Journal of Algebraic Combinatorics **6** (1997), 59–87

Applications of the Frobenius Formulas for the Characters of the Symmetric Group and the Hecke Algebras of Type A

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Received ; Revised September 13, 1995

Abstract. We give a simple combinatorial proof of Ram's rule for computing the characters of the Hecke Algebra. We also establish a relationship between the characters of the Hecke algebra and the Kronecker product of two irreducible representations of the Symmetric Group which allows us to give new combinatorial interpretations to the Kronecker product of two Schur functions evaluated at a Schur function of hook shape or a two row shape. We also give a formula for the regular representation of the Hecke algebra.

Keywords: character, symmetric group, Hecke algebra, Kronecker product

1. Introduction

Frobenius began the study of the representation theory and character theory of the symmetric group S_f at the turn of the century [5]. There is one irreducible representation of S_f corresponding to each partition λ of f. Frobenius gave the following remarkable formula for the irreducible characters of the symmetric group. If p_μ denotes the power symmetric function and s_λ is the Schur function, then

$$p_\mu = \sum_{\lambda \vdash f} \chi_{S_f}^\lambda(\mu) s_\lambda, \tag{1}$$

where $\chi_{S_f}^\lambda(\mu)$ is the value of the irreducible character $\chi_{S_f}^\lambda(\mu)$ evaluated at a permutation of cycle type μ ([13] I Section 7 and [12] contain proofs of this formula which are essentially the same as that of Frobenius). This formula can be used to give a combinatorial rules, often called the Murnaghan-Nakayama rule, for computing the irreducible characters of the symmetric group.

*Supported by NSF-Postdoctoral Fellowship.
†Partially supported by NSF-grant #DMS 93-06427.

A Frobenius type formula for the characters of the Hecke algebra $H_f(q)$ was derived in [14] by studying the Schur-Weyl type duality between the Hecke algebra and the quantum group $U_q(s\ell(n))$. The q-extension of the Murnaghan-Nakayama rule was also given in [14]. It was derived there through a connection between the irreducible characters of the Hecke algebras and Kronecker products of symmetric group representations. For each partition μ of f there is a symmetric function $\bar{q}_\mu$ (depending on q) such that for certain special elements $T_{\gamma_\mu} \in H_f$

$$\bar{q}_\mu = \sum_{\mu \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) s_\lambda, \tag{2}$$

where $\chi^\lambda_{H_f}$ denotes the irreducible character of the Hecke algebra and s_λ is the ordinary Schur function. By specializing $q = 1$ in (2) one gets the classical Frobenius formula (1).

In this paper we begin with the Frobenius formula (2) derived in [14]. Using this formula we give a direct proof of the combinatorial algorithm for computing the irreducible characters of the Hecke algebra by using the Remmel-Whitney rule for multiplying Schur functions. The Remmel-Whitney rule is a version of the Littlewood-Richardson rule which is particularly nice for our purposes.

Following the proof of the combinatorial rule for the characters of the Hecke algebra, we derive explicitly the connection between the Hecke algebra characters and Kronecker products of symmetric group representations which came into play in [14]. By understanding this connection one gets a combinatorial rule for computing Kronecker coefficients $\kappa_{\lambda\mu\nu}$ where ν is the partition $(1^{f-m}m)$, for some m. Furthermore one finds that this approach can be generalized to compute Kronecker coefficients for other cases. We work this out explicitly to give a combinatorial algorithm for computing $\kappa_{\lambda\mu\nu}$ in the case where $\nu = (f-m, m)$. In the most general form, this approach gives a new proof of the Littlewood-Garsia-Remmel formula [6] which is particularly painless.

In the final section of this paper we give two further applications of the Frobenius formula:

(1) We compute explicitly the character of the regular representation R of the Hecke algebra. The formula is

$$\chi^R(T_{\gamma_\mu}) = \frac{f!(q-1)^{f-k}}{\mu_1!\mu_2!\cdots\mu_k!},$$

(2) We compute explicitly the generic degrees of $\mathrm{GL}_n(\mathbb{F}_q)$.

A combinatorial proof of the rule for computing Hecke algebra characters has also been given by van der Jeugt [20] by using the version of the Littlewood-Richardson rule given in [13]. One can also give a combinatorial proof of the rule for computing Hecke algebra characters which avoids the use of the Littlewood-Richardson (see the remark in Section 2). Some of the methods used in this paper have been used in [18] to obtain further results on Kronecker product decompositions. The formula for the trace of the regular representation of the Hecke algebra is, to our knowledge, new. The generic degrees of $\mathrm{GL}_n(\mathbb{F}_q)$ are well known, only the approach is new. For further background on generic degrees see [3] and [8].

A portion of the work on this paper was done during a visit of the first author to the Mathematisches Institut of Universitat Basel in Switzerland. The first author would like to thank Professor H. Kraft and the Universitat Basel for their generous hospitality. We would also like to thank Professor A. Garsia for many stimulating discussions.

2. The Frobenius formulas and Murnaghan-Nakayama rules

We will use the notations in [13] for partitions and symmetric functions except that we will use the French notation for partitions. In particular, if λ is a partition, $\lambda = (0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell)$, then the length of λ, $\ell(\lambda)$, is the number of parts λ_i, the weight of λ, $|\lambda|$, is the sum of the parts, and we write $\lambda \vdash f$ to denote that λ is a partition of f, i.e., $|\lambda| = f$. We let F_λ denote the Ferrer's diagrams of λ where F_λ is the set of left justified rows of cells or boxes with λ_i cells in the ith row from the top for $i = 1, \ldots, \ell$. λ' denotes the conjugate partition to λ. If $\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_\ell)$ and $\mu = (0 \leq \mu_1 \leq \cdots \leq \mu_k)$, then we write $\lambda \subseteq \mu$ if $\ell \leq k$ and $\lambda_{\ell-i} \leq \mu_{k-i}$ for $i = 0, \ldots, \ell - 1$. If $\lambda \subseteq \mu$, then $\mu - \lambda$ is the set of boxes in the Ferrers diagram of μ that are not contained in the Ferrers diagram of λ. $|\mu - \lambda|$ is the number of boxes contained in $\mu - \lambda$.

Let S_f denote the symmetric group of permutations of f symbols and denote the group algebra of the symmetric group over $\mathbb{C}$ by $\mathbb{C}S_f$. $\mathbb{C}S_f$ can be defined as the algebra over $\mathbb{C}$ generated by $s_1, s_2, \ldots, s_{f-1}$, with relations

$$s_i s_j = s_j s_i, \quad \text{if } |i - j| > 1, \tag{3}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{4}$$

$$s_i^2 = 1. \tag{5}$$

Here s_i may be thought of as an element of S_f by identifying s_i with the transposition $(i, i + 1)$. The irreducible representations of S_f are indexed by partitions λ of f and we shall denote the corresponding irreducible characters by $\chi^\lambda_{S_f}$.

The Hecke algebra $H_f(q)$ is the algebra over $\mathbb{C}(q)$, the field of rational functions in a variable q, generated by $g_1, g_2, \ldots, g_{f-1}$ with relations

$$g_i g_j = g_j g_i, \quad \text{if } |i - j| > 1 \tag{6}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{7}$$

$$g_i^2 = (q - 1) g_i + q. \tag{8}$$

The irreducible representations of $H_f(q)$ are also indexed by partitions of f and we shall denote the corresponding characters by $\chi^\lambda_{H_f}$.

Let $\sigma \in S_f$. A reduced decomposition of σ is an expression $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ with k minimal. k is called the length of σ and denoted by $\ell(\sigma)$. To each permutation $\sigma \in S_f$ we associate an element $T_\sigma = g_{i_1} g_{i_2} \cdots g_{i_f} \in H_f(q)$, where $\sigma = s_{i_1} s_{i_2} \cdots s_{i_f}$ is a reduced decomposition of σ. It is well known that each element T_σ is independent of the reduced decomposition of σ and that the set of elements $\{T_\sigma\}_{\sigma \in S_f}$ form a basis of $H_f(q)$.

Let γ_r be the permutation in S_r given by $\gamma_r = s_{r-1} s_{r-2} \cdots s_1$. Thus in cycle notation, $\gamma_r = (r, r - 1, \ldots, 1)$. For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of f one has a natural imbedding of $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$ into S_f under which we can view the element

$$\gamma_\mu = \gamma_{\mu_1} \times \gamma_{\mu_2} \times \cdots \times \gamma_{\mu_k} \tag{9}$$

as an element of S_f. Thus in cycle notation

$$\gamma_\mu = (\mu_1, \ldots, 1)(\mu_2 + \mu_1, \ldots, 1 + \mu_1) \cdots \left(f, \ldots, 1 + \sum_{i<k} \mu_i \right).$$

T_{γ_μ} is the corresponding element of $H_f(q)$. Since any permutation $\sigma \in S_f$ is conjugate to a γ_μ for some partition μ, we have that for any character χ_{S_f} of S_f, χ_{S_f} is completely determined by the values $\chi_{S_f}(\gamma_\mu)$. We shall sometimes write $\chi_{S_f}(\mu)$ for $\chi_{S_f}(\gamma_\mu)$. The following theorem is proved in [14].

Theorem 1 *Any character χ_{H_f} of $H_f(q)$ is completely determined by the values $\chi_{H_f}(T_{\gamma_\mu})$.*

Let $x_1, x_2, \ldots, x_n$, $(n > f)$, be independent commuting variables. A column strict tableau of shape λ is a filling of the Ferrers diagram of λ with numbers from the set $\{1, 2, \ldots, n\}$ such that the numbers are weakly increasing in the rows from left to right and strictly increasing in the columns from bottom to top. Similarly, a row strict tableau of shape λ is a filling of F_λ with numbers from $\{1, \ldots, n\}$ such that the numbers are weakly increasing in columns from bottom to top and strictly increasing in rows from left to right. The weight of a column strict tableau T is given by the product

$$x^T = \prod_{i=1}^{n} x_i^{t_i}$$

where t_i is the number of i's appearing in the tableau T. The Schur function s_λ is defined by

$$s_\lambda = \sum_T x^T,$$

where the sum is over all column strict tableaux of shape λ, and x^T denotes the weight of the tableau T.

For each integer $r > 0$ define the power symmetric function, p_r, by

$$p_r = p_r(x_1, x_2, \ldots, x_n) = x_1^r + x_2^r + \cdots + x_n^r,$$

and for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ define

$$p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k}.$$

The *Frobenius formula* for S_f is

$$p_\mu = \sum_{\lambda \vdash f} \chi_{S_f}^{\lambda}(\mu) s_\lambda. \tag{10}$$

Define, for each integer $r > 0$,

$$\bar{q}_r = \bar{q}_r(x_1, \ldots, x_n : q) = \sum_{\vec{i}=(i_1,\ldots,i_r)} q^{N_=(\vec{i})}(q-1)^{N_<(\vec{i})} x_{i_1} x_{i_2} \cdots x_{i_r} \tag{11}$$

where the sum runs over all weakly increasing sequences $\vec{i} = 1 \leq i_1 \leq \cdots \leq i_r \leq n$ and $N_=(\vec{i}) = |\{j < r : i_j = i_{j+1}\}|$ and $N_<(\vec{i}) = |\{j < r : i_j < i_{j+1}\}|$. For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, let

$$\bar{q}_\mu = \bar{q}_{\mu_1} \bar{q}_{\mu_2} \cdots \bar{q}_{\mu_k}. \tag{12}$$

Note that for $q = 1$, $\bar{q}_r = p_r$ and $\bar{q}_\mu = p_\mu$. The *Frobenius formula* for the irreducible characters of $H_f(q)$ is

$$\bar{q}_\mu = \sum_{\lambda \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) s_\lambda \tag{13}$$

(see [14]).

The following algorithm for computing the values $\chi^\lambda_{S_f}(\mu)$, called the Murnaghan-Nakayama rule, can be derived from the Frobenius formula ([13] I Section 7 Ex. 9, [12]).

$$\chi^\lambda_{S_f}(\mu) = \sum_T wt(T), \tag{14}$$

where the sum is over all μ-rim hook tableaux T of shape λ. Here a rim hook of λ is a sequence of cells along the north-east boundary of F_λ so that any two consecutive cells in h share an edge and the removal from F_λ of the cells in h leaves one with a Ferrers diagram of another partition. See figure 1 for a picture of all rim hooks of length 3 for $\lambda = (2, 2, 2, 3, 4)$.

If $\mu = (0 < \mu_1 \leq \cdots \leq \mu_k)$, a μ-rim hook tableau T of shape λ is a sequence of partitions

$$T = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda)$$

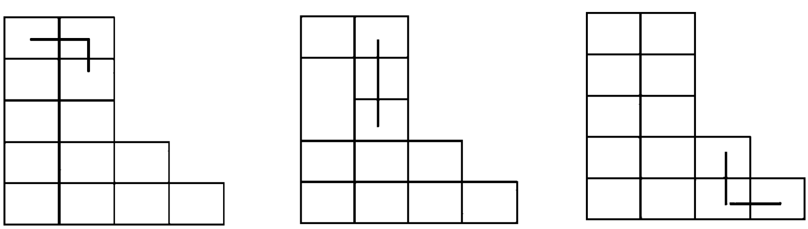


Figure 1. Rim hooks.

such that for each $1 \le i \le k$, $\lambda^{(i)} - \lambda^{(i-1)}$ is a rim hook for $\lambda^{(i)}$ of length μ_i. The weight of a rim hook h is

$$wt(h) = (-1)^{r(h)-1} \tag{15}$$

where $r(h)$ is the number of rows in h and the weight of T is

$$wt(T) = \prod_{i=1}^{k} wt(\lambda^{(i)} - \lambda^{(i-1)}). \tag{16}$$

There is also a q-extension of the Murnagham-Nakayama rule giving a combinatorial rule for computing the values $\chi^{\lambda}_{H_f}(T_{\gamma_\mu})$ derived in [14].

$$\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right) = \sum_{T} wt_q(T) \tag{17}$$

where the sum is over all μ-broken rim hook tableaux T of shape λ. Here a broken rim hook b of λ is a sequence of rim hooks $(h_1, \ldots, h_d)$ of λ (starting from the bottom) such that for all $1 \le i < d$, h_i and h_{i+1} do not have any cells in common nor are there cells $c_1 \in h_i$ and $c_2 \in h_{i+1}$ such that c_1 and c_2 meet along an edge. We let $n(b)$ denote the number of rim hooks in b. See figure 2 for a picture of a broken rim hook b of $\lambda = (2, 2, 3, 3, 7)$ where $n(b) = 3$.

Note that any rim hook of λ is a broken rim hook b of λ where $n(b) = 1$. Then if $\mu = (\mu_1, \ldots, \mu_k)$, a μ-broken rim hook tableau T is a sequence of partitions

$$T = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)})$$

such that for each $1 \le i \le k$, $\lambda^{(i)} - \lambda^{(i-1)}$ is a broken rim hook of $\lambda^{(i)}$ of total length μ_i. In this case the weight of a rim hook h is

$$wt_q(h) = (-1)^{r(h)-1} q^{c(h)-1} \tag{18}$$

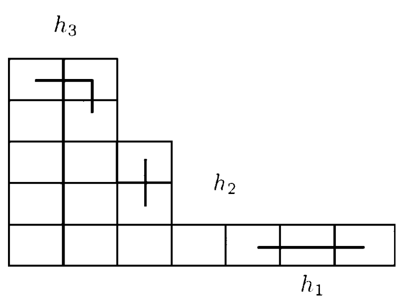


Figure 2. A broken rim hook.

where $c(h)$ is the number of columns of h. The weight of a broken rim hook b is

$$wt_q(b) = (q-1)^{n(b)-1} \prod_{\text{rim hooks } h \in b} wt_q(h). \tag{19}$$

For example the weight of the broken rim hook tableau pictured in figure 2 is $(q-1)^2 (-q)(-1)(q^2) = (q-1)^2 q^3$. Finally the weight of T is

$$wt_q(T) = \prod_{i=1}^{k} wt_q(\lambda^{(i)} - \lambda^{(i-1)}). \tag{20}$$

We note that a more succinct way to describe broken rim hooks and rim hooks is the following. A skew shape $\lambda - \mu$ is a broken rim hook if $\lambda - \mu$ contains no 2×2 block of boxes and $\lambda - \mu$ is a rim hook if $\lambda - \mu$ contains no 2×2 block of boxes and it is connected in the sense that any two consecutive cells of $\lambda - \mu$ share an edge.

Finally, we note that if we set $q = 1$ in (19), then the weight of a broken rim hook b is nonzero only if b is a rim hook. Thus when $q = 1$, the righthand side of (17) reduces to the righthand side of (14) and hence the q-extension of the Murnagham-Nakayama rule reduces to the Murnagham-Nakayama rule.

3. The combinatorial rule for the irreducible characters of $H_f(q)$

In this section we will give a proof of the combinatorial rule described in (17) for computing the irreducible characters of the Hecke algebra by using the Frobenius formula and the Remmel-Whitney rule for multiplying Schur functions.

The Remmel-Whitney algorithm [19] for expanding the product the Schur functions s_λ and s_μ as a sum of Schur functions is the following. Place the shapes μ and ν end to end so that the lower right corner of ν is touching the upper left corner of μ. Fill the resulting diagram, which we shall call D, from right to left and bottom to top with the numbers 1 to $|\mu| + |\nu|$. For example, in the case where $\mu = (2, 4, 4)$ and $\nu = (1, 3, 3)$, D is pictured in figure 3.

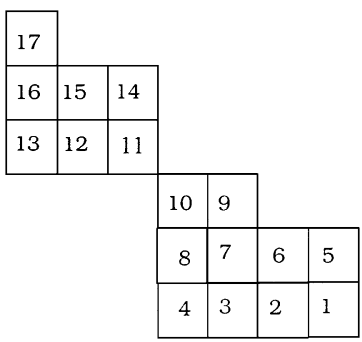


Figure 3. Filling for $\mu = (2, 4, 4)$ and $\lambda = (1, 3, 3)$.

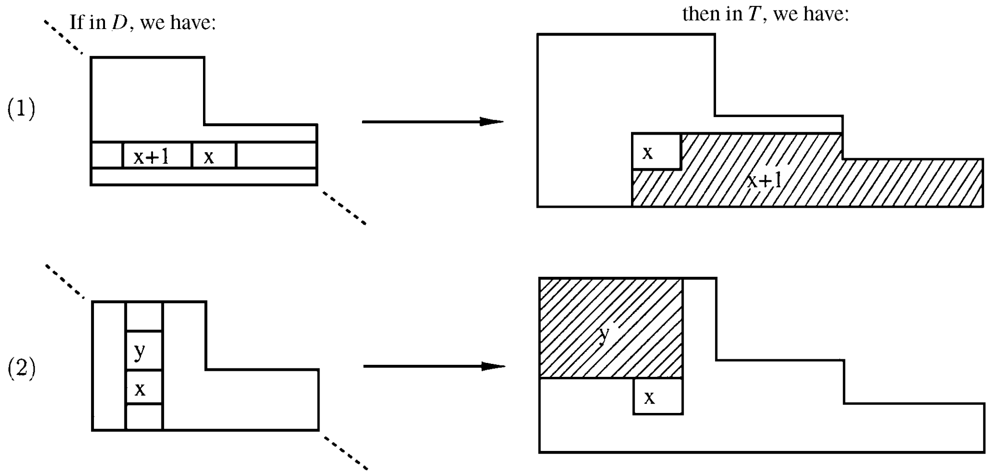


Figure 4. R-W conditions.

This given, one constructs all tableaux T (fillings of Ferrers diagrams with the numbers 1 to $|\mu| + |\nu|$) that satisfy the following rules.

(1) x is weakly below and strictly to the right of y in T if y is immediately to the left of x in D.
(2) y is strictly above and weakly to the left of x in T if y is immediately above x in D.

Any standard tableaux T satisfying (1) and (2) is called D-compatible and the number of D-compatible tableaux T of shape λ is the coefficient of s_λ in $s_\mu s_\nu$ which we denote by $c^\lambda_{\mu,\nu}$.

The two conditions (1) and (2) may be conveniently pictured as in figure 4.
One further remark about the Remmel-Whitney algorithm is to note that the rules (1) and (2) will completely force the placement of the numbers in the lower Ferrers diagram of D. That is, if $\mu = (\mu_1 \leq \cdots \leq \mu_k)$, then in all D-compatible tableaux, $1, \ldots, \mu_k$ lie in the first row, $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ lie in second row, etc. Hence the numbers $1, \ldots, |\mu|$ will fill a diagram of shape μ in all D-compatible tableaux.

The first step in proving (17) is to give the expansion of the function $\bar{q}_r$ defined by (11) as a sum of Schur functions.

Theorem 2 *Let s_λ denote the Schur function and let $\bar{q}_r$ be as defined in* (11). *Then*

$$\bar{q}_r = \sum_{m=1}^{r} (-1)^{r-m} q^{m-1} s_{(1^{r-m},m)}. \tag{21}$$

Proof: Define a marked increasing sequence of length r to be a sequence $I = (i_1, i_2, \ldots, i_r)$, $1 \leq i_1 \leq i_2 \cdots \leq i_r \leq n$ such that each i_j is either marked or unmarked according to the following rules.

(1) i_1 is unmarked.
(2) If $i_j = i_{j+1}$ then i_{j+1} is unmarked.
(3) If $i_j < i_{j+1}$ then i_{j+1} may be either marked or unmarked. □

Given a marked increasing sequence $I = (i_1, i_2, \ldots, i_r)$, let $U(I) = \#$ of unmarked elements of I and $M(I) = \#$ of marked elements of I. Then we define the weight of I to be

$$wt(I) = q^{U(I)-1}(-1)^{M(I)}x_{i_1}x_{i_2}\cdots x_{i_r}.$$

It is easy to see from (11) that

$$\bar{q}_r = \sum_I wt(I),$$

where the sum is over all marked increasing sequences of length r.

To each marked increasing sequence I with m unmarked elements, we associate the column strict tableau T of shape $(1^{r-m}, m)$ containing

(1) i_1 in the corner square.
(2) the unmarked elements of I in the horizontal portion of $(1^{r-m}, m)$, and
(3) the marked elements of I in the vertical portion of $(1^{r-m}, m)$.

See figure 5 for an example of this correspondence where we have underlined the marked elements.

This gives a bijection between marked increasing sequences of length r and column strict tableaux of shapes $(1^{r-m}m)$. We have

$$\begin{aligned}\bar{q}_r &= \sum_I wt(I) = \sum_{m=1}^{r} (-1)^{r-m}q^{m-1}\sum_T x^T \\ &= \sum_{m=1}^{r} (-1)^{r-m}q^{m-1}s_{(1^{r-m},m)}\end{aligned}$$

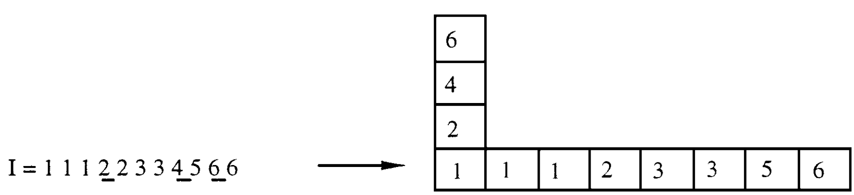


Figure 5. The column strict tableau of a marked sequence.

where the inner sum in the second line is over all column strict tableau T of shape $(1^{r-m}, m)$. □

Theorem 3 *The irreducible characters of the Hecke algebra are given by*

$$\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right) = \sum_{T} wt_q(T),$$

where T_{γ_μ} is the element of $H_f(q)$ described in Section 1 *and the sum is over all μ-broken rim hook tableaux T of shape λ, and $wt_q(T)$ is as defined in* (20).

Proof: Let ν be a partition. We use the Remmel-Whitney rule for multiplying Schur functions and the formula (21)

$$\bar{q}_r = \sum_{m=1}^{r} (-1)^{r-m} q^{m-1} s_{(1^{r-m},m)},$$

to compute the product $\bar{q}_r s_\nu$. In order to compute the product $(-1)^{r-m} q^{m-1} s_{(1^{r-m},m)} s_\nu$ easily, modify the Remmel-Whitney rule slightly so that the boxes in the vertical part of $(1^{r-m}, m)$ have a weight of -1 and the boxes in the horizontal part have weight q. Let the corner box of $(1^{r-m}, m)$ have weight 1. □

Now compute the coefficient of s_λ in $s_{(1^{r-m},m)} s_\nu$. Note that by our remarks following figure 4, when one applies the Remmel-Whitney rule, every D-compatible T must contain the shape of ν. Moreover, it is easy to see that the R-W conditions (1) and (2) corresponding to the elements of D in the hook $(1^{r-m}, m)$ force that $\lambda - \nu$ does not contain any 2×2 block. Thus the coefficient of s_λ in $s_{(1^{r-m},m)} s_\nu$ is zero unless $\lambda \supseteq \nu$ and $\lambda - \nu$ is a broken rim hook. Moreover if T is a D-compatible tableau of shape λ and θ denotes the elements of T which lie in the shape $\lambda - \nu$, then

(i) any box in θ which has a box to its right must be filled with an element in the horizontal part of $(1^{r-m}, m)$ in D,
(ii) any box in θ which has a box in θ under it must be filled with an element which lies in the vertical part of $(1^{r-m}, m)$ in D,
(iii) the lowest and rightmost box in θ must be filled with the element in the corner box of $(1^{r-m}, m)$ in D,
(iv) any box of θ which has neither a box of θ below it or to its right could be filled with either a box from the horizontal or the vertical part of $(1^{r-m}, m)$ in D depending on the value of m and the placement of the other elements.

In fact, it is easy to see that if we place 1 c (for corner), $m-1$ h's (for horizontal), and $r-m$ v's (for vertical) in the diagram of θ following rules (i)–(iv) above, then we can easily reconstruct θ by filling the box with a c with the element in the corner element of $(1^{r-m}, m)$ in D, filling the boxes with h's from right to left with the elements in the horizontal part of

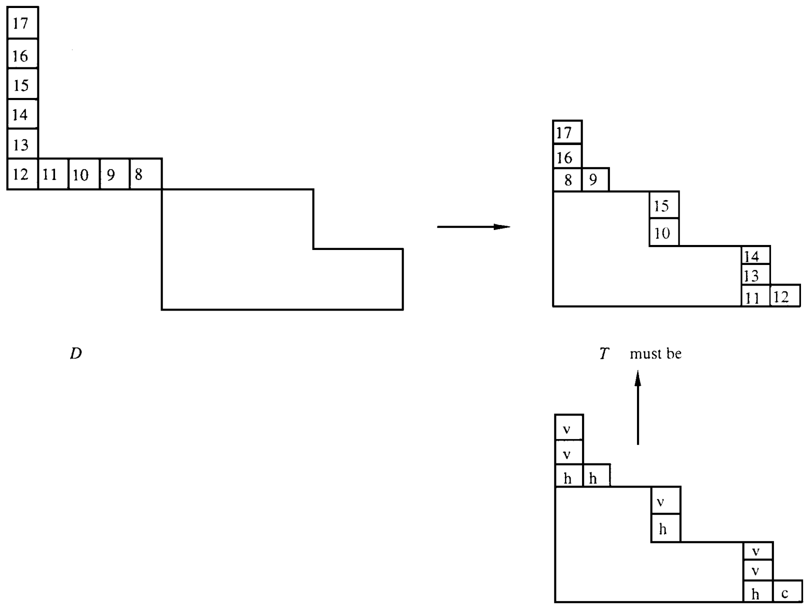


Figure 6. Correspondence for $(-1)^{r-m}\, q^{m-1}\, s_{(1^{r-m},m)}\, s_{\nu}$.

$(1^{m-n}, n)$ in D, and filling in the boxes with v's from bottom to top with the elements in vertical part of $(1^{m-r}, r)$ in D. See figure 6 for an example.

Now suppose $\lambda \supseteq \nu$ and $\lambda - \nu$ is a broken rim hook. It then follows that if we compute the coefficient c_λ of s_λ in $\bar{q}_r s_\nu = \sum_{m=1}^{r} (-1)^{r-m} q^{m-1} s_{(1^{r-m},m)} s_\nu$, then c_λ equals the number of all fillings of $\lambda - \nu$ with h's, v's, and 1 c such that

(I) any box of $\lambda - \nu$ with a box to its right must be filled with an h and hence have weight q,
(II) any box of $\lambda - \nu$ with a box of $\lambda - \nu$ below must be filled with a v and hence have weight -1,
(III) the lowest and rightmost box must be filled with c and hence have weight 1, and
(IV) any box of $\lambda - \nu$ with neither a box of $\lambda - \nu$ below it or to its right can be filled with either an h or a v and hence contributes a factor of $q - 1$ to c_λ.

Note that the boxes of $\lambda - \nu$ which satisfy condition IV above are precisely the lowest and rightmost cell in a rim hook h_i which lies strictly above the lowest rim hook h_1 of $\lambda - \nu$. It thus follows that if $\lambda - \nu = (h_1, \ldots, h_k)$ where $h_1, \ldots, h_k$ are the consecutive rim hooks of $\lambda - \nu$ starting from the bottom, then

$$\begin{aligned} c_\lambda &= (-1)^{r(h_1)-1}(q)^{c(h_1)-1} \prod_{j=2}^{k} (q-1)(-1)^{r(h_j)-1} q^{c(h_j)-1} \\ &= (q-1)^{n(\lambda-\nu)-1} \prod_{\text{rim hook } h \in \lambda - \nu} (-1)^{r(h)-1}(q)^{c(h)-1} = \mathrm{wt}_q(\lambda - \nu). \end{aligned}$$

Thus we have proved that

$$\bar{q}_r s_\nu = \sum_\lambda wt_q(\lambda - \nu) s_\lambda, \tag{22}$$

where the sum is over all partitions λ such that $\lambda - \nu$ is a broken rim hook of length r and the weight $\mathrm{wt}_q(\lambda - \nu)$ of the broken hook $\lambda - \nu$ is as in (19).

We know that

$$\bar{q}_\mu = \sum_{\lambda \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) s_\lambda, \tag{23}$$

and that

$$\bar{q}_\mu = \bar{q}_{\mu_1} \bar{q}_{\mu_2} \cdots \bar{q}_{\mu_k}.$$

The theorem follows by induction on the length of μ. □

Remark The proof of (17) given in this section is probably the most straightforward combinatorial proof if we allow the use of the Littlewood-Richardson rule. However one can avoid the use of the Littlewood-Richardson rule and use only Pieri's rules for expanding the products $s_r s_\lambda$ and $s_{(1^r)} s_\lambda$ as a sum of Schur functions by using the identity

$$s_{(1^{r-m}, m)} = \sum_{k=m}^{r} (-1)^{r-m} h_k e_{r-k} \tag{24}$$

which implies that

$$\bar{q}_r = \frac{1}{q-1} \sum h_m e_{r-m} (-1)^{r-m} q^m. \tag{25}$$

using (25) to express the product $\bar{q}_{\mu_1} \bar{q}_{\mu_2} \cdots \bar{q}_{m_k}$, one can easily derive formula (6.4) of [14] and then follow the proof of [14] to derive Theorem 3. Moreover one can derive Theorem 3 without any use of Pieri's rules or the Littlewood-Richardson rule by λ-ring manipulations, see [15].

4. λ-ring notation for symmetric functions

In this section we introduce the λ-ring notation for symmetric functions. This notation is the primary tool for deriving the connection between the Hecke algebra characters and Kronecker products of symmetric group representations. See [10] and [11] for more details on λ-rings.

An alphabet is a sum of commuting variables, so that, for example, $X = x_1 + x_2 + \cdots + x_n$ is the set of commuting variables $x_1, x_2, \ldots, x_n$. In this notation, if $X = x_1 + x_2 + \cdots + x_n$ and $Y = y_1 + y_2 + \cdots + y_n$ then XY represents the alphabet of variables $\{x_i y_j\}_{1 \le i, j \le n}$.

For each integer $r > 0$ the power symmetric function is given by

$$\begin{aligned} p_r(0) &= 0, \\ p_r(x) &= x^r, \\ p_r(X+Y) &= p_r(X) + p_r(Y), \\ p_r(XY) &= p_r(X)p_r(Y), \end{aligned}$$

where x is any single variable and X and Y are any two alphabets. For each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ define

$$p_\mu(X) = p_{\mu_1}(X)p_{\mu_2}(X)\cdots p_{\mu_k}(X).$$

Note that the above relations imply

$$\begin{aligned} p_r(-X) &= -p_r(X), \\ p_\mu(XY) &= p_\mu(X)p_\mu(Y), \end{aligned}$$

where r is a positive integer, μ is a partition and X and Y are arbitrary alphabets.

If ρ is a partition and m_i is the number parts of ρ equal to i, then we let $z_\rho = 1^{m_1}2^{m_2}\cdots m_1!m_2!\cdots$ and define the Schur function by

$$s_\lambda(X) = \sum_{\rho \vdash |\lambda|} \frac{\chi^\lambda_{S_f}(\rho)}{z_\rho}\, p_\rho(X) \tag{26}$$

Note that (26) is a generalized Frobenius formula. Define the skew Schur function $s_{\lambda/\mu}(X)$ by

$$s_{\lambda/\mu}(X) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(X). \tag{27}$$

where $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients computed by the Remmel-Whitney rule in Section 2. Then we have the following properties of Schur functions, see [13].

$$s_\mu(X)s_\mu(X) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(X) \tag{28}$$

$$s_\lambda(X+Y) = \sum_{\mu\subseteq\lambda} s_\mu(X)s_{\lambda/\mu}(Y) \quad \text{(sum rule)} \tag{29}$$

$$s_\lambda(-X) = (-1)^{|\lambda|}s_{\lambda'}(X) \quad \text{(duality)} \tag{30}$$

$$s_\lambda(XY) = \sum_{\mu,\nu} \kappa_{\lambda\mu\nu} s_\mu(X)s_\nu(Y) \quad \text{(product rule)} \tag{31}$$

In (31), $\kappa_{\lambda\mu\nu}$ is the Kronecker coefficient which is equal to the multiplicity of the irreducible representation A^λ of the symmetric group in the Kronecker product, $A^\mu \times A^\nu$, of the irreducible representations A^μ and A^ν and is defined by

$$\kappa_{\lambda\mu\nu} = \sum_{\rho \vdash f} \frac{\chi^\lambda_{S_f}(\rho)\chi^\mu_{S_f}(\rho)\chi^\nu_{S_f}(\rho)}{z_\rho}$$

Define the homogeneous symmetric function by

$$h_r(X) = s_{(r)}(X),$$

for integers $r > 0$, and

$$h_\mu(X) = h_{\mu_1}(X)h_{\mu_2}(X)\cdots h_{\mu_k}(X),$$

for partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. For each pair of partitions λ and μ define numbers $K^{-1}_{\mu\lambda}$ by

$$s_\lambda(X) = \sum_\mu h_\mu(X)K^{-1}_{\mu\lambda}. \tag{32}$$

The numbers $K^{-1}_{\mu\lambda}$ have the following combinatorial description (see [4]):

Given partitions $\mu \subset \lambda$, we say that $\lambda - \mu$ is a special rim hook if $\lambda - \mu$ is a rim hook and $\lambda - \mu$ contains a box from the first column of λ. A special rim hook tableau T of shape λ is a sequence of partitions

$$T = (\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda)$$

such that for each $1 \le i \le k$, $\lambda^{(i)} - \lambda^{(i-1)}$ is a special rim hook of $\lambda^{(i)}$. The type of the special rim hook tableau T is the partition determined by the integers $|\lambda^{(i)} - \lambda^{(i-1)}|$. The weight of a special rim hook $h_i = \lambda^{(i)} - \lambda^{(i-1)}$ is defined to be $\mathrm{wt}(h_i) = (-1)^{r(h_i)-1}$ as in (15) and the weight of T is defined to be

$$wt(T) = \prod_{i=1}^{k} wt(\lambda^{(i)} - \lambda^{(i-1)}). \tag{33}$$

Then

$$K^{-1}_{\mu\lambda} = \sum_T wt(T), \tag{34}$$

where the sum is over all special rim hook tableaux T of shape λ and type μ.

By (31) and the fact that $\kappa_{\lambda\mu(r)} = \delta_{\lambda\mu}$,

$$h_r(XY) = \sum_{\mu \vdash r} s_\mu(X)s_\mu(Y), \tag{35}$$

and from (29)

$$h_r(X+Y) = \sum_{m=0}^{r} h_m(X)h_{r-m}(Y).$$

5. Kronecker products

We now have the machinery to develop the connection between the characters of the Hecke algebra and Kronecker product decompositions. We recall two lemmas from [14]. Our first lemma easily follows from the sum formula (29).

Lemma 4 *Let t be a variable and λ a partition of f. Then, in λ-ring notation,*

$$s_\lambda(1-t) = \begin{cases} (1-t)(-t)^{f-m}, & \text{if } \lambda = (1^{f-m}, m) \quad \text{for some} \quad m \geq 1; \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Lemma 5 *In λ-ring notation*

$$\bar{q}_\mu(X;q) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} h_\mu(X(1-q^{-1})), \tag{37}$$

where h_μ denotes the homogeneous symmetric function.

Proof: Combining (35) and (36) we have

$$\begin{aligned} h_r(X(1-q^{-1})) &= \sum_{\mu \vdash r} s_\mu(X) s_\mu(1-q^{-1}) \\ &= \sum_{m=1}^{r} s_{(1^{r-m},m)}(X)(-q^{-1})^{r-m}(1-q^{-1}). \end{aligned}$$

If we multiply both sides by q^r and divide by $q-1$, then by (21)

$$\frac{q^r}{q-1} h_r(X(1-q^{-1})) = \sum_{m=1}^{r} (-1)^{r-m} q^{m-1} s_{(1^{r-m},m)}(X) = \bar{q}_r(X;q).$$

The lemma then follows from the definitions of h_μ and $\bar{q}_\mu$. □

We note that in light of Lemma 5, we can derive an alternative way to compute $\chi^\lambda_{H_f}(T_{\gamma_\mu})$. Let

$$q_\mu(X;q) = h_\mu(X(1-q)). \tag{38}$$

$q_\mu(X;q)$ is the Hall-Littlewood q-function of [13]. Using (29) and (30), we have

$$\begin{aligned} q_r(X;q) &= h_r(X(1-q)) \\ &= s_{(r)}(X-qX) \\ &= \sum_{p=0}^{r} s_{(p)}(X)(-q)^{r-p}s_{(1^{r-p})}(X). \end{aligned}$$

Thus if $\mu = (0 \le \mu_1 \le \cdots \le \mu_k)$, then

$$\begin{aligned} q_\mu(X;q) = \sum_{p_1=0}^{\mu_1}\cdots\sum_{p_k=0}^{\mu_k} &(-q)^{|\mu|-\sum p_i} s_{(p_1)}(X)\cdots s_{(p_k)}(X)s_{(1^{\mu_1-p_1})}(X) \\ &\times \cdots s_{(1^{\mu_k-p_k})}(X). \end{aligned} \tag{39}$$

Now let

$$\bar{K}_{\lambda,\mu}(q) = \sum_{r=0}^{|\mu|} (-q)^r \bar{K}^r_{\lambda,\mu} \tag{40}$$

where $\bar{K}^r_{\lambda,\mu}$ is the number of pairs column strict tableaux (T, S) such that T is of shape ν where $\nu \subseteq \lambda$ and content $1^{a_1}\cdots k^{a_k}$, S is of shape $\lambda' - \nu'$ and content $1^{b_1}\cdots k^{b_k}$, $|\lambda - \nu| = r$, and $a_j + b_j = \mu_j$ for $j = 1,\dots,k$. Here we say a column strict tableau P has content $1^{c_1}\cdots n^{c_n}$ if there are exactly c_i occurrences of i in P for $i = 1,\dots,n$. Another way to view the pairs (T, S) is to replace S by S' where S' results from S by transposing S about the main diagonal and then replacing each number i in S by i'. Then $P = T + S'$ is a filling of F_λ with regular numbers plus primed numbers such that the regular numbers form a column strict tableau of shape $\nu \subseteq \lambda$, the primed numbers form a row strict tableau of shape $\lambda - \nu$, and for any i, the total number of occurrences of i and i' in P is μ_i. Such tableaux P are called (k, k)-semistandard tableau of type μ by Berele and Regev [1]. For example, if $\mu = (2, 2)$ and $\lambda = (1, 3)$, figure 7 pictures the 12 $(2, 2)$ semistandard tableau of shape λ and type μ along with their associated power of q and shows that $\bar{K}_{\lambda,\mu} = -2q + 5q^2 - 4q^3 + q^4$.

We note that clearly $\bar{K}_{\lambda,\mu}(0) = K_{\lambda,\mu}$ where $K_{\lambda,\mu}$ is the Kostka number which is equal to the number of column strict tableaux of shape λ and content $1^{\mu_1}\cdots k^{\mu_k}$.

This given, one can apply Pieri's rules or the Remmel-Whitney rule to expand the right-hand side of (39) and derive the following.

Theorem 6

$$q_\mu(X,q) = \sum_{\mu} \bar{K}_{\lambda,\mu}(q)s_\lambda(X). \tag{41}$$

Combining Lemma 5, Theorem 6, and the Frobenius formula, we have the following.

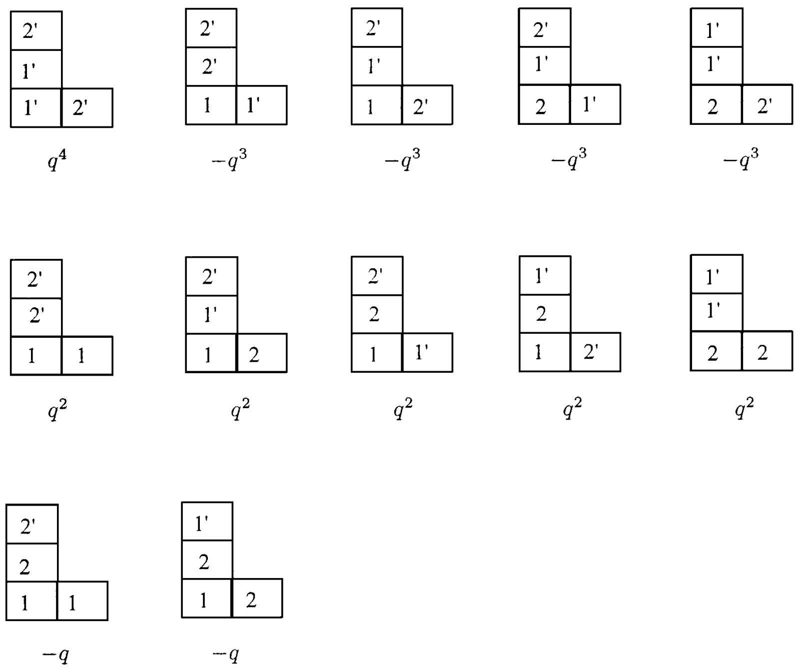


Figure 7. (1, 3) semistandard tableaux of type (2, 2).

Theorem 7

$$\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\bar{K}_{\lambda,\mu}(q^{-1}) = \frac{1}{(q-1)^{\ell(\mu)}}\sum_{r=0}^{|\mu|}(-1)^r q^{|\mu|-r}\bar{K}^r_{\lambda,\mu}.$$

Proof: By (2), (37), (38), and (41),

$$\sum_{\lambda\vdash f}\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right)s_\lambda(X) = \bar{q}_\mu(X,q) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}q_\mu(X,q^{-1})$$
$$= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\sum_{\lambda}\bar{K}_{\lambda,\mu}(q^{-1})s_\lambda(X).$$

The theorem then follows by taking the coefficient of $s_\lambda(X)$ and using the definition of $\bar{K}_{\lambda,\mu}$. □

Theorem 8 *Let $\kappa_{\lambda\mu\nu}$ denote the Kronecker coefficient, $K^{-1}_{\mu\nu}$ the inverse Kostka number given in* (33), *and $\chi^{\lambda}_{H_f}$ the irreducible character of the Hecke algebra. Then*

$$\sum_{m=1}^{f} \kappa_{\lambda\nu(1^{f-m}m)}(-1)^{f-m}q^{m-1} = \sum_{\mu\vdash f}(q-1)^{\ell(\mu)-1}\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right)K^{-1}_{\mu\nu}. \tag{42}$$

Proof: Using Lemma 5, (33), and the Frobenius formula (14) we have

$$\begin{aligned} s_\nu(X(1-q^{-1})) &= \sum_{\mu\vdash f} h_\mu(X(1-q^{-1}))K^{-1}_{\mu\nu} \\ &= \sum_{\mu\vdash f} \frac{(q-1)^{\ell(\mu)}}{q^{|\mu|}}\bar{q}_\mu(X;q)K^{-1}_{\mu\nu} \\ &= \sum_{\mu\vdash f}\sum_{\lambda\vdash f} \frac{(q-1)^{\ell(\mu)}}{q^{|\mu|}}\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right)s_\lambda(X)K^{-1}_{\mu\nu}. \end{aligned}$$

On the other hand by the product rule and Lemma (4),

$$\begin{aligned} s_\nu(X(1-q^{-1})) &= \sum_{\lambda,\gamma} \kappa_{\nu\lambda\gamma}s_\lambda(X)s_\gamma(1-q^{-1}) \\ &= \sum_{\lambda\vdash f}\sum_{m=1}^{f} \kappa_{\nu\lambda(1^{f-m}m)}s_\lambda(X)(-q^{-1})^{f-m}(1-q^{-1}). \end{aligned}$$

Setting these two equal and taking the coefficient of $s_\lambda(X)$ on each side, we have

$$\sum_{\mu\vdash f} \frac{(q-1)^{\ell(\mu)}}{q^{|\mu|}}\chi^{\lambda}_{H_f}\left(T_{\gamma_\mu}\right)K^{-1}_{\mu\nu} = \sum_{m=1}^{f} \kappa_{\nu\lambda(1^{f-m}m)}(-q^{-1})^{f-m}(1-q^{-1}).$$

The theorem follows by multiplying each side by $q^f = q^{|\mu|}$ and dividing each side by $q-1$. □

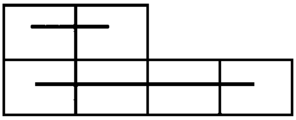
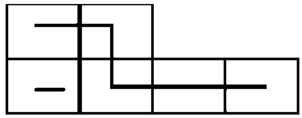
Recalling that the values $\chi^{\lambda}_{H_f}(T_{\gamma_\mu})$ and $K^{-1}_{\mu\nu}$ have combinatorial interpretations, given in Theorem 3 and (34) respectively, we get the following.

Corollary 9 *The Kronecker coefficient $\kappa_{\nu\lambda(1^{f-m}m)}$ is equal to the coefficient of q^{m-1} in*

$$\sum_{\mu\vdash f}(-1)^{f-m}(q-1)^{\ell(\mu)-1}\sum_{T,R} wt_q(T)wt(R) \tag{43}$$

where the inner sum is over all pairs (T, R) consisting of a μ-broken rim hook tableau T of shape λ and a special rim hook tableau R of shape ν and type μ. The weights $wt_q(T)$, $wt(R)$ are as in (20) *and* (33).

We note that calculating the polynomial in (42) gives an efficient way to compute the multiplicity of the irreducible components corresponding to all hook shapes in the Kronecker product of the irreducible modules corresponding to λ and μ. For example, if $\lambda = (1, 2, 3)$ and $\nu = (2, 4)$, then there are two special rim hook tableaux of shape ν.

$R_1 =$  $R_2 =$ 

$wt(R_1) = +1$ $wt(R_2) = -1$

Now R_1 is of type $(2, 4)$ and there are two broken rim hook tableaux $T = (\emptyset \subset \lambda^{(1)} \subset \lambda^{(2)} = \lambda)$ of type $(2, 4)$ where below we indicate the shape $\lambda^{(1)}$ by placing 1's in the boxes of $\lambda^{(1)}$ and 2's in the boxes of $\lambda^{(2)} - \lambda^{(1)}$.

$T_1^{(1)} =$

2		
2	2	
1	1	2

$T_2^{(1)} =$

2		
1	2	
1	2	2

$wt_q(T_1^{(1)}) = q[(q-1)(-q)]$ $wt_q(T_2^{(1)}) = (-1)[(q-1)(-q)].$

There is one broken rim hook tableau $T_1^{(2)}$ of type $(1, 5)$.

$T_1^{(2)} =$

2		
2	2	
1	2	2

$$wt_q(T_1^{(2)}) = (1)[q^2].$$

Thus for our given λ and ν,

$$\begin{aligned} \sum_{\mu \vdash f} (q-1)^{\ell(\mu)-1} \chi_{H_f}^{\lambda}\left(T_{\gamma_\mu}\right) K_{\mu\nu}^{-1} &= (q-1)[-q^2(q-1) + q(q-1)] - (q-1)q^2 \\ &= -q^4 + 2q^3 - 2q^2 + q \qquad (44) \\ &= \sum_{m=1}^{6} \kappa_{\lambda,\nu(1^{6-m},m)}(-1)^{6-m} q^{m-1}. \end{aligned}$$

We can then read off from (44) that

$$\kappa_{\lambda,\nu,(1^6)} = 0, \quad \kappa_{\lambda,\nu,(1^4,2)} = 1, \quad \kappa_{\lambda,\nu,(1^3,3)} = 2$$
$$\kappa_{\lambda,\nu,(1^2,4)} = 2, \quad \kappa_{\lambda,\nu(1^2,5)} = 1, \quad \text{and} \quad \kappa_{\lambda,\nu,(6)} = 0.$$

Theorem (8) thus explicitly gives the connection between the irreducible characters $\chi^{\lambda}_{H_f}$ of the Hecke algebra and the Kronecker coefficients $\kappa_{\nu\lambda(1^{f-m},m)}$. Corollary 9 uses this result to give a combinatorial algorithm for computing the Kronecker coefficients $\kappa_{\nu\lambda(1^{f-m}m)}$.

One finds that the same type of approach can be used to compute other Kronecker coefficients. In some sense the Frobenius formula says that $\bar{q}_\mu(X; q)$ is a generating function for the irreducible characters of the Hecke algebra. In Lemma 5 we found that $\bar{q}_\mu(X; q)$ can be described via a homogeneous symmetric function in the alphabet $X(1-q^{-1})$. The idea is to use a homogeneous symmetric function in a different alphabet to compute different Kronecker coefficients.

We shall work out the case where the alphabet is $X(1 + t)$. This example gives a combinatorial algorithm for computing the coefficients $\kappa_{\nu\lambda\gamma}$ where γ is a partition with two rows. The analogous results to (36), (22) and (43) follow in (45), (46) and (49). We shall, for the most part, omit the proofs as they are so similar to the previous case.

In λ-ring notation,

$$s_\lambda(1+t) = \begin{cases} t^m + t^{m+1} + \cdots + t^{f-m}, & \text{if } \lambda = (m, f-m) \text{ for some} \\ & \qquad 0 \le m \le \lfloor f/2 \rfloor; \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

Using the sum rule,

$$\begin{aligned} h_r(X(1+t)) &= h_r(X + tX) \\ &= \sum_{m=0}^{r} t^m h_m(X) h_{r-m}(X). \end{aligned}$$

It follows then, from Pieri's rule or the Remmel-Whitney rule, that

$$h_r(X(1+t))s_\nu(X) = \sum_{\nu\subset\mu\subset\lambda} t^{|\mu-\nu|} s_\lambda(X), \tag{46}$$

where the sum is over all sequences of partitions $\nu \subset \mu \subset \lambda$ such that $\lambda - \mu$ and $\mu - \nu$ are both horizontal strips and such that $|\lambda - \nu| = r$. Here a horizontal strip is a set of boxes such that no two boxes are in the same column. This leads us to define a μ-double strip tableau T of shape λ to be a sequence of partitions

$$T = (\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(2k)} = \lambda)$$

such that for each $1 \leq i \leq 2k$, $\lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip and such that, for each $1 \leq j \leq k$, $|\lambda^{(2j)} - \lambda^{(2_j-2)}| = \mu_j$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. Define

$$w_d^t(T) = \prod_{j=1}^{k} t^{|\lambda^{(2j-1)} - \lambda^{(2j-2)}|}. \tag{47}$$

Then, for $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$,

$$h_\mu(X(1+t)) = \sum_{\lambda \vdash f} \left[\sum_T w_d^t(T) \right] s_\lambda(X), \tag{48}$$

where the inner sum is over all μ-double strip tableaux T of shape λ.

Evaluating $s_\nu(X(1+t))$ by way of the product rule and then using (45), one has

$$\begin{aligned} s_\nu(X(1+t)) &= \sum_{\lambda,\gamma} \kappa_{\nu\lambda\gamma} s_\lambda(X) s_\gamma(1+t) \\ &= \sum_{\lambda \vdash f} \sum_{m=0}^{\lfloor f/2 \rfloor} \kappa_{\nu\lambda(m,\, f-m)} s_\lambda(X) \sum_{j=m}^{f-m} t^j \\ &= \frac{1}{1-t} \sum_\lambda \sum_{m=0}^{\lfloor f/2 \rfloor} s_\lambda(X)(t^m - t^{f+1-m}) \kappa_{\nu\lambda(m,\, f-m)}. \end{aligned}$$

From this one gets that for $m \leq \lfloor \frac{f}{2} \rfloor$,

$$s_\nu(X(1+t))|_{t^m} - s_\nu(X(1+t))|_{t^{m-1}} = \sum_\lambda \kappa_{\nu\lambda(m,\, f-m)} s_\lambda(X), \tag{49}$$

where $s_\nu(X(1+t))|_{t^m}$ denotes the coefficient of t^m in $s_\nu(X(1+t))$. On the other hand, from (48),

$$\begin{aligned} s_\nu(X(1+t)) &= \sum_\mu h_\mu(X(1+t)) K_{\mu\nu}^{-1} \\ &= \sum_\mu \sum_\lambda \sum_T w_d^t(T) s_\lambda(X) K_{\mu\nu}^{-1}, \end{aligned} \tag{50}$$

where, as before, the inner sum is over all μ-double strip tableaux T of shape λ. Taking the coefficient of $s_\lambda(X)$ in (49) and (50) gives

$$\kappa_{\nu\lambda(m,\, f-m)} = \sum_\mu \left(\sum_T w_d^t(T) \right) K_{\mu\nu}^{-1}\Big|_{t^m} - \sum_\mu \left(\sum_T w_d^t(T) \right) K_{\mu\nu}^{-1}\Big|_{t^{m-1}}$$

from which we get the following combinatorial algorithm for computing the coefficients $\kappa_{\nu\lambda(f-m,m)}$.

Theorem 10 *Let $c_m(\lambda, \nu)$ denote the coefficient of t^m in*

$$\sum_{\mu \vdash f} \sum_{(T,R)} w_d^t(T) wt(R), \tag{51}$$

where the sum is over all pairs (T, R) consisting of a μ-double strip tableau T of shape λ and a special hook tableau R of shape ν and type μ. The weights $w_d^t(T)$ and $wt(R)$ are as in (47) *and* (33) *respectively. Then*

$$\kappa_{\nu\lambda(m, f-m)} = c_m(\lambda, \nu) - c_{m-1}(\lambda, \nu).$$

In view of these examples one would like to find a general method for computing arbitrary Kronecker coefficients. The general result coming out of this approach turns out to be equivalent to a theorem of Littlewood and Garsia-Remmel, in fact this approach gives a particularly nice proof of the Littlewood-Garsia-Remmel result.

The Littlewood-Garsia-Remmel result is as follows. Define an operation $\otimes$ on symmetric functions by defining

$$s_\lambda(X) \otimes s_\mu(X) = \sum_\nu \kappa_{\lambda\mu\nu} s_\nu(X),$$

where the $\kappa_{\lambda\mu\nu}$ are the Kronecker coefficients. Extend linearly so that $\otimes$ is defined on all symmetric functions. In view of (31), we have that

$$s_\lambda(X) \otimes s_\mu(X) = \text{coefficient of } \ s_\mu(Y) \ \text{ in } \ s_\lambda(XY),$$

and, more generally, for any symmetric function $A(X)$,

$$A(X) \otimes s_\mu(X) = \text{coefficient of } \ s_\mu(Y) \ \text{ in } \ A(XY).$$

Theorem 11 (Littlewood-Garsia-Remmel) *If h_μ and s_λ denote the homogeneous symmetric function and the Schur function respectively*, *then*

$$h_\mu(X) \otimes s_\rho(X) = \sum_{(\rho)} \prod_{i=1}^{k} s_{\rho^{(i)}/\rho^{(i-1)}}(X), \tag{52}$$

where the sum is over all sequences of partitions $(\rho) = (\emptyset = \rho^{(0)} \subset \rho^{(1)} \subset \cdots \subset \rho^{(k)} = \rho)$ such that $|\rho^{(i)} - \rho^{(i-1)}| = \mu_i$.

Proof: The proof is by induction on the length of μ. In view of formula (35),

$$h_r(XY) = \sum_{\rho \vdash r} s_\rho(X) s_\rho(Y),$$

so that $h_r(X) \otimes s_\rho(X) = s_\rho(X)$ proving the formula when $\ell(\mu) = 1$, $(\mu = (r))$. □

Now, let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ and let $\hat{\mu} = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ and assume that

$$h_{\hat{\mu}}(XY) = \sum_{\hat{\rho} \vdash |\hat{\mu}|} s_{\hat{\rho}}(Y) \sum_{\hat{\rho}} \prod_{i=1}^{k-1} s_{\hat{\rho}^{(i)}/\hat{\rho}^{(i-1)}}(X),$$

where the inner sum is over all sequences $(\hat{\rho}) = (\emptyset = \hat{\rho}^{(0)} \subset \hat{\rho}^{(1)} \subset \cdots \subset \hat{\rho}^{(k-1)} = \hat{\rho})$ such that $|\rho^{(i)} - \rho^{(i-1)}| = \mu_i$, $1 \le i \le k-1$. Then

$$\begin{aligned} h_\mu(XY) &= h_{\mu_k}(XY) h_{\hat{\mu}}(XY) \\ &= \left(\sum_{\tau \vdash \mu_k} s_\tau(X) s_\tau(Y) \right) \left[\sum_{\hat{\rho} \vdash |\hat{\mu}|} s_{\hat{\rho}}(Y) \sum_{(\hat{\rho})} \prod_{i=1}^{k-1} s_{\hat{\rho}^{(i)} / \hat{\rho}^{(i-1)}}(X) \right]. \end{aligned}$$

Using formula (28) to multiply $s_\tau(Y)$ and $s_{\hat{\rho}}(Y)$, we have

$$h_\mu(XY) = \sum_{\tau \vdash \mu_k} \sum_{\hat{\rho} \vdash |\hat{\mu}|} \sum_{\rho \supset \hat{\rho}} c^{\rho}_{\tau \hat{\rho}} s_\rho(Y) s_\tau(X) \sum_{(\hat{\rho})} \prod_{i=1}^{k-1} s_{\hat{\rho}^{(i)}/\hat{\rho}^{(i-1)}}(X).$$

Then, using (27) to rewrite $\sum_\tau \; c^{\rho}_{\tau\hat{\rho}} s_\tau(X)$ and recalling that $\hat{\rho} = \hat{\rho}^{(k-1)}$, one obtains

$$\begin{aligned} h_\mu(XY) &= \sum_{\rho} s_\rho(Y) \sum_{\hat{\rho}^{(k-1)} \subset \rho} s_{\rho/\hat{\rho}^{(k-1)}}(X) \sum_{(\hat{\rho})} \prod_{i=1}^{k-1} s_{\hat{\rho}^{(i)}/\hat{\rho}^{(i-1)}}(X) \\ &= \sum_{\rho} s_\rho(Y) \sum_{(\rho)} \prod_{i=1}^{k} s_{\rho^{(i)}/\rho^{(i-1)}}(X) \end{aligned}$$

and the theorem follows. □

In order to state this result in a fashion similar to the results in (43) and (51) define a general μ-skew tableau of shape λ to be a sequence of partitions

$$T = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda)$$

such that $|\lambda^{(i)} - \lambda^{(i-1)}| = \mu_i$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. Define the weight of a general μ-skew tableau T to be

$$wt_s(T) = \sum_{i=1}^{k} s_{\lambda^{(i)}/\lambda^{(i-1)}}(X). \tag{53}$$

Then we have the following corollary of Theorem 10.

Corollary 12 *Let $\kappa_{\nu\lambda\gamma}$ be the Kronecker coefficient. Then $\kappa_{\nu\lambda\gamma}$ is the coefficient of $s_\gamma(X)$ in*

$$\sum_{\mu\vdash f}\sum_{(T,R)} wt_s(T)wt(R)$$

where the inner sum is over all pairs (T, R) consisting of a general μ-skew tableau T of shape λ and a special hook tableau R of shape ν and type μ. The weights $wt_s(T)$ and $wt(R)$ are as in (53) *and* (33) *respectively.*

Proof: The product rule for Schur functions is

$$s_\nu(XY) = \sum_{\lambda,\gamma} \kappa_{\nu\lambda\gamma} s_\lambda(Y) s_\gamma(X).$$

On the other hand we have, from Theorem 46, that

$$\begin{aligned} s_\nu(XY) &= \sum_\mu h_\mu(XY) K^{-1}_{\mu\nu} \\ &= \sum_\mu \sum_\lambda s_\lambda(Y) \sum_{(\lambda)} \prod_{i=1}^{k} s_{\lambda^{(i)}/\lambda^{(i-1)}}(X) K^{-1}_{\mu\nu}, \end{aligned}$$

where the inner sum is over all sequences $(\lambda) = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda)$ such that, for each $1 \le i \le k$, $|\lambda^{(i)} - \lambda^{(i-1)}| = \mu_i$. Taking the coefficient of $s_\lambda(Y)$ in each of these two expressions we have that

$$\sum_\gamma \kappa_{\nu\lambda\gamma} s_\gamma(X) = \sum_\mu \sum_T wt(T) K^{-1}_{\mu\nu},$$

where the inner sum is over all general μ-skew tableaux T of shape λ. The theorem follows from (34). □

We would like to point out that although this gives an algorithm for computing the coefficients $\kappa_{\nu\lambda\gamma}$, the complexity of these computations can be enormous. In many cases, these computations can be greatly simplified. For example, Remmel has found explicit formulas for $\kappa_{(1^{r-m},m),(1^{r-p},p),\lambda}$ and $\kappa_{(1^{r-m}m,),(p,r-p),\lambda}$ in [16] and [17] starting basically from Corollary 7. Whitehead and Remmel [18], starting from Theorem 10, have developed an algorithm to compute $\kappa_{(m,r-m),(p,r-p),\lambda}$ for any λ, which will compute such coefficients for r in the thousands.

6. *q*-analogues of the regular representation of S_f

In this section we use the Frobenius formula to give an explicit formula for the character of the regular representation of $H_f(q)$ and to compute the generic degrees of $\mathrm{GL}_n(\mathbb{F}_q)$. Each of these computations involves a character of the Hecke algebra which is a q-analogue of the character of the regular representation of S_f. First we consider the trace of the regular representation of $H_f(q)$.

Let χ^R denote the character of the regular representation of $H_f(q)$, i.e., the trace of the action of $H_f(q)$ on itself by left multiplication. We want to compute the values $\chi^R(T_{\gamma_\mu})$, $\mu \vdash f$ where T_{γ_μ} is as in Section 1. We will need two facts from the representation theory of semisimple algebras.

(1) The multiplicity of a given irreducible representation H^λ in the regular representation is equal to the dimension of the representation H^λ.
(2) The dimension of a representation H with character χ is given by $\chi(1)$ where 1 is the identity element.

Denote the identity element of $H_f(q)$ by 1_H and the identity element of S_f by 1_S. In the notation of section 1, $1_H = T_{\gamma_{(1^f)}}$ and $1_S = \gamma_{(1^f)}$. One can easily see from the combinatorial rules for computing characters that the dimension of the irreducible representation of $H_f(q)$ corresponding to λ is

$$d_\lambda = \chi^\lambda_{H_f}(1_H) = \chi^\lambda_{S_f}(1_S).$$

Thus, from (1) above, for the character of the regular representation χ^R,

$$\begin{aligned}\chi^R(T_{\gamma_\mu}) &= \sum_{\lambda \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) d_\lambda \\ &= \sum_{\lambda \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) \chi^\lambda_{S_f}(1).\end{aligned} \tag{54}$$

Now, the classical Frobenius formula gives that

$$s_\lambda(X) = \sum_{\rho \vdash f} \frac{\chi^\lambda_{S_f}(\rho)}{z_\rho} p_\rho(X).$$

Since $z_\rho = 1^f f! = f!$ when ρ is the partition (1^f)

$$s_\lambda(X)|_{p_{1^f}(x)} = \frac{\chi^\lambda_{S_f}(1_s)}{f!} \tag{55}$$

where $s_\lambda(X)|_{p_{1^f}}$ denotes the coefficient of p_{1^f} in $s_\lambda(X)$. Combining (54) and (55) we have that

$$\chi^R(T_{\gamma_\mu}) = \sum_{\lambda \vdash f} \chi^\lambda_{H_f}(T_{\gamma_\mu}) f! s_\lambda(X)|_{p_{1^f}}.$$

However, by the Frobenius formula this is

$$\chi^R(T_{\gamma_\mu}) = f! \bar{q}_\mu(X; q)|_{p_{1^f}}. \tag{56}$$

Theorem 13 *If χ^R denotes the character of the regular representation of $H_f(q)$ then, for each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of f.*

$$\chi^R\big(T_{\gamma_\mu}\big) = \frac{f!(q-1)^{f-k}}{\mu_1!\mu_2!\cdots\mu_k!}.$$

Proof: Lemma 5 gives that

$$\bar{q}_\mu(X;q) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} h_\mu(X(1-q^{-1})).$$

By the Frobenius formula (27)

$$h_r(X(1-q^{-1})) = s_{(r)}(X(1-q^{-1})) = \sum_{\nu\vdash r} \chi_{S_f}^{(r)}(\gamma_\nu)\frac{p_\nu(X(1-q))}{z_\nu}.$$

From the combinatorial rule for computing the characters of S_f one can easily see that $\chi_{S_f}^{(r)}(\gamma_\nu) = 1$ for all $\nu \vdash r$. Thus

$$\begin{aligned}
\bar{q}_\mu(X;q) &= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} h_{\mu_i}(X(1-q^{-1})) \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \sum_{\nu^{(i)}\vdash\mu_i} \frac{p_{\nu^{(i)}}(X(1-q^{-1}))}{z_{\nu^{(i)}}} \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \sum_{\nu^{(i)}\vdash\mu_i} \frac{p_{\nu^{(i)}}(X)p_{\nu^{(i)}}(1-q^{-1})}{z_{\nu^{(i)}}} \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \sum_{\substack{\nu^{(i)}\vdash\mu_i \\ 1\le i\le\ell(\mu)=k}} \\
&\quad \times \frac{p_{\nu^{(1)}}(X)p_{\nu^{(2)}}(X)\cdots p_{\nu^{(k)}}(X)p_{\nu^{(1)}}(1-q^{-1})p_{\nu^{(2)}}(1-q^{-1})\cdots p_{\nu^{(k)}}(1-q^{-1})}{z_{\nu^{(1)}}z_{\nu^{(2)}}\cdots z_{\nu^{(k)}}}.
\end{aligned}$$

Since $p_{\nu^{(1)}}(X)p_{\nu^{(2)}}(X)\cdots p_{\nu^{(k)}}(X) = p_{(1^f)}(X)$ if and only if $\nu^{(i)} = 1^{\mu_i}$ for all $1 \le i \le \ell(\mu)$ and $z_{\nu^{(i)}} = \mu_i!$ when $\nu^{(i)} = 1^{\mu_i}$,

$$\begin{aligned}
q_\mu(X;q)|_{p_{1^f}} &= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \frac{p_{1^f}(1-q^{-1})}{\mu_1!\mu_2!\cdots\mu_k!} \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \frac{(1-q^{-1})^f}{\mu_1!\mu_2!\cdots\mu_k!}.
\end{aligned}$$

Since $q^{|\mu|} = q^f$ and $\ell(\mu) = k$ we have that

$$\chi^R\big(T_{\gamma_\mu}\big) = \frac{f!(q-1)^{f-k}}{\mu_1!\mu_2!\cdots\mu_k!}.$$

□

Generic degrees of $GL_n(\mathbb{F}_q)$

Let χ^I be the character of the Hecke algebra $H_n(q)$ given by

$$\chi^I(T_{\gamma_\mu}) = \begin{cases} [n]!, & \text{if } \mu = (1^n); \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

where $[i] = 1 + q + q^2 + \cdots + q^{i-1}$ and $[n]! = [n][n-1]\cdots[1]$. The generic degrees of $\mathrm{GL}_n(\mathbb{F}_q)$ are the integers m_λ such that for every $\mu \vdash n$

$$\chi^I(T_{\gamma_\mu}) = \sum_{\lambda \vdash n} \chi^\lambda_{H_n}(T_{\gamma_\mu}) m_\lambda.$$

Background on generic degrees can be found in [3] and [8]. Although we do not need to know the origin of the character χ^I for our computations we would like to motivate, very briefly, their definition. Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the finite field with q elements. Let B be a Borel subgroup of G (for example, the upper triangular matrices in $\mathrm{GL}_n(\mathbb{F}_q)$). Let I be the G module given by inducing the trivial representation of B to G. There is an action of $H_n(q)$ on I such that the actions of $H_n(q)$ and of G on I each generate the full centralizer of the action of the other in $\mathrm{End}(I)$. The character χ^I is the trace of the action of the Hecke algebra on I and the m_λ are the dimensions of the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ appearing in I.

The trick to computing the m_λ by using the Frobenius formula is to recognize that χ^I is a Markov or Ocneanu trace for the Hecke algebra. For each $z \in \mathbb{C}$ there is an Ocneanu trace χ^z on $H_n(q)$, defined inductively using the inclusions $H_1(q) \subset H_2(q) \subset \cdots$, by

$$\begin{aligned} \chi^z(1_H) &= 1, \\ \chi^z(g_k h) &= z\chi^z(h) \quad \text{if } h \in H_k(q). \end{aligned}$$

Here 1_H denotes the identity element of the Hecke algebra and the g_k, $1 \le k \le n-1$, are the generators of $H_n(q)$.

King and Wybourne [9] and Gyojia [7] have proved the following result (see also [14] for a proof). Let μ be a partition of n. Then, in λ-ring notation

$$\chi^z(T_{\gamma_\mu}) = z^{n-\ell(\mu)} = \bar{q}_\mu\left(\frac{w-z}{1-q}\right), \tag{58}$$

where $w = 1 - q + z$.

Applying the Frobenius formula to (59) we have that

$$\begin{aligned} \chi^z(T_{\gamma_\mu}) &= \bar{q}_\mu\left(\frac{w-z}{1-q}\right) \\ &= \sum_{\lambda \vdash n} \chi^\lambda_{H_n}(T_{\gamma_\mu}) s_\lambda\left(\frac{w-z}{1-q}\right). \end{aligned} \tag{59}$$

It is shown in [13] I Section 3 Ex. 3 that

$$s_\lambda\left(\frac{w-z}{1-q}\right) = \prod_{(i,j)\in\lambda} \frac{wq^{i-1} - zq^{j-1}}{1-q^{h(i,j)}}, \tag{60}$$

where $(i, j) \in \lambda$ denotes the box in the ith row and the jth column of the Ferrers diagram of λ and

$$h(i, j) = \lambda_i - i + \lambda'_j - j + 1,$$

λ_i being the length of the ith row of λ and λ'_j the length of the jth column of λ. For each partition λ define the polynomial $H^\lambda(q)$, a q-analogue of the product of the hooks, by

$$H^\lambda(q) = \prod_{(i,j)\in\lambda} \frac{1-q^{h(i,j)}}{1-q}.$$

Theorem 14 *The generic degrees of $GL_n(\mathbb{F}_q)$ are given by*

$$m_\lambda = \frac{q^{n(\lambda)}[n]!}{H^\lambda(q)}$$

where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$.

Proof: From the definitions of the χ^I and the Ocneanu trace we have

$$\chi^I(T_{\gamma_\mu}) = [n]!\chi^0(T_{\gamma_\mu})$$

for all μ partitions of n, χ^0 being the Ocneanu trace for $z = 0$. Thus, from (60) and (61)

$$\begin{aligned}\chi^I(T_{\gamma_\mu}) &= [n]! \sum_{\lambda\vdash n} \chi^\lambda_{H_n}(T_{\gamma_\mu}) \prod_{(i,j)\in\lambda} \frac{(1-q+0)q^{i-1} - 0q^{j-1}}{1-q^{h(i,j)}} \\ &= \sum_{\lambda\vdash n} \chi^\lambda_{H_n}(T_{\gamma_\mu})[n]! \prod_{(i,j)\in\lambda} \frac{q^{i-1}(1-q)}{1-q^{h(i,j)}}. \qquad \square\end{aligned}$$

Remark It follows easily from (57) and (58) that, for the characters χ^R and χ^I of $H_f(q)$,

$$\lim_{q\to 1} \chi^R(T_{\gamma_\mu}) = \lim_{q\to 1} \chi^I(T_{\gamma_\mu}) = \begin{cases} f!, & \text{if } \mu = (1^f); \\ 0, & \text{otherwise.}\end{cases}$$

This shows that the regular representation R of $H_f(q)$ and the representation I of $H_f(q)$ on the induced representation from a Borel B to $G = \mathrm{GL}_f(\mathbb{F}_q)$ are both q-analogues (different q-analogues!) of the regular representation of the symmetric group S_f.

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