that some solutions of modular invariance equations with respect to those P yield spin models which are not tensor products of cyclic models. For instance, let $X = (\mathbf{Z}/2\mathbf{Z})^2$ and consider the eigenmatrix

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

where the columns correspond to $A_{(0,0)}, A_{(0,1)}, A_{(1,0)}, A_{(1,1)}$ in that order. Let $\Delta = \mathrm{Diag}[1, 1, 1, -1]$. Then $(P\Delta)^3 = 8I$. However Δ does not correspond to a tensor product of two cyclic models, since otherwise Δ would be of the form $\mathrm{Diag}[t_0t_0', t_0t_1', t_1t_0', t_1t_1']$.

5) Relation with the Kac-Wakimoto construction of spin models

In [20] spin models are constructed using even rational lattices. More precisely, a *rational lattice* L is a finitely generated free Abelian group with a symmetric $\boldsymbol{Q}$-valued bilinear form $\langle , \rangle$. A rational lattice L is called *even* if the integral lattice

$$M = \{\alpha \in L \mid \langle \alpha, \beta \rangle \in \mathbf{Z} \text{ for all } \beta \in L\}$$

is even, i.e., $\langle \alpha, \alpha \rangle \in 2\boldsymbol{Z}$ for all $\alpha \in M$. Note that $X = L/M$ is a finite Abelian group. Since $\langle \alpha, \alpha \rangle \equiv \langle \beta, \beta \rangle (\mathrm{mod}\ 2\boldsymbol{Z})$ holds for any α and $\beta \in L$ with $\alpha \equiv \beta (\mathrm{mod}\ M)$, we can define complex numbers $\lambda_\alpha (\alpha \in X)$ and μ by

(49) $\lambda_\alpha = \varepsilon(\alpha) e^{\pi\sqrt{-1}\langle \alpha, \alpha \rangle}, \alpha \in X$

and

(50) $\mu = \sum_{\alpha \in X} \lambda_\alpha,$

where ε is a character of X. Let $\chi_\alpha(\beta) = e^{-2\pi\sqrt{-1}<\alpha,\beta>}$ for α and $\beta \in X$, and let P, $Q \in M(X)$ be the matrices defined by $P_{\alpha,\beta} = \chi_\alpha(\beta)$ and $Q_{\alpha,\beta} = \overline{\chi_\alpha(\beta)}$. Lemma 2.1 in [20] implies

$$\sum_{\beta \in X} e^{-2\pi\sqrt{-1}\langle \alpha - \gamma, \beta \rangle} = n\delta_{\alpha,\gamma},$$

where $n = |X|$, which is equivalent to

$$\sum_{\beta \in X} \chi_\alpha(\beta)\overline{\chi_\beta(\gamma)} = n\delta_{\alpha,\gamma}.$$

This implies that P and $Q(=\bar{P})$ satisfy $PQ = nI$ and are the eigenmatrices of the Bose-Mesner algebra of the scheme on $X = L/M$ (with respect to some ordering of the idempotents). Lemma 2.4 in [20] implies

(51) $\mu\bar{\mu} = n$.

Let $|X| = n = D^2$ (i.e., D is one of the square roots of n). Fix an element γ in X and let $t_0^{(\gamma)}$ be one of the square roots of $D\mu^{-1}\lambda_\gamma$, i.e.,

(52) $(t_0^{(\gamma)})^2 = D\mu^{-1}\lambda_\gamma$.

((51) shows that $\mu \neq 0$ and this definition is valid.) Define $t_\alpha^{(\gamma)}$ by

(53) $t_\alpha^{(\gamma)} = t_0^{(\gamma)}\lambda_\gamma^{-1}\lambda_{\gamma+\alpha}$ for $\alpha \in X$.

Then we have the following.

Theorem 4.5
(54) $\overline{\chi_\alpha(\beta)}t_\alpha^{(\gamma)}t_\beta^{(\gamma)} = t_0^{(\gamma)}t_{\alpha+\beta}^{(\gamma)}$ *for* $\alpha, \beta \in X$.
(55) $\sum_{\alpha \in X} t_\alpha^{(\gamma)} = D(t_0^{(\gamma)})^{-1}$.
(56) *Let* $W_{(\gamma)}^+ = D\sum_{\alpha \in X}(t_\alpha^{(\gamma)})^{-1}E_\alpha$ *and* $W_{(\gamma)}^- = \sum_{\alpha \in X}(t_\alpha^{(\gamma)})^{-1}A_\alpha$. *Then* $(X, W_{(\gamma)}^+, W_{(\gamma)}^-)$ *is a spin model with loop variable* D.
(57) *Let* $\Delta = Diag[(t_\alpha^{(\gamma)})^{-1}]_{\alpha \in X}$. *Then* $(P\Delta)^3 = (t_0^{(\gamma)})^{-1}D^3I$.

Proof: (54) and (55) are easy to check from the definitions (they are also proved in the original Japanese version of [20]). Then (56) follows from Corollary 4.3, and the following equality (58) follows from (20).

(58) $W_{(\gamma)}^+ = \sum_{\alpha \in X} t_{-\alpha}^{(\gamma)}A_\alpha$.

In view of (58), (56) is essentially equivalent to Theorem 2.1 in [20]. Finally (54), (55) and Proposition 4.2 imply (57). □

In the following we consider the converse of the construction by Kac and Wakimoto. For any finite Abelian group X we construct an even lattice L in the following way. We use the notation given in 4.4).

Since $\chi_{a_i}(a_j)^{d_{i,j}} = 1$, there exists a rational number $\alpha_{i,j} = \alpha_{j,i}$ satisfying $\chi_{a_i}(a_j) = e^{-2\pi\sqrt{-1}\alpha_{i,j}}$. Moreover if $d_{i,j}(= (n_i, n_j))$ is odd, then we can choose $\alpha_{i,j} = k_{i,j}/d_{i,j}$ with some even integer $k_{i,j}$. Therefore $n_i\alpha_{i,j} \in 2\mathbf{Z}$ if n_i is odd. If n_i is even, then $n_j\alpha_{i,j} \in \mathbf{Z}$

and $n_i n_j \alpha_{i,j} \in 2\mathbf{Z}$. Hence $n_i n_j \alpha_{i,j} \in 2\mathbf{Z}$ for all i and j in $\{1, 2, \ldots, r\}$. Let L be a free Abelian group of finite rank r generated by $e_1, e_2, \ldots, e_r$. Define a bilinear form on L by $\langle e_i, e_j \rangle = \alpha_{i,j}$ for $i, j \in \{1, 2, \ldots, r\}$. Let $M = \{\alpha \in L | \langle \alpha, \beta \rangle \in \mathbf{Z}$ for all $\beta \in L\}$. The next proposition is essentially equivalent to Lemma 2.5 in [20].

Proposition 4.6
(i) *L is an even lattice.*
(ii) *$L/M \cong X$.*

Proof: We will first show that

(59) $M = \sum_{i=1}^{r} \mathbf{Z} n_i e_i$.

Since $\langle n_i e_i, \sum_{j=1}^{r} x_j e_j \rangle = \sum_{j=1}^{r} x_j n_i \alpha_{i,j} \in \mathbf{Z}$ for any i in $\{1, 2, \ldots, r\}$ and $x_1, x_2, \ldots, x_r$ in $\mathbf{Z}$, we have $\sum_{i=1}^{r} \mathbf{Z} n_i e_i \subseteqq M$. Conversely let $x_L = \sum_{j=1}^{r} x_j e_j \in M$. Then $\langle x_L, e_l \rangle \in \mathbf{Z}$ for any l in $\{1, 2, \ldots, r\}$. Let $x_X = \sum_{i=1}^{r} x_i a_i \in X$. Then

$$\begin{aligned}\chi_{x_X}(a_l) &= \prod_{i=1}^{r} \chi_{a_i}(a_l)^{x_i} = \prod_{i=1}^{r} e^{-2\pi\sqrt{-1}\alpha_{i,l} x_i} \\ &= e^{-2\pi\sqrt{-1}\sum_{i=1}^{r}\alpha_{i,l} x_i} = e^{-2\pi\sqrt{-1}\langle x_L, e_l \rangle} = 1\end{aligned}$$

for any l in $\{1, 2, \ldots, r\}$. This implies that $\chi_{x_X}(y) = 1$ for all y in X. Therefore $x_X = 0$ in X and $x_i \equiv 0 \pmod{n_i}$ for $i = 1, 2, \ldots, r$. This implies $M \subseteqq \sum_{i=1}^{r} \mathbf{Z} n_i e_i$ and completes the proof of (59). Since $\langle n_i e_i, n_j e_j \rangle = n_i n_j \alpha_{i,j} \in 2\mathbf{Z}$ for all i, j in $\{1, 2, \ldots, r\}$, we have $\langle \alpha, \beta \rangle \in 2\mathbf{Z}$ for all α and β in M. Hence L is an even lattice. (ii) is immediate from (59). □

From now on we identify e_i and a_i $(i = 1, 2, \ldots, r)$. Let us define ε by the following equation:
For $x = \sum_{i=1}^{r} x_i a_i \in X$,

$$\begin{aligned}\varepsilon(x) &= \prod_{i=1}^{r} \eta_i^{-x_i} e^{-\pi\sqrt{-1}x_i\alpha_{i,i}} \\ &= \prod_{i=1}^{r} \eta_i^{-x_i} e^{-\pi\sqrt{-1}x_i\langle a_i, a_i \rangle}.\end{aligned}$$

Since

$$\begin{aligned}\left(\eta_i^{-1} e^{-\pi\sqrt{-1}\alpha_{i,i}}\right)^{n_i} &= \chi_{a_i}(a_i)^{\frac{n_i(n_i-1)}{2}} e^{-\pi\sqrt{-1}n_i\alpha_{i,i}} && \text{(by (48))} \\ &= e^{-\pi\sqrt{-1}n_i^2\alpha_{i,i}} = 1 && \text{(by the definition of } \alpha_{i,i}),\end{aligned}$$

this definition is valid and ε is a character of X. Define λ_α, μ and $t_\alpha^{(\gamma)}$ for $\alpha, \gamma \in X$ by (49), (50), (52) and (53). Then one easily checks that the formula (45) of the solution of the modular invariance equation $(P\Delta)^3 = t_0 D^3 I$, $\Delta = \mathrm{Diag}[t_x]_{x\in X}$ is expressed by

(60) $t_x = (t_x^{(0)})^{-1}$, $x \in X$,

provided that $t_0 = (t_0^{(0)})^{-1}$. Finally note that formula (46), which expresses (43), becomes (55).

Remark The Kac-Wakimoto construction essentially gives all the solutions of the modular invariance equation with respect to the given eigenmatrix P. However Theorem 4.4 shows that to obtain all the solutions it is enough to consider the case $\gamma = 0$.

6) Dualities of Abelian group schemes

In this section we classify the dualities of Abelian group schemes. We use the same notation as given in 4).

Define a duality of $\mathfrak{A}$ with the eigenmatrix P given by

(61) $P_{x,y} = \prod_{i=1}^{r} \zeta_i^{x_i y_i}$,

where $x = \sum_{i=1}^{r} x_i a_i$, $y = \sum_{i=1}^{r} y_i a_i \in X$ and ζ_i is a primitive n_ith root of unity. Since the set of all the Hadamard (or ordinary) idempotents in $\mathfrak{A}$ is uniquely determined and an eigenmatrix P' corresponds to a duality if and only if it is symmetric (see Section 4.1), there is a one to one correspondence between the set of all the dualities of $\mathfrak{A}$ and the set of all the permutations σ on X satisfying

(62) $\sigma(0) = 0$ and $P_{x,\sigma(y)} = P_{\sigma(x),y}$ for any x and y in X.

Let σ be such a permutation and let $\sigma(a_j) = \sum_{i=1}^{r} \sigma_{i,j} a_i$, where $\sigma_{i,j} \in \mathbf{Z}$. For any $x \in X$, let $\sigma(x) = \sum_{i=1}^{r} (x)_i a_i$, with $(x)_i \in \mathbf{Z}$. Then by (62) we have $P_{a_i,\sigma(a_j)} = P_{\sigma(a_i),a_j}$ and hence

(63) $\zeta_i^{\sigma_{i,j}} = \zeta_j^{\sigma_{j,i}}$

for any $i, j \in \{1, 2, \ldots, r\}$. Then by (62)

$$\begin{aligned}
\zeta_j^{(x)_j} = P_{a_j,\sigma(x)} &= P_{\sigma(a_j),x} \\
&= \prod_{i=1}^{r} \zeta_i^{x_i \sigma_{i,j}} \\
&= \prod_{i=1}^{r} \zeta_j^{x_i \sigma_{j,i}} \qquad \text{(by (63))} \\
&= \zeta_j^{\sum_{i=1}^{r} x_i \sigma_{j,i}},
\end{aligned}$$

that is, $(x)_j \equiv \sum_{i=1}^{r} x_i \sigma_{j,i}$ and hence the permutation σ is given by

(64) $\sigma(x) = \sum_{j=1}^{r} (\sum_{i=1}^{r} x_i \sigma_{j,i}) a_j$.

We can easily check that (63) and (64) imply (62) and that the expression (64) determines the permutation σ on X uniquely without depending on the choice of the $\sigma_{j,i} \in \mathbf{Z}$. Hence we have the following result.

Theorem 4.7 *The set of dualities of the Abelian group scheme $\mathfrak{A}$ has a one to one correspondence with the set of $r \times r$ nonsingular matrices $(\sigma_{i,j})$ satisfying $\sigma_{i,j} \in \mathbf{Z}$ and $\zeta_i^{\sigma_{i,j}} = \zeta_j^{\sigma_{j,i}}$ for any i and j in $\{1, 2, \ldots, r\}$. (Here such a matrix is said to be nonsingular if it induces a permutation on X.)*

Corollary 4.8 *Let p be a prime number. The set of dualities of the scheme of the elementary abelian group $\mathbf{Z}_p \times \cdots \times \mathbf{Z}_p$ of order p^r has a one to one correspondence with the set of nonsingular symmetric $r \times r$ matrices over the finite field $\mathbf{Z}_p$.*

Remark Corollary 4.8 for $p = 2$ was originally proved by Yamada [33].

5. Examples

Contrary to the Abelian group case, spin models at the algebraic level generally do not give actual spin models. In this section we give two such examples without proof.

1) Spin models on strongly regular graphs

We use the notations of [15].
Consider a strongly regular graph G with first eigenmatrix

$$P = \begin{pmatrix} 1 & k & n-k-1 \\ 1 & s & -s-1 \\ 1 & r & -r-1 \end{pmatrix}$$

We assume that G is formally self-dual, i.e., $P^2 = nI$, and this reduces to the equalities $k = r^2 + r - rs$, $n = (r-s)^2$. Let $D = \varepsilon(r-s)$ with $\varepsilon \in \{+1, -1\}$. We assume $n \geq 2$. Let $\Delta = \mathrm{Diag}[t_0, t_1, t_2] (t_i \neq 0)$. Then we have the following result.

Theorem 5.1 *With the notation given as above the modular invariance equation $(P\Delta)^3 = t_0 D^3 I$ holds if and only if one of the following (1) or (2) is satisfied.*
(1) $t_1 \neq t_2, t_1 t_2 = \varepsilon, t_0 = -st_1 + \varepsilon(r+1)t_1^{-1}$ *and* $t_0^{-1} = -st_1^{-1} + \varepsilon(r+1)t_1$.
(2) $n = 4, D = -2\varepsilon, t_1 = t_2, t_1^2 = \varepsilon, t_0 = -t_1$.

Proof: First assume that the modular equivalence equation holds. Since the equation is equivalent to $P\Delta P = t_0 D \Delta^{-1} P \Delta^{-1}$ we have

(i) $\sum_{l=0}^{2} P_{i,l} P_{l,j} t_l = t_0 D t_i^{-1} t_j^{-1} P_{i,j}$ for all i, j.

Putting $j = 0, i = 1$ and $j = 0, i = 2$, we have

$$t_0 + st_1 - (s+1)t_2 = Dt_1^{-1}$$
$$t_0 + rt_1 - (r+1)t_2 = Dt_2^{-1},$$

respectively. Hence $(s-r)t_1 - (s-r)t_2 = D(t_1^{-1} - t_2^{-1})$ and then $(t_1 - t_2)(s - r + Dt_1^{-1}t_2^{-1}) = 0$. Therefore if $t_1 \neq t_2$ we have $t_1 t_2 = D(r-s)^{-1} = \varepsilon$. It follows that $t_0 = -st_1 + (s+1)\varepsilon t_1^{-1} + Dt_1^{-1} = -st_1 + \varepsilon(r+1)t_1^{-1}$. Putting $j = i = 0$ in (i) we have

$$t_0 + kt_1 + (n-k-1)t_2 = Dt_0^{-1}$$

and hence

$$t_0^{-1} = D^{-1}\big(-st_1 + \varepsilon(r+1)t_1^{-1} + kt_1 + (n-k-1)\varepsilon t_1^{-1}\big).$$

Easy calculations give $t_0^{-1} = -st_1^{-1} + \varepsilon(r+1)t_1$ and (1) holds.

If $t_1 = t_2$, then

(ii) $t_0 - t_1 = Dt_1^{-1}$.

Putting $j = i = 0$ in (i) we have

(iii) $t_0 + (n-1)t_1 = Dt_0^{-1}$.

By (ii) and (iii) we have $t_1 + Dt_1^{-1} + (n-1)t_1 = D(t_1 + Dt_1^{-1})^{-1}$ and then $(t_1^{-1} + Dt_1)$ $(t_1 + Dt_1^{-1}) = 1$. Hence $D = -t_1^2 - t_1^{-2}$ and (by (ii)) $t_0 = -t_1^{-3}$. Next, taking $j = 1$ and $i = 1, i = 2$ in (i), we obtain:

(iv) $kt_0 + s^2 t_1 - (s+1)rt_1 = Dst_0 t_1^{-2}$
(v) $kt_0 + rst_1 - (r+1)rt_1 = Drt_0 t_1^{-2}$

The difference of these two equations gives $t_1^{-3} = (s-r)D^{-1}t_0^{-1} = \varepsilon t_1^3$. Hence $(t_1^2)^3 = \varepsilon$ and $t_1^2 \in \{\varepsilon,\ e^{\frac{2\pi\sqrt{-1}}{3}}\varepsilon, e^{-\frac{2\pi\sqrt{-1}}{3}}\varepsilon\}$. If $t_1^2 \neq \varepsilon$, $D = -\varepsilon(e^{\frac{2\pi\sqrt{-1}}{3}} + e^{-\frac{2\pi\sqrt{-1}}{3}}) = \varepsilon$ and $n = 1$, which we have ruled out in our hypothesis. If $t_1^2 = \varepsilon$, then $t_0 = -t_1$, $D = -2\varepsilon$, $n = 4$ and (2) holds.

Conversely assume that t_0, t_1, and t_2 satisfy the condition (1) or (2).
Let $W^+ = t_0^{-1}I + t_1^{-1}A_1 + t_2^{-1}A_2$ and $W^- = t_0 I + t_1 A_1 + t_2 A_2$. If Ψ is the duality with matrix P in the basis $\{E_0, E_1, E_2\}$, its matrix in the basis $\{A_0, A_1, A_2\}$ is also P. We can easily check that

$$P\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = D\begin{pmatrix} t_0^{-1} \\ t_1^{-1} \\ t_2^{-1} \end{pmatrix}$$

and then $\Psi(W^-) = DW^+$ holds. Let us denote t_0^{-1} by a. We have clearly (20) and (23) and by applying Ψ we get (21) and (24). Using this, one easily checks that $\Psi(M) = D^{-1}aW^+(W^- \circ (W^+M))$ when $M = I, M = J$, or $M = W^-$. Assuming that $t_1 \neq t_2$ (case (1)), these three matrices form a basis of the Bose-Mesner algebra $\mathfrak{A}$, and hence $\Psi(M) = D^{-1}aW^+(W^- \circ (W^+M))$ for all M in $\mathfrak{H}$. It then follows from Theorem 4.4. that $(P\Delta)^3 = a^{-1}D^3I$. For case (2) an explicit computation gives the modular equivalence equation. □

In other words, W^+, W^-, given in the proof of Theorem 5.1, define a spin model at the algebraic level. However it is shown in [15] that W^+, W^- will define an actual spin model if and only if G is primitive when $n \geq 5$ and G is triply regular (i.e., both constituents with respect to each vertex are strongly regular). Very few graphs meet this last requirement and this yields many examples of spin models at the algebraic level which are not actual spin models (for instance, consider Paley graphs with at least 13 vertices, or graphs associated with bilinear forms, alternating bilinear forms or hermitian forms (see [7]).

2) Modular invariance of the association scheme on 2-Sylow subgroup of Suzuki simple group Sz(8)

Let $n \geq 1$, $q = 2^{2n+1}$. Let θ be a generator of the Galois group of the finite field extension $GF(q)/GF(2)$ defined by $\alpha^\theta = \alpha^{2^n}$ for $\alpha \in GF(q)$. For $\alpha, \beta \in GF(q)$, let

$$S(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha^\theta & 1 & 0 & 0 \\ \beta & \alpha & 1 & 0 \\ (\alpha,\beta)_1 & (\alpha,\beta)_2 & \alpha^\theta & 1 \end{pmatrix},$$

where $(\alpha, \beta)_1 = \alpha^{2\theta+1} + \alpha^\theta\beta + \beta^{2\theta}$ and $(\alpha, \beta)_2 = \alpha^{\theta+1} + \beta$. Let $U = \{S(\alpha, \beta) \mid \alpha, \beta \in GF(q)\}$. Since $S(\alpha, \beta)S(\gamma, \delta) = S(\alpha+\gamma, \alpha\gamma^\theta+\beta+\delta)$ and $S(\gamma, \delta)S(\gamma, \gamma^{\theta+1}+\delta) = S(0, 0)$ (the identity element of $GL(4, q)$), U is a subgroup of $GL(4, q)$ of order $|U| = q^2$. It is known that U is isomorphic to the 2-Sylow subgroup of the Suzuki simple group $Sz(q)$(see [29, 30]). The center of U is given by $Z(U) = \{S(0, \beta) \mid \beta \in GF(q)\}$ which is of order q. For any γ and δ in $GF(q)$, we have $S(\gamma, \gamma^{\theta+1}+\delta)S(\alpha, \beta)S(\gamma, \delta) = S(\alpha, \beta+\gamma\alpha^\theta+\gamma^\theta\alpha)$. Therefore $S(\alpha, \beta)$ and $S(\alpha_1, \beta_1)$ are conjugate to each other if and only if $\alpha_1 = \alpha$ and $\beta_1 = \beta + \gamma\alpha^\theta + \gamma^\theta\alpha$ with some $\gamma \in GF(q)$. Let $GF(q)^*$ be the set of all the nonzero elements in $GF(q)$. For $\alpha \in GF(q)^*$, let $H_\alpha = \{S(0, \gamma\alpha^\theta + \gamma^\theta\alpha) \mid \gamma \in GF(q)\}$. Then H_α is a subgroup of $Z(U)$ of index 2. We can easily check that $S(0, \alpha^{\theta+1}) \notin H_\alpha$. Therefore $S(\alpha, 0)H_\alpha$, $S(\alpha, \alpha^{\theta+1})H_\alpha, \alpha \in GF(q)^*$ and $\{S(0, \alpha)\}, \alpha \in GF(q)$ form the complete set of conjugate classes of U.

Let Λ denote the following set of representatives of the conjugacy classes: $\Lambda = \{S(\alpha, 0), S(\alpha, \alpha^{\theta+1}) \mid \alpha \in GF(q)^*\} \cup \{S(0, \alpha) \mid \alpha \in GF(q)\}$. Then $|\Lambda| = 3q - 2$. For each $\lambda \in \Lambda$, C_λ denotes the conjugacy class containing λ, and A_λ denotes the matrix in $M(U)$ defined by

$$A_\lambda[x, y] = \begin{cases} 1 & \text{if } yx^{-1} \in C_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then properties (1)–(5) are satisfied (see [8] Section 2.2.). For convenience we replace the index set $\{0, 1, \ldots, d\}$ by Λ. Note that $A_{(0,0)} = A_0 = I$, ${}^tA_{(0,\alpha)} = A_{(0,\alpha)}$, and ${}^tA_{(\alpha,0)} = A_{(\alpha,\alpha^{\theta+1})}$. Thus $\mathfrak{X} = (A_\lambda, \lambda \in \Lambda)$ is an association scheme of class $3q - 3$.

In the following we assume $q = 2^3$. Then U is of order 64. Let $\Lambda = \Lambda_0 \cup \Lambda_1$, where $\Lambda_0 = \{S(0,\alpha) | \alpha \in GF(8)\} \subseteq \Lambda$ and $\Lambda_1 = \{S(\alpha, 0), S(\alpha, \alpha^{\theta+1}) \mid \alpha \in GF(8)^*\} \subseteq \Lambda$. Then we can easily show that $\mathfrak{X}$ has self-dual structures for the orderings of the ordinary idempotents $\{E_\lambda, \lambda \in \Lambda\}$ which give the matrices P defined below as the first eigen matrices of $\mathfrak{X}$:

$P_{i,0} = 1$ for all $i \in \Lambda$.
$P_{i,j} = 1$ for $i, j \in \Lambda_0$.
$P_{i,j} = 1$ for $i \in \{S(\alpha, 0), S(\alpha, \alpha^{\theta+1})\}$, $j \in H_\alpha$, and $\alpha \neq 0$.
$P_{i,j} = -1$ for $i \in \{S(\alpha, 0), S(\alpha, \alpha^{\theta+1})\}$, $j \notin H_\alpha$, and $\alpha \neq 0$.
$P_{i,j} = P_{i,j'} = 4P_{j,i} = 4P_{j',i}$ for $i \in \Lambda_0$, $j \in \Lambda_1$
$P_{i,i} = 4\varepsilon$ and $P_{i,i'} = P_{i',i} = -4\varepsilon_i$ for $i \in \Lambda_1$, where $\varepsilon_i = \varepsilon_{i'} = \pm\sqrt{-1}$.
$P_{i,j} = P_{j,i} = 0$ for $i, j \in \Lambda_1$ and $j \neq i, j \neq i'$.

We have the following result.

Theorem 5.2 *The Bose-Mesner algebra of a* 2*-Sylow subgroup* U *of the Suzuki simple group* $Sz(8)$ *has the modular invariance property* $(P\Delta)^3 = t_0 D^3 I$ *with an invertible diagonal matrix* $\Delta = Diag[t_\lambda]_{\lambda\in\Lambda}$ *if and only if the following conditions are satisfied.* (*Note that* $D = \pm 8$.)

(i) $t_\lambda = t_0$ *for* $\lambda \in \Lambda_0$.
(ii) $t_i + t_{i'} = 0$ *and* $t_i = \pm\frac{1-\varepsilon}{\sqrt{2}} t_0$ *for* $i \in \Lambda_1$.
(iii) $t_0^2 = D/8$.

Remark

(i) The proof of Theorem 5.2 is done by straightforward but tedious computations.
(ii) By Theorem 3.6, (U, W^+, W^-) with $W^+ = D\sum_{x\in\Lambda} t_x E_x$ and $W^- = \sum_{x\in\Lambda} t_x A_x$ is a spin model at the algebraic level. However we have checked that it is not an actual spin model, in spite of the fact that in a sense this scheme is very close to an Abelian group scheme.

In the original version of this paper and somewhere else we announced that a similar statement like the one given in Theorem 5.2 is also true for any $q = 2^{2n+1}$. However this is not true in general for $n \geq 2$ as was pointed out by Hanaki and Okuyama in private communication. Namely, the group association scheme may not be self-dual if $n \geq 2$. (Previously, there were miscalulations of the character tables of U.) We thank Hanaki and Okuyama for pointing out this mistake by showing that the groups for the case $n = 2$ are not self-dual.

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