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An Algebra Associated with a Spin Model

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Abstract. To each symmetric $n \times n$ matrix W with non-zero complex entries, we associate a vector space $\mathcal{N}$, consisting of certain symmetric $n \times n$ matrices. If W satisfies

$$\sum_{x=1}^{n} \frac{W_{a,x}}{W_{b,x}} = n\delta_{a,b} \quad (a,\ b = 1, \dots, n),$$

then $\mathcal{N}$ becomes a commutative algebra under both ordinary matrix product and Hadamard product (entry-wise product), so that $\mathcal{N}$ is the Bose-Mesner algebra of some association scheme. If W satisfies the star-triangle equation:

$$\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \frac{W_{a,x} W_{b,x}}{W_{c,x}} = \frac{W_{a,b}}{W_{a,c} W_{b,c}} \quad (a,\ b,\ c = 1, \dots, n),$$

then W belongs to $\mathcal{N}$. This gives an algebraic proof of Jaeger's result which asserts that every spin model which defines a link invariant comes from some association scheme.

Keywords: spin model, association scheme, Bose-Mesner algebra

1. Introduction

A spin model is one of the statistical mechanical models which were introduced by V. Jones [12] to construct invariants of knots and links. A *spin model* is a triple $S = (X,\ W^+,\ W^-)$, where $X = \{1, \dots, n\}$, and $W^\pm$ are symmetric $n \times n$ matrices with complex number entries such that $W^+_{b,c} W^-_{b,c} = 1$ for all $b, c \in X$.

Jones gave the following conditions, under which the normalized partition function of a spin model $S = (X, W^\pm)$ is invariant under Reidemeister moves of Types II and III.

$$\text{Type II.} \quad \sum_{x \in X} W^+_{a,x} W^-_{b,x} = n\delta_{a,b}, \qquad (a,\ b \in X).$$

$$\text{Type III.} \quad \sum_{x \in X} W^+_{a,x} W^+_{b,x} W^-_{c,x} = \sqrt{n}\, W^+_{a,b} W^-_{a,c} W^-_{b,c}, \qquad (a, b, c \in X).$$

In this paper, we associate a vector space $\mathcal{N}$ to each spin model $S = (X, W^+, W^-)$ as follows. For each $b, c \in X$, we consider an n-dimensional column vector $\mathbf{u}_{b,c}$ of size n with x-entry $W^+_{b,x} W^-_{c,x}$. Then $\mathcal{N}$ will be the set of all symmetric $n \times n$ matrices A such that $\mathbf{u}_{b,c}$ is an eigenvector of A for all $b, c \in X$.

When S satisfies the Type II condition, $\mathcal{N}$ is closed under both ordinary matrix product and Hadamard (entry-wise) product, and $\mathcal{N}$ becomes a commutative algebra (with unity element) with respect to each of these two products. This implies that $\mathcal{N}$ is the Bose-Mesner algebra of some association scheme. Definitions of an association scheme and its Bose-Mesner algebra will be given in Section 3. Spin models with the Type II condition are of special importance for the study of subfactors in the theory of von Neumann algebras (see [1, 6]).

When S satisfies the Type III condition, $\mathcal{N}$ contains W^+. When S satisfies both Type II and Type III conditions, $\mathcal{N}$ is the Bose-Mesner algebra of some association scheme and contains W^+, W^-. This gives an algebraic proof of a result by Jaeger [11], which was obtained by the method of "tangles".

In Section 2, we show that $\mathcal{N}$ is an algebra with respect to both ordinary product and Hadamard product when S satisfies Type II condition. In Section 3, we consider relations between the algebra $\mathcal{N}$ and Bose-Mesner algebras of an association scheme.

For general references about association schemes and their Bose-Mesner algebras, see [4, 5]. For spin models and related link invariants, see [7, 12]. For spin models constructed from association schemes, see [2, 3, 8, 9, 10, 14, 15].

2. The Algebra $\mathcal{N}$

Throughout this note, we fix a spin model $S = (X,\ W^+, W^-)$, where $X = \{1, \ldots, n\}$.

For each $b,\ c \in X$, we consider an n-dimensional column vector $\mathbf{u}_{b,c}$ of size n with x-entry

$$(\mathbf{u}_{b,c})_x = W^+_{b,x} W^-_{c,x}.$$

Let us define $\mathcal{N}$ to be the set of all symmetric $n \times n$ matrices A such that $\mathbf{u}_{b,c}$ is an eigenvector of A for all $b,\ c \in X$. Let $\lambda^A_{b,c}$ denote the eigenvalue of A on $\mathbf{u}_{b,c}$:

$$A\mathbf{u}_{b,c} = \lambda^A_{b,c}\, \mathbf{u}_{b,c}.$$

Clearly $\mathcal{N}$ is a subspace of the full matrix algebra $\mathcal{M}_n(\mathbf{C})$, and $I \in \mathcal{N}$. For $A,\ B \in \mathcal{N}$ and $\alpha \in \mathbf{C}$, the eigenvalues of $A + B$ and αA are given by $\lambda^{A+B}_{b,c} = \lambda^A_{b,c} + \lambda^B_{b,c}$ and $\lambda^{\alpha A}_{b,c} = \alpha\lambda^A_{b,c}$.

From now on, *we assume that $S = (X, W^+, W^-)$ satisfies the Type II condition.*

We need the following well-known fact to show that $\mathcal{N}$ is an algebra with respect to ordinary matrix product. Here we give a proof to emphasize that this fact follows from the Type II condition.

Lemma 1 *For a fixed $b \in X$, the n vectors $\mathbf{u}_{b,x}$, $x \in X$, are linearly independent.*

Proof: The matrix $U = (\mathbf{u}_{b,1}, \ldots, \mathbf{u}_{b,n})$ can be written as $U = \Delta W^-$, where Δ denotes the diagonal matrix with diagonal entries $W_{b,1}, \ldots, W_{b,n}$. Since we have $W^+W^- = nI$ from the Type II condition, W^- is non-singular, and hence U is non-singular. $\square$

In particular, the vectors $\mathbf{u}_{b,c}$, $b,\ c \in X$, span the n-dimensional space. Now it is easy to prove that $\mathcal{N}$ is closed under ordinary matrix product.

Lemma 2 *For A, B in $\mathcal{N}$, $AB = BA \in \mathcal{N}$. The eigenvalue of AB is given by $\lambda^{AB}_{b,c} = \lambda^{A}_{b,c}\lambda^{B}_{b,c}$.*

Proof: We have $A\mathbf{u}_{b,c} = \lambda^{A}_{b,c}\mathbf{u}_{b,c}$ and $B\mathbf{u}_{b,c} = \lambda^{B}_{b,c}\mathbf{u}_{b,c}$. So $(AB)\mathbf{u}_{b,c} = A(B\mathbf{u}_{b,c}) = A(\lambda^{B}_{b,c}\mathbf{u}_{b,c}) = \lambda^{B}_{b,c}(A\mathbf{u}_{b,c}) = \lambda^{B}_{b,c}(\lambda^{A}_{b,c}\mathbf{u}_{b,c})$. In the same way, we have $(BA)\mathbf{u}_{b,c} = \lambda^{A}_{b,c}(\lambda^{B}_{b,c}\mathbf{u}_{b,c})$. Therefore $(AB)\mathbf{u}_{b,c} = \lambda^{A}_{b,c}\lambda^{B}_{b,c}\mathbf{u}_{b,c} = (BA)\mathbf{u}_{b,c}$ holds for all b, $c \in X$. This implies $AB = BA$ since the vectors $\mathbf{u}_{b,c}$ span the n-dimensional space. Since A and B are symmetric and commute, AB is also symmetric, and so $AB \in \mathcal{N}$. □

We have shown that $\mathcal{N}$ is an algebra with unity element I under ordinary product. Next we show that $\mathcal{N}$ is closed under Hadamard product. We need the following Lemma. For two matrices A, B (of any sizes), let $A \circ B$ denote the Hadamard product of A and B, defined by $(A \circ B)_{b,c} = A_{b,c}B_{b,c}$.

Lemma 3 *For all $n \times n$ matrices A and B, and for all b, $c \in X$,*

$$(A \circ B)\,\mathbf{u}_{b,c} = \frac{1}{n}\sum_{x\in X}(A\mathbf{u}_{b,x}) \circ (B\mathbf{u}_{x,c}).$$

Proof: The a-entry of $(A \circ B)\mathbf{u}_{b,c}$ is

$$\begin{aligned}((A \circ B)\mathbf{u}_{b,c})_a &= \sum_{x\in X} A_{a,x}B_{a,x}W^{+}_{b,x}W^{-}_{c,x}\\ &= \sum_{x\in X}\sum_{y\in X}\delta_{x,y}A_{a,x}B_{a,y}W^{+}_{b,x}W^{-}_{c,y}.\end{aligned}$$

From the Type II condition, we have

$$\delta_{x,y} = \frac{1}{n}\sum_{z\in X}W^{+}_{y,z}W^{-}_{x,z}.$$

So the above equation implies

$$\begin{aligned}((A \circ B)\mathbf{u}_{b,c})_a &= \sum_{x\in X}\sum_{y\in X}\frac{1}{n}\sum_{z\in X}W^{+}_{y,z}W^{-}_{x,z}A_{a,x}B_{a,y}W^{+}_{b,x}W^{-}_{c,y}\\ &= \frac{1}{n}\sum_{z\in X}\sum_{x\in X}A_{a,x}W^{+}_{b,x}W^{-}_{z,x}\sum_{y\in X}B_{a,y}W^{+}_{z,y}W^{-}_{c,y}\\ &= \frac{1}{n}\sum_{z\in X}\sum_{x\in X}A_{a,x}(\mathbf{u}_{b,z})_x\sum_{y\in X}B_{a,y}(\mathbf{u}_{z,c})_y\\ &= \frac{1}{n}\sum_{z\in X}(A\mathbf{u}_{b,z})_a\,(B\mathbf{u}_{z,c})_a\\ &= \frac{1}{n}\sum_{z\in X}((A\mathbf{u}_{b,z}) \circ (B\mathbf{u}_{z,c}))_a.\end{aligned}$$

This proves the assertion. □

Lemma 4 *For* $A,\ B \in \mathcal{N}$, *and for* $b,\ c \in X$,

$$(A \circ B)\, \mathbf{u}_{b,c} = \lambda_{b,c}^{A\circ B} \mathbf{u}_{b,c}$$

holds, *where*

$$\lambda_{b,c}^{A\circ B} = \frac{1}{n} \sum_{x \in X} \lambda_{b,x}^{A} \lambda_{x,c}^{B}.$$

Proof: For $x \in X$, we have

$$(A\mathbf{u}_{b,x}) \circ (B\mathbf{u}_{x,c}) = \left(\lambda_{b,x}^{A} \mathbf{u}_{b,x}\right) \circ \left(\lambda_{x,c}^{B}\, \mathbf{u}_{x,c}\right) = \lambda_{b,x}^{A} \lambda_{x,c}^{B}\, (\mathbf{u}_{b,x} \circ \mathbf{u}_{x,c}).$$

Here $\mathbf{u}_{b,x} \circ \mathbf{u}_{x,c} = \mathbf{u}_{b,c}$ holds by the definition of $\mathbf{u}_{b,c}$. Then Lemma 3 implies

$$(A \circ B)\, \mathbf{u}_{b,c} = \frac{1}{n} \sum_{x \in X} \lambda_{b,x}^{A} \lambda_{x,c}^{B} \mathbf{u}_{b,c}. \qquad \square$$

Let J denote the $n \times n$ matrix, all entries of which are equal to 1.

Lemma 5 $J \in \mathcal{N}$.

Proof: The a-entry of $J\mathbf{u}_{b,c}$ is given by

$$(J\mathbf{u}_{b,c})_a = \sum_{x \in X} J_{a,x}\, (\mathbf{u}_{b,c})_x = \sum_{x \in X} W_{b,x}^{+} W_{c,x}^{-} = n\delta_{b,c}.$$

So $J\mathbf{u}_{b,c} = \lambda_{b,c}^{J} \mathbf{u}_{b,c}$ holds with $\lambda_{b,c}^{J} = n\delta_{b,c}$. $\square$

We have shown that $\mathcal{N}$ is a commutative algebra under Hadamard product with unity element J.

Theorem 6 *If the spin model* $S = (X,\ W^{+},\ W^{-})$ *satisfies the Type II condition*, *then* $\mathcal{N}$ *is closed under both ordinary matrix product and Hadamard product*, *and* $I,\ J \in \mathcal{N}$.

3. Association schemes

A d-class (symmetric) *association scheme* on X is a partition of $X \times X$ into $d + 1$ subsets R_i, $i = 0, \ldots, d$, where $R_0 = \{(x, x) \mid x \in X\}$, which satisfies the following conditions:

(1) If $(x, y) \in R_i$, then $(y, x) \in R_i$ $(i = 0, \ldots, d)$.

(2) For every i, j, k in $\{0, \ldots, d\}$, there is an integer p_{ij}^k such that, for every x, y in X with (x, y) in R_k,

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}|.$$

Define $n \times n$ matrices $A_i, i = 0, \ldots, d$ whose (x, y)-entry $(A_i)_{x,y}$ equals 1 if $(x, y) \in R_i$ and equals 0 otherwise. Then the above definition can be written as follows.

(3) $A_i \neq 0$, $A_i \circ A_j = \delta_{ij} A_i$, ${}^tA_i = A_i$,

(4) $A_0 = I$,

(5) $\sum_{i=0}^d A_i = J$,

(6) $A_i A_j = A_j A_i = \sum_{k=0}^d p_{ij}^k A_k$.

Let $\mathcal{A}$ be the vector space spanned by the matrices A_i, $i = 0, \ldots, d$. From the above conditions, $\mathcal{A}$ is a commutative algebra under both ordinary matrix product and Hadamard product, which is called the *Bose-Mesner algebra* of the association scheme.

It is known (see [5] 2.6.1) that a vector space $\mathcal{A}$ of symmetric $n \times n$ matrices is the Bose-Mesner algebra of an association scheme on $X = \{1, \ldots, n\}$ if and only if $\mathcal{A}$ is closed under both ordinary matrix product and Hadamard product, and $I, J \in \mathcal{A}$. Theorem 6 together with this characterization of Bose-Mesner algebras implies the following corollary.

Corollary 7 *If the spin model $S = (X, W^+, W^-)$ satisfies the Type II condition, then $\mathcal{N}$ is the Bose-Mesner algebra of an association scheme on X.*

Now we consider the Type III condition.

Lemma 8 *If S satisfies the Type III condition, then $W^+ \in \mathcal{N}$.*

Proof: Type III condition can be written as $(W^+\mathbf{u}_{b,c})_a = \sqrt{n}\, W_{b,c}^-(\mathbf{u}_{b,c})_a$, and so

$$W^+\mathbf{u}_{b,c} = \sqrt{n}\, W_{b,c}^-\mathbf{u}_{b,c}$$

holds for all $b, c \in X$. This means that $\mathbf{u}_{b,c}$ is an eigenvector of W^+ with the eigenvalue $\sqrt{n}\, W_{b,c}^-$, so that $W^+ \in \mathcal{N}$. □

Remark It is known (see [12]) that, assuming the Type II condition, the Type III condition is equivalent to

$$\sum_{x \in X} W_{a,x}^- W_{b,x}^- W_{c,x}^+ = \sqrt{n}\, W_{a,b}^- W_{a,c}^+ W_{b,c}^+.$$

As in the above proof, this equation shows that $\mathbf{u}_{c,b}$ is an eigenvector of W^- with the eigenvalue $\sqrt{n}\, W_{c,b}^+$. So, if $S = (X, W^+, W^-)$ satisfies both conditions of Types II and III, then W^- belongs to $\mathcal{N}$.

Theorem 6 and Lemma 8 imply the following result by Jaeger.

Corollary 9 (Jaeger [11]) *If a spin model $S = (X, W^+, W^-)$ satisfies the conditions of Types II and III, then there is an association scheme whose Bose-Mesner algebra contains W^+ and W^-.*

References

1. R. Bacher, P. de la Harpe, and V.F.R. Jones, *Tours de Centralisateurs Pour Les Paires d'Algèbres*, *Modèles à Spins et Modèles à Vertex*, in preparation.
2. E. Bannai and E. Bannai, "Spin models on finite cyclic groups," *J. Alg. Combin.* **3** (1994), 243–259.
3. E. Bannai, E. Bannai, and F. Jaeger, "On spin models, modular invariance, and duality," to appear.
4. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, 1984.
5. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.
6. F. Goodman, P. de la Harpe, and V. Jones, *Coxeter Graphs and Towers of Algebras*, Springer, 1989.
7. P. de la Harpe and V. Jones, "Graph invariants related to statistical mechanical models: examples and problems," *J. Combin. Theory Ser. B* **57** (1993), 207–227.
8. F. Jaeger, "Strongly regular graphs and spin models for the Kauffman polynomial," *Geom. Dedicata* **44** (1992), 23–52.
9. F. Jaeger, "On spin models, triply regular association schemes, and duality," *J. Alg. Combin.* **4** (1995), 103–144.
10. F. Jaeger, "New constructions of models for link invariants," *Pac. J. Math.*, to appear.
11. F. Jaeger, "Towards a classification of spin models in terms of association schemes," *Advanced Studies in Pure Math.* **24** (1996), 197–225.
12. V. Jones, "On knot invariants related to some statistical mechanical models," *Pac. J. Math.* **137** (1989), 311–336.
13. K. Nomura, "Spin models constructed from Hadamard matrices," *J. Combin. Theory Ser. A* **68** (1994), 251–261.
14. K. Nomura, "Spin models on bipartite distance-regular graphs," *J. Combin. Theory Ser. B* **64** (1995), 300–313.
15. K. Nomura, "Spin models on triangle-free connected graphs," *J. Combin. Theory Ser. B* **67** (1996), 284–295.