Journal of Algebraic Combinatorics **6** (1997), 253–257

Aztec Diamonds, Checkerboard Graphs, and Spanning Trees

DONALD E. KNUTH
Department of Computer Science, Stanford University, Stanford, CA 94305-9045

Received October 25, 1994; Revised January 23, 1995

Abstract. This note derives the characteristic polynomial of a graph that represents nonjump moves in a generalized game of checkers. The number of spanning trees is also determined.

Keywords: Aztec diamond, spanning tree, graph spectra, enumeration

Consider the graph on mn vertices $\{(x, y) \mid 1 \le x \le m,\ 1 \le y \le n\}$, with (x, y) adjacent to (x', y') if and only if $|x - x'| = |y - y'| = 1$. This graph consists of disjoint subgraphs

$$EC_{m,n} = \{(x, y) \mid x + y \text{ is even}\},$$
$$OC_{m,n} = \{(x, y) \mid x + y \text{ is odd}\},$$

having respectively $\lceil mn/2 \rceil$ and $\lfloor mn/2 \rfloor$ vertices. When mn is even, $EC_{m,n}$ and $OC_{m,n}$ are isomorphic. The special case $OC_{2n+1,2n+1}$ has been called an *Aztec diamond of order n* by Elkies et al. [6], who gave several interesting proofs that it contains exactly $2^{n(n+1)/2}$ perfect matchings. Richard Stanley recently conjectured [11] that $OC_{2n+1,2n+1}$ contains exactly 4 times as many spanning trees as $EC_{2n+1,2n+1}$, and it was his conjecture that motivated the present note. We will see that Stanley's conjecture follows from some even more remarkable properties of these graphs.

In general, if G and H are arbitrary bipartite graphs having parts of respective sizes (p, q) and (r, s), their *weak direct product* $G \times H$ has $(p + q)(r + s)$ vertices (u, v), with (u, v) adjacent to (u', v') if and only if u is adjacent to u' and v to v'. This graph $G \times H$ divides naturally into even and odd subgraphs

$$E(G, H) = \{(u, v) \mid u \in G \text{ and } v \in H \text{ are in corresponding parts}\},$$
$$O(G, H) = \{(u, v) \mid u \in G \text{ and } v \in H \text{ are in opposite parts}\},$$

which are disjoint. Notice that $E(G, H)$ and $O(G, H)$ have $pr + qs$ and $ps + qr$ vertices, respectively. Our graphs $EC_{m,n}$ and $OC_{m,n}$ are just $E(P_m, P_n)$ and $O(P_m, P_n)$, where P_n denotes a simple path on n points.

Let $P(G; x)$ be the characteristic polynomial of the adjacency matrix of a graph G. The eigenvalues of $E(G, H)$ and $O(G, H)$ turn out have a simple relation to the eigenvalues of G and H:

Theorem 1 *The characteristic polynomials $P(E(G,H);x)$ and $P(O(G,H);x)$ satisfy*

$$P(E(G,H);x)P(O(G,H);x)=\prod_{j=1}^{p+q}\prod_{k=1}^{r+s}(x-\mu_j\lambda_k); \tag{1}$$

$$P(E(G,H);x)=x^{(p-q)(r-s)}P(O(G,H);x). \tag{2}$$

Proof: This theorem is a consequence of more general results proved in [7], as remarked at the top of page 67 in that paper, but for our purposes a direct proof is preferable.

Let A and B be the adjacency matrices of G and H. It is well known [2; 12] that the adjacency matrix of $G\times H$ is the Kronecker product $A\otimes B$, and that the eigenvalues of $A\otimes B$ are $\mu_j\lambda_k$ when A and B are square matrices having eigenvalues μ_j and λ_k, respectively [10, page 24]. Since the left side of (1) is just $P(G,H;x)$, equation (1) is therefore clear.

Equation (2) is more surprising, because the graphs $E(G,H)$ and $O(G,H)$ often look completely different from each other. But we can express A and B in the form

$$A=\begin{pmatrix}O_p & C\\ C^T & O_q\end{pmatrix},\qquad B=\begin{pmatrix}O_r & D\\ D^T & O_s\end{pmatrix}, \tag{3}$$

where C and D have respective sizes $p\times q$ and $r\times s$, and where O_k denotes a $k\times k$ matrix of zeroes. It follows that the adjacency matrices of $E(G,H)$ and $O(G,H)$ are respectively

$$\begin{pmatrix}O_{pr} & C\otimes D\\ C^T\otimes D^T & O_{qs}\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}O_{ps} & C\otimes D^T\\ C^T\otimes D & O_{qr}\end{pmatrix}. \tag{4}$$

We want to show that these matrices have the same eigenvalues, except for the multiplicity of 0.

One way to complete the proof is to observe that the kth powers of both matrices have the same trace, for all k. When $k=2l$ is even, both matrix powers have trace $(\mathrm{tr}(CC^T)^l+\mathrm{tr}(C^TC)^l)(\mathrm{tr}(DD^T)^l+\mathrm{tr}(D^TD)^l)$ by [10, pages 8, 18]; and when k is odd the traces are zero. The coefficients $a_1, a_2, \ldots$ of $P(G;x)=x^{|G|}(1-a_1x^{-1}+a_2x^{-2}-\cdots)$ are completely determined by the traces of powers of the adjacency matrix of any graph G, via Newton's identities; therefore (2) holds. □

Corollary 1 *The characteristic polynomials $P(EC_{m,n};x)$ and $P(OC_{m,n};x)$ satisfy*

$$P(EC_{m,n};x)P(OC_{m,n};x)=\prod_{j=1}^{m}\prod_{k=1}^{n}\left(x-4\cos\frac{j\pi}{m+1}\,\cos\frac{k\pi}{n+1}\right); \tag{5}$$

$$P(EC_{m,n};x)=x^{mn \bmod 2}\,P(OC_{m,n};x). \tag{6}$$

Proof: It is well known [9, problem 1.29; or 3, page 73], that the eigenvalues of the path graph P_m are

$$\left\{2\cos\frac{\pi}{m+1},\,2\cos\frac{2\pi}{m+1},\,\ldots,\,2\cos\frac{m\pi}{m+1}\right\}. \tag{7}$$

Therefore (1) and (2) reduce to (5) and (6). □

Theorem 2 *If $m \geq 2$ and $n \geq 2$, the number of spanning trees of $EC_{m,n}$ is $P(OC_{m-2,n-2}; 4)$, and the number of spanning trees of $OC_{m,n}$ is $P(EC_{m-2,n-2}; 4)$.*

Proof: Both $EC_{m,n}$ and $OC_{m,n}$ are connected planar graphs, so they have exactly as many spanning trees as their duals [9, problem 5.23]. The dual graph $EC^*_{m,n}$ has vertices (x, y) where $1 < x < m$ and $1 < y < n$ and $x + y$ is odd, corresponding to the face centered at (x, y); it also has an additional vertex ∞ corresponding to the exterior face. All its non-infinite vertices have degree 4, and when $EC^*_{m,n}$ is restricted to those vertices it is just $OC_{m-2,n-2}$. Therefore the submatrix of the Laplacian of $EC^*_{m,n}$ that we obtain by omitting row ∞ and column ∞ is just $4I - M$, where M is the adjacency matrix of $OC_{m-2,n-2}$. And the number of spanning trees of $EC^*_{m,n}$ is just the determinant of this matrix, according to the Matrix Tree Theorem [1; 9, problem 4.9; 3, page 38].

A similar argument enumerates the spanning trees of $OC_{m,n}$. The basic idea of this proof is due to Cvetković and Gutman [4]; see also [5, pages 85–88]. □

Combining Theorem 2 with Eq. (6) now yields a generalization of Stanley's conjecture [11].

Corollary 2 *When m and n are both odd, $OC_{m,n}$ contains exactly 4 times as many spanning trees as $EC_{m,n}$.*

Another corollary that does not appear to be obvious a priori follows from Theorem 2 and Eq. (5):

Corollary 3 *When m and n are both even, $EC_{m,n}$ contains an odd number of spanning trees.*

Proof: The adjacency matrix of P_m is nonsingular mod 2 when m is even. Hence the adjacency matrix of $EC_{m,n} \cup OC_{m,n}$ is nonsingular mod 2. Hence $P(EC_{m,n}; 4) \equiv 1 \pmod 2$. □

Stanley [11] tabulated the number of spanning trees in $OC_{2n+1,2n+1}$ for $n \leq 6$ and observed that the numbers consisted entirely of small prime factors. For example, the Aztec diamond graph $OC_{13,13}$ has exactly $2^{32} \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 73^2 \cdot 193^2$ spanning trees. One way to account for this is to note that the number of spanning trees in $OC_{2n+1,2n+1}$ is

$$4^{2n-1} \prod_{j=1}^{n-1} \prod_{k=1}^{n-1} \left(4 - 4\cos\frac{j\pi}{2n}\cos\frac{k\pi}{2n}\right)\left(4 + 4\cos\frac{j\pi}{2n}\cos\frac{k\pi}{2n}\right)$$
$$= 4^{2n-1} \prod_{j=1}^{n-1} \prod_{k=1}^{n-1} (4 - (\omega^j + \omega^{-j})(\omega^k + \omega^{-k}))(4 + (\omega^j + \omega^{-j})(\omega^k + \omega^{-k})), \quad (8)$$

where $\omega = e^{\pi i/2n}$ is a primitive $4n$th root of unity. Thus each factor such as $4 - (\omega^j + \omega^{-j})(\omega^k + \omega^{-k})$ is an algebraic integer in a cyclotomic number field, and all of its conjugates $4 - (\omega^{jt} + \omega^{-jt})(\omega^{kt} + \omega^{-kt})$ appear. Each product of conjugate factors is therefore an integer factor of (8).

Let us say that the edge from (x, y) to (x', y') in the graph is positive or negative, according as $(x - x')(y - y')$ is $+1$ or -1. The authors of [6] showed that the generating function for perfect matchings in $OC_{2n+1,2n+1}$ is $(u^2 + v^2)^{n(n+1)/2}$, in the sense that the coefficient of $u^k v^l$ in this function is the number of perfect matchings with k positive edges and l negative ones. It is natural to consider the analogous question for spanning trees: What is the generating function for spanning trees of $EC_{m,n}$ and $OC_{m,n}$ that use a given number of positive and negative edges? A careful analysis of the proof of Theorem 2 shows that the generating function for cotrees (the complements of spanning trees) in $OC_{m,n}$ is $P(EC_{m-2,n-2}; 2u + 2v)$, where P now represents the characteristic polynomial of the weighted adjacency matrix with positive and negative edges represented respectively by u and v. There are $\lceil (m-1)(n-1)/2 \rceil$ positive edges and $\lfloor (m-1)(n-1)/2 \rfloor$ negative edges altogether, so we get the generating function for trees instead of cotrees by replacing u and v by u^{-1} and v^{-1}, then multiplying by $u^{\lceil (m-1)(n-1)/2 \rceil} v^{\lfloor (m-1)(n-1)/2 \rfloor}$. A similar approach works for $EC_{m,n}$.

Unfortunately, however, the polynomial P does not seem to simplify nicely for general u and v, as it does when $u = v = 1$. In the case $m = n = 3$, the results look reasonably encouraging because we have

$$P(EC_{3,3}; x) = x^3(x^2 - 2(u^2 + v^2)),$$
$$P(OC_{3,3}; x) = (x + u + v)(x - u - v)(x + u - v)(x - u + v).$$

But when n increases to 5 we get

$$P(EC_{3,5}; x) = x^4(x^2 - 2(u^2 + uv + v^2))(x^2 - 2(u^2 - uv - v^2)),$$
$$P(OC_{3,5}; x) = x(x^2 - (u^2 + v^2))(x^4 - 3(u^2 + v^2)x^2 + 2(u^2 - v^2)).$$

The quartic factor of $P(OC_{3,5}; x)$ cannot be decomposed into quadratics having the general form $(x^2 - (\alpha u^2 + \beta uv + \gamma v^2))(x^2 - (\alpha' u^2 + \beta' uv + \gamma' v^2))$, so it is unclear how to proceed. Some simplification may be possible, because additional factors do appear when we set $x = 2u + 2v$:

$$\begin{aligned}
P(EC_{3,5}; 2u + 2v) &= 64(u + v)^4(u^2 + 3uv + v^2)(u^2 + 5uv + v^2)\\
P(OC_{3,5}; 2u + 2v) &= 4(u + v)^3(3u^2 + 8uv + 3v^2)(3u^2 + 14uv + 3v^2)\\
P(EC_{5,5}; 2u + 2v) &= 32(u + v)^5(3u^2 + 8uv + 2v^2)(2u^2 + 8uv + 3v^2)\\
&\quad \times (2u^4 + 24u^3v + 53u^2v^2 + 24uv^3 + 2v^4)\\
P(OC_{5,5}; 2u + 2v) &= 5(u + v)^4(u^2 + 4uv + v^2)(3u^2 + 8uv + 3v^2)\\
&\quad \times (15u^2 + 10uv + v^2)(u^2 + 10uv + 15v^2).
\end{aligned}$$

However, these factors are explained by the symmetries of $EC_{m,n}$ and $OC_{m,n}$.

Acknowledgments

I thank an anonymous referee for suggesting the present form of Theorem 1, which is considerably more general (and easier to prove) than the special case I had observed in the first draft of this note. Noam Elkies gave me the insight about algebraic conjugates, when

I was studying the related problem of spanning trees in grids [8]. After writing this note, I learned from Richard Stanley that the eigenvalues of the adjacency matrices of $OC_{2n+1,2n+1}$ and $EC_{2n+1,2n+1}$ were independently discovered by Timothy Chow. In May, 1996, Dr. Chow wrote me that he has succeeded in generalizing the results to the two connected components of the tensor product of any two connected bipartite graphs.

References

1. C.W. Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Journal für die reine und angewandte Mathematik* **57** (1860), 111–121.
2. Karel Čulik, "Zur theorie der graphen," *Časopis pro Pěstování Matematiky* **83** (1958), 133–155.
3. Dragoš M. Cvetković, Michael Doob, and Horst Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.
4. D. Cvetković and I. Gutman, "A new spectral method for determining the number of spanning trees," *Publications de l'Institut Mathématique*, Beograd, **29**(43) (1981), 49–52.
5. Dragoš M. Cvetković, Michael Doob, Ivan Gutman, and Aleksandar Torgašev, "Recent Results in the Theory of Graph Spectra," *Annals of Discrete Mathematics* **36** (1988).
6. Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp, "Alternating-sign matrices and domino tilings," *J. Alg. Combin.* **1** (1992), 111–132 and 219–234.
7. C. Godsil and B. McKay, "Products of graphs and their spectra," in *Combinatorial Mathematics IV*, A. Dold and B. Eckmann (Eds.), Lecture Notes in Mathematics **560** (1975), 61–72.
8. Germain Kreweras, "Complexité et circuits eulériens dans les sommes tensorielles de graphes," *Journal of Combinatorial Theory* B**24** (1978), 202–212.
9. László Lovász, *Combinatorial Problems and Exercises*, 2nd Edition, Budapest, Akadémiai Kiadó, 1993.
10. Marvin Marcus and Henrik Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Boston, Allyn and Bacon, 1964.
11. Richard P. Stanley, "Spanning trees of Aztec diamonds," open problem presented at a DIMACS meeting on *Formal Power Series and Algebraic Combinatorics*, Piscataway, NJ, May 23–27, 1994.
12. Paul M. Weichsel, "The Kronecker product of graphs," *Proceedings of the American Mathematical Society* **13** (1962), 47–52.