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The Irreducible Modules of the Terwilliger Algebras of Doob Schemes

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Abstract. Let Y be any commutative association scheme and we fix any vertex x of Y. Terwilleger introduced a non-commutative, associative, and semi-simple $\mathbf{C}$-algebra $T = T(x)$ for Y and x in [4]. We call T the Terwilliger (or subconstituent) algebra of Y with respect to x.

Let $W(\subset \mathbf{C}^{|X|})$ be an irreducible $T(x)$-module. W is said to be thin if W satisfies a certain simple condition. Y is said to be thin with respect to x if each irreducible $T(x)$-module is thin. Y is said to be thin if Y is thin with respect to each vertex in X.

The Doob schemes are direct product of a number of Shrikhande graphs and some complete graphs K_4. Terwilliger proved in [4] that Doob scheme is not thin if the diameter is greater than two. I give the irreducible $T(x)$-modules of Doob schemes.

1. Introduction

1.1. *The Terwilliger algebras of association schemes*

In this section we give the definition of the Terwilliger algebra, and some of its properties. See [4] for more information.

Definition 1 Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative association scheme, and A_i be the ith associate matrix of Y $(0 \le i \le D)$. Now fix any $x \in X$. For each integer i $(0 \le i \le D)$, let $E_i^*(x)$ be the diagonal matrix in $\mathrm{Mat}_X(\mathbf{C})$ defined by

$$(E_i^*(x))_{yy} := \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise,} \end{cases} \qquad (y \in X)$$

where $\mathrm{Mat}_X(\mathbf{C})$ is the $\mathrm{Mat}_{|X|}(\mathbf{C})$ whose rows and columns are indexed by X. Throughout this paper, we adopt the convention that $E_i^*(x) := 0$ for any integer i such that $i < 0$ or $i > D$.

Let $T(x)$ be the subalgebra of $\mathrm{Mat}_X(\mathbf{C})$ generated by A_i $(0 \le i \le D)$ and $E_i^*(x)$ $(0 \le i \le D)$. We call $T(x)$ the *Terwilliger* (*or subconstituent*) *algebra of* Y *with respect to* x.

Definition 2 Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative association scheme, and E_i $(0 \le i \le D)$ be the primitive idempotents of Y. X Pick any $x \in X$ and write $E_i^* = E_i^*(x)$

$(0 \le i \le D)$, $T = T(x)$. T acts on $V := \mathbf{C}^{|X|}$ by left multiplication. We call V the *standard module*. Let $W(\subset V)$ be an irreducible T-module and define

$$W_s := \{i \mid 0 \le i \le D, E_i W \ne 0\}.$$

We call W_s the *support* of W and $|W_s| - 1$ the *diameter* of W. Now define

$$W_\sigma := \{i \mid 0 \le i \le D, E_i^* W \ne 0\}.$$

We call W_σ the *dual-support* of W and $|W_\sigma| - 1$ the *dual-diameter* of W. W is said to be *thin* if

$$\dim E_i^* W \le 1 \text{ for all } i \ (0 \le i \le D).$$

W is said to be *dual-thin* if

$$\dim E_i W \le 1 \text{ for all } i \ (0 \le i \le D).$$

Y is said to be *thin* (resp. *dual-thin*) *with respect to* x if each irreducible $T(x)$-module is thin (resp. *dual thin*). Y is said to be *thin* (resp. *dual-thin*) if Y is thin (resp. *dual thin*) with respect to each vertex $x \in X$.

Definition 3

(1) Assume the association scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ is a P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$ of the associate matrices. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. Let W be an irreducible T-module. We set

$$\nu := \min\{i \mid 0 \le i \le D, \ E_i^* W \ne 0\}.$$

We call ν the *dual-endpoint* of W.

(2) Assume the association scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ is a Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. Let W be an irreducible T-module. We set

$$\mu := \min\{i \mid 0 \le i \le D, \ E_i W \ne 0\}.$$

We call μ the *endpoint* of W.

Lemma 1 ([4], Lemmas 3.4, 3.9 and 3.12) *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative association scheme.*

(1) *Pick any $x \in X$. Then $T(x)$ is a semi-simple algebra.*

(2) *Assume Y is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_D$ of the associate matrices. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. Let W be an irreducible T-module with the dual-endpoint ν and the dual-diameter d^*. Then*

- *T is generated by $A := A_1$ and E_i^* $(0 \le i \le D)$.*
- *The dual-support $W_\sigma = \{\nu, \ \nu + 1, \ldots, \nu + d^*\}$.*

(3) *Assume Y is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Pick any $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$. Let W be an irreducible T-module with the endpoint μ and the diameter d. Then*

$$\text{The support } W_s = \{\mu,\ \mu+1, \ldots, \mu+d\}.$$

1.2. Doob schemes

Definition 4 The *direct product* (=: Γ) of t graphs $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$ is defined by as follows.

$$\begin{aligned}\text{The vertex set of } \Gamma &:= V\Gamma \\ &:= V\Gamma^1 \times V\Gamma^2 \times \cdots \times V\Gamma^t,\end{aligned}$$

where $V\Gamma^i$ is the vertex set of Γ^i $(1 \leq i \leq t)$. And for $x = (x_1, x_2, \ldots, x_t)$, $y = (y_1, y_2, \ldots, y_t) \in V\Gamma$, x is adjacent to y if and only if there is m $(1 \leq m \leq t)$ such that x_m is adjacent to y_m and $x_i = y_i$ for all $i \neq m$.

Definition 5 ([2, 3])

(1) A graph is called *complete* when any two of its vertices are adjacent. The complete graph on n vertices is denoted by K_n.
(2) Shrikhande graph S is defined as follows. The vertex set of S is $\{(i, j) \mid 1 \leq i \leq 4, 1 \leq j \leq 4\}$ and two vertices (i, j) and (k, l) are defined to be adjacent if and only if the following three condition hold:
   (i) $i \neq k$.
   (ii) $j \neq l$.
   (iii) $i - j \not\equiv k - l \pmod 4$.
(3) The Doob schemes are direct product of a number of Shrikhande graphs and some complete graphs K_4.

Doob schemes are distance-regular graphs, that is, P-polynomial association schemes. The eigenvalues of the 1-st associate matrix of Doob scheme are $\{3D - 4i \mid 0 \leq i \leq D\}$, where D is the diameter of the Doob scheme. Doob schems are Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$, where E_i is the primitive idempotent corresponding to the eigenvalue $3D - 4i$ $(0 \leq i \leq D)$. Terwilleger proved in [4] that Doob scheme is not thin if the diameter is greater than two.

2. The irreducible modules of Terwilliger algebras of Doob schemes

Let $D_1 \geq 1$ and $D_2 \geq 0$. We denote the Doob scheme as

$$\Gamma(D_1, D_2) = \underbrace{S \times \cdots \times S}_{D_1} \times \underbrace{K_4 \times \cdots \times K_4}_{D_2},$$

where S is Shrikhande graph and K_4 is the complete graph of 4-vertices.

For any x, y (vertices of $\Gamma(D_1, D_2)$), there exist a $\varphi \in \mathrm{Aut}(\Gamma(D_1, D_2))$ such that $\varphi(x) = y$. Hence $T(x) \cong T(y)$. So we may pick $x = (\underbrace{x_0, \ldots, x_0}_{D_1}, \underbrace{y_0, \ldots, y_0}_{D_2})$ as the fixed point of Terwilliger algebra, where x_0 is a vertex of S and y_0 is a vertex of K_4. Let $E_i^* = E_i^*(x)$ be the ith dual idempotent of $\Gamma(D_1, D_2)$ and $T = T(x)$ be the Terwilliger algebra of $\Gamma(D_1, D_2)$ with respect to x.

For any integers n, m $(0 \le n \le D_1,\ 0 \le m \le D_2,\ n + m > 0)$, we consider the Terwilliger algebra of $\Gamma(n, m)$ with respect to $x(n, m) = (\underbrace{x_0, \ldots x_0}_{n}, \underbrace{y_0, \ldots, y_0}_{m})$.

Let $E_i^*(n, m) = E_i^*(n, m)(x(n, m))$ be the ith dual idempotent of $\Gamma(n, m)$ $(0 \le i \le 2n + m)$, $A(n, m)$ be the 1st associate matrix, and $T(n, m) = T(n, m)(x(n, m))$ be the Terwilliger algebra of $\Gamma(n, m)$ with respect to $x(n, m)$.

We find

$$\begin{aligned} A(n,m) = &\left(\sum_{i=1}^{n} I_{16} \otimes \cdots \otimes I_{16} \otimes \overset{i}{\smile}{A(1,0)} \otimes I_{16} \otimes \cdots \otimes I_{16}\right) \otimes I_{4^m} \\ &+ I_{16^n} \otimes \left(\sum_{j=1}^{m} I_4 \otimes \cdots \otimes I_4 \otimes \overset{j}{\smile}{A(0,1)} \otimes I_4 \otimes \cdots \otimes I_4\right), \end{aligned}$$

$$E_i^*(n,m) = \sum_{\substack{i_1,\ldots,i_n \in \{0,1,2\},\\ j_1,\ldots,j_m \in \{0,1\},\\ i_1+\cdots+i_n+j_1+\cdots+j_m = i.}} E_{i_1}^*(1,0) \otimes \cdots \otimes E_{i_n}^*(1,0) \otimes E_{j_1}^*(0,1) \otimes \cdots \otimes E_{j_m}^*(0,1),$$

$$0 \le i \le 2n + m,$$

and for any k, l $(1 \le k \le n - 1,\ 1 \le l \le m - 1)$,

$$\begin{aligned} A(n,m) &= A(k,0) \otimes I_{16^{n-k}4^m} + I_{16^k} \otimes A(n-k, m) \\ &= A(n,l) \otimes I_{4^{m-l}} + I_{16^n 4^l} \otimes A(0, m-l), \end{aligned}$$

$$\begin{aligned} E_i^*(n,m) &= \sum_{\substack{r,s \ge 0\\ r+s=i}} E_r^*(k,0) \otimes E_s^*(n-k, m) \\ &= \sum_{\substack{r,s \ge 0\\ r+s=i}} E_r^*(n,l) \otimes E_s^*(0, m-l). \end{aligned}$$

$T(D_1, D_2)$ naturally acts on $\mathbf{C}^{16^{D_1}4^{D_2}} \cong \underbrace{\mathbf{C}^{16} \otimes \cdots \otimes \mathbf{C}^{16}}_{D_1} \otimes \underbrace{\mathbf{C}^4 \otimes \cdots \otimes \mathbf{C}^4}_{D_2}$. We identify the standard module and $\underbrace{\mathbf{C}^{16} \otimes \cdots \otimes \mathbf{C}^{16}}_{D_1} \otimes \underbrace{\mathbf{C}^4 \otimes \cdots \otimes \mathbf{C}^4}_{D_2}$. First of all we consider $T(1, 0)$-modules and $T(0, 1)$-modules.

Proposition 1

(1)

$$\mathbf{C}^{16} \cong U_0 \oplus U_1^{\oplus 2} \oplus U_2^{\oplus 2} \oplus U_3 \oplus U_4 \oplus U_5^{\oplus 3},$$

as $T(1,0)$-modules, where $U_0,\ U_1,\ U_2,\ U_3,\ U_4,\ U_5$ are the following irreducible $T(1,0)$-modules. The notation $[\alpha]_\beta$ is the matrix representation α with respect to the basis β.

- *U_0 has a basis (a_0, a_1, a_2) such that $a_i \in E_i^*(1,0)\mathbf{C}^{16}$ $(i = 0, 1, 2)$ and*

$$[A(1,0)]_{(a_0,\ a_1,\ a_2)} = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

- *U_1 has a basis $(b_1,\ b_2)$ such that $b_i \in E_i^*(1,0)\mathbf{C}^{16}$ $(i = 1, 2)$ and*

$$[A(1,0)]_{(b_1,\ b_2)} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

- *U_2 has a basis $(c_1,\ c_2)$ such that $c_i \in E_i^*(1,0)\mathbf{C}^{16}$ $(i = 1, 2)$ and*

$$[A(1,0)]_{(c_1,\ c_2)} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

- *U_3 has a basis d_1 such that $d_1 \in E_1^*(1,0)\mathbf{C}^{16}$ and $A(1,0)d_1 = -2d_1$.*
- *U_4 has a basis e_2 such that $e_2 \in E_2^*(1,0)\mathbf{C}^{16}$ and $A(1,0)e_2 = 2e_2$.*
- *U_5 has a basis f_2 such that $f_2 \in E_2^*(1,0)\mathbf{C}^{16}$ and $A(1,0)f_2 = -2f_2$.*

(2)

$$\mathbf{C}^4 \cong V_0 \oplus V_1^{\oplus 2},$$

as $T(0,1)$-modules, where $V_0,\ V_1$ are the following irreducible $T(0,1)$-modules.

- *V_0 has a basis $(g_0,\ g_1)$ such that $g_i \in E_i^*(0,1)\mathbf{C}^4$ $(i = 0, 1)$ and*

$$[A(0,1)]_{(g_0,\ g_1)} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

- *V_1 has a basis h_1 such that $h_1 \in E_1^*(0,1)\mathbf{C}^4$ and $A(0,1)h_1 = -h_1$.*

Proof: Straightforward. □

Proposition 2

(1) *Let $U_1', \ldots, U_{D_1}' \subset \mathbf{C}^{16}$ be $T(1,0)$-modules and $V_1', \ldots, V_{D_2}' \subset \mathbf{C}^4$ be $T(0,1)$-modules. Then $U_1' \otimes \cdots \otimes U_{D_1}' \otimes V_1' \otimes \cdots \otimes V_{D_2}'$ is a $T(D_1, D_2)$-module.*

(2) *Let $U_1^1, \ldots, U_{D_1}^1,\ U_1^2, \ldots, U_{D_1}^2 \subset \mathbf{C}^{16}$ be $T(1,0)$-modules and $V_1^1, \ldots, V_{D_2}^1,\ V_1^2, \ldots, V_{D_2}^2 \subset \mathbf{C}^4$ be $T(0,1)$-modules. Assume $U_i^1 \cong U_i^2$ $(1 \le i \le D_1)$ and $V_i^1 \cong V_i^2$ $(1 \le i \le D_2)$. Then*

$$U_1^1 \otimes \cdots \otimes U_{D_1}^1 \otimes V_1^1 \otimes \cdots \otimes V_{D_2}^1 \cong U_1^2 \otimes \cdots \otimes U_{D_1}^2 \otimes V_1^2 \otimes \cdots \otimes V_{D_2}^2$$

as $T(D_1, D_2)$-modules.

(3) *Let* $U_1', U_2' \subset \mathbf{C}^{16^p}$ *be* $T(p,0)$*-modules, and* $V_1', V_2' \subset \mathbf{C}^{16^q 4^r}$ *be* $T(q,r)$*-modules. Then* $U_1' \otimes V_1', U_2' \otimes V_2'$ *are* $T(p+q, r)$*-modules. Moreover if* $U_1' \cong U_2'$ *and* $V_1' \cong V_2'$*, then* $U_1' \otimes V_1' \cong U_2' \otimes V_2'$*.*
Let $U_1'', U_2'' \subset \mathbf{C}^{16^p 4^q}$ *be* $T(p,q)$*-modules, and* $V_1'', V_2'' \subset \mathbf{C}^{4^r}$ *be* $T(0,r)$*-modules. Then* $U_1'' \otimes V_1'', U_2'' \otimes V_2''$ *are* $T(p, q+r)$*-modules. Moreover if* $U_1'' \cong U_2''$ *and* $V_1'' \cong V_2''$*, then* $U_1'' \otimes V_1'' \cong U_2'' \otimes V_2''$*.*

(4) *Let* $U_1', \ldots, U_{D_1}'$ *be elements of* $\{U_0, U_1, U_2, U_3, U_4, U_5\}$*,* $V_1', \ldots, V_{D_2}'$ *be elements of* $\{V_0, V_1\}$ *and set* $N_j := \#\{i \mid U_i' = U_j\}$ $(j = 0,1,2,3,4,5)$*,* $M_k := \#\{i \mid V_i' = V_k\}$ $(k = 0, 1)$*. Then*

$$\left(\bigotimes_{i=1}^{D_1} U_i'\right) \otimes \left(\bigotimes_{j=1}^{D_2} V_j'\right) \cong \left(\bigotimes_{i=0}^{5} U_i^{\otimes N_i}\right) \otimes \left(\bigotimes_{j=0}^{1} V_j^{\otimes M_j}\right).$$

Proof: Straightforward. □

It is known that all irreducible $T(D_1, D_2)$-modules arise in the irreducible decomposition of the standard module $\mathbf{C}^{16^{D_1}4^{D_2}}$ up to isomorphism. From now on we decompose the standard module into irreducible $T(D_1, D_2)$-modules and then get all irreducible $T(D_1, D_2)$-modules.

From the above proposition, we get

$$\mathbf{C}^{16^{D_1}4^{D_2}} \cong \bigoplus_{\substack{N_0,\ldots,N_5\geq 0,\\ N_0+\cdots+N_5=D_1}} \bigoplus_{\substack{M_0,M_1\geq 0,\\ M_0+M_1=D_2}} \left(\left(\bigotimes_{i=0}^{5} U_i^{\otimes N_i}\right) \otimes \left(\bigotimes_{j=0}^{1} V_j^{\otimes M_j}\right)\right)^{\oplus \frac{D_1!}{N_0!\cdots N_5!}\frac{D_2!}{M_0!M_1!}2^{N_1}2^{N_2}3^{N_5}2^{M_1}}$$

as $T(D_1, D_2)$-modules. Hence it is sufficient to consider irreducible decompositions of $T(D_1, D_2)$-module $(\bigotimes_{i=0}^{5} U_i^{\otimes N_i}) \otimes (\bigotimes_{j=0}^{1} V_j^{\otimes M_j})$, where

$$N_0, \ldots, N_5, M_0, M_1 \geq 0, \qquad N_0 + \cdots + N_5 = D_1, \qquad M_0 + M_1 = D_2.$$

Proposition 3 *Let* n, m, ν, d, p, t *be elements of* $\mathbf{Z}$ *and* $n, m, \nu, d, p \geq 0$*.*

Let $W = W(n, m;\ \nu, d, p, t)$ $(\subset \mathbf{C}^{16^n 4^m})$ *be a* $T(n,m)$*-module with dual-endpoint* ν*, dual-diameter* $d+p$*, dimension* $(d+1)(p+1)$*, and a basis* w_{ij} $(0 \leq i \leq d,\ 0 \leq j \leq p)$ *satisfying*

$$\begin{aligned}
w_{ij} &\in E^*_{\nu+i+j}(n,m)W,\\
A(n,m)w_{ij} &= 3(d-i+1)w_{i-1,j} + (p-j+1)w_{i,j-1}\\
&\quad + (t + 2(i-j))w_{ij}\\
&\quad + 3(j+1)w_{i,j+1} + (i+1)w_{i+1,j},\\
w_{ij} &:= 0, \text{ if } i \notin \{0,\ldots,d\} \text{ or } j \notin \{0,\ldots,p\}.
\end{aligned}$$

Then

(1) *W is an irreducible $T(n,m)$-module.*
(2)

$$\dim E^*_{\nu+k}(n,m)W = \begin{cases} k+1, & \text{if } 0 \le k \le \min\{d,p\}, \\ \min\{d,p\}+1, & \text{if } \min\{d,p\} < k \le \max\{d,p\}, \\ d+p+1-k, & \text{if } \max\{d,p\} < k \le d+p. \end{cases}$$

(3) *W is thin if and only if $dp = 0$.*
(4) *Let μ be the endpoint of W. Then*

$$\mu = \frac{3(2n+m) - t - 3d - p}{4},$$
$$\dim E_{\mu+k}(n,m)W = \dim E^*_{\nu+k}(n,m)W,$$
$$\textit{the diameter of } W = \textit{the dual-diameter of } W.$$

(5) *For $W' = W(n,m;\ \nu', d', p', t')$, $W \cong W'$ if and only if $(\nu, d, p, t) = (\nu', d', p', t')$.*

Proof: (2) From the condition of the basis w_{ij} $(0 \le i \le d,\ 0 \le j \le p)$ of W,

$$\dim E^*_{\nu+k}(n,m)W = \#\{(i,j) \mid i+j = k\},\ 0 \le k \le d+p.$$

We get (2).
(3) is clear from (2).
(1) We decompose W into irreducible modules

$$W \cong W_1 \oplus W_2 \oplus \cdots, (W_1,\ W_2, \ldots \text{ are irreducible } T(n,m)\text{-modules.})$$

From the condition for the basis w_{ij} $(0 \le i \le d,\ 0 \le j \le p)$, $E^*_\nu(n,m)W = \mathbf{C}w_{00}$. So we may assume $w_{00} \in W_1$. We will show that

$$(*) \quad w_{ij} \in W_1 \quad \text{for any } i, j \ge 0 \text{ such that } i+j = k.$$

by induction on $k (0 \le k \le d+p)$. If $k = 0$, $(*)$ holds by the assumption. Next we assume $(*)$ holds for $k-1$.
We define

$$R_i := E^*_{\nu+i+1}(n,m)A(n,m)E^*_{\nu+i}(n,m),\ 0 \le i \le d+p-1,$$
$$F_i := E^*_{\nu+i}(n,m)A(n,m)E^*_{\nu+i}(n,m),\ 0 \le i \le d+p,$$
$$L_i := E^*_{\nu+i-1}(n,m)A(n,m)E^*_{\nu+i}(n,m),\ 1 \le i \le d+p.$$

By the assumption of the induction, $R_{k-1}w_{ij}, F_k R_{k-1}w_{ij} \in W_1$ for any $i, j \ge 0$ $(i+j = k-1)$. The followings hold.

For k $(1 \le k \le \min\{d, p\})$,

$$\{w_{ij} \mid 0 \le i \le d,\ 0 \le j \le p,\ i+j=k\} = \{w_{0k},\ w_{1,k-1}, \dots,\ w_{k0}\},$$

$$(F_k R_{k-1} w_{0,k-1},\ R_{k-1} w_{0,k-1},\ R_{k-1} w_{1,k-2}, \dots,\ R_{k-1} w_{k-1,0})$$

$$= (w_{0k},\ w_{1,k-1},\ \dots,\ w_{k0}) \begin{bmatrix} 3k(t-2k) & 3k & 0 & \cdots & 0 \\ t-2k+4 & 1 & 3(k-1) & \ddots & \vdots \\ 0 & 0 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 3 \\ 0 & 0 & \cdots & 0 & k \end{bmatrix}$$

For k $(\min\{d, p\} < k \le \max\{d, p\})$,

- Case $p \le d$,

$$\{w_{ij} \mid 0 \le i \le d, 0 \le p, i+j=k\} = \{w_{k-p,p}, w_{k-p+1,p-1}, \dots, w_{k0}\},$$

$$(R_{k-1} w_{k-1-p,p}, R_{k-1} w_{k-p,p-1}, \dots, R_{k-1} w_{k-1,0})$$

$$= (w_{k-p,p}, w_{k-p+1,p-1}, \dots, w_{k0}) \begin{bmatrix} k-p & 3p & 0 & \cdots & 0 \\ 0 & k-p+1 & 3(p-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 3 \\ 0 & \cdots & \cdots & 0 & k \end{bmatrix}$$

- Case $d < p$,

$$\{w_{ij} \mid 0 \le i \le d, 0 \le j \le p, i+j=k\} = \{w_{0k}, w_{1,k-1}, \dots, w_{d,k-d}\},$$

$$(R_{k-1} w_{0,k-1}, R_{k-1} w_{1,k-2}, \dots, R_{k-1} w_{d,k-1-d})$$

$$= (w_{0k}, w_{1,k-1}, \dots, w_{d,k-d}) \begin{bmatrix} 3k & 0 & \cdots & \cdots & 0 \\ 1 & 3(k-1) & \ddots & & \vdots \\ 0 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d & 3(k-d) \end{bmatrix}$$

For k $(\max\{d, p\} < k \le d+p)$,

$$\{w_{ij} \mid 0 \le i \le d, 0 \le j \le p, i+j=k\} = \{w_{k-p,p}, w_{k-p+1,p-1}, \dots, w_{d,k-d}\},$$

$$(R_{k-1} w_{k-1-p,p}, R_{k-1} w_{k-p,p-1}, \dots, R_{k-1} w_{d,k-1-d})$$

$$= (w_{k-p,p}, w_{k-p+1,p-1}, \dots, w_{d,k-d}) \begin{bmatrix} k-p & 3p & 0 & \cdots & 0 \\ 0 & k-p+1 & 3(p-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d & 3(k-d) \end{bmatrix}$$

The elements of the left hand sides are elements of W_1 by the assumption of the induction. And since the rank of each matrix is $\#\{(i, j) \mid i + j = k\}$, $w_{ij} \in W_1$ for any $i, j \geq 0$ $(i + j = k)$.

(4)

$$[A(n, m)]_{(w_{ij} \mid\, 0\leq i\leq d,\ 0\leq j\leq p)} \cong B \otimes I_{p+1} + I_{d+1} \otimes C + tI_{d+1} \otimes I_{p+1},$$

where

$$B := \begin{bmatrix} 0 & 3d & 0 & \cdots & 0 \\ 1 & 2 & 3(d-1) & \ddots & \vdots \\ 0 & 2 & 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 3 \\ 0 & \cdots & 0 & d & 2d \end{bmatrix}, \quad C := \begin{bmatrix} 0 & p & 0 & \cdots & 0 \\ 3 & -2 & p-1 & \ddots & \vdots \\ 0 & 6 & -4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 3p & -2p \end{bmatrix}$$

The eigenvalues of B are $\{3d - 4i \mid 0 \leq i \leq d\}$ and the eigenvalues of C are $\{p - 4j \mid 0 \leq j \leq p\}$. Therefore the eigenvalues of $[A(n, m)]_{(w_{ij} | 0\leq i\leq d, 0\leq j\leq p)}$ are

$$\{3d + p + t - 4(i + j) \,|\, 0 \leq i \leq d,\ 0 \leq j \leq p\},$$

with multiplicity.

The eigenvalues of $A(n, m)$ are $\{3(2n + m) - 4k \mid 0 \leq k \leq 2n + m\}$. By the definition of the endpoint and the diameter of W,

$$3(2n + m) - 4\mu = 3d + p + t,$$
$$\text{the diameter of } W = d + p,$$

Therefore $\mu = \frac{3(2n+m)-3d-p-t}{4}$, and the diameter of W is equal to the dual-diameter of W.

And for $k(0 \leq k \leq d + p)$,

$$\begin{aligned} \dim E_{\mu+k}(n, m)W &= \#\{(i, j) \,|\, 0 \leq i \leq d,\ 0 \leq j \leq p,\ 3(2n + m) - 4(\mu + k) \\ &= 3d + p + t - 4(i + j)\}, \\ &= \#\{(i, j) \,|\, 0 \leq i \leq d,\ 0 \leq j \leq p,\ k = i + j\} \\ &= \dim E^*_{\nu+k}(n, m)W \end{aligned}$$

(5) The "if" part is clear. Conversely, we assume $W \cong W'$. Let φ be an isomorphism from W to W'. Let w'_{ij} $(0 \leq i \leq d',\ 0 \leq j \leq p')$ be a basis of W' satisfying the condition in this proposition.

$$0 \neq \varphi(E^*_\nu(n, m)W) = E^*_\nu(n, m)\varphi(W) \subset E^*_\nu(n, m)W'.$$

By the definition of the dual-endpoint, we get $\nu \geq \nu'$. Similarly we get $\nu' \geq \nu$, therefore $\nu = \nu'$. Since $\dim E^*_\nu(n, m)W = \dim E^*_\nu(n, m)W' = 1$,

$$\varphi(w_{00}) = aw'_{00}\ , \quad \text{where } a \in \mathbf{C},\ a \neq 0.$$

Since $\frac{1}{a}\varphi$ is an isomorphism from W to W', we may assume $\varphi(w_{00}) = w'_{00}$.

$$tw'_{00} = \varphi(F_0 w_{00}) = F_0\varphi(w_{00}) = t'w'_{00}.$$

Hence $t = t'$.

Suppose $dp \neq 0$. Since $\varphi(E^*_{\nu+1}(n,m)W) = E^*_{\nu+1}(n,m)\varphi(W) \subset E^*_{\nu+1}(n,m)W'$, we can write

$$\begin{aligned}\varphi(w_{01}) &= \alpha w'_{01} + \gamma w'_{10},\\ \varphi(w_{10}) &= \beta w'_{01} + \delta w'_{10},\end{aligned}$$

where α, $\beta\gamma$, $\delta \in$ **C**.

$$\begin{aligned}&(w'_{01},\ w'_{10})\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}t-2 & 0\\ 0 & t+2\end{bmatrix}\\ &= (\varphi(w_{01}),\ \varphi(w_{10}))\begin{bmatrix}t-2 & 0\\ 0 & t+2\end{bmatrix}\\ &= (\varphi(F_1 w_{01}),\ \varphi(F_1 w_{10}))\\ &= (F_1\varphi(w_{01}),\ F_1\varphi(w_{10}))\\ &= (w'_{01},\ w'_{10})\begin{bmatrix}t-2 & 0\\ 0 & t+2\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix},\end{aligned}$$

then we get $\beta = \gamma = 0$.

$$\begin{aligned}3\alpha w'_{01} + \delta w'_{10} &= \varphi(3w_{01} + w_{10})\\ &= \varphi(R_0 w_{00})\\ &= R_0\varphi(w_{00})\\ &= 3w'_{01} + w'_{10}.\end{aligned}$$

Hence $\alpha = \delta = 1$.

$$\begin{aligned}pw'_{00} &= \varphi(L_1 w_{01}) = L_1\varphi(w_{01}) = p'w'_{00}.\\ 3dw'_{00} &= \varphi(L_1 w_{10}) = L_1\varphi(w_{10}) = 3d'w'_{00}.\end{aligned}$$

Hence $p = p'$ and $d = d'$.

If $dp = 0$, then $d = d'$ and $p = p'$ clearly. This completes the proof. □

Lemma 2

$$\begin{aligned}&U_0 \cong W(1,0;0,2,0,0), && V_0 \cong W(0,1;0,1,0,0),\\ &U_1 \cong W(1,0;1,1,0,-1), && V_1 \cong W(0,1;1,0,0,-1),\\ &U_2 \cong W(1,0;1,0,1,1), &&\\ &U_3 \cong W(1,0;1,0,0,-2), &&\\ &U_4 \cong W(1,0;2,0,0,2), &&\\ &U_5 \cong W(1,0;2,0,0,-2). &&\end{aligned}$$

Proof: Straightforward. □

Lemma 3 *Let* $n, m, \nu, d, p, t \in \mathbf{Z}$ *and* $n, m, \nu, d, p \geq 0$.

(1)

$$W(n,0;\ \nu,d,p,t)\otimes U_0 \cong \begin{cases} W(n+1,0;\ \nu,d+2,p,t)\oplus W(n+1,0;\ \nu+1,d,p,t+2) & \\ \quad \oplus W(n+1,0;\ \nu+2,d-2,p,t+4), & \text{if } d\geq 2,\\ W(n+1,0;\ \nu,3,p,t)\oplus W(n+1,0;\ \nu+1,1,p,t+2), & \text{if } d=1,\\ W(n+1,0;\ \nu,2,p,t), & \text{if } d=0.\end{cases}$$

(2)

$$W(n,0;\ \nu,d,p,t)\otimes U_1 \cong \begin{cases} W(n+1,0;\ \nu+1,d+1,p,t-1) & \\ \quad \oplus W(n+1,0;\ \nu+2,d-1,p,t+1), & \text{if } d\geq 1,\\ W(n+1,0;\ \nu+1,1,p,t-1), & \text{if } d=0.\end{cases}$$

(3)

$$W(n,0;\ \nu,d,p,t)\otimes U_2 \cong \begin{cases} W(n+1,0;\ \nu+1,d,p+1,t+1) & \\ \quad \oplus W(n+1,0;\ \nu+2,d,p-1,t-1), & \text{if } p\geq 1,\\ W(n+1,0;\ \nu+1,d,1,t+1), & \text{if } p=0.\end{cases}$$

(4)

$$W(n,0;\nu,d,p,t)\otimes U_3 \cong W(n+1,0;\nu+1,d,p,t-2).$$

(5)

$$W(n,0;\nu,d,p,t)\otimes U_4 \cong W(n+1,0;\nu+2,d,p,t+2).$$

(6)

$$W(n,0;\nu,d,p,t)\otimes U_5 \cong W(n+1,0;\nu+2,d,p,t-2).$$

(7)

$$W(n,m;\ \nu,d,p,t)\otimes V_0 \cong \begin{cases} W(n,m+1;\ \nu,d+1,p,t) & \\ \quad \oplus W(n,m+1;\ \nu+1,d-1,p,t+2), & \text{if } d\geq 1,\\ W(n,m+1;\ \nu+1,1,p,t), & \text{if } d=0.\end{cases}$$

(8)

$$W(n,m;\nu,d,p,t)\otimes V_1 \cong W(n,m+1;\nu+1,d,p,t-1).$$

Proof: (1) Let w_{ij} $(0 \le i \le d,\ 0 \le j \le p)$ be a basis of $W = W(n, 0; \nu, d, p, t)$ satisfying the condition in Proposition 3. Let $\alpha, \beta, \gamma\ \in \mathbf{C}$. Using Propositions 1, 3, the followings hold.

$$\begin{aligned}
&E_k^*(n,0)(\alpha w_{ij}\otimes a_0+\beta w_{i-1,j}\otimes a_1+\gamma w_{i-2,j}\otimes a_2)\\
&\quad=(E_k^*(n-1,0)\otimes E_0^*(1,0)+E_{k-1}^*(n-1,0)\otimes E_1^*(1,0)\\
&\qquad+E_{k-2}^*(n-1,0)\otimes E_2^*(1,0))\\
&\qquad\times(\alpha w_{ij}\otimes a_0+\beta w_{i-1,j}\otimes a_1+\gamma w_{i-2,j}\otimes a_2)\\
&\quad=\alpha(E_k^*(n-1,0)w_{ij}\otimes E_0^*(1,0)a_0+E_{k-1}^*(n-1,0)w_{ij}\otimes E_1^*(1,0)a_0\\
&\qquad+E_{k-2}^*(n-1,0)w_{ij}\otimes E_2^*(1,0)a_0)\\
&\qquad+\beta(E_k^*(n-1,0)w_{i-1,j}\otimes E_0^*(1,0)a_1+E_{k-1}^*(n-1,0)w_{i-1,j}\otimes E_1^*(1,0)a_1\\
&\qquad+E_{k-2}^*(n-1,0)w_{i-1,j}\otimes E_2^*(1,0)a_1)\\
&\qquad+\gamma(E_k^*(n-1,0)w_{i-2,j}\otimes E_0^*(1,0)a_2+E_{k-1}^*(n-1,0)w_{i-2,j}\otimes E_1^*(1,0)a_2\\
&\qquad+E_{k-2}^*(n-1,0)w_{i-2,j}\otimes E_2^*(1,0)a_2)\\
&\quad=\alpha E_k^*(n-1,0)w_{ij}\otimes E_0^*(1,0)a_0+\beta E_{k-1}^*(n-1,0)w_{i-1,j}\otimes E_1^*(1,0)a_1\\
&\qquad+\gamma E_{k-2}^*(n-1,0)w_{i-2,j}\otimes E_2^*(1,0)a_2\\
&\quad=\delta_{k,i+j}(\alpha w_{ij}\otimes a_0+\beta w_{i-1,j}\otimes a_1+\gamma w_{i-2,j}\otimes a_2),
\end{aligned}$$

where

$$\delta_{uv} := \begin{cases} 1, & \text{if } u = v,\\ 0, & \text{otherwise.}\end{cases}$$

And

$$\begin{aligned}
&A(n,0)(\alpha w_{ij}\otimes a_0+\beta w_{i-1,j}\otimes a_1+\gamma w_{i-2,j}\otimes a_2)\\
&\quad=(A(n-1,0)\otimes I_{16}+I_{16^{n-1}}\otimes A(1,0))(\alpha w_{ij}\otimes a_0+\beta w_{i-1,j}\otimes a_1+\gamma w_{i-2,j}\otimes a_2)\\
&\quad=\alpha(A(n-1,0)w_{ij}\otimes a_0+w_{ij}\otimes A(1,0)a_0)\\
&\qquad+\beta(A(n-1,0)w_{i-1,j}\otimes a_1+w_{i-1,j}\otimes A(1,0)a_1)\\
&\qquad+\gamma(A(n-1,0)w_{i-2,j}\otimes a_2+w_{i-2,j}\otimes A(1,0)a_2)\\
&\quad=\alpha\{3(d-i+1)w_{i-1,j}+(t+2(i-j))w_{ij}+(i+1)w_{i+1,j}\}\otimes a_0\\
&\qquad+\alpha\{(p-j+1)w_{i,j-1}\otimes a_0+3(j+1)w_{i,j+1}\otimes a_0\}\\
&\qquad+\alpha w_{ij}\otimes a_1\\
&\qquad+\beta\{3(d-(i-1)+1)w_{i-2,j}+(t+2(i-1-j))w_{i-1,j}+iw_{ij}\}\otimes a_1\\
&\qquad+\beta\{(p-j+1)w_{i-1,j-1}\otimes a_1+3(j+1)w_{i-1,j+1}\otimes a_1\}\\
&\qquad+\beta w_{i-1,j}\otimes(6a_0+2a_1+2a_2)\\
&\qquad+\gamma\{3(d-(i-2)+1)w_{i-3,j}+(t+2(i-2-j))w_{i-2,j}+(i-1)w_{i-1,j}\}\otimes a_2\\
&\qquad+\gamma\{(p-j+1)w_{i-2,j-1}\otimes a_2+3(j+1)w_{i-2,j+1}\otimes a_2\}\\
&\qquad+\gamma w_{i-2,j}\otimes(3a_1+4a_2)
\end{aligned}$$

$$\begin{aligned}
&= (3(d-i+1)\alpha + 6\beta)w_{i-1,j} \otimes a_0 + (3(d-i+2)\beta + 3\gamma)w_{i-2,j} \otimes a_1 \\
&\quad + 3(d-i+3)\gamma w_{i-3,j} \otimes a_2 \\
&\quad + (p-j+1)(\alpha w_{i,j-1} \otimes a_0 + \beta w_{i-1,j-1} \otimes a_1 + \gamma w_{i-2,j-1} \otimes a_2) \\
&\quad + (t+2(i-j))(\alpha w_{ij} \otimes a_0 + \beta w_{i-1,j} \otimes a_1 + \gamma w_{i-2,j} \otimes a_2) \\
&\quad + (i+1)\alpha w_{i+1,j} \otimes a_0 + (\alpha + \beta i)w_{ij} \otimes a_1 \\
&\quad + (2\beta + \gamma(i-1))w_{i-1,j} \otimes a_2 \\
&\quad + 3(j+1)(\alpha w_{i,j+1} \otimes a_0 + \beta w_{i-1,j+1} \otimes a_1 + \gamma w_{i-2,j+1} \otimes a_2).
\end{aligned}$$

Here we define

$$\begin{aligned}
v^0_{ij} &:= w_{ij} \otimes a_0 + w_{i-1,j} \otimes a_1 + w_{i-2,j} \otimes a_2, && 0 \le i \le d+2,\ 0 \le j \le p, \\
v^1_{ij} &:= 2(i+1)w_{i+1,j} \otimes a_0 + (2i-d)w_{ij} \otimes a_1 \\
&\qquad + 2(i-d-1)w_{i-1,j} \otimes a_2, && 0 \le i \le d,\ 0 \le j \le p, \\
v^2_{ij} &:= (i+2)(i+1)w_{i+2,j} \otimes a_0 \\
&\qquad + (i+1)(1-d+i)w_{i+1,j} \otimes a_1 \\
&\qquad + (i-d)(1-d+i)w_{ij} \otimes a_2, && 0 \le i \le d-2,\ 0 \le j \le p.
\end{aligned}$$

$\{v^0_{ij} \mid 0 \le i \le d+2,\ 0 \le j \le p\} \cup \{v^1_{ij} \mid 0 \le i \le d,\ 0 \le j \le p,\ v^1_{ij} \ne 0\} \cup \{v^2_{ij} \mid 0 \le i \le d-2,\ 0 \le j \le p,\ v^2_{ij} \ne 0\}$ is a basis of $W(n, 0;\ \nu, d, p, t) \otimes U_0$.

Using the above formula,

$$\begin{aligned}
v^0_{ij} &\in E^*_{\nu+i+j}(n+1, 0)(W \otimes U_0), \\
A(n,0)v^0_{ij} &= 3(d+2-i+1)v^0_{i-1,j} + (p-j+1)v^0_{i,j-1} + (t+2(i-j))v^0_{ij} \\
&\quad + (i+1)v^0_{i+1,j} + 3(j+1)v^0_{i,j+1}, \\
v^1_{ij} &\in E^*_{\nu+1+i+j}(n+1, 0)(W \otimes U_0), \\
A(n,0)v^1_{ij} &= 3(d-i+1)v^1_{i-1,j} + (p-j+1)v^1_{i,j-1} + (t+2+2(i-j))v^1_{ij} \\
&\quad + (i+1)v^1_{i+1,j} + 3(j+1)v^1_{i,j+1}, \\
v^2_{ij} &\in E^*_{\nu+2+i+j}(n+1, 0)(W \otimes U_0), \\
A(n,0)v^2_{ij} &= 3(d-2-i+1)v^2_{i-1,j} + (p-j+1)v^2_{i,j-1} + (t+4+2(i-j))v^2_{ij} \\
&\quad + (i+1)v^2_{i+1,j} + 3(j+1)v^2_{i,j+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=0}^{d+2}\sum_{j=0}^{p} \mathbf{C}v^0_{ij} \cong W(n+1, 0;\ \nu, d+2, p, t), \\
&\sum_{i=0}^{d}\sum_{j=0}^{p} \mathbf{C}v^1_{ij} \cong W(n+1, 0;\ \nu+1, d, p, t+2), \qquad \text{if } d \ge 1, \\
&\sum_{i=0}^{d-2}\sum_{j=0}^{p} \mathbf{C}v^2_{ij} \cong W(n+1, 0;\ \nu+2, d-2, p, t+4), \qquad \text{if } d \ge 2.
\end{aligned}$$

We get (1).

(2) We define

$$v^0_{ij} := w_{ij} \otimes b_1 + w_{i-1,j} \otimes b_2, \qquad 0 \le i \le d+1,\ 0 \le j \le p,$$
$$v^1_{ij} := (i+1)w_{i+1,j} \otimes b_1 - (d-i)w_{ij} \otimes b_2, \quad 0 \le i \le d-1,\ 0 \le j \le p,$$

and can prove (2) as same as (1).

(3) We define

$$v^0_{ij} := w_{ij} \otimes c_1 + w_{i,j-1} \otimes c_2, \qquad 0 \le i \le d,\ 0 \le j \le p+1,$$
$$v^1_{ij} := (j+1)w_{i,j+1} \otimes c_1 - (p-j)w_{ij} \otimes c_2, \quad 0 \le i \le d,\ 0 \le j \le p-1,$$

and can prove (3) as same as (1).

For (7) it is sufficient to take a basis which are the same form as (2).

(4), (5), (6), (8) are easy calculation. □

Proposition 4 *For* $N_0, N_1, N_2, N_3, N_4, M_0 \ge 0$ *such that*

$$N_0 + N_1 + N_2 + N_3 + N_4 \le D_1 \text{ and } M_0 \le D_2,$$

$(\otimes^5_{i=0} U_i^{\otimes N_i}) \otimes (\otimes^1_{j=0} V_j^{\otimes M_j})$ *decomposes into irreducible modules which are isomorphic to*

$$\begin{aligned} &W(D_1, D_2;\ r+s+N_1+N_3+2N_4+2N_5+N_2-M_1,\ 2N_0+N_1+M_0-2r,\\ &\quad N_2-2s,\ 2r-2s-N_1+N_2-2N_3+2N_4-2N_5+M_1),\\ &= W(D_1, D_2;\ r+s-2N_0-N_1-N_2-N_3-M_0+2D_1+D_2,\\ &\quad 2N_0+N_1+M_0-2r, N_2-2s,\\ &\quad 2r-2s+2N_0+N_1+3N_2+4N_4+M_0-2D_1-D_2), \end{aligned}$$

$$\begin{cases} r = 0, & \text{if } N_0 = 1 \text{ and } N_1 = M_0 = 0,\\ r = 0, 1, \dots, N_0 + \lfloor \frac{N_1+M_0}{2} \rfloor, & \text{otherwise}, \end{cases}$$

$$s = 0,\ 1, \dots,\ \left\lfloor \frac{N_2}{2} \right\rfloor,$$

where $N_5 = D_1 - N_0 - N_1 - N_2 - N_3 - N_4,\ M_1 = D_2 - M_0$ *and the notation*

$$\lfloor \alpha \rfloor := \max\{\, i \in \mathbf{Z} \mid i \le \alpha\},\ \alpha \in \mathbf{R}.$$

Proof: Using Proposition 2(3) and Lemma 2, 3, we get the result. □

From now on we will observe the range which $\nu,\ d,\ p,\ t$:

$$\begin{aligned}
\nu &:= \nu(N_0, N_1, N_2, N_3, N_4, M_0, r, s)\\
&:= r + s - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2,\\
d &:= d(N_0, N_1, N_2, N_3, N_4, M_0, r, s)\\
&:= 2N_0 + N_1 + M_0 - 2r,\\
p &:= p(N_0, N_1, N_2, N_3, N_4, M_0, r, s)\\
&:= N_2 - 2s,\\
t &:= t(N_0, N_1, N_2, N_3, N_4, M_0, r, s)\\
&:= 2r - 2s + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2.
\end{aligned}$$

move where the $N_0, N_1, N_2, N_3, N_4, M_0, r, s$ move in G:

$$G := \Bigg\{(N_0, N_1, N_2, N_3, N_4, M_0, r, s) \mid N_0, \dots, N_4, M_0 \geq 0,\\
N_0 + \cdots + N_4 \leq D_1,\ \ M_0 \leq D_2,\\
\begin{cases} r = 0, & \text{if } N_0 = 1 \text{ and } N_1 = M_0 = 0,\\ r = 0, 1, \dots, N_0 + \left\lfloor \frac{N_1+M_0}{2} \right\rfloor, & \text{otherwise,} \end{cases}\\
s = 0,\ 1, \dots, \left\lfloor \frac{N_2}{2} \right\rfloor \Bigg\}.$$

We define $H := \{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, s) \in G\}$.

Lemma 4

(1)

$$H = \{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G\}.$$

(2)

$$H = \{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G, N_1 \in \{0, 1\}\}.$$

(3)

$$\begin{aligned}
&\{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G, N_1 \in \{0, 1\}, d \geq 1\}\\
&\quad = \{(\nu, d, p, t) \mid (N_0, 0, N_2, N_3, N_4, M_0, r, 0) \in G, d \geq 1\}.
\end{aligned}$$

(4) *Suppose* $D_2 \geq 2$.

$$\begin{aligned}
&\{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G, N_1 \in \{0, 1\}, d = 0\}\\
&\quad = \{(\nu, d, p, t) \mid (N_0, 0, N_2, N_3, N_4, M_0, r, 0) \in G, d = 0\}.
\end{aligned}$$

(5) *For* $\gamma \in \mathbf{Z}\,(0 \le \gamma \le 2D_1 + D_2)$,

$$\{(\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G, 2N_0 + M_0 = \gamma,\ N_1 \in \{0,1\}\}$$
$$= \Bigg\{ (\nu, d, p, t) \mid (N_0, N_1, N_2, N_3, N_4, M_0, r, 0) \in G,\ N_1 \in \{0,1\},$$
$$M_0 = \begin{cases} \min\{\gamma,\ D_2\}, & \text{if } \gamma - D_2 \text{ is even},\\ \min\{\gamma,\ D_2 - 1\}, & \text{if } \gamma - D_2 \text{ is odd}, \end{cases}$$
$$N_0 = \frac{\gamma - M_0}{2} \Bigg\}.$$

Proof: For each case, it is clear that the set of left hand side includes the set of the right hand side. Conversely,
(1) For

$$\begin{aligned} \nu &= r + s - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2,\\ d &= 2N_0 + N_1 + M_0 - 2r,\\ p &= N_2 - 2s,\\ t &= 2r - 2s + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2, \end{aligned}$$

we define

$$N_2' := N_2 - 2s,\ N_3' := N_3 + s,\ N_4' := N_4 + s.$$

Then $(N_0, N_1, N_2', N_3', N_4', r, 0) \in G$ and

$$\begin{aligned} \nu &= r - N_1 - N_2' - N_3' - M_0 + 2D_1 + D_2,\\ d &= 2N_0 + N_1 + M_0 - 2r,\\ p &= N_2',\\ t &= 2r + 2N_0 + N_1 + 3N_2' + 4N_4' + M_0 - 2D_1 - D_2. \end{aligned}$$

(2) Using (1), it is sufficient to consider the case $s = 0$. For

$$\begin{aligned} \nu &= r - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2,\\ d &= 2N_0 + N_1 + M_0 - 2r,\\ p &= N_2,\\ t &= 2r + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2, \end{aligned}$$

we define $N_0', N_1', N_3', N_4', r'$ as follows.

$$\begin{cases} N_0' := 0,\ N_1' := 0,\ N_3' := N_3+1,\ N_4' := N_4+1,\ r' := 0, \\ \qquad\qquad\qquad\qquad \text{if } N_0 = 0,\ N_1 = 2,\ \text{and } r = 1, \\ N_0' := \lfloor \frac{2N_0+N_1}{2} \rfloor + r,\ N_3' := N_3,\ N_4' := N_4,\ r' := r, \\ N_1 := \begin{cases} 1, \text{ if } 2N_0+N_1 \text{ is odd}, \\ 0, \text{ if } 2N_0+N_1 \text{ is even}, \end{cases} \\ \qquad\qquad\qquad\qquad \text{otherwise}. \end{cases}$$

Then $(N_0', N_1', N_3', N_4', M_0, r, 0) \in G$, $N_1' \in \{0, 1\}$ and

$$\begin{aligned} \nu &= r' - 2N_0' - N_1' - N_2 - N_3' - M_0 + 2D_1 + D_2, \\ d &= 2N_0' + N_1' + M_0 - 2r', \\ p &= N_2, \\ t &= 2r' + 2N_0' + N_1' + 3N_2 + 4N_4' + M_0 - 2D_1 - D_2. \end{aligned}$$

(3) For

$$\begin{aligned} &N_1 \in \{0,\ 1\}, \\ &\nu = r - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2, \\ &d = 2N_0 + N_1 + M_0 - 2r, \\ &p = N_2, \\ &t = 2r + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2, \end{aligned}$$

we define

$$\begin{cases} N_0' := N_0,\ M_0' := 0, & \text{if } N_1 = 0, \\ N_0' := N_0 + 1,\ M_0' := M_0 - 1, & \text{if } N_1 = 1 \text{ and } M_0 > 0, \\ N_0' := N_0,\ M_0' := 1, & \text{if } N_1 = 1 \text{ and } M_0 = 0. \end{cases}$$

Then $(N_0',\ 0,\ N_2, N_3, N_4, M_0', r, 0) \in G$

$$\begin{aligned} \nu &= r - 2N_0' - N_2 - N_3 - M_0' + 2D_1 + D_2, \\ d &= 2N_0' + M_0' - 2r, \\ p &= N_2, \\ t &= 2r + 2N_0' + 3N_2 + 4N_4 + M_0' - 2D_1 - D_2. \end{aligned}$$

(4) For

$$\begin{aligned} &N_1 \in \{0,\ 1\}, \\ &\nu = r - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2, \\ &0 = 2N_0 + N_1 + M_0 - 2r, \\ &p = N_2, \\ &t = 2r + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2, \end{aligned}$$

we find that if $N_1 = 1$, then M_0 is odd. We define

$$\begin{cases} N_0' := N_0,\ M_0' := 0, & \text{if } N_1 = 0, \\ N_0' := N_0 + 1,\ M_0' := M_0 - 1, & \text{if } N_1 = 1 \text{ and } M_0 \geq 3, \\ N_0' := N_0,\ M_0' := 2, & \text{if } N_1 = 1 \text{ and } M_0 = 1. \end{cases}$$

Then $(N_0',\ 0,\ N_2, N_3, N_4, M_0', r, 0) \in G$

$$\begin{aligned} \nu &= r - 2N_0' - N_2 - N_3 - M_0' + 2D_1 + D_2, \\ 0 &= 2N_0' + M_0' - 2r, \\ p &= N_2, \\ t &= 2r + 2N_0' + 3N_2 + 4N_4 + M_0' - 2D_1 - D_2. \end{aligned}$$

(5) For

$$\begin{aligned} \gamma &= 2N_0 + M_0, \\ \nu &= r - 2N_0 - N_1 - N_2 - N_3 - M_0 + 2D_1 + D_2, \\ d &= 2N_0 + N_1 + M_0 - 2r, \\ p &= N_2, \\ t &= 2r + 2N_0 + N_1 + 3N_2 + 4N_4 + M_0 - 2D_1 - D_2, \end{aligned}$$

we define

$$\begin{aligned} M_0' &:= \begin{cases} \min\{\gamma,\ D_2\}, & \text{if } \gamma - D_2 \text{ is even}, \\ \min\{\gamma,\ D_2 - 1\}, & \text{if } \gamma - D_2 \text{ is odd}, \end{cases} \\ N_0' &:= \frac{\gamma - M_0'}{2}. \end{aligned}$$

Then $(N_0', N_1, N_2, N_3, N_4, M_0', r, 0) \in G$ and

$$\begin{aligned} \nu &= r - 2N_0' - N_2 - N_3 - M_0' + 2D_1 + D_2, \\ d &= 2N_0' + M_0' - 2r, \\ p &= N_2, \\ t &= 2r + 2N_0' + 3N_2 + 4N_4 + M_0' - 2D_1 - D_2. \end{aligned}$$

□

Here we define the following notations.

$$\alpha := 2N_0 + 2N_3 + N_1 + M_0,\ \ \beta := N_4 - N_3.$$

Then

$$\begin{aligned} \nu &= -\frac{d + \alpha}{2} - p + 2D_1 + D_2, \\ t &= -4\nu - 3d - p + 6D_1 + 3D_2 + 4\beta. \end{aligned}$$

We will observe the range the d, p, α, β move ($d = 2N_0 + N_1 + M_0 - 2r,\ p = N_2$) where $N_0, N_1, N_2, N_3, N_4, M_0, r$ move in G_1:

$$G_1 := \left\{ (N_0, N_1, N_2, N_3, N_4, M_0, r) \mid N_0, \dots, N_4, M_0 \geq 0, \quad N_1 \in \{0, 1\} \right.$$
$$N_0 + \cdots + N_4 \leq D_1,\ M_0 \leq D_2,$$
$$\left. \begin{cases} r = 0, & \text{if } N_0 = 1 \text{ and } N_1 = M_0 = 0, \\ r = 0, 1, \dots, N_0 + \left\lfloor \frac{N_1 + M_0}{2} \right\rfloor, & \text{otherwise.} \end{cases} \right\},$$

from Lemma 4(2).

Lemma 5

(1) *Suppose d, p such that*

$$\begin{cases} 0 \leq d \leq D_2, & \textit{if } D_2 \geq 2, \\ 0 < d \leq D_2, & \textit{otherwise,} \end{cases}$$
$$0 \leq p \leq D_1.$$

are given. Then $\alpha, \beta \in \mathbf{Z}$ behave as follows.

$$\alpha - d \textit{ is even and } d \leq \alpha \leq \begin{cases} 2(D_1 - p) + D_2, & \textit{if } d - D_2 \textit{ is even}, \\ 2(D_1 - p) + D_2 - 1, & \textit{if } d - D_2 \textit{ is odd}, \end{cases}$$
$$\max\left\{ -\frac{\alpha - d}{2},\ -D_1 + p \right\} \leq \beta \leq \min\left\{ D_1 - p - \left\lfloor \frac{\alpha - D_2 + 1}{2} \right\rfloor,\ D_1 - p \right\}.$$

(2) *Suppose d, p ($D_2 < d \leq 2D_1 + D_2,\ 0 \leq p \leq D_1 - \lfloor \frac{d - D_2 + 1}{2} \rfloor$) are given. Then $\alpha, \beta \in$ $\mathbf{Z}$ behave as follows.*

$$\alpha - d \textit{ is even and } d \leq \alpha \leq \begin{cases} 2(D_1 - p) + D_2, & \textit{if } d - D_2 \textit{ is even}, \\ 2(D_1 - p) + D_2 - 1, & \textit{if } d - D_2 \textit{ is odd}, \end{cases}$$
$$-\frac{\alpha - d}{2} \leq \beta \leq D_1 - p - \left\lfloor \frac{\alpha - D_2 + 1}{2} \right\rfloor.$$

(3) *Suppose $D_2 = 1$, and d, p ($d = 0,\ 0 \leq p \leq D_1$) are given. Then $\alpha,\ \beta \in \mathbf{Z}$ behave as follows.*

$$\alpha \textit{ is even and } 0 \leq \alpha \leq 2(D_1 - p),$$
$$-\frac{\alpha}{2} \leq \beta \leq D_1 - p - \frac{\alpha}{2}.$$

(4) *Suppose $D_2 = 0$, and d, p ($d = 0,\ p = D_1$) are given. Then $\alpha,\ \beta \in \mathbf{Z}$ behave as follows.*

$$(\alpha, \beta) = (0, 0).$$

(5) *Suppose* $D_2 = 0$, *and* d, p $(d = 0,\ 0 \le p \le D_1 - 1)$ *are given. Then* $\alpha, \beta \in \mathbf{Z}$ *behave as follows.*

$$\alpha \text{ is even and } 0 \le \alpha \le 2(D_1 - p),$$
$$\begin{cases} -1 \le \beta \le D_1 - p - 2, & \text{if } \alpha = 2, \\ -D_1 + p \le \beta \le 0 \text{ and } \beta \ne -D_1 + p + 1, & \text{if } \alpha = 2(D_1 - p), \\ -\dfrac{\alpha}{2} \le \beta \le D_1 - p - \dfrac{\alpha}{2}, & \text{otherwise.} \end{cases}$$

Proof:

(1) From Lemma 4(3), (4), it is sufficient to consider the case $N_1 = 0$. $(2N_0 + M_0,\ r)$ can take the following values in this case.

$$(2N_0 + M_0,\ r) \in \left\{ \left(k,\ \frac{k-d}{2}\right) \middle| \ k - d \text{ is even}, \right.$$
$$\left. d \le k \le \begin{cases} 2(D_1 - p) + D_2, & \text{if } D_2 - d \text{ is even}, \\ 2(D_1 - p) + D_2 - 1, & \text{if } D_2 - d \text{ is odd}, \end{cases} \right\}$$

So it is sufficient to consider $N_0,\ M_0,\ r$ take the following values from Lemma 4(5).

$$\begin{aligned} &(N_0,\ M_0,\ r) \\ &\in \left\{ \left(0,\ i,\ \frac{i-d}{2}\right) \middle| \ i - d \text{ is even}, d \le i \le \begin{cases} D_2, & \text{if } D_2 - d \text{ is even}, \\ D_2 - 1, & \text{if } D_2 - d \text{ is odd}, \end{cases} \right\} \\ &\cup \left\{ \left(j,\ i,\ j + \frac{i-d}{2}\right) \middle| \ i = \begin{cases} D_2, & \text{if } D_2 - d \text{ is even}, \\ D_2 - 1, & \text{if } D_2 - d \text{ is odd}, \end{cases} \ 0 \le j \le D_1 - p \right\}. \end{aligned}$$

For each $N_0,\ M_0$, and r, N_3 and N_4 can behave as follows.

$$\begin{aligned} &0 \le N_3 \le D_1 - p - N_0 - N_1, \\ &0 \le N_4 \le D_1 - p - N_0 - N_1 - N_3. \end{aligned}$$

For $\alpha = 2N_0 + 2N_3 + N_1 + M_0,\ \beta = N_4 - N_3$, if we range the parameters N_4, N_3, N_0, M_0 with this ordering, we get the result.

(2) From Lemma 4 (3),(5), it is sufficient to consider the case $N_1 = 0$ and $N_0,\ M_0,\ r$ take the following values.

$$\begin{aligned} &(N_0,\ M_0,\ r) \\ &\in \left\{ \left(j,\ i,\ j + \frac{i-d}{2}\right) \middle| \ i = \begin{cases} D_2, & \text{if } D_2 - d \text{ is even}, \\ D_2 - 1, & \text{if } D_2 - d \text{ is odd}, \end{cases} \ \frac{d-i}{2} \le j \le D_1 - p \right\}. \end{aligned}$$

We can get the result as same as (1).

(3) From Lemma 4 (5), it is sufficient to consider the case N_0, N_1, M_0, r take the following values.

$$(N_0, N_1, M_0, r) \in \{(j,\ 0,\ 0,\ j) \mid 0 \le j \le D_1 - p,\ j \ne 1\} \\ \cup \{(j,\ 1,\ 1,\ j+1) \mid 0 \le j \le D_1 - p - 1\}.$$

We can get the result as same as (1).

(4) $M_0 = 0$ since $D_2 = 0$. N_0, N_1, N_3, N_4, r take values

$$N_0 = N_1 = N_3 = N_4 = r = 0.$$

So we get the result.

(5) $M_0 = 0$ since $D_2 = 0$. From Lemma 4 (5), it is sufficient to consider the case N_0, N_1, r take the following values.

$$(N_0,\ N_1,\ r) \in \{(j,\ 0,\ j) \mid 0 \le j \le D_1 - p,\ j \ne 1\}.$$

We can get the result as same as (1). □

Theorem 1 *The irreducible modules of $T(D_1, D_2)$ are $W(D_1, D_2;\ \nu, d, p, t)$, where the parameters $\nu, d, p, t \in \mathbf{Z}$ behave as follows.*

(1) *In the case $D_2 > 0$,*

$$0 \le d \le D_2, \qquad 0 \le p \le D_1,$$
$$\left\lfloor \frac{D_2 - d + 1}{2} \right\rfloor + D_1 \le \nu \le -d - p + 2D_1 + D_2,$$
$$t \equiv \max\{d + 3p - 2D_1 - D_2,\ -4\nu - 3d + 3p + 2D_1 + 3D_2\} \pmod 4 \ \textit{and}$$
$$\begin{cases} \max\{d + 3p - 2D_1 - D_2,\ -4\nu - 3d + 3p + 2D_1 + 3D_2\} \le t \\ \quad \le \min\{-d - p + 2D_1 + D_2,\ -4\nu - 3d - 5p + 10D_1 + 3D_2\}, \\ \qquad\qquad \textit{if } d - D_2 \textit{ is even}, \\ \max\{d + 3p - 2D_1 - D_2,\ -4\nu - 3d + 3p + 2D_1 + 3D_2\} \le t \\ \quad \le \min\{-d - p + 2D_1 + D_2 - 2,\ -4\nu - 3d - 5p + 10D_1 + 3D_2\}, \\ \qquad\qquad \textit{if } d - D_2 \textit{ is odd}, \end{cases}$$

and

$$D_2 < d \le 2D_1 + D_2, \qquad 0 \le p \le D_1 - \left\lfloor \frac{d - D_2 + 1}{2} \right\rfloor,$$
$$\left\lfloor \frac{D_2 - d + 1}{2} \right\rfloor + D_1 \le \nu \le -d - p + 2D_1 + D_2,$$
$$t \equiv d + 3p - 2D_1 - D_2 \pmod 4 \ \textit{and}$$
$$\begin{cases} d + 3p - 2D_1 - D_2 \le t \le -d - p + 2D_1 + D_2, & \textit{if } d - D_2 \textit{ is even}, \\ d + 3p - 2D_1 - D_2 \le t \le -d - p + 2D_1 + D_2 - 2, & \textit{if } d - D_2 \textit{ is odd}. \end{cases}$$

(2) *In the case* $D_2 = 0$,

$$\begin{aligned}
&d = 0, \qquad 0 \le p \le D_1 - 1, \qquad D_1 \le \nu \le -p + 2D_1,\\
&t \equiv 3p - 2D_1 \pmod 4 \text{ and}\\
&\begin{cases} 3p - 2D_1 \le t \le -4 - p + 2D_1, & \text{if } \nu = -1 - p + 2D_1,\\ 3p - 2D_1 \le t \le -p + 2D_1 \text{ and } t \ne 3p - 2D_1 + 4, & \text{if } \nu = D_1,\\ 3p - 2D_1 \le t \le -p + 2D_1, & \text{otherwise}. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&0 \le d \le 2D_1,\\
&\begin{cases} 0 \le p \le D_1 - \left\lfloor \frac{d+1}{2} \right\rfloor, & \text{if } d > 0,\\ p = D_1, & \text{if } d = 0, \end{cases}\\
&\left\lfloor \frac{-d+1}{2} \right\rfloor + D_1 \le \nu \le -d - p + 2D_1,\\
&t \equiv d + 3p - 2D_1 \pmod 4 \text{ and}\\
&\begin{cases} d + 3p - 2D_1 \le t \le -d - p + 2D_1, & \text{if } d \text{ is even},\\ d + 3p - 2D_1 \le t \le -d - p + 2D_1 - 2, & \text{if } d \text{ is odd}. \end{cases}
\end{aligned}$$

Proof: $d = 2N_0 + N_1 + M_0 - 2r$, $p = N_2$, where N_0, N_1, N_2, M_0, r move in the following set:

$$\begin{aligned}
&\{(N_0, N_1, N_2, M_0, r) \mid (N_0, N_1, N_2, N_3, M_4, M_0, r) \in G_1\}\\
&= \Bigg\{ (N_0, N_1, N_2, M_0, r) \mid N_0,\ N_1,\ N_2,\ M_0 \ge 0,\\
&\quad N_0 + N_1 + N_2 \le D_1,\ M_0 \le D_2,\ N_1 \in \{0, 1\},\\
&\begin{cases} r = 0, & \text{if } N_0 = 1 \text{ and } N_1 = M_0 = 0,\\ 0 \le r \le N_0 + \left\lfloor \frac{N_1 + M_0}{2} \right\rfloor, & \text{otherwise}. \end{cases} \Bigg\}
\end{aligned}$$

So we get

$$0 \le d \le 2D_1 + D_2,$$

$$\begin{cases} 0 \le p \le D_1, & \text{if } 0 \le d \le D_2,\\ 0 \le p \le D_1 - \left\lfloor \frac{d - D_2 + 1}{2} \right\rfloor, & \text{if } D_2 < d \le 2D_1 + D_2. \end{cases}$$

Using Lemma 5 and

$$\begin{aligned}
\nu &= -\frac{d + \alpha}{2} - p + 2D_1 + D_2,\\
t &= -4\nu - 3d - p + 6D_1 + 3D_2 + 4\beta,
\end{aligned}$$

we get the result. □

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