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Flocks of Infinite Hyperbolic Quadrics

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Abstract. Let K be a field containing a nonsquare γ and $F = K(\sqrt{\gamma})$ a quadratic extension. Let σ denote the unique involutory automorphism of F fixing K pointwise. For every field K such that the nonzero squares of K do not form an index 1 or 2 subgroup of $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^-$, a construction is given which produces large numbers of infinite nearfield and non nearfield flocks of an infinite hyperbolic quadric in PG$(3, K)$.

Keywords: flock, quadric, Bol translation plane

1. Introduction

A flock of a hyperbolic quadric H in PG$(3, K)$, where K is a field, is a set of mutually disjoint conics whose union covers H. K can be either finite or infinite but only the finite case has been extensively studied.

When K is finite and isomorphic to GF(q), major results of Thas [20, 21] and work of Bader, Lunardon [2] completely classify the flocks.

In this case, corresponding to a flock is a translation plane with spread S in PG$(3, q)$ such that S is the union of a set of reguli which mutually share two lines (see [1, 12]).

Furthermore, it is shown in Johnson [12] that a translation plane with spread in PG$(3, q)$ that admits an affine homology group one of whose component orbits union the axis and coaxis is a regulus also produces a flock of a hyperbolic quadric.

The major result which allows the classification of flocks of hyperbolic quadrics in the finite case is that of Thas [20] (theorem 2) which shows that given a flock in PG$(3, q)$, q odd, and a conic of the flock, there is an involutory homology fixing the conic pointwise which leaves the flock invariant.

Translating the action of the involutory homologies over to the corresponding translation plane, it turns out that, for each component of the plane, there is a central involutory homology fixing this component pointwise and inverting two particular fixed components L and M.

A Bol translation plane is one which admits a left coordinatizing quasifield Q that has the Bol axiom: $a(b \cdot ac) = (a \cdot ba)c$ for all a, b, c in Q. We recall the result of Burn:

Theorem 1.1 (Burn [7]) *A translation plane is a Bol plane if and only if there exist components L and M such that for each component N distinct from L and M there is an involutory perspectivity with axis N that inverts L and M.*

In a series of articles (see e.g., [10, 13, 14]), Kallaher and Kallaher and Hanson show that with two possible exceptional orders (3^4 and 3^6), the only finite Bol planes are nearfields. Actually, combining this with some work of Bonisoli [6], it also follows that the only Bol planes with spreads in $PG(3, q)$ are nearfield planes.

The flocks corresponding to the regular nearfield planes with spreads in $PG(3, q)$ have been constructed with geometric methods by Thas [19] and are therefore called the Thas flocks.

There are three other nearfields (irregular nearfields) of orders $11^2, 23^2, 59^2$ which are, of course, Bol quasifields and which produce flocks of hyperbolic quadrics. These were independently discovered by Bader [1] and Johnson [12] and for order 11^2 and 23^2 by Baker and Ebert [3]. The corresponding flocks are sometimes called the Bader-Baker-Ebert-Johnson flocks (BBEJ) (see e.g., [21]) or merely the irregular nearfield flocks.

So, by a result of Thas, the corresponding translation planes are Bol planes and by the work of Kallaher and Bonisoli, these planes are all nearfields planes. The translation of the requisite theory from the flocks to the translation planes is accomplished in Bader-Lunardon [2] (see pp. 179–181). Furthermore, Thas has shown that there can be no nonlinear flock of a hyperbolic quadric of even order in $PG(3, 2^r)$.

Hence,

Theorem 1.2 (Thas, Bader-Lunardon) *A flock of a hyperbolic quadric in* $PG(3, q)$ *is either*

(1) *linear*,

(2) *a Thas flock*, *or*

(3) *a BBEJ flock of order* p^2 *for* $p = 11, 23,$ *or* 59.

Now we consider what can be said for flocks of infinite hyperbolic quadrics.

It has been an open question whether the results on flocks of finite hyperbolic quadrics may be extended to the infinite case.

In particular, is it true that corresponding to an infinite flock is an infinite translation plane? Furthermore, if there is a translation plane, is the plane Bol?

In Section 2, we show algebraically the connections between flocks of hyperbolic quadrics in $PG(3, K)$, K a field, and translation planes with spreads in $PG(3, K)$ composed of a set of reguli that share two components.

Hence, corresponding to an infinite flock is an infinite translation plane exactly as in the finite case. However, even if the translation plane would turn out to be Bol, there is no theory which could then be utilized to show that the translation plane is a nearfield plane.

Actually, Burn [7] has constructed some Bol planes which are not nearfield planes with spread in $PG(3, Q)$ where Q is the field of rational numbers. We show that these planes produce infinite non nearfield flocks of a hyperbolic quadric.

The main ingredient which specifies translation planes that produce flocks is that there is what might be called a “regulus inducing” homology group.

In the finite case, a nearfield flock plane which is not of order $11^2, 23^2, 59^2$ is an André plane. In fact, a finite André plane which admits the regulus inducing homology group must be a nearfield with applying the classification theorem of Thas, Bader-Lunardon.

So, a natural place to look for examples of flocks in the infinite case which might not quite fit the restrictive pattern of the finite case would be to consider the infinite André planes which admit regulus inducing homology groups.

In Section 3, we completely determine the set of André planes which produce the type of translation plane corresponding to a flock of a hyperbolic quadric. All of these planes are Bol planes.

Recall that, in the finite case, all such planes are nearfield planes and there is a unique nontrivial nearfield plane of each order.

In the infinite case, we see that the situation is much more complex and different.

In fact, there are fields K such that there are infinitely many mutually nonisomorphic nearfield planes with spreads in $PG(3, K)$.

So, there are infinitely many mutually nonisomorphic flocks of a infinite hyperbolic quadric in $PG(3, K)$.

As mentioned, a major unsolved problem in the general case is whether all hyperbolic flocks are Bol flocks in the sense that the associated translation planes are Bol planes.

Recently, Riesinger [16] considered spreads in $PG(3, K)$, K a field, that consist of a set of reguli that share two lines.

Furthermore, Riesinger provides a class of examples which produce 4-dimensional translation planes with 6-dimensional collineation group when the planes are considered as topological projective planes.

As we show in Section 2 that translation planes with spreads of the indicated type correspond to flocks of a hyperbolic quadric in $PG(3, K)$, then there are some new flocks which we call the flocks of Riesinger.

In Section 6, we point out that these flocks are not Bol flocks.

2. The correspondence

Theorem 2.1

(1) *Let F be a flock of the hyperbolic quadric $x_1x_4 = x_2x_3$ in $PG(3, K)$ represented by homogeneous coordinates (x_1, x_2, x_3, x_4) where K is a field. Then the set of planes which contain the conics in F may be represented as follows*:

$$\rho_o : x_2 = x_3,$$

$\pi_t : x_1 - tx_2 + f(t)x_3 - g(t)x_4 = 0$ *for all t in K where f and g are functions of K such that f is bijective.*

(2) *Corresponding to the flock F is a translation plane π_F with spread in $PG(3, K)$ written over the corresponding 4-dimensional vector space V_4 over K as follows*: *Let $V_4 = (x, y)$ where x and y are 2-vectors over K. Then the spread may be represented as follows*:

$$y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}, \quad y = x\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}, \quad x = 0, \text{ for all } t, v \text{ and } u \neq 0 \text{ in } K.$$

Define

$$R_t = \left\{ y = x \begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}, x = 0 \mid u \ in \ K \right\},$$

$$R_\infty = \left\{ y = x \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \middle| \ v \ in \ K \right\}.$$

Then $\{R_t, R_\infty\}$ *is a set of reguli that share two lines* (*components* $x = 0$, $y = 0$).
The translation plane admits the collineation group

$$\left\langle \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \middle| \ v, u \ in \ K - \{0\} \right\rangle$$

which contains two affine homology groups whose component orbits union the axis and coaxis define the reguli (*regulus nets*).

(3) *A translation plane with spread in* $PG(3, K)$ *which is the union of reguli that share two components may be represented in the form* (2).
Equivalently, *a translation plane with spread in* $PG(3, K)$ *which admits a homology group one of whose component orbits union the axis and coaxis is a regulus may be represented in the form* (2). *In either case*, *such a translation plane produces a flock of a hyperbolic quadric in* $PG(3, K)$.

Proof: This result is known in the case that K is finite and can be found in Johnson [12]. Furthermore, one can use the Klein quadric to verify the translation back and forth between the flocks and the planes (see Section 6). The only possible question with this construction is whether a cover of the vector space produces a cover of the quadric and conversely. We shall provide an algebraic proof that a translation plane with the required properties produces a hyperbolic flock and leave the proof that the flock gives rise to the translation plane to the reader.

Suppose that a translation plane with spread in PG$(3, K)$ admits an affine homology group one of whose component orbits union the axis and coaxis is a regulus R in PG$(3, K)$. Choose a representation so that the axis is $y = 0$, the coaxis $x = 0$ and $y = x$ is a component (line) of the regulus R. Then R is represented by the partial spread $x = 0$, $y = x\left[\begin{smallmatrix} v & 0 \\ 0 & v \end{smallmatrix}\right]$ for all v in K. Moreover, the homology group takes the matrix form:

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \middle| \ u \text{ is in } K - \{0\} \right\rangle.$$

There are functions f and g on K and components of the following matrix form:

$$y = x \begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix} \text{ for all elements } t \text{ of } K.$$

Note that, in particular, this says that the function f is 1−1 as otherwise, differences of certain corresponding matrices are singular and nonzero contrary to the assumption that the components form a unique cover of the vector space. The homology group maps these components into $y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}$ for all nonzero u in K. Hence, the regulus R and these components for all $v, t, u \neq 0$ in K define the spread in PG(3, K).

Take any value a in K and consider the vector $(1, -a, 0, 1)$. Since this vector is not on $x = 0$ or $y = xvI$ and we are assuming a "cover", there is a unique pair (u, t) with u nonzero such that $(1, -a, 0, 1)$ is incident with the component $y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}$. Hence, we have $f(t)u - au = 0$ and $g(t)u - atu = 1$. In particular, since u is nonzero, we must have $f(t) = a$. Hence, f is "onto".

In order to see that the planes listed in the theorem intersected with the hyperbolic quadric in PG(3, K) form a unique cover of the hyperbolic quadric and hence define a hyperbolic flock, we must show that for all points (a, b, c, d) for $b \neq c$ and $ad = bc$, there is a unique t in K such that the point is on the plane π_t. Since we have a cover of the 4-dimensional vector space, we know that for a vector (e, h, m, n) where not both e and h are zero and $\langle(m, n)\rangle$ is not in $\langle(e, h)\rangle$, there is a unique ordered pair (t, u) such that (e, h, m, n) is on the component $y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}$.

To distinguish between points of PG(3, K) that relate to the flock and vectors of V_4 which relate to the translation plane, we shall use the terms "points" and "vectors" respectively.

That is, for all e, h, m, n such that not both e and h are zero and the vector (m, n) is not in the 1-space generated by (e, h), there is a unique ordered pair (t, u) such that

$$ef(t)u + hu = m \quad \text{and} \quad eg(t)u + htu = n. \tag{1}$$

The point (a, b, c, d) is on π_t if and only if

$$a - bt + f(t)c - g(t)d = 0. \tag{2}$$

First assume that $bc \neq 0$. Then, without loss of generality, we may take $b = 1$ so that $ad = c$ (recall that the point is considered homogeneously).

Hence, we require that the point $(cd^{-1}, 1, c, d)$ for $c \neq 1$ is contained in a unique plane π_t. This is equivalent to the following equation having a unique solution:

$$c - dt + f(t)cd - g(t)d^2 = 0. \tag{3}$$

Consider the vector $(1, d^{-1}, 1, cd^{-1})$. Since $(1, cd^{-1})$ is in $\langle(1, d^{-1})\rangle$ if and only if $c = 1$, there is a unique ordered pair (t_o, u) such that (2, 2) is satisfied with $(e, h, m, n) = (1, d^{-1}, 1, cd^{-1})$ so that

$$f(t_o)u + d^{-1}u = 1 \quad \text{and} \quad g(t_o)u + d^{-1}t_ou = cd^{-1}. \tag{4}$$

Hence, we must have

$$cd^{-1}(f(t_o) + d^{-1})u = (g(t_o) + d^{-1}t)u \quad \text{so that}$$
$$c - dt_o + f(t_o)cd - g(t_o)d^2 = 0.$$

Now to show uniqueness. First assume that $f(t_o)d+1 = 0 = z$. Then $g(t_o)d+t = 0 = w$ and the vector $(1, d^{-1}, 0, 0)$ is on the component $y = x\left[\begin{smallmatrix} f(t_o)d & g(t_o)d \\ d & t_o d \end{smallmatrix}\right]$ and $y = 0$, which is a contradiction. Hence, $zw \neq 0$.

So, $w = zcd^{-1}$ and the vector $(1, d^{-1}, z, zcd^{-1})$ is on the component $y = x\left[\begin{smallmatrix} f(t_o)d & g(t_o)d \\ d & t_o d \end{smallmatrix}\right]$.

Now assume that there exists another element s_o such that

$$c - ds_o + f(s_o)cd - g(s_o)d^2 = 0.$$

Then $f(s_o)d + 1 = z^* \neq 0$ and there exists an element v in K such that $z^*v = z$.

Then the vector $(1, d^{-1}, z, zcd^{-1}) = (1, d^{-1}, z^*v, z^*vcd^{-1})$ is also on $y = x\left[\begin{smallmatrix} f(s_o)dv & g(s_o)dv \\ dv & s_o dv \end{smallmatrix}\right]$. By uniqueness of the vector space cover, it follows that $(t_o, d) = (s_o, dv)$. Hence, there is a unique plane π_t containing the point (a, b, c, d) such that $b \neq c$ and $ad = bc$ where bc is nonzero.

Now assume that $bc = 0$. If $b = 0$ and $d = 0$ then without loss of generality, we may take $c = 1$ so we are considering the point $(a, 0, 1, 0)$. We need to determine a t in K such that $a + f(t) = 0$. Since f is 1-1 and onto as noted above, there exists a unique value t which solves this equation and hence a unique plane π_t containing the point $(a, 0, 1, 0)$.

If $b = 0$ and $a = 0$ and $c = 1$, it is required to uniquely cover the point $(0, 0, 1, d)$ by a plane so we require a unique solution to the equation $f(t) - g(t)d = 0$. Note that the vector $(1, 0, d, 1)$ must be incident with $y = x\left[\begin{smallmatrix} f(t_1)u & g(t_1)u \\ u & t_1 u \end{smallmatrix}\right]$ for some ordered pair (t_1, u). This implies that $f(t_1)u = d$ and $g(t_1)u = 1$ so that $f(t_1) = g(t_1)d$. Moreover, $f(t_1) = z_1 \neq 0$ as otherwise the spread would contain $y = x\left[\begin{smallmatrix} 0 & 0 \\ 1 & t_1 \end{smallmatrix}\right]$. Hence, the vector $(1, 0, z_1, z_1/d)$ is on $y = x\left[\begin{smallmatrix} f(t_1) & g(t_1) \\ 1 & t_1 \end{smallmatrix}\right]$. If there exists another solution s_1 then $f(s_1) = z_1^*$ and there exists an element w of K such that $z_1^* w = z_1$. Hence, the previous vector also belongs to $y = x\left[\begin{smallmatrix} f(s_1)w & g(s_1)w \\ w & s_1 w \end{smallmatrix}\right]$ which, by uniqueness of the vector space cover, implies that $(t_1, 1) = (s_1, w)$.

If $c = 0$ then $a = 0$ or $d = 0$ and $b = 1$ without loss of generality. We are trying to show that there is a unique solution to $a - t + g(t)d = 0$. If $d = 0$ this is trivial. Thus, assume that $a = 0$.

The vector $(d^2, -, d, 1, 0)$ must be incident with $y = x\left[\begin{smallmatrix} f(t_2)u & g(t_2)u \\ u & t_2 u \end{smallmatrix}\right]$, for some unique pair (t_2, u). Thus, there is a solution t_2 to $d(f(t_2)d - 1)u = 1$ and $d(g(t_2)d - t)u = 0$. Let $f(t_2)\,d - 1 = z_2 \neq 0$. Then the vector $(1, -d^{-1}, z_2, 0)$ is on the component $y = x\left[\begin{smallmatrix} f(t_2)d & g(t_2)d \\ d & t_2 d \end{smallmatrix}\right]$ so clearly $z_2 \neq 0$.

If there is another solution s_2 then let $f(s_2)d - 1 = z_2^*$ so that there exists an element w such that $z_2^* w = z_2$. Hence the previous point is also on the component $y = x\left[\begin{smallmatrix} f(s_2)dw & g(s_2)dw \\ dw & s_2 dw \end{smallmatrix}\right]$ so by uniqueness, we must have $(t_2, d) = (s_2, dw)$. Hence, a translation plane with spread in $PG(3, K)$ which admits an affine homology group of the type listed above produces a flock of a hyperbolic quadric.

To complete the proof of part (3), we must show that if a translation plane has its spread in $PG(3, K)$ and the spread is a union of reguli sharing two components, then there is a homology group of the type mentioned above. We coordinatize so that a given regulus net has the standard form $x = 0$, $y = x\left[\begin{smallmatrix} v & 0 \\ 0 & v \end{smallmatrix}\right]$ for all v in K. Let $y = x\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be a component not in this regulus net. Since the component is in a regulus net, change bases by $(x, y) \to (x, y\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1})$. After the basis change, the second regulus must have standard form. Now mapping back with the inverse basis change, it follows that the second regulus must have the basic form $x = 0$, $y = 0$, $y = x\begin{bmatrix} a & b \\ c & d \end{bmatrix}uI_2$ for all $u \neq 0$ in K. Hence, it follows that the translation plane must admit the indicated homology group with axis $y = 0$ and coaxis $x = 0$. This proves (3).

To prove (1), we may choose a basis so that a given plane of the flock has equation $x_2 = x_3$. From here, it is fairly direct that we may represent the flock in the form given. The function $f(t)$ is 1-1 to avoid intersections and must be onto in order to ensure a cover.

The proof of (2) follows along the lines of the proof (3) and is left to the reader. □

3. André quasifields of flock type

In this Section, we completely determine the André planes with spreads in PG$(3, K)$ which produce or correspond to flocks of hyperbolic quadrics in PG$(3, K)$.

Let K be a field which contains nonsquares. Let γ be a nonsquare and $F = K(\sqrt{\gamma})$.

Let Σ_F denote the Pappian affine plane coordinatized by F and write the components of the plane as $x = 0$, $y = xm$ for all m in F. We consider the construction of the André planes (quasifields) with kernel containing K. Let σ denote the automorphism of order 2 which fixes K pointwise.

We propose to construct all of the André planes that admit the Pappian collineation group $H\ \langle (x, y) \to (xv, yu) \mid u, v \text{ in } K^* \rangle$ and which contain the standard regulus net. This is equivalent to constructing translation planes whose spread is in PG$(3, K)$ and which is the union of reguli sharing two components. We have seen in Section 2 that such a translation plane is equivalent to a flock of a hyperbolic quadric in PG$(3, K)$. We call such translation planes **hyperbolic flock planes**.

Let $R_\delta = \{y = xm \mid m^{1+\sigma} = \delta\}$, δ in K^*. Let $K(\sqrt{\gamma})^{*(\sigma+1)} = K^-$. Let S denote the subgroup of nonzero squares in K and note that S is a subgroup of K^-. We call such a partial spread (or net generated by this partial spread) an André partial spread (or André net). The replacement or derivation of the André net is accomplished by replacing R_δ by the opposite regulus net $R_\delta^* = \{y = x^\sigma m \mid m^{1+\sigma} = \delta\}$.

We define an André multiplication:

$$x * m = x^{\sigma^{(m^{1+\sigma} g)}} m \text{ where } g \text{ is any mapping from } K^- \text{ into } Z_2$$
$$\text{(or GF(2)) such that } 1g = 0.$$

In order that this produces a multiplication for which the elements of K are in the center, and we have that $x * m = xm$ for all x in K (juxtaposition shall denote multiplication in F), $m^{1+\sigma} g = 0$ for all m in K^*. This is accomplished if and only if $\alpha^2 g = 0$ for all α in K^*.

If we consider this by the replacement or nonreplacement of various André nets then we do not replace any André net R_δ where δ is a square in K^-.

Consider the image of R_β under $H : y = xm \to y = xmw$ for all w in K. And, since $(mw)^{1+\sigma} = m^{1+\sigma} w^2$, it follows that whenever R_β is replaced by R_β^*, we also must replace $R_{\beta\alpha^2}$ by $R_{\beta\alpha^2}^*$ for all α in K^*.

Hence, in order to obtain non-Pappian translation planes of this type, we must have that K^- properly contains the subgroup of nonzero squares S in K^*. For example, if K is the field of real numbers and σ maps $m = \alpha + i\beta$ onto $\alpha - i\beta$ for α, β in K then $m^{\sigma+1} = \alpha^2 + \beta^2$ which is positive so a square. In other words, the group H acts transitively on the set of all André nets in this case. Since we have agreed not to replace R_{α^2}, we do not obtain a non-Pappian plane. Hence, K cannot be the field of real numbers.

We note that the images of $y = xm$ or $y = x^\sigma m$ under H union the components $x = 0$ and $y = 0$ form reguli in PG$(3, K)$.

Consider the quotient group K^-/S. Since each element of this group has order 2 or 1, it follows that this group is an elementary Abelian 2-group. Hence, we may consider this group as a vector space over GF(2).

When we choose the set of André nets $\{R_\beta\}$ to replace, we must replace all corresponding nets $R_{\beta\alpha^2}$. This corresponds to the selection of a subset λ of K^-/S which we map under g to 1 and all other elements of the vector space map to 0. We have the condition that λ does not contain the identity element or rather that we do not replace the André square nets R_{α^2} in order to obtain the central property that we require. The property that we obtain using the group H in the associated André plane is equivalent to having K in the right nucleus (that is, $(a * b) * \alpha = (a * (b * \alpha)$ for all a, b in the quasifield and all α in K).

Hence, we obtain the following:

Theorem 3.1

(1) *The set of André quasifields constructable from a field $F = K(\sqrt{\gamma})$ with K in the intersection of the center and right nucleus* (*i.e.*, *the André quasifields of hyperbolic flock type*) *are obtained by any mapping from K^-/S to GF*(2) *such that the identity* (*zero vector*) *is mapped to* 0 *where $K^- = (K(\sqrt{\gamma})^*)^{\sigma+1}$ and σ is the involutory automorphism fixing K pointwise.*

(2) *The set of André nearfields of hyperbolic flock type are obtained by the choice of a linear functional of K^-/S considered as a GF*(2)*-vector space. Hence, there is a* 1-1 *correspondence between the set of André nearfields of hyperbolic flock type and the dual space of K^-/S.*

Proof: We have noted that any André quasifield of the type constructed above has the required properties of having K in the intersection of the center and right nucleus. By the previous section in which the equivalence of spreads in PG$(3, K)$ which are unions of reguli sharing two components and spreads containing reguli and an affine homology group one of whose component orbits union the axis and coaxis is a regulus is shown, it follows that the above procedure is the only way to produce André quasifields with the required properties. Hence, we have the proof to (1).

An André nearfield is produced exactly when the multiplication defines a group. This translates to having the mapping g above a homomorphism from K^- into Z_2. When considered as acting on K^-/S, the required mapping induces a homomorphism from K^-/S into GF(2). That is, we have a linear mapping from a vector space over GF(2) into its associated scalar field GF(2). In other words, each nearfield of hyperbolic flock type corresponds exactly to a linear functional of K^-/S so that the nearfields are in 1-1 correspondence with the dual space of K^-/S. □

Theorem 3.2

(1) *Let* $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^-$. *Suppose the dimension of* K^-/S *is finite. Then the number of ways of constructing André planes of hyperbolic flock type from a given quadratic extension field is exactly* $2^{|K^-/S|-1}$. *Furthermore, the zero map corresponds to the Pappian plane.*

(2) *If the dimension is* 1 (*order* 2), *there are exactly two André quasifields of hyperbolic flock type.*

Since any mapping of Z_2 *onto* Z_2 *which maps* 0 *to* 0 *is either trivial or a homomorphism, it follows that the two André quasifields of hyperbolic flock type are the field* F *itself and a nearfield.* (*For example, in the finite field case of odd order, this is precisely the situation.*)

(3) *Any field which is an algebraic extension of a finite field of odd order but not a set of quadratic extensions of quadratic extensions will also produce exactly one nontrivial nearfield of hyperbolic flock type.*

(4) *The number of non nearfield André quasifields of hyperbolic flock type is exactly* $2^{|K^-/S|-1} - 2^d$ *where* $d =$ *the dimension of* $K^-/S = \log_2 |K^-/S|$.

(5) *If the dimension of* K^-/S *is infinite, there are infinitely many André quasifields of hyperbolic flock type which are not nearfields.*

Hence, if the dimension of $K^-/S \geq 2$ *then there exist André quasifields of hyperbolic flock type which are not nearfields.*

Proof (3): We need to show only that the subgroup of nonzero squares is of index 2 in $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^-$. Let a and b be nonsquares. Since a, b generate a finite field over the given field, it follows that the product of these two elements is a square. It is only required that there exist nonsquares in the field since it follows that $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^- = K^*$ in this case. For example, note that $(et + u)^{\sigma+1} = u^2 - \gamma t^2$ for u, t in K and $\{e, 1\}$ a K basis. Restricted to a finite field isomorphic to GF(q) containing γ, $u^2 - \gamma t^2$ takes on both squares and nonsquares and is GF$(q)^*$. If the nonsquares do not remain nonsquares in K then K is a series of quadratic extensions. Since the set of squares in K forms an index two subgroup in the case under question, then $K^- = K^*$.

(4) and (5) follows directly from the above results and (2.1). □

Theorem 3.3 *Let* K *be a field,* S *the set of nonzero squares of* K *and* $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^-$. *Assume that the dimension of* $K^-/S \geq 1$.

(1) *Then each of the André quasifields constructed from a given quadratic extension field which have the property that the center and right nucleus contain* K *is a Bol quasifield and constructs a flock of a hyperbolic quadric in* $PG(3, K)$.

(2) *If the dimension of* $K^-/S \geq 2$ *then there exist infinite flocks of a hyperbolic quadric in* $PG(3, K)$ *which are not nearfield flocks.*

Proof: By (3.2) and (2.1), it remains only to show that the André quasifields constructed as in (3.1) are Bol quasifields.

We mentioned the Bol identity in Section 1. When considering the Bol identity in the form presented, components are written in the general form $y = m \cdot x$. Since we are writing multiplication on the opposite side, the Bol identity takes the form:

$$((c * a) * b) * a = c * ((a * b) * a) \text{ for all elements } a, b, c \text{ of the quasifield.}$$

Write $x * y = x^{\sigma(y)}y$ where $\sigma(y) = \sigma^{(y^{1+\sigma}g)}$. Then the Bol identity takes the following form:

$$c^{\sigma(a)\sigma(b)\sigma(a)}a^{\sigma(b)\sigma(a)}b^{\sigma(c)}a = c^{\sigma(aba)}a^{\sigma(b)\sigma(a)}b^{\sigma(c)}a \text{ since } \sigma(x^{\sigma(z)}) = \sigma(x) \text{ for all } x, z.$$

(See also [7] (2.6) for the same calculation in the finite case.)

Hence, we must check that $c^{\sigma(a)\sigma(b)\sigma(a)} = c^{\sigma(aba)}$. Thus, we have to verify that

$$\sigma^{(2a)^{1+\sigma}g+b^{1+\alpha}g} \text{ is equivalent to } \sigma^{(a^2b)^{1+\sigma}g}$$

or equivalently, that

$$b^{1+\sigma}g \equiv (a^2b)^{1+\sigma}g \bmod 2.$$

Whenever we replace an André net R_δ , we also replace the set of André nets $R_{\delta\alpha^2}$ for all α in K. Letting $b^{1+\sigma} = \beta$ and $a^{1+\sigma} = \alpha$, the last congruence becomes $\beta g \equiv \alpha^2\beta\, g \bmod 2$ which is the congruence statement of our replacement procedure.

Hence, all of the André quasifields constructed above are Bol quasifields. □

4. The flocks and isomorphism

From Section 3, given a field K and multiplicative subgroup S of nonzero squares, if K^-/S has dimension ≥2, we may construct at least one non nearfield flock of a hyperbolic quadric in PG$(3, K)$.

In this section, we consider possible isomorphisms between the flocks. We consider two flocks within the same projective space to be isomorphic if and only if there exists an element of $P\Gamma L(4, K)$ which preserves the hyperbolic quadric and which maps the conics of one flock onto the conics of the second flock. From the standpoint of the associated translation plane, we may consider two translation planes defined on the same vector space and sharing the two components which are common to the set of reguli of each spread. There is a corresponding isomorphism which will either fix or interchange the two common components and be in $\Gamma L(4, K)$. Conversely, for the planes constructed in Section 3, we shall see later than any isomorphism of planes permutes the regulus nets associated with the flock and hence induces an isomorphism of flocks.

Theorem 4.1 *Two flocks of a hyperbolic quadric in PG*$(3, K)$ *constructed as in Section* 3 *are isomorphic if and only if there is an isomorphism of the corresponding translation planes which fixes the two common components of the base regulus nets*, *permutes the base regulus nets*, *and belongs to* $\Gamma L(4, K)$.

Proof: Since each of the planes constructed in Section 3 are Bol planes (with respect to the lines $x = 0, y = 0$ or rather infinite points (0) and (∞)), it follows from Kallaher [13] (Corollary 3.2.2) that the points (∞) and (0) are fixed or interchanged by the full collineation group of the plane. Moreover, considering there are collineations interchanging the two indicated infinite points, we have the proof to (4.1). □

We also note that any Desarguesian plane constructed as in Section 3 is actually Pappian.

Theorem 4.2 *If π is a Desarguesian plane with spread in $PG(3, K)$ for K a field which contains a K-regulus then π is Pappian.*

Proof: If the spread contains a regulus and the regulus net is coordinatized in the standard manner then the coordinate quasifield Q contains K in its center. Let $\{1, e\}$ be a basis for Q over K as a vector space. Assume that Q is a skewfield. Then, for $\alpha, \beta, \delta, \rho$ in K, $(\alpha + \beta e)(\delta + \rho e) = \alpha\delta + (\beta\delta + \alpha\rho)e + \beta\rho e^2$ and since K is a field, it then easily follows that, in this case, the quasifield must be a field provided it is a skewfield. □

We recall that a linear flock is one where the planes of the conics of the flock share a line.

Theorem 4.3

(1) *A linear hyperbolic flock in $PG(3, K)$ corresponds to a Pappian plane coordinatized by a quadratic field extension F of K.*
(2) *Two linear hyperbolic flocks in $PG(3, K)$ are isomorphic if and only if the corresponding quadratic extension fields are isomorphic.*
(3) *There exist fields K such that there are infinitely many mutually nonisomorphic linear hyperbolic flocks in $PG(3, K)$.*

Proof: Using the notation of Section 2, we may assume that there is a common line of the form $\langle(1, 0, 0, b), (a, 1, 1, c)$ where b is not zero and a is not equal to $c\rangle$ where a, b, c are elements of K.

If $(1, 0, 0, b)$ is common to the planes denoted by π_t then it follows that $g(t) = b^{-1}$ for all t in K.

Similarly, if $(a, 1, 1, c)$ is common to the planes π_t then $f(t) = t + b^{-1}c - a = t + d$.

The corresponding translation plane has components of the form

$$y = x\begin{bmatrix}(t+d)u & b^{-1}u \\ u & tu\end{bmatrix} \text{ and } y = x\begin{bmatrix}v & 0 \\ 0 & v\end{bmatrix} \text{ for all } t, u, v \text{ in } K \text{ and } u \neq 0.$$

It follows easily that the spread is additive and multiplicative so that the spread is Desarguesian and hence Pappian by the above note (4.2). Moreover, the coordinate fields are quadratic extension fields of K.

Two linear flocks are isomorphic provided the corresponding Pappian planes are isomorphic if and only if the corresponding coordinate fields are isomorphic. This proves (1) and (2).

To prove (3), we note that there are infinitely many mutually nonisomorphic quadratic extensions of the field of rationals. For example, take the set M of all integer primes. Then $Q(\sqrt{p})$ is not isomorphic to $Q(\sqrt{q})$ since mapping $\sqrt{p}$ to $a\sqrt{q}$ for a in Q implies that $a^2 = p/q$ which cannot be the case. □

Lemma 4.4 *Let π_1 and π_2 be non-Pappian Bol planes constructed from a given Pappian plane Σ coordinatized by F(a 2-dimensional field extension of K) as in Section 3. Let ν be an isomorphism of π_1 onto π_2.*

Then ν may be represented by a K-semilinear mapping of the form $(x, y) \to (x^\rho A, y^\rho B)$ for 2×2 K-matrices A, B and $x^\rho = (x_1, x_2)^\rho = (x_1^\rho, x_2^\rho)$, where ρ is an automorphism of K.

The reader should note the difference between x^ρ defined above and x^σ which is the image of an element of the field F under the automorphism σ.

Definition 4.5 In the planes under consideration, there is a Pappian affine plane Σ coordinatized by a quadratic extension F of K.

The lines of the constructed translation planes have the form $x = c, y = xm + b$ or $y = x^\sigma m + b$ for m, b, c in F where σ is the unique involution in $\mathrm{Gal}_K F$.

The set of lines without a superscript σ shall be called the unreplaced net U and the set of lines corresponding to those with a superscript σ shall be called the replaceable net R.

The net replacing R (consisting of the lines $y = x^\sigma m + b$ for $y = xm$ in R) shall be denoted by R^* and called the replacing net.

There is a multiplication $*$ defined as follows: $x * m = x^{\sigma^g} m$ where $g = 0$ or 1 if and only if $y = xm$ is in U or R respectively.

Remark 1 The André nets R are regulus nets with opposite regulus net R^*.

Proof: Note that $y = x^\sigma n$ meets $y = xm$ for $n^{\sigma+1} = m^{\sigma+1} = \alpha$ if and only if there exists a solution to $xm = x^\sigma n$ which is valid if and only if $x^{1-\sigma} = (m/n)$. Since $(m/n)^{\sigma+1} = 1$, it follows by Hilbert's Theorem 90 that there exists an element v such that $v^{1-\sigma} = m/n$.

Thus, the line $y = x^\sigma n$ meets every line $y = xm$ and is contained in the union of such lines. □

Proposition 4.6 *All of the planes constructed as in Section 3 admit the following collineation groups:*

H: $\langle (x, y) \to (ax, by)$ *where* $a^{-1}b$ *is in* $K^* \rangle$,

B: $\langle \tau_a : (x, y) \to (y * a, x * a^{-1}) \rangle$.

Both groups leave invariant R, R^*, and U.

Furthermore, the full collineation group of the plane normalizes the group N: $\langle (x, y) \to (xv, yu)$ for all u, v in $K \rangle$.

Proof: Note that juxtaposition denotes multiplication in the field F and $*$ denotes quasifield multiplication in the associated constructed André quasifield.

The replaced net R^*consists of a set of K-regulus nets defined by Baer subplanes of Σ. The kernel homology group defined by the mappings $(x, y) \to (ax, ay)$ for all a in F then acts as a collineation group of any constructed translation plane. Since by the construction, the planes also admit the group whose elements are defined by the mappings $(x, y) \to (xu, yv)$ for all u, v in K, it follows that the planes admit the collineation group H. Since we have shown that the planes are Bol planes, it follows that the planes admit the group B (see Burn [7]). However, we wish to show that the indicated nets are left invariant.

First assume that $y = xc$ is in U and note that it follows by construction that $y = xc^{-1}$ is also in U. Then, in the constructed plane, $w * c = wc$ and $w * c^{-1} = wc^{-1}$. Then under the mapping τ_c, we have $y = xm \to y = xm^{-1}c^{-2}$, and $y = x^\sigma m \to y = x^\sigma m^{-\sigma} c^{-(1+\sigma)}$. Recall that when we replace an André regulus net R_δ then we also replace the set of André nets $R_{\delta\alpha^2}$ for all α in K^*. Hence, it is clear that τ_c is a collineation of the plane when $y = xc$ is in U. Similarly, when $y = xc$ is in R, then the form of τ_c becomes $(x, y) \to (y^\sigma c, x^\sigma c^{-1})$ and $y = xm$ maps to $y = xm^{-\sigma}c^{-2}$ and $y = x^\sigma m$ maps to $y = x^\sigma m^{-1}c^{-(1+\sigma)}$.

It remains to show that the group N is normal in the full collineation group assuming that the plane is non-Pappian. Clearly, τ_1 normalizes N. Hence, we may assume a collineation f fixes $x = 0$ and $y = 0$ and has the basic form $(x, y) \to (x^\rho A, y^\rho B)$ where A, B are 2×2 K-matrices as in (4.4). It follows that since A and B commute with uI_2 and u^ρ is in K, f clearly normalizes the group N. Hence, this completes the proof of (4.6). □

Lemma 4.7 *In a plane constructed as above, if a collineation h maps $y = x$ into a component of R^* then U and R^* are interchanged by h.*

Proof: By (4.6), h either fixes or interchanges $x = 0$ and $y = 0$. If h interchanges $x = 0$ and $y = 0$ then $h\tau_1$ fixes $x = 0$ and $y = 0$ and still maps $y = x$ into a component of R^* as the group B fixes R^*.

Hence, we may assume without loss of generality that h fixes $x = 0$ and $y = 0$.

We note that the group H of (4.6) acts transitively on the nonzero points of $y = 0$ and leaves each of the nets R, R^*, U invariant. Hence, we may assume that h fixes a given nonzero point say $(0, 1, 0, 0)$ on $y = 0$.

By (4.4), we may represent h as $(x, y) \to (x^\rho A, y^\rho B)$ for 2×2 nonsingular matrices with elements in K. Note that $(0, 1)^\rho A = (0, 1)$ if and only if $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$. Moreover, $y = x$ maps to $y = x^\sigma m$ for some element m of F. Hence, recalling the notation developed in Section 3, we have $y = x^\sigma m$ represented as $y = x\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u & \gamma t \\ t & u \end{bmatrix}$ for some u, t in K where t is nonzero. It then follows that $A^{-1}B = \begin{bmatrix} -u & -\gamma t \\ t & u \end{bmatrix}$.

Since we are trying to show that U and R^* are interchanged by h, we assume that there is an element $y = xn = x\begin{bmatrix} w & \gamma s \\ s & w \end{bmatrix}$ in U which maps back into U. Note that we assume that s is nonzero as otherwise, this is merely an element of the regulus net containing $y = x, y = 0, x = 0$ which must map into R^* due to the existence of the normal group N.

The image of $y = xn$ is

$$y = xA^{-1}\begin{bmatrix} w^\rho & (\gamma s)^\rho \\ s^\rho & w^\rho \end{bmatrix} B = \begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w^\rho & (\gamma s)^\rho \\ s^\rho & w^\rho \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -u & -\gamma t \\ t & u \end{bmatrix}$$

which is the matrix $(**)$:

$$\begin{bmatrix} u(bs^\rho - w^\rho) + ta^{-1}s^\rho(\gamma^\rho - b^2) & \gamma t(bs^\rho - w^\rho) + ua^{-1}s^\rho(\gamma^\rho - b^2) \\ t(bs^\rho + w^\rho) - uas^\rho & u(bs^\rho + w^\rho) - as^\rho t\gamma \end{bmatrix}.$$

We note that the general form for the components is $y = x\left[\begin{smallmatrix} \pm v & \pm\gamma k \\ k & v \end{smallmatrix}\right]$ for elements v, k of K where $\pm$ is $+$ if and only if the component is in U. Hence, we may equate the $(1, 1)$ and $(2, 2)$ elements of the previous matrix and obtain the relation $(1, 2) = \gamma(2, 1)$.

This results in the following two equations:

$$2uw^\rho = s^\rho t(a\gamma + a^{-1}(\gamma^\rho - b^2)) \tag{5}$$

and

$$2\gamma tw^\rho = us^\rho(a\gamma + a^{-1}(\gamma^\rho - b^2)). \tag{6}$$

If $a\gamma + a^{-1}(\gamma^\rho - b^2) \neq 0$ then since st is nonzero we may divide (5) by (6) to obtain

$$u/\gamma t = t/u \tag{7}$$

which is valid if and only if $u^2 = \gamma t^2$ which is contrary to the assumption that γ is nonsquare.

Hence, we must obtain $(a\gamma + a^{-1}(\gamma^\rho - b^2)) = 0$ which in turn forces $w = 0$.

Now certainly there exist components of U of the general form $y = x\left[\begin{smallmatrix} w & \gamma s \\ s & w \end{smallmatrix}\right]$ for ws nonzero since for example we are not replacing any component such that $w^2 - \gamma s^2 = 1$ (or square).

By the above note, none of these components can map into U so must map into R^*. This means that in the above equation the matrix $(**)$ forces the entry equations: $(1, 1) = -(2, 2)$ and $-\gamma(2, 1) = (1, 2)$. Simplifying, we obtain the following two equations:

$$2ubs^\rho = ts^\rho(a\gamma - a^{-1}(\gamma^\rho - b^2)) \tag{8}$$

and

$$2\gamma tbs^\rho = us^\rho(a\gamma - a^{-1}(\gamma^\rho - b^2)). \tag{9}$$

From above, we know that $a\gamma + a^{-1}(\gamma^\rho - b^2) = 0$ so $a\gamma - a^{-1}(\gamma^\rho - b^2) \neq 0$.

Since $ts \neq 0$, dividing (8) by (9) forces $u/\gamma t = t/u$ which is a contradiction as before.

Hence, we have a contradiction to assuming that once $y = x$ maps into R^* then some element of U maps back into U. Hence, every element of U must map into R^*, and by using the inverses of the elements above, every element of R^* must map into U. That is, U and R^* are interchanged by the collineation. □

Theorem 4.8 (The interchange theorem) *In a non-Pappian Bol André plane constructed as in Section* 3*, the unreplaced net and replacing net are either both fixed or interchanged by a collineation of the plane.*

Proof: Let U and R^* denote the unreplaced and replacing nets respectively. Let j be a collineation of the plane. Suppose that some component $y = xm$ of U maps into R^*. Change bases by the mapping h: $(x, y) \to (x, ym^{-1})$. Then an isomorphic plane is obtained with corresponding unreplaced and replacing nets Uh and R^*h respectively.

Let

$$U^- = \{n^{1+\sigma} \mid y = xn \text{ is in } U\} \text{ and } R^{*-} = \{n^{1+\sigma} \mid y = x^\sigma n \text{ is in } R^*\}.$$

Recall, that δ in U^-implies that $\delta\alpha^2$ is in U^- and β in R^{*-} implies that $\beta\alpha^2$ is in R^{*-} for all α in K.

Let $m^{1+\sigma} = \alpha_o$. Then $(Uh)^- = U^-\alpha_o$ and $(R^*h)^- = R^{*-}\alpha_o$ (use the analogous definitions for the indicated subsets of K). So, δ in $(Uh)^-$ implies that $\delta\alpha^2$ is in $(Uh)^-$ and β in $(R^*h)^-$ implies that $\beta\alpha^2$ is in $(R^*h)^-$. This shows that the isomorphic plane has exactly the same groups acting on it and in the same representation as H and N above in the statement of (4.6).

By the previous lemma, $h^{-1}jh$ interchanges Uh and R^*h so that j interchanges U and R^*. □

This argument is actually more general and proves the following isomorphism theorem.

We shall denote a translation plane constructed from a given Desarguesian plane by replacement of R and nonreplacement of U by $U \cup R^*$.

Theorem 4.9 *Let $\pi_1 = U_1 \cup R_1^*$ and $\pi_2 = U_2 \cup R_2^*$ be isomorphic non-Pappian André planes constructed as in Section* 3.

Then an isomorphism from π_1 onto π_2 either maps U_1 onto U_2 and R_1^* onto R_2^* or maps U_1 onto R_2^* and R_1^* onto U_2.

Proof: We consider the two planes to share the components $x = 0$, $y = 0$, $y = x$. Any isomorphism must fix or interchange $x = 0$ and $y = 0$ or otherwise one of the planes will be Desarguesian by Kallaher [13] (3.2.1) or (3.2.2). Since the planes are Bol, we may assume that the isomorphism fixes $x = 0$ and $y = 0$ and thus has the form of the collineation of a Bol plane used in the proof of (4.8). Because the general form (components $y = x\begin{bmatrix} \pm v & \pm\gamma k \\ k & v \end{bmatrix}$) of the components of either Bol planes is the same, we may use the argument of (4.8) to prove (4.9). □

Corollary 4.10 *Let π be a non-Pappian André nearfield plane constructed as in Section* 3 *from a Desarguesian plane Σ coordinatized by the field extension F of K. Let U and R^* denote the unreplaced and replacing nets so that $\pi = U \cup R^*$.*

Then there is a collineation which interchanges U and R^*.

Proof: Certainly there is a homology group with axis $x = 0$ and coaxis $y = 0$ which acts regularly on the points on the line at infinity distinct from (0) and (∞). By (4.9), the conclusion follows immediately. □

We require a proposition on the determination of fields with large intersections.

Proposition 4.11 *Let Σ and Δ be Pappian planes coordinatized by quadratic extension fields F_Σ and F_Δ respectively of a given field K. We consider the two spreads within $PG(3, K)$ so the planes may be considered as defined on the same points.*

If the two Pappian planes share a net which properly contains a K-regulus net then the two planes are equal and consequently the two fields are identical.

Proof: We take the regulus net to be defined in the standard way as the net defined by the partial spread $x = 0, y = 0, y = xk$ where k is in K . If a net properly contains this regulus net then we may define a common subfield of the two fields in question. It is trivial to verify that any subfield of a quadratic extension of K and properly containing K is the field itself. Hence, $F_\Sigma = F_\Delta$. □

Theorem 4.12 *The isomorphism classes of André nearfield planes constructed as in Section 3 from a given Pappian plane Σ coordinatized by a field extension F of K are in 1-1 correspondence with the set of orbits of subgroups of index two of K^-/S under the automorphism group of K where S is the subgroup of nonzero squares of K^* and $K^- = F^{*(\sigma+1)}$.*

The automorphism group of K induces a natural action on the dual space of K^-/S as a $GF(2)$-vector space and the isomorphism classes of non-Pappian André nearfield planes of hyperbolic flock type are in 1-1 correspondence with the orbits different from the zero vector of the automorphism group acting on the dual space.

Proof: Let π_1 and π_2 be isomorphic and non-Desarguesian André nearfield planes constructed from Σ as in Section 3 and let ρ be an isomorphism from π_1 onto π_2. By the use of the collineation group, we may assume that ρ fixes $x = 0$ and $y = 0$. By (4.9), we may assume that if $\pi_i = U_i \cup R_i^*$ then ρ maps U_1 onto U_2, maps R_1^* onto R_2^* and hence maps R_1 onto R_2. In particular, by (4.11), ρ is a collineation of the associated Pappian plane Σ. We note that since a non-Pappian nearfield plane is obtained by a homomorphism of K^-/S onto GF(2), it follows that the kernel of the homomorphism is U_1^-/S.

Thus, we may represent ρ in the form $(x, y) \to (x^\omega a, y^\omega b)$, where a, b are in F and ω denotes an automorphism of K and extend to F. Note that $y = xm$ of U_1 maps to $y = xm^\omega(a^{-1}b)$ and some image must be $y = x$. Furthermore, for any automorphism ω of K, extend to an automorphism of F so that if $k = n^{\sigma+1}$ then $k^\omega = n^{\omega(\sigma+1)}$. Note that if $(et + u)^{\sigma+1} = u^2 - \gamma t^2$ then $(et^\omega + u^\omega)^{\sigma+1} = u^{2\omega} - \gamma t^{2\omega}$ so that ω acts on $(K(\sqrt{\gamma})^*)^{\sigma+1} = K^-$.

Hence, it follows that $(a^{-1}b)^{1+\sigma} m^{\omega(1+\sigma)} = 1$ so that $(a^{-1}b)^{1+\sigma} = k$ is in $U_1^{\omega-}$ (recall if k is in U_1^- so is k^{-1} and also $(U_1^\omega)^-$ is denoted by $U_1^{\omega-}$).

Using the notation developed in (4.8), we must have U_1^- map to U_2^-, so that the above implies that as $y = xn$ in U_1 maps to $y = xn^\omega a^{-1}b$ then n in U_1^- implies $n^{\omega(1+\sigma)}$ in $U_1^{\omega-}$. Since $U_1^{\omega-}$ is a subgroup of K^- as noted above, and $(a^{-1}b)^{1+\sigma}$in $U_1^{\omega-}$ implies that $(n^\omega a^{-1}b)^{1+\sigma}$ is in $U_1^{\omega-}$, so it follows that $U_2^- = U_1^{\omega-}$. Hence, an isomorphism of the two planes is uniquely determined by an automorphism of K.

Now if η is in the dual space of K^-/S, and ω an automorphism of K, we define η^ω as the mapping which takes zS onto $(z^\omega S)\eta$. A homomorphism η with kernel U_1^-/S then defines a homomorphism η^{ω} with kernel $U_1^{\omega-}/S$. Hence, the set of isomorphism classes of non-Pappian André nearfield planes of the type under consideration is in 1-1 correspondence with the orbits of the automorphism group of K acting on the dual space of K^-/S. □

Corollary 4.13 *If the automorphism group of a field K is trivial and there is a quadratic field extension F of K then the set of isomorphism classes of André nearfield planes of hyperbolic flock type constructed from the Pappian plane coordinatized by F is in* 1-1 *correspondence with the dual space of K^-/S as a $GF(2)$-vector space.*

Theorem 4.14 *Let K be a field which has quadratic field extensions F_1 and F_2. Assume that π_1 and π_2 are non-Pappian nearfield planes of hyperbolic flock type constructed from the Pappian planes Σ_1 and Σ_2 coordinatized by F_1 and F_2 respectively.*

If the planes π_1 and π_2 are isomorphic then the fields F_1 and F_2 are isomorphic.

Proof: First of all, note that we may consider both Pappian planes as defined on the same points as the associated spreads are both in PG$(3, K)$. Let ρ be an isomorphism from $\pi_1 = U_1 \cup R_1^*$ onto $\pi_2 = U_2 \cup R_2^*$. The components represented by $x = 0$ and $y = 0$ in the Pappian planes are not necessarily the same but it is clear from previous arguments that any isomorphism must map the set of these two components of the first plane onto the set of these same two components of the second plane. Moreover, we may also assume that ρ actually fixes $x = 0$ and $y = 0$ with the obvious interpretation. And, we may assume that U_1 maps into U_2. Hence, it also follows that R_1^* maps to R_2^* so that R_1 maps to R_2. It follows by (4.11) that ρ is an isomorphism from the Pappian coordinatized by F_1 onto the Pappian plane coordinatized by F_2 which implies that the two fields are isomorphic. □

Actually, (4.14) is not stated in its most general form. If we consider a Pappian plane Σ of the type considered in the statement of the result, there is a set of André nets (corresponding to reguli in PG$(3, K)$) which cover the components except for $x = 0$ and $y = 0$. If all of these nets are derived (replaced by the partial spread net of Baer subplanes) then another Pappian plane Σ^* is obtained which also may be coordinatized by the same field. Recall, in the arguments above, the components of the plane in question are represented in the form $y = x\left[\begin{smallmatrix} \pm v & \pm\gamma k \\ k & v \end{smallmatrix}\right]$ where v, k are in K. The components are in Σ exactly when $\pm = +$ and in Σ^* exactly when $\pm = -$. A basis change of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

then shows that the components of Σ^* may be represented by components of the form

$$x = 0, y = x \begin{bmatrix} v & -\gamma k \\ -k & v \end{bmatrix} \text{ for all } u, k \text{ in } K.$$

Effectively, this amounts to choosing two different points to represent $(1, 1)$ in a given Pappian plane once it has been decided what subspaces to be called $x = 0$ and $y = 0$. Note that the two subspaces with equation $y = x$ do not belong to the same Pappian plane as components. In the case of Σ and Σ^*, the component $y = x$ of Σ (Σ^*) is a Baer subplane of Σ^* (Σ).

In the above proof, if we would not assume that the isomorphic planes are nearfield planes then by the interchange theorem (extended to the isomorphism case), it would be possible that ρ might map U_1 onto R_2^* and R_1^* onto U_2. By (4.11), it then follows that ρ is an isomorphism of the Pappian plane used in the construction of π_1 onto the Pappian plane $\Sigma_{F_2}^*$ "derived" from the Pappian plane Σ_{F_2} used in the construction of the plane π_2. Hence, it follows that the fields coordinatizing these two Pappian planes are isomorphic so that the fields F_1 and F_2 are isomorphic.

Thus, we have:

Theorem 4.15 *Let F_1 and F_2 be quadratic extension fields of the field K. Let π_1 and π_2 be non-Pappian André planes of hyperbolic flock type constructed from the Pappian planes coordinatized by the fields F_1 and F_2 respectively.*

If π_1 is isomorphic to π_2 then F_1 is isomorphic to F_2.

There are therefore a vast number of mutually nonisomorphic André planes of hyperbolic flock type and hence a vast number of mutually nonisomorphic hyperbolic flocks of a hyperbolic quadric in $PG(3, K)$ for certain fields K.

Theorem 4.16

(1) *There exist infinitely many finite algebraic extensions of the rationals Q which admit trivial automorphism groups.*

For example, let p be prime and n an integer then $x^n - p$ is irreducible and for n odd, there is exactly one real root which we denote by $p^{1/n}$. Then $Q[p^{1/n}]$ has trivial automorphism group.

(2) *For a field K of part (1), there exists a set of integer primes such that square roots and quotients of distinct square roots are not in K so there exist infinitely many mutually nonisomorphic quadratic extensions.*

(3) *For each of the fields K of part (1) and for each quadratic extension of part (2), there exists a set of nearfield planes whose isomorphism class is in 1-1 correspondence with the dual space of K^-/S as a $GF(2)$-vector space where S is the set of squares of nonzero elements of K.*

Hence, there are infinitely many mutually nonisomorphic nearfield planes of hyperbolic flock type which are not Pappian planes.

(4) *For each such field K of part (1), there exists an infinite number of mutually nonisomorphic nonlinear hyperbolic flocks in $PG(3, K)$ corresponding to nearfield planes.*

Proof:

1. Note that the polynomial $x^n - p$ has exactly one real root.
2. There exits a set M of square roots of integer primes such that no element or quotient of two distinct elements are contained in any given field $Q[p^{1/n}]$. Then it follows that $Q[p^{1/n}][\sqrt{q}]$ and $Q[p^{1/n}][\sqrt{k}]$ for distinct primes q, k whose square roots are in M are not isomorphic.

The proof of (3) follows immediately from (4.14).

For each field K of part (1), there are infinitely many mutually nonisomorphic quadratic extensions. For each of these quadratic extensions, there are $|K^-/S|-1$ non-Pappian and mutually nonisomorphic nearfield planes. It is easy to verify that $|K^-/S|>1$ in the situation under consideration. For example, for a K basis $\{e,1\}$ where $e^2=\gamma$ a nonsquare in K then $(et+u)^{\sigma+1}=u^2-\gamma t^2$ is not always a square in any of the fields K in question. □

We may also consider the more general setting where we construct an André plane of hyperbolic flock type which is not a nearfield plane. For example, such are always possible when the dimension of K^-/S is larger than 1. The rather technical isomorphism result is as follows:

Theorem 4.17 *Let Σ be a Pappian plane and construct planes π_i, $i=1,2$ as non-Pappian André Bol planes as above with spreads in $PG(3,K)$. Let $\pi_1=U_1\cup R_1^*$, be isomorphic to $\pi_2=U_2\cup R_2^*$ by a collineation g.*

Then, without loss of generality, g may be represented in the following form:

$$(x,y)\to\left(x^\rho\begin{bmatrix}a&0\\0&1\end{bmatrix},y^\rho\begin{bmatrix}\pm a&0\\0&1\end{bmatrix}d\right)$$

where d is an element of the field extension F coordinatizing Σ and ρ is an automorphism of K where $x^\rho=(x_1^\rho,x_2^\rho)$ where x_i is in K, $i=1,2$.

And, representing the field elements in the form $\left[\begin{smallmatrix}u&\gamma t\\t&u\end{smallmatrix}\right]$ for all u,t in K, if $\pm=+$ then $a=1$ and if $\pm=-$ then $\gamma^{\rho-1}=a^2$, and $a^{\rho+1}=1$.

Proof: We know that the unreplaced and replacing nets from one plane are either mapped to the unreplaced and replacing nets respectively or to the replacing and unreplaced nets respectively.

If the unreplaced net maps to the corresponding unreplaced net then we may use the argument as above to show that the isomorphism is a collineation of the Pappian plane Σ and the existence of the collineation group allows that the collineation be as above where in this case $\pm=+$ and $a=1$.

If the unreplaced net maps to the corresponding replaced net then we may represent the isomorphism in the form $(x,y)\to(x^\rho A,y^\rho B)$ for 2×2 K matrices A, B. By following the argument of (4.7) and we see that if the unreplaced net maps to the corresponding replacing net then the Eqs. (5) and (6) of (4.7) are changed slightly. Note that, in this case, the $\pm$ in the matrices is $-$ and we equate the entries in the image matrix $(**)$ under g as $(1,1)=-(2,2)$ and $(1,2)=-\gamma(2,1)$ (that is, $-$ instead of $+$). The analogous equations are then:

$$\text{(see (5) and (7) of (4.7))}: 2ubs^\rho=s^\rho t(a\gamma-a^{-1}(\gamma^\rho-b^2)) \tag{10}$$

and

$$\text{(see (6) and (8) of (4.7))}: 2\gamma tbs^\rho=s^\rho u(a\gamma-a^{-1}(\gamma^\rho-b^2)). \tag{11}$$

If $a\gamma - a^{-1}(\gamma^{\rho} - b^2) \neq 0$ then we obtain $u^2 = \gamma t^2$, a contradiction.

Hence, $a\gamma - a^{-1}(\gamma^{\rho} - b^2) = 0$ which forces $b = 0$ so that $a^2 = \gamma^{\rho-1}$. This allows that we may choose $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. Since g must also map the replacing net into the corresponding unreplaced net, this shows that g^2 must be a collineation of Σ. This condition forces $a^{\rho+1} = 1$. Since we note that the matrix $B = A\begin{bmatrix} -u & -\gamma t \\ t & u \end{bmatrix}$, we have the proof of the result. □

Corollary 4.18 *Assume the conditions of* (4.17) *and further assume that the field K has a trivial automophism group. We shall assume the notation previously established.*

Then, under an isomorphism, U_1^- maps to $U_1^- k$ and $R_1^- = (R_1^-)^*$ maps to $R_1^- k$ for some nonzero element k in K^-. Note that $U_1^- k$ can be either U_2^- or R_2^-.

Proof: We must now have $a^2 = 1$ in (11) of (4.17) so that $a = \pm 1$. Recall $x^{\sigma} = x\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence the isomorphism either has the form $(x, y) \to (x^{\sigma^g}, yd)$ or $(x, y) \to (x, y^{\sigma^g} d)$ where $g = 0$ or 1. It is easy to check that a component $y = xm$ will either map to $y = x^{\sigma^g} md$ or $y = x^{\sigma^g} m^{\sigma} d$. Since $m^{\sigma(1+\sigma)} = m^{1+\sigma}$, the result follows. □

Corollary 4.19 *Let K be a field with trivial automorphism group. Let S denote the subgroup of nonzero squares of K. Let $C(S) = \{T$ where T is any subset of K^- which is closed under multiplication by elements of $S\}$.*

The set of isomorphism classes of André planes of hyperbolic flock type obtained from a given quadratic extension field F of K is in 1-1 *correspondence with the set of orbits of $C(S)$ under multiplication by elements of K^-.*

In particular, if U_1^-, and U_2^- are subgroups such that U_i^-/S is an index two subgroup of K^-/S for $i = 1, 2$ then the sets are in distinct orbits. These sets correspond to nonisomorphic nearfield planes as seen above.

Note that the empty set corresponds to the Pappian plane Σ^* obtained from the Pappian plane Σ coordinatized by F by the derivation of all of the André nets defined by the unique automorphism of the field which fixes K pointwise. The set K^- similarly corresponds to Σ.

Proof: If a set is a group then the image under k is a group only if k^{-1} and hence k is in the original group. □

5. The flocks of Burn

As mentioned in the introduction, Burn [7] (p. 356) gives an example of a class of Bol quasifields which are not nearfields. All of these Bol quasifields produce hyperbolic flocks and are André quasifields of hyperbolic flock type and thus appear in Sections 3 and 4. We shall give these examples an interpreted in our notation.

Let K be the field Q of rationals. Let $F = Q(\sqrt{d})$ for some nonsquare d in Q. Let p be any prime in Q and write any element k of Q in the form $(-1)^{\beta} p^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ where $\{p, p_1, p_2, \ldots, p_t\}$ are distinct primes in Q, $\beta = 0$ or 1, α is an integer, and α_i is a nonzero integer for $i = 1, 2, \ldots, t$. Note that Q^- properly contains the subgroup S of nonzero squares of Q.

Consider a component $y = xm$ of the corresponding Pappian plane coordinatized by F and let σ denote the unique involutory automorphism which fixes $K = Q$ pointwise.

If $k = m^{1+\sigma}$ has the representation as above where $\alpha_i \equiv 0 \bmod 2$, then $y = xm$ is not replaced so is in U. If, with respect to k, some $\alpha_j \equiv 1 \bmod 2$ then replace $y = xm$ by $y = x^{\sigma}m$ if and only if $\alpha + \Sigma\alpha_i \equiv 1 \bmod 2$.

Burn shows that these examples provide Bol quasifields which are not nearfields.

In order to bring these examples into our notation, we first note that if the element k in $K = Q$ is a square then each of the exponents in any representation are congruent to 0 mod 2 so that none of the corresponding components are replaced. This is equivalent to requiring that S is always contained in the unreplaced set U^-. Now assume that k has representation so that some exponent $\alpha_j \equiv 1 \bmod 2$, and further that the sum of the exponents not equal to the exponent of -1 also is congruent to 1 mod 2. Then $k\delta^2$ must have representation so that the sum of the exponents not equal to the exponent of -1 also is congruent to 1 mod 2. In other words, once we agree to replace the André net R_ρ then we also must replace the André nets $R_{\rho\beta^2}$ for all β in $K = Q$.

Since this type of replacement does not produce groups U^- such that U^-/S is a subgroup of K^-/S, it follows that these examples do not produce nearfields.

Hence,

Theorem 5.1 *The flocks of a hyperbolic quadric of Burn are non nearfield Bol flocks.*

6. The flocks of Riesinger

Recently, Riesinger [16] gave some conditions by which a spread in PG$(3, K)$, K a field, exists which consists of the union of a set of reguli which share two lines. Furthermore, Riesinger gives an example where K is the field of real numbers which is not Pappian. Here the emphasis is on topological planes and the example constructed gives a 4-dimensional translation plane with 6-dimensional group.

By Section 2, there is a corresponding flock of a hyperbolic quadric. In Sections 3 and 4, we have constructed many classes of flocks of hyperbolic quadrics all of which are what might be called Bol flocks in that the corresponding translation planes are Bol planes. And, all finite hyperbolic flocks are Bol flocks.

In this section, we show that the flocks of Riesinger are not Bol flocks. In order to do this, we shall translate the construction of Riesinger into the notation developed in Section 2 and then verify that the required involutions do not exist in the translation plane.

We developed the connections between the translation planes and the flocks algebraically without the use of the Klein quadric. However, by applying the arguments of Johnson [12] (Section 4), we may also use a Klein quadric as follows:

We assume that the Klein quadric is given by $x_o x_5 - x_1 x_4 + x_2 x_3 = 0$ within the projective 5-space PG$(5, K)$ given by homogeneous coordinates $(x_o, x_1, x_2, x_3, x_4, x_5)$. If $\{e_1, e_2, e_3, e_4\}$ is a basis for the underlying 4-dimensional vector space over K, we choose $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ as a basis for the underlying 6 -dimensional vector space over K where $\wedge$ denotes exterior product so that $e_i \wedge e_j = -e_j \wedge e_j$.

A component of the form $y = x\left[\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right] = M$ then corresponds to the point $(1, m_3, m_4, -m_1, -m_2, \Delta M)$ where ΔM denotes the determinant of M.

The component $x = 0$ corresponds to $(0, 0, 0, 0, 0, 1)$.

The flock is obtained from a spread which is a union of reguli sharing two lines in PG(3, K) as follows. The theorem stated has been proved in Johnson [12] for the finite case. The infinite case follows in a similar manner with appropriate changes.

Theorem 6.1 (See [12] (2.7)) *Let π be a translation plane with spread in $PG(3, K)$, for K a field. Let the spread consist of a set $\{R_i \mid i \text{ in } \lambda\}$ of reguli R_i which share two lines. Embed the spread in $PG(5, K)$ as a set of decomposable vectors (points) in the associated vector space V_6. Let Q denote the Klein quadric.*

Then the reguli R_i correspond to a set of 3-spaces π_i^* in V_6 whose polar planes π_i all lie in a 4-space Σ such that $\pi_i \cap \pi_j \cap Q = \phi$ and $\cup_i$ in $\lambda(\pi_i \cap Q)$ is a hyperbolic quadric in Σ.

Also, $\{\pi_i \cap Q \mid i \text{ in } \lambda\}$ is a flock of a hyperbolic quadric in $PG(3, K)$.

If the components initially have the form $x = 0$, $y = x\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$, $y = x\begin{bmatrix} f(t)u & g(t)u \\ u & tu \end{bmatrix}$ for all $v, t, u \neq 0$ in K then it turns out by following along the arguments in Section 4 of [12] that the planes have the equations as follows: We may take λ as K, PG(3, K) as (x_1, x_2, x_3, x_4), the hyperbolic quadric as $x_1x_4 = x_2x_3$ and the planes as ρ: $x_2 = x_3$, π_t: $x_1 - tx_2 + f(t)x_3 - g(t)x_4 = 0$ for all t in K (see Section 2).

In the paper by Riesinger, the Klein quadric is taken using the equation $x_0x_3 + x_1x_4 + x_2x_5 = 0$ and a basis for the 6-dimensional space as $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3\}$ but, elements are represented right to left as opposed to left to right.

If we write out the matrix $y = x\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} = M$ in terms of this latter basis as $(1, 0, m_1, m_2) \wedge (0, 1, m_3, m_4)$ and represented right to left, we obtain the 6-vector: $(-m_1, m_2, \Delta M, m_4, m_3, 1)$.

In Riesinger [16] (Satz (3.5) p. 146) (3.5.11), there is a representation of a spread consisting of a union of reguli sharing two components. Here the field K is the field of real numbers.

The spread as a set of vectors in 6-dimensional vector space over the reals is:

$$\{(0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1),$$
$$(T_0(f, g), T_1(f, g), T_2(f, g), T_3(f, g), T_4(f, g), T_5(f, g)),$$
$$\text{for all } f, g \text{ in the field of real numbers}\},$$

where

$$T_0(f, g) = (f^3 + \alpha f^2 g + fg^2)/(f^2 + (1 + \alpha)g^2),$$
$$T_1(f, g) = (f^2 g + \alpha f g^2 + g^3)/(f^2 + (1 + \alpha)g^2),$$
$$T_2(f, g) = (f^4 + \alpha f^3 g + 2 f^2 g^2 + \alpha f g^3 + g^4)/(f^2 + (1 + \alpha)g^2),$$
$$T_3(f, g) = -f,$$
$$T_4(f, g) = -g,$$
$$T_5(f, g) = 1, \text{ where } \alpha \text{ is a real number such that } |\alpha| < 0.08.$$

In our representation, the first two vectors correspond to $y = 0$ and $x = 0$ respectively.

Now let $g = 0$ to obtain the vector $(f, 0, f^2, -f, 0, 1)$. This vector corresponds in our representation to the component $y = x\begin{bmatrix} -f & 0 \\ 0 & -f \end{bmatrix}$ for all f in the reals.

If g is not 0, let $-g = u$ and $-f = ut$. If we work out what the form of the vector becomes, and translate this back to our component representation, we obtain the following corresponding component:

$$y = x\begin{bmatrix} (t(t^2+\alpha t+1)/(t^2+1+\alpha))u & -((t^2+\alpha t+1)/(t^2+1+\alpha))u \\ u & ut \end{bmatrix}$$

for all $t, u \neq 0$ in the field of real numbers.

We may now apply Theorem (2.1) to obtain the flocks of Riesinger.

Theorem 6.2 *Let K be the field of real numbers and let α be a real number of absolute value less than* 0.08. *Let $PG(3, K)$ be represented by homogeneous coordinates (x_1, x_2, x_3, x_4). Let $x_1x_4 = x_2x_3$ represent a hyperbolic quadric.*

Then the following is a flock F_α of the hyperbolic quadric:

ρ: $x_2 = x_3$,

π_t: $x_1 - tx_2 + (t(t^2+\alpha t+1)/(t^2+1+\alpha))x_3 + ((t^2+\alpha t+1)/(t^2+1+\alpha))x_4 = 0$, *for all t in K.*

Note that when $\alpha = 0$, we obtain a linear flock and a corresponding Pappian plane.

Theorem 6.3 *A Riesinger translation plane F_α for $\alpha \neq 0$ does not admit an involutory collineation interchanging $x = 0$ and $y = 0$ with axis $y = x$ (using the notation of* (6.2)).

Proof: A collineation which fixes $y = x$ pointwise is linear over the field K of real numbers and if it interchanges $y = 0$ and $x = 0$ has the form $(x, y) \to (yA, xB)$ where A, B are 2×2 matrices with entries in K. Since $y = x$ is fixed pointwise, $A = B$.

Assume that a component has the form $y = x\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix}$. Then the involution maps this component onto $y = x\begin{bmatrix} t/\Delta & -g(t)/\Delta \\ -1/\Delta & f(t)/\Delta \end{bmatrix}$ where Δ denotes the determinant of the indicated matrix.

The plane admits the collineation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\Delta & 0 \\ 0 & 0 & 0 & -\Delta \end{bmatrix}$$

which maps the latter component onto $y = x\begin{bmatrix} -t & g(t) \\ 1 & -f(t) \end{bmatrix}$. Hence, it follows that

$$f(-f(t)) = -t \text{ and } g(-f(t)) = g(t) \text{ for all } t \text{ in } K.$$

In the planes in question, we have

$$f(t) = t(t^2+\alpha t+1)/(t^2+\alpha+1)$$

and

$$g(t) = -(t^2 + \alpha t + 1)/(t^2 + \alpha + 1).$$

Let $t = 1$ so that $f(1) = (2+\alpha)/(2+\alpha) = 1$. Then $f(-1) = -(2-\alpha)/(2+\alpha) = -1$ if and only if $2\alpha = 0$ if and only if $\alpha = 0$. □

Theorem 6.4 *The nonlinear flocks of Riesinger are not Bol flocks.*

Proof: By Riesinger [16](3.5.13), the full collineation group of the associated translation plane leaves the set of components $\{x = 0 \text{ and } y = 0\}$ invariant. Hence, if the translation plane is a Bol plane, it is a Bol plane with respect to the infinite points (0) and (∞). And, by the theorem of Burn, for each component, there is an involutory central collineation interchanging these two points with axis the given component. Taking the component to be $y = x$, the form for the Bol quasifield is as represented in the previous Section 3 and so there is an involution of the form $(x, y) \to (y, x)$. However, we have seen that this cannot represent a collineation[1]. □

7. Infinite flocks of quadric sets

In this article, we have constructed infinite nonlinear flocks of hyperbolic quadrics in PG$(3, K)$ for K an infinite field. It is also possible to construct infinite nonlinear flocks of quadratic cones and infinite nonlinear flocks of elliptic quadrics in PG$(3, K)$.

For example, the reader might like to consult De Clerck and Van Maldeghem [8], Jha-Johnson [11], and Biliotti-Johnson [4] for results about and constructions of infinite flocks of quadratic cones.

By Thas [17] for even order, and Orr [15] for odd order, there can be no nonlinear finite flocks of elliptic quadrics in PG$(3, q)$. However, Dembowski [9] gives an example of a nonlinear flock of an elliptic quadric in PG$(3, R)$ where R is the field of real numbers. Also, this example is generalized in Biliotti-Johnson [5].

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