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On Flat Flag-Transitive $c.c^*$-Geometries

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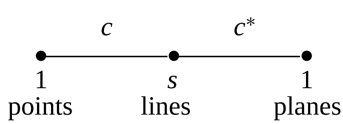
Abstract. We study flat flag-transitive $c.c^*$-geometries. We prove that, apart from one exception related to Sym(6), all these geometries are gluings in the meaning of [6]. They are obtained by gluing two copies of an affine space over GF(2). There are several ways of gluing two copies of the n-dimensional affine space over GF(2). In one way, which deserves to be called the canonical one, we get a geometry with automorphism group $G = 2^{2n} \cdot L_n(2)$ and covered by the truncated Coxeter complex of type D_{2^n}. The non-canonical ways give us geometries with smaller automorphism group ($G \leq 2^{2n} \cdot (2^n - 1)n$) and which seldom (never ?) can be obtained as quotients of truncated Coxeter complexes.

Keywords: diagram geometry, semi-biplane, amalgam of group

1. Introduction

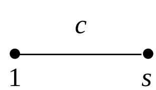
We follow [21] for the terminology and notation of diagram geometry, except that we use the symbol $\mathrm{Aut}(\Gamma)$ instead of $\mathrm{Aut}_s(\Gamma)$ to denote the group of type-preserving automorphisms of a geometry Γ.

A $c.c^*$*-geometry* is a geometry with diagram as follows:

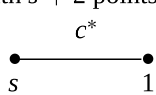


$(c.c^*)$

where s is a positive integer, called the *order* of the geometry. We recall that the stroke



means the class of circular spaces with $s + 2$ points and



has the dual meaning. We also recall that a circular space is a complete graph with at least three vertices, viewed as a geometry of rank 2 with vertices and edges as points and lines, respectively. Thus, given a set V of size $|V| \geq 3$, a group of permutations of V is flag-transitive on the circular space with set of points V if and only if it is doubly-transitive on V.

A $c.c^*$-geometry Γ is said to be *flat* if all points of Γ are incident with all planes of Γ. In this paper we shall focus on flat $c.c^*$-geometries admitting a flag-transitive automorphism group.

Getting control on these geometries turns out to be useful to aquire information on universal covers of other geometries. The reader may see [20] (Section 5.3) for an example of this.

The paper is organized as follows. In Sections 2 and 3 we survey some examples of $c.c^*$-geometries which we need to have at hand in this paper. We will focus on flat ones, but some non-flat examples will be considered, too. The Main Theorem of the paper is stated and proved in Section 4. Our Theorem does not finish the investigation of flat $c.c^*$-geometries. Rather, it points at a number of problems. We study some of them in Section 5.

2. Examples by gluing

2.1. On 1-factorizations of complete graphs

We need to recall some facts on 1-factorizations of complete graphs before describing the gluing construction.

Let $\Gamma = (V, E)$ be a finite complete graph of valency $k \geq 1$, with set of vertices V and set of edges E. A 1-*factorization* of Γ is a mapping χ from E to a set I of size k, called *the set of colours of* χ, such that, for every vertex $a \in V$, the restriction of χ to the set E_a of edges containing a is a bijection from E_a to I. That is, denoted by $\|$ the equivalence relation on E defined by "being in the same fiber of χ", $\|$ is a parallelism of the circular space Γ, in the meaning of [6]. According to the notation of [6], we denote the set of colours I by Γ^∞ and, given an edge $e \in E$, we write $\infty(e)$ for $\chi(e)$. We call $\infty(e)$ the *point at infinity* of e.

We recall that a complete graph of valency k admits a 1-factorization if and only if k is odd (see [16]).

Let χ_1, χ_2 be 1-factorizations of a complete graph $\Gamma = (V, E)$, with the same set of colours Γ^∞. An *isomorphism* from χ_1 to χ_2 is a permutation f of V that maps the fibers of χ_1 onto the fibers of χ_2. That is, a permutation f of V is an isomorphism from χ_1 to χ_2 if and only if there is a permutation α of Γ^∞ such that $\chi_2(f(e)) = \alpha(\chi_1(e))$ for every edge $e \in E$. Clearly, such a permutation α, if it exists, is unique. We call it the *action at infinity* of the isomorphism f and we set $f^\infty = \alpha$.

In particular, given a 1-factorization χ of Γ, the isomorphisms from χ to χ are called *automorphisms* of χ. We denote the automorphism group of χ by $\mathrm{Aut}(\chi)$.

The function mapping $f \in \mathrm{Aut}(\chi)$ onto $f^\infty \in \mathrm{Aut}(\Gamma^\infty)$ is a homomorphism from $\mathrm{Aut}(\chi)$ to $\mathrm{Aut}(\Gamma^\infty)$. We denote its image by A^∞ and its kernel by K, in slight variation to [6]. We call A^∞ the *action at infinity* of A and K the *translation group* of χ.

Clearly, K acts semi-regularly on the set V of vertices of Γ and, given a vertex $a \in V$, its stabilizer A_a in A acts faithfully on Γ^∞. It is not difficult to see that, if K is transitive (hence regular) on V, then $A^\infty = A_a^\infty$ ($\cong A_a$). In this case the extension $A = K \cdot A^\infty$ splits, and A is doubly-transitive on V if and only if A^∞ is transitive on Γ^∞.

Let $\Gamma = (V, E)$ be a finite complete graph of odd valency k. When $k = 2^n - 1$ and when $k = 5$, 11 or 27, a 1-factorization χ can be defined on Γ in such a way that $\mathrm{Aut}(\chi)$ is doubly-transitive on V. We shall describe these 1-factorizations in detail, since we will refer to their properties later on.

(1) *Let* $k = 2^n - 1$. Then Γ can be viewed as the point-line system of the n-dimensional affine geometry $\mathrm{AG}(n, 2)$ over $\mathrm{GF}(2)$. The case of $n = 1$ is too trivial to be worth a discussion. Thus, we assume $n > 1$.

Take the points of $\mathrm{PG}(n-1, 2)$ as colours and let χ be the function mapping every line of $\mathrm{AG}(n, 2)$ onto its point at infinity. Clearly, χ is a 1-factorization of Γ and $\mathrm{Aut}(\chi) = 2^n : L_n(2)$, the n-dimensional affine linear group over $\mathrm{GF}(2)$, doubly-transitive on the set V of points of $\mathrm{AG}(n, 2)$. The translation group K of χ is just the translation group of $\mathrm{AG}(n, 2)$, and $A^\infty = L_n(2)$.

$\mathrm{Aut}(\chi)$ also contains proper subgroups doubly-transitive on V. When $n \neq 7$, all of them have the following form (see [9]): $G = K \cdot X$, with X a proper subgroup of $L_n(2)$ transitive on Γ^∞ (for instance, a Singer cycle, or its normalizer). On the other hand, when $k = 7$ (that is, $n = 3$) an exceptional phenomenon also occurs. We have $L_2(7) \cong L_3(2)$ (see [10]) and there is bijective mapping φ from the set V of points of $\mathrm{AG}(3, 2)$ to the set of points of $\mathrm{PG}(1, 7)$ such that the group $G = \{\varphi^{-1} g \varphi \mid g \in L_2(7)\} \cong L_2(7) \cong L_3(2)$ is contained in the 3-dimensional affine linear group over $\mathrm{GF}(2)$ (see [9]). That is, $G \leq \mathrm{Aut}(\chi)$. As $L_2(7)$ is doubly-transitive on $\mathrm{PG}(1, 7)$, G is also doubly-transitive on V. However, $G \cap K = 1$.

It will be useful to have a symbol and a name for the pair (Γ, χ) with Γ and χ as above. We will denote it by $\mathrm{AS}(n, 2)$ and we call it the *n-dimensional affine space over* $\mathrm{GF}(2)$, keeping the symbol $\mathrm{AG}(n, 2)$ for the n-dimensional affine geometry over $\mathrm{GF}(2)$, viewed as a geometry of rank n.

(2) *Let* $k = 5$. Then Γ admits just one 1-factorization χ, which can be constructed as follows ([7, 17]).

We can assume that $V = H$, for a hyperoval H of $\mathrm{PG}(2, 4)$. As set of colours we take a line L of $\mathrm{PG}(2, 4)$ external to H and, given any two distinct points $a, b \in H$, we define $\chi(\{a, b\})$ as the meet point of L with the line of $\mathrm{PG}(2, 4)$ joining a with b.

The stabilizer of H in $P\Gamma L_3(4)$ is $\mathrm{Sym}(6)$, the full permutation group on the six points of H. The stabilizer of L in this group is $\mathrm{Sym}(5)$, acting doubly-transitively and faithfully both on H and on L (it acts as $\mathrm{PGL}_2(5)$ on the six points of H and as $P\Gamma L_2(4)$ on L). Hence $\mathrm{Aut}(\chi) = \mathrm{Sym}(5)$, $K = 1$ and $A^\infty = \mathrm{Aut}(\chi)$.

The group $\mathrm{Alt}(5) \leq \mathrm{Aut}(\chi) \cong L_2(5) \cong L_2(4)$ also acts doubly-transitively on H. It is the only proper subgroup of $\mathrm{Aut}(\chi)$ with this property.

(3) *Let* $k = 11$. We can now assume that $V = C$, with C a nondegenerate conic of $\mathrm{PG}(2, 11)$. Thus, the edges of Γ can be viewed as the secant lines of C. The stabilizer of C in $L_3(11)$ is $\mathrm{PGL}_2(11)$, doubly-transitive on C. Its commutator subgroup $L_2(11)$ is also doubly-transitive on C and acts imprimitively on the 66 secant lines of C, with 11 classes of size 6. Furthermore, it is doubly-transitive on that set of imprimitivity

classes [7]). Since the secant lines of C are the edges of Γ, we can take those imprimitivity classes as the fibers of a 1-factorization χ of Γ. We have $\mathrm{Aut}(\chi) = L_2(11)$ (see [7]), doubly-transitive both on $V = C$ and Γ^∞ and faithful on Γ^∞. Thus, $K = 1$. No proper subgroup of $\mathrm{Aut}(\chi)$ is doubly-transitive on V (see [7]; also [8]).

(4) *Finally, let* $k = 27$. As vertices of Γ we can take the 28 points of the Ree unital $U_R(3)$. There are nine subgroups $X = 2^3 : 7$ in $L_2(8) = R(3)'$, forming a complete conjugacy class $\mathcal{X}$ both in $R(3)'$ and in $R(3) = L_2(8) \cdot 3$ (see [10]). An $X \in \mathcal{X}$ is maximal in $R(3)'$, whereas it has index 3 in its normalizer in $R(3)$, which is maximal in $R(3)$. A group $X \in \mathcal{X}$ is transitive on $V = U_R(3)$, with point stabilizer of order 2, contained in the maximal subgroup $Y = 2^3$ of X (see [10]). Therefore X acts imprimitively on V, with seven imprimitivity classes of size 4. Let C be one of those classes and let X_a be the stabilizer in X of a point $a \in X$. Since Y is abelian, X_a is normal in Y. Furthemore, Y transitively permutes the four points of C. Hence X_a fixes all points of C and Y acts as 2^2 on C. That is, viewing C as a copy of $\mathrm{AG}(2, 2)$, Y acts on C as the translation group of $\mathrm{AG}(2, 2)$. Therefore, if $\{L_1, L_2\}$ is a partition of C in two pairs, $\{L_1, L_2\}$ has seven images by X, one for each of the imprimitivity classes of X on V. These seven pairs give us a partition of V in 14 pairs. We call this partition a *parallel class contributed by* X. Since C can be partitioned in pairs in three ways, X contributes three parallel classes. Clearly, it stabilizes each of them. Let now X vary in $\mathcal{X}$. Thus we obtain $3 \times 9 = 27$ parallel classes, which can be taken as the fibers of a 1-factorization χ of Γ. It is clear by the above construction that $R(3)'$ is not transitive on the set of fibers of χ, but it has three orbits on it, each of size 9 (note that $R(3)'$ is transitive, but not doubly-transitive on V). For every $X \in \mathcal{X}$, the three parallel classes contributed by X belong to distinct orbits. However, $R(3)$ permutes the fibers of χ. Indeed, in order to get $R(3)$ from $R(3)'$ we only need a 3-element belonging to the normalizer in $R(3)$ of some $X \in \mathcal{X}$, and that element cyclically permutes the three parallel classes contributed by X. This also shows that $R(3)$ is transitive on the set of fibres of χ. This amounts to say that $R(3)$ is doubly-transitive on V (compare [8]). It is clear from [8] that no group of permutations of V properly containing $R(3)$ preserves χ. Hence $\mathrm{Aut}(\chi) = R(3)$, doubly-transitive on V.

$R(3)'$ is the only proper nontrivial normal subgroup of $R(3)$. Therefore $K = 1$. Note also that no proper subgroup of $R(3)$ is doubly-transitive on V (see [8]).

(The above construction is due to Cameron and Korchmaros [9]. The exposition they give for it in [9] is fairly concise. We have expanded it a bit.)

Proposition 1 (Cameron and Korchmaros [9]) *Let* $\Gamma = (V, E)$ *be a complete graph of odd valency* k *and let* χ *be a* 1*-factorization of* Γ *such that* $\mathrm{Aut}(\chi)$ *is doubly-transitive on* V. *Then* $k = 2^n - 1$, 5, 11 *or* 27 *and* χ *is as in the above Examples* (1)–(4).

2.2. Gluings

Let $\Gamma = (V, E)$ be a complete graph of odd valency $k > 1$ and let χ_1, χ_2 be 1-factorizations of Γ with the same set of colours $\Gamma^\infty = \Gamma_1^\infty = \Gamma_2^\infty$. Let α be a permutation of Γ^∞. We define a $c.c^*$-geometry Γ as follows.

We take $V \times \{1\}$ (respectively, $V \times \{2\}$) as the set of *points* (*planes*) of Γ. As *lines* we take the pairs $(e_1, e_2) \in E \times E$ with $\alpha(\chi_2(e_2)) = \chi_1(e_1)$. We state that all points of Γ are incident with all planes of Γ. A point or a plane (a, i) (where $i = 1$ or 2) and a line (e_1, e_2) are declared to be incident when $a \in e_i$.

It is not difficult to check that Γ is in fact a $c.c^*$-geometry of order $s = k - 1$ and it is clear by the definiton that Γ is flat. We call it the *gluing* of (Γ, χ_1) with (Γ, χ_2) via α (also the α-gluing of χ_1 with χ_2, for short), and we denote it by the symbol $\mathrm{Gl}_\alpha(\chi_1, \chi_2)$.

The above construction is in fact a special case of a more general construction described in [6]. The properties we shall mention in what follows are also specializations of properties proved in [6] (Section 3.4).

For $i = 1, 2$, let K_i be the translation group of χ_i and let A_i^∞ be the action at infinity of $A_i = \mathrm{Aut}(\chi_i)$. Every type-preserving automorphism g of $\mathrm{Gl}_\alpha(\chi_1, \chi_2)$ induces on V an automorphism g_i of χ_i, $i = 1, 2$. As $\mathrm{Aut}(\mathrm{Gl}_\alpha(\chi_1, \chi_2))$ acts on the lines of the gluing, we have $g_1^\infty = \alpha g_2^\infty \alpha^{-1}$. On the other hand, given $g_1 \in A_1$ and $g_2 \in A_2$ such that $g_1^\infty = \alpha g_2^\infty \alpha^{-1}$, the function g that maps $(v, 1)$ onto $(g_1(v), 1)$ and $(v, 2)$ onto $(g_2(v), 2)$ defines an automorphism of $\mathrm{Gl}_\alpha(\chi_1, \chi_2)$. Thus we may identify K_1 (K_2) with the automorphism group of the gluing that induces K_1 (K_2) on the points (planes) and the trivial automorphism on the planes (points). Therefore

$$\mathrm{Aut}(\mathrm{Gl}_\alpha(\chi_1, \chi_2)) = (K_1 \times K_2) \cdot \left(A_1^\infty \cap \alpha A_2^\infty \alpha^{-1}\right) \tag{1}$$

The following is an obvious consequence of this description of $\mathrm{Aut}(\mathrm{Gl}_\alpha(\chi_1, \chi_2))$.

Proposition 2 *Let K_1 and K_2 be transitive on V. Then $Gl_\alpha(\chi_1, \chi_2)$ is flag-transitive if and only if $A_1^\infty \cap \alpha A_2^\infty \alpha^{-1}$ is transitive on Γ^∞.*

Assume that both K_1 and K_2 are transitive on V. Chosen a vertex $a \in V$, we can identify A_1^∞ with $(A_1)_a$ and A_2^∞ with $(A_2)_a$, and α can be viewed as a permutation of $V \backslash \{a\}$. Thus, the group $X_{\alpha,a} = (A_1)_a \cap \alpha(A_2)_a \alpha^{-1}$, which is the stabilizer in $\mathrm{Aut}(\mathrm{Gl}_\alpha(\chi_1, \chi_2))$ of the flag $\{(a, 1), (a, 2)\}$, is isomorphic with $A_1^\infty \cap \alpha A_2^\infty \alpha^{-1}$ and the extension (1) splits:

$$\mathrm{Aut}(\mathrm{Gl}_\alpha(\chi_1, \chi_2)) = (K_1 \times K_2) : X_{\alpha,a} \tag{2}$$

Given $g \in X_{\alpha,a}$ and $x \in V$, we have

$$g((x, 1)) = (g(x), 1) \quad \text{and} \quad g((x, 2)) = (\alpha^{-1} g \alpha(x), 2) \tag{3}$$

Assume $\chi_1 = \chi_2 = \chi$, say. The following holds (see [6], Theorem 3.9):

Proposition 3 *Given two permutations α, β of Γ^∞, we have $Gl_\alpha(\chi, \chi) \cong Gl_\beta(\chi, \chi)$ if and only if $\alpha \in A^\infty \beta A^\infty$.*

Therefore

Corollary 4 *The number of non-isomorphic gluings of χ with itself is equal to the number of double cosets of A^∞ in the group of all permutations of Γ^∞.*

A gluing $\mathrm{Gl}_\alpha(\chi, \chi)$ is said to be *canonical* if $\alpha \in A^\infty$. In particular, $\mathrm{Gl}_\iota(\chi, \chi)$ is canonical, where ι denotes the identity permutation of Γ^∞.

By Proposition 3, the canonical gluings of χ with itself are pairwise isomorphic. Thus, if $\mathrm{Gl}_\alpha(\chi, \chi)$ is canonical, then we can assume that $\alpha = \iota$. By (1) we have the following:

$$\mathrm{Aut}(\mathrm{Gl}_\iota(\chi, \chi)) = (K \times K) \cdot A^\infty \tag{4}$$

In short, the automorphism group of a canonical gluing is as large as possible.

2.3. Gluing two copies of AS(n, 2)

The canonical gluing of the affine space $\mathrm{AS}(n, 2)$ with itself (see 2.1.2(1)) is flag-transitive. Its automorphism group has the following structure

$$(2^n \times 2^n) \cdot L_n(2)$$

where $L_n(2)$ acts in the natural way on both factors isomorphic to 2^n.

By Corollary 4, the number of non-isomorphic gluings of two copies of $\mathrm{AS}(n, 2)$ equals the number of double cosets of $L_n(2)$ in $\mathrm{Sym}(2^n - 1)$. When $n = 2$ we have $L_2(2) = \mathrm{Sym}(3)$, hence only one gluing is possible, namely the canonical one.

Let $n = 3$. Exploiting the information given on $L_3(2)$ and Alt(7) in [10] and [5] (p. 69), it is not difficult to check that $L_3(2)$ has four double cosets in $\mathrm{Sym}(7)$, corresponding to elements $\alpha, \beta, \gamma, \delta$ with

$$\begin{aligned} \alpha &\in L_3(2), \\ L_3(2) \cap \beta L_3(2)\beta^{-1} &\cong \mathrm{Frob}(21), \\ L_3(2) \cap \gamma L_3(2)\gamma^{-1} &\cong \mathrm{Alt}(4), \\ L_3(2) \cap \delta L_3(2)\delta^{-1} &\cong \mathrm{Sym}(4). \end{aligned}$$

Thus, we have three non-canonical ways of gluing two copies of $\mathrm{AS}(3, 2)$. Only one of these gluings is flag-transitive, namely the gluing via β. Indeed $\mathrm{Frob}(21)$ is transitive on the set Γ^∞ of points of $\mathrm{PG}(2, 2)$ (it is even flag-transitive on $\mathrm{PG}(2, 2)$), whereas no subgroup of $\mathrm{Sym}(7)$ isomorphic to $\mathrm{Sym}(4)$ or to $\mathrm{Alt}(4)$ can be transitive on Γ^∞.

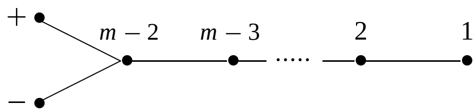
Needless to say, the larger n is, the more ways exist of gluing $\mathrm{AS}(n, 2)$ with itself. Most of these gluings are not flag-transitive. However, flag-transitive non-canonical gluings exist for every $n > 2$, as we will see in Section 5.

3. More examples

In this section we describe a few more $c.c^*$-geometries we shall deal with in this paper.

3.1. *The truncated Coxeter complex of type D_m*

Let Δ_m be a Coxeter complex of type D_m ($m > 3$). We take $+, -, m-2, m-3, \ldots, 2, 1$ as types, as follows:



A $c.c^*$-geometry of order $m-2$ is obtained from Δ_m by removing all elements of type $i = 1, 2, \ldots, m-3$. We denote this geometry by $\mathrm{Tr}(\Delta_m)$ and we call it *the truncated Coxeter complex of type D_m*. (In [3] $\mathrm{Tr}(\Delta_m)$ is called *the two-coloured hypercube*). $\mathrm{Tr}(\Delta_m)$ is simply connected (see [3], p. 327).

Theorem 5 *The universal cover of the canonical gluing of $AS(n, 2)$ with itself is $Tr(\Delta_m)$, with $m = 2^n$.*

Proof: Let Γ be the canonical gluing of $\mathrm{AS}(n, 2)$ with itself. Since we consider a canonical gluing, α can be assumed to be the identity in (3) of Section 2.2. Thus, we can apply Corollary 3.5 of [3] and we get the result. □

3.1.1. Quotients of Tr(Δ_m). We firstly recall some properties of Δ_m. The elements of Δ_m of type 1 and 2 from a complete m-partite graph $\Delta_m^{1,2}$, with the elements of type 1 as vertices and those of type 2 as edges. The elements of Δ_m of type $i = 3, 4, \ldots, m-2$ are the i-cliques of this graph, and those of type $+$ and $-$ are the maximal cliques. The maximal cliques of $\Delta_m^{1,2}$ have size m and two maximal cliques X, Y are of the same type when $m - |X \cap Y|$ is even. The blocks of $\Delta_m^{1,2}$ have size 2.

Given a maximal clique $A = \{a_1, a_2, \ldots, a_m\}$ of $\Delta_m^{1,2}$, let $B = \{b_1, b_2, \ldots, b_m\}$ be the (unique) maximal clique of $\Delta_m^{1,2}$ disjoint from A, with indices chosen in such a way that a_i and b_j are joined in $\Delta_m^{1,2}$ if and only if $i \neq j$.

For $J \subseteq I = \{1, 2, \ldots, m\}$, let e_J be the automorphism of $\Delta_m^{1,2}$ interchanging a_j with b_j for all $i \in J$ and fixing the other vertices of $\Delta_m^{1,2}$. We call $|J|$ the *weight* of e_J.

For every permutation $\sigma \in \mathrm{Sym}(m)$, let g_σ be the automorphism of $\Delta_m^{1,2}$ that maps a_i onto $a_{\sigma(i)}$ and b_i onto $b_{\sigma(i)}$, for $i \in I$.

The elements e_J of even weight form an elementary abelian 2-group E of order 2^{m-1}, whereas $S = \{g_\sigma\}_{\sigma \in \mathrm{Sym}(m)}$ is a copy of $\mathrm{Sym}(m)$. The Coxeter group of type D_m is $E : S$. This is also the automorphism group of $\mathrm{Tr}(\Delta_m)$. Indeed Δ_m can be recovered from $\mathrm{Tr}(\Delta_m)$ (the graph $\Delta_m^{1,2}$ uniquely determines Δ_m, the elements of $\mathrm{Tr}(\Delta_m)$ are the maximal cliques and the $(m-2)$-cliques of $\Delta_m^{1,2}$, and $\Delta_m^{1,2}$ can be recovered from these cliques).

Comparing the conditions given in Section 11.1 of [21] for a group to define a quotient, it is not difficult to see that a subgroup X of E defines a quotient of $\mathrm{Tr}(\Delta_m)$ if and only if all non-identity elements of X have weight at least four.

We shall now describe a subgroup $\bar{X} \leq E$ for which $\mathrm{Tr}(\Delta_m)/\bar{X}$ is the canonical gluing of two copies of $\mathrm{AS}(n, 2)$. (The subgroups with this property are pairwise conjugated in $E : S$, by a well known property of universal covers.)

As $m = 2^n$, we can take $I = \{1, 2, \ldots, m\}$ as the set of points of a model $\mathcal{A}$ of AG$(n, 2)$. Let $\mathcal{I}$ be the set of affine subspaces of $\mathcal{A}$ of dimension ≥ 2 and let $\bar{X} = \{e_J\}_{J \in \mathcal{I} \cup \{\emptyset\}}$. It is not difficult to check that $\bar{X}$ is a linear subspace of E and that all non-zero vectors of $\bar{X}$ have weight ≥ 4. Hence $\bar{X}$ defines a quotient of Tr(Δ_m). Furthermore, $E/\bar{X} \cong V(n, 2)$. Consequently the quotient Tr$(\Delta_m)/\bar{X}$ is flat. The normalizer of $\bar{X}$ in $E : S$ is $E : \mathrm{AS}L(n, 2) = (2^n \times 2^n)L_n(2)$. By the Main Theorem of this paper (Section 4) the quotient Tr$(\Delta_m)/\bar{X}$ is the canonical gluing of two copies of AS$(n, 2)$.

3.1.2. A special case: $n = 2$. Let $n = 2$. The center of $E : S$ is the unique non-trivial subgroup of E defining a quotient. This quotient is the canonical gluing of two copies of AS$(2, 2)$.

Note that a model of Tr(Δ_4) can also be constructed as follows: given a plane π of PG$(3, 2)$ and a point $p \in \pi$, remove π and the star of p. By a result of Levefre-Percsy and Van Nypelseer [18], what remains is isomorphic to Tr(Δ_4). The center of E is generated by the elation of PG$(3, 2)$ of center p and axis π.

3.1.3. The case of $n = 3$. Let $n = 3$ and let $\bar{X} = \{e_J\}_{J \in \mathcal{I} \cup \{\emptyset\}}$ be the subgroup of E such that Tr$(\Delta_m)/\bar{X}$ is the canonical gluing of two copies of AS$(n, 2)$, as in Section 3.1.1. (Note that the elements of $\mathcal{I}$ are the set I and hyperplanes of $\mathcal{A}$).

The normalizer of $\bar{X}$ in S contains a subgroup $L \cong L_3(2)$ which is doubly-transitive on I (see Section 2.1, Example (1)). Hence the automorphism group of Tr$(\Delta_8)/H$ contains a flag-transitive subgroup G with the following properties:

(i) $G \cong 2^3 : L_3(2)$;
(ii) $G_a \cong L_3(2)$ for every element a of Tr$(\Delta_m)/H$ of type $+$ (or $-$). Furthermore, the action of G_a on the 8 elements of type $-$ (respectively $+$) incident to a is the doubly-transitive action of $L_3(2)$ on the 8 points of AG$(3, 2)$.

On the other hand, Tr$(\Delta_8)/\bar{X}$ is the only flat quotient of Tr(Δ_8) admitting a flag-transitive automorphism group like that. Indeed, let $X \leq E : S$ define a flat quotient of Tr$(\Delta_8)/X$ with Aut(Tr$(\Delta_8)/X$) admitting a flag-transitive subgroup G with the above properties (i) and (ii).

As Tr(Δ_8) is flat, X has order 16. Its normalizer N in $E : S$ contains $X \cdot G = 2^4(2^3 \cdot L_3(2))$, flag-transitive on Tr$(\Delta_8)$ because G is flag-transitive on Tr$(\Delta_8)/X$. Let $L = S \cap X \cdot G$ be the stabilizer in $X \cdot G$ of the maximal clique A of $\Delta_8^{1,2}$. By (ii), $L \cong L_3(2)$, doubly-transitive on A. It is now clear that X must be a subgroup of E. Since it defines a quotient of Tr(Δ_8) its non-identity elements have weight at least 4. If one of them has weight 6, then we get 24 elements of weight 6 in X, by the doubly-transitive action of L on A and because L normalizes X. This is impossible, because $|X| = 16$. It is now clear that X contains 14 elements of weight 4 and one element of weight 8. By (ii), the action of L on A is a copy of the doubly-transitive action of $L_3(2)$ on the 8 points of AG$(3, 2)$. Thus, the 14 elements of X of weight 4 represent the 14 planes of AG$(3, 2)$. That is, $X = \bar{X}$ (up to conjugacy in S).

3.2. The two JVT-geometries

Let p and π be a point and a plane of PG$(3, 4)$, with $p \notin \pi$. Let O be a hyperoval of π. We can define a rank 3 geometry $\Gamma(p, O)$, as follows.

Let $\mathcal{C}$ be the set of line of PG(3, 4) joining p with points of O and let $C = \bigcup_{L\in\mathcal{C}} L$. We take $P = C\backslash(\{p\} \cup O)$ as the set of points of $\Gamma(p, O)$. As planes we take the planes u of PG(3, 4) such that $p \notin u$ and $u \cap O = \emptyset$. Two points of P not on the same line of $\mathcal{C}$ are said to form a line of $\Gamma(p, O)$. The incidence relation is the natural one, inherited from PG(3, 4). It is straightforward to check that $\Gamma(p, O)$ is a flag-transitive $c.c^*$-geometry of order 4.

We have $\mathrm{Aut}(\Gamma(p, O)) = H \cdot \mathrm{Sym}(6)$, where $H = Z_3$ is the group of homologies of PG(3, 4) of center p and axis π. (Note that $H \cdot \mathrm{Alt}(6)$ also acts flag-transively on $\Gamma(p, O)$.) It follows from [4] (Theorem B, (3) (ii)) that $\Gamma(p, O)$ is simply connected.

$\Gamma(p, O)$ can be factorized by H and $\Gamma(p, O)/H$ is flat and flag-transitive, with $\mathrm{Aut}(\Gamma(p, O)) = \mathrm{Sym}(6)$ (but Alt(6) also acts flag-transitively on it).

We call $\Gamma(p, O)$ the *non-flat JVT-geometry*, after its discoverers Janko and van Trung [14] (but they gave a different description for this geometry). The quotient $\Gamma(p, O)/H$ will be called the *flat JVT-geometry*.

The flat JVT-geometry is not a gluing. Indeed there is a unique way of gluing two complete graphs with six vertices, but that gluing is not flag-transitive ([6], Section 6.2.4, p. 385).

4. The main theorem

Theorem 6 (Main theorem) *Let Γ be a flag-transitive flat $c.c^*$-geometry. Then Γ is one of the following*:

(i) *the flat JVT-geometry*;

(ii) *the canonical gluing of two copies of $AS(n, 2)$, $n \geq 2$*;

(iii) *a non-canonical gluing of two copies of $AS(n, 2)$, $n \geq 3$, with $Aut(\Gamma) \leq (K_1 \times K_2) \cdot F$, where $K_1 \cong K_2 \cong 2^n$ and $F \leq \Gamma L_1(2^n)$.*

In case (i), $\mathrm{Aut}(\Gamma) = \mathrm{Sym}(6)$ and the universal cover of Γ is the non-flat JVT-geometry (see Section 3.2). In case (ii), the universal cover of Γ is the truncated Coxeter complex of type D_m with $m = 2^n$ (Theorem 5), and $\mathrm{Aut}(\Gamma) = (2^n \times 2^n) \cdot L_n(2)$ (see Section 2.2).

We shall prove the above theorem in the next subsection. The following corollary is easily got by assembling Theorems 5 and 6:

Corollary 7 *A flag-transitive flat $c.c^*$-geometry is the canonical gluing of two copies of $AS(n, 2)$ if and only if its automorphism group is a quotient of the Coxeter group of type D_m, with $m = 2^n$.*

4.1. Proof of Theorem 6

Let Γ be a flat $c.c^*$-geometry of order s. Since Γ is flat, there are just $s+2$ points and $s+2$ planes in Γ. Furthermore, given any two distinct points (planes) x and y and any plane (point) z, there is just one line incident with x, y and z. Therefore, given any two distinct points (planes), there are $(s+2)/2$ lines incident with them both. (By the way, this forces s to be even).

Let Γ be flag-transitive and let G a flag-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Given an element x of Γ, we denote the stabilizer of x in G by G_x. If x is a point or a plane, then G_x acts faithfully on the residue Γ_x of x, whereas, if x is a line, then the kernel K_x of the action of G_x on Γ_x is the stabilizer of any of the four chambers containing x, and $G_x/K_x = 2^2$ (by [3], Lemma 3.1).

Given two lines l and m of Γ, if l and m are incident with the same pair of planes (points), then we write $l \parallel^+ m$ (resp., $l \parallel^- m$). Clearly, $\parallel^+$ and $\parallel^-$ are equivalence relations on the set of lines of Γ and, if $l \parallel^+ m$ (resp. $l \parallel^- m$), then l and m have no points in common (are not incident with any common plane).

For every plane (point) x, we denote by $\parallel_x$ the equivalence relation induced by $\parallel^+$ (resp. $\parallel^-$) on the set of lines incident to x.

Lemma 8 *For every plane or point x, the classes of $\parallel_x$ are the fibers of a 1-factorization of the complete graph Γ_x.*

Proof: Let x be a plane, to fix ideas. If l, m are lines of Γ_x such that $l \parallel_x m$, then l and m have no points in common. On the other hand, given a plane $y \neq x$, there are just $(s+2)/2$ lines incident with both x and y. The lemma is now obvious. □

Corollary 9 *If Γ is not the flat JVT-geometry, then $s = 2^n - 2$ for some $n \geq 2$ and the following hold, with x any point or plane of Γ:*
(i) *Γ_x, equipped with $\parallel_x$, is a model of $AS(n, 2)$;*
(ii) *G_x is a doubly-transitive subgroup of $AGL_n(2)$ and either it contains the translation subgroup of $AGL_n(2)$, or $n = 3$ and $G_x \cong L_3(2)$.*

Proof: By Lemma 8 and Proposition 1, either $s = 2^n - 2$ and (i), (ii) hold, or we have one of the following:

(a) $s = 4$ and $G_x = \mathrm{Sym}(5)$ or $\mathrm{Alt}(5)$ (see Section 2.1, Example (2));
(b) $s = 10$ and $G_x = L_2(11)$ (see Section 2.1, Example (3));
(c) $s = 26$ and $G_x = R(3)$ (see Section 2.1, Example (4)).

In case (a) the universal cover of Γ is the non-flat JVT-geometry, by Theorem B of [4]. In this case Γ is the flat JVT-geometry.

Case (b) is impossible by Theorem B of [4]. Assume we have (c). Let K be the stabilizer in G of all points of Γ. By Lemma 3.1 of [3], K is semi-regular on the set of planes of Γ. Thus, $|K|$ is a divisor of 28, since Γ has 28 planes. However, $G_x = R(3)$ for every point x, and $R(3)$ does not contain any normal subgroup of order 2, 4, 7, 14 or 28. Therefore $K = 1$. Consequently, G acts faithfully on the 28 points of Γ. It is also doubly-transitive on them and it has order $|G| = 28 \cdot |R(3)| = 2^5 3^3 7$. However, no doubly-transitive group of degree 28 exists with that order (see [8]). Thus, (c) is impossible. □

Lemma 10 *Let $s = 6$ and $G_x \cong L_3(2)$ for a point or a plane x. Then Γ is the canonical gluing of two copies of $AS(3, 2)$ (hence $G = 2^3 \cdot L_3(2)$ is a proper subgroup of $\mathrm{Aut}(\Gamma) = 2^6 : L_3(2)$).*

Proof: The universal cover of Γ is $\mathrm{Tr}(\Delta_8)$, by Theorem A of [4]. The statement follows from what we said in Section 3.1.3. $\square$

Henceforth we assume that $s = 2^n - 2$. Hence (i) and (ii) of Corollary 9 hold. The case of $n = 3$ with $G_x = L_3(2)$ (x a point or a plane) has been examined in Lemma 10. Thus, when $n = 3$ we also assume that $G_x \not\cong L_3(2)$, for any point or plane x. Therefore, for any point or plane x, the pair $(\Gamma_x, \|_x)$ is a model of $\mathrm{AS}(n, 2)$ and G_x contains the translation group T_x of the affine space $(\Gamma_x, \|_x)$.

Lemma 11 *We have $T_x = T_y$ for any two planes or two points x, y of Γ.*

Proof: Let x be a plane (a point) of Γ. Since T_x fixes all classes of $\|_x$, it also fixes all planes (points) of Γ, since those classes bijectively correspond to the planes (points) of Γ distinct from x. Therefore $T_x \leq G_y$ for every plane (point) y of Γ. Let y be any of them. Since T_x fixes all planes (points) of Γ, it also fixes all classes of $\|_y$. Hence $T_x = T_y$. $\square$

Given a pair $e = \{x, y\}$ of distinct points (planes) and a plane (a point) z, we denote by l_e^z the line of Γ_z incident to both x and y. Given two pairs of distinct points (planes) e_1, e_2 and a plane (point) z, if $l_{e_1}^z \|_z l_{e_2}^z$ then we write $e_1 \|_{[z]} e_2$.

Lemma 12 *We have $\|_{[x]} = \|_{[y]}$ for any two planes (points) x and y.*

Proof: For every plane (point) x, the classes of $\|_{[x]}$ are the orbits of T_x on the set of points (planes) of Γ. The conclusion follows from Lemma 11. $\square$

We write $\|_1$ or $\|_2$ for $\|_{[x]}$, according to whether x is a plane or a point. (This notation is consistent, by the previous lemma.) We also denote by Γ_1 (resp. Γ_2) the complete graph with the points (planes) of Γ as vertices. Thus $(\Gamma_1, \|_1)$ (resp. $(\Gamma_2, \|_2)$) is a model of $\mathrm{AS}(n, 2)$.

Given a line l of Γ, we denote by $\sigma_1(l)$ (resp. $\sigma_2(l)$) the pair of points (planes) incident to l.

Lemma 13 *Given any two lines l, m of Γ, we have $\sigma_1(l) \|_1 \sigma_1(m)$ if and only if $\sigma_2(l) \|_2 \sigma_2(m)$.*

Proof: Let $\sigma_1(l) = \{a, a'\}$, $\sigma_2(l) = \{u, u'\}$, $\sigma_1(m) = \{b, b'\}$ and $\sigma_2(m) = \{v, v'\}$. Assume $\{u, u'\} \|_2 \{v, v'\}$, to fix ideas. This means that $l \|_a m'$, with m' the line of Γ_a joining v with v'. We have $\sigma_1(m') = \sigma_1(l)$, by the definition of $\|_a$. On the other hand, $\sigma_2(m') = \sigma_2(m) = \{v, v'\}$, by the choice of m'. Hence $m' \|_v m$. Therefore $\sigma_1(m') \|_1 \sigma_1(m)$. That is, $\{a, a'\} \|_1 \{b, b'\}$. $\square$

Lemma 14 *The geometry Γ is a gluing of two copies of $AS(n, 2)$.*

Proof: Fix a point a and a plane u of Γ. For $i = 1, 2$ and for every edge e of Γ_i, let $\chi_i(e)$ be the line $l \in \Gamma_{a,u}$ such that $\sigma_i(l) \|_i e$. It is clear that χ_i is a 1-factorization of

Γ_i, with the classes of $\|_i$ as its fibers and $\Gamma_{a,u}$ as set of colours. By Lemma 13, we have $\chi_1(\sigma_1(l)) = \chi_2(\sigma_2(l))$ for every line l of Γ. It is now clear that Γ is the gluing $\mathrm{Gl}_\alpha(\chi_1, \chi_2)$ of (Γ_1, χ_1) with (Γ_2, χ_2) with $\alpha = 1$. On the other hand, both (Γ_1, χ_1) and (Γ_2, χ_2) are isomorphic to $\mathrm{AS}(n, 2)$. The statement follows. □

Thus, Γ is the gluing of two copies S_1, S_2 of $\mathrm{AS}(n, 2)$ via some permutation α of the set Γ^∞ of the points of $\mathrm{PG}(n-1, 2)$. Modulo replacing Γ with some of its isomorphic copies if necessary, we can assume that $S_1 = S_2 = S$.

For x a point or a plane and for G a flag-transitive automorphism group of Γ the stabilizer G_x acts doubly-transitively on the planes or points in its residue Γ_x, respectively. Moreover we have $G = (V_1 \times V_2)X$, with $X = G_{a,u}$, a a point of Γ, u a plane of Γ incident to a, $V_1 = O_2(G_a) = K_a$ and $V_2 = O_2(G_u) = K_u$. Note that $X = L_n(2) \cap \alpha L_n(2)\alpha^{-1}$ (see Section 2.2, (1)).

Lemma 15 *Let $\alpha \notin L_n(2)$. Then $n \geq 3$ and $X \leq \Gamma L_1(2^n)$.*

Proof: We have $n \geq 3$ because $L_2(2) = \mathrm{Sym}(3)$. We can assume that a and u are the same element of S, say p_0, and we can take the elements of $S^\infty := S \backslash \{p_0\}$ as points of $\mathrm{PG}(n-1, 2)$. Both V_1 and V_2 act regularly on S. Given x, let x_1 (x_2) be the element of V_1 (V_2) mapping p_0 onto x. Given $g \in X$, we denote by $g(x)$ and $g[x]$ the images of p_0 by x_1^g and x_2^g respectively. Thus $g(p_0) = g[p_0]$ and $g(x) = g^\alpha[x]$ for every $x \in S^\infty$.

Clearly, $X \leq \Gamma L_m(q)$ with $q = 2^{n/m}$, for some divisor m of n (possibly, $m = 1$ or $m = n$). Since G_a is an affine doubly-transitive permutation group over $\mathrm{GF}(2)$, by [19] either $m = 1$ or X contains a normal subgroup Y isomorphic to $\mathrm{SL}_m(q)$, $Sp_m(q)$ (m even), $G_2(q)'$, A_6 or A_7, with $m = 6$ when $Y \cong G_2(q)'$ and $m = n = 4$ when $Y \cong A_6$ or A_7.

We need to prove that $m = 1$. Assume $m > 1$, by contradiction. Let Ξ be a natural geometry for the action of Y on V_2. The elements of Ξ are linear subspaces of V_2 (in fact, they are subspaces of $V(m, q)$). Thus they can be viewed as subsets (possibly, points) of S^∞, via the one-to-one correspondence we have stated between V_2 and S. Given $p \in S^\infty$, we will denote by $\langle p \rangle$ the point of Ξ containing p.

The group Y is transitive on S^∞. Furthermore, Y^α is contained in $L_n(2)$, as $Y \leq X = L_n(2) \cap \alpha L_n(2)\alpha^{-1}$. On the other hand, there is exactly one conjugacy class in $L_n(2)$ of subgroups isomorphic to Y and transitive on S^∞ (see [1] (21.6)(1) and [15]). This means that there exists an element $\varphi \in \mathrm{Aut}(V_2) = L_n(2)$ such that $Y^{\alpha\varphi} = Y$. The permutation $\psi = \alpha\varphi$ of S^∞ induces an automorphism of X. As X is transitive on S^∞, by multiplying by some element of X if necessary we can also assume that ψ stabilizes some element $p \in S^\infty$.

We claim that there is a $g \in \mathrm{Aut}(V_2)$, such that ψg centralizes Y. Assume the contrary. If $Y \cong SL_m(q)$, $Sp_m(q)$ ($(m, q) \neq (4, 2)$, $Sp_4(2)' \cong A_6$ or $G_2(q)'$, then ψ induces some graph automorphism on Y. On the other hand ψ, stabilizing p, also normalizes the stabilizer Y_p of p in Y and maps stabilizers of points of Ξ onto stabilizers of maximal subspaces of Ξ, since it acts as a graph automorphisms on Y. Therefore, Y_p stabilizes $\langle p \rangle$ and some maximal subspace of Ξ. However, this is impossible. (Note that $Y_p Z(\mathrm{GL}_m(q))$ contains the stabilizer of $\langle p \rangle$ in Y.) This contradiction forces $Y \cong A_7$, $\langle Y, \psi \rangle \cong S_7$ and

$Y_p \cong L_3(2)$. This gives again a contradiction as $N_{S_7}(L_3(2)) = L_3(2)$, [10]. Hence there is some $g \in \mathrm{Aut}(V_2)$, such that ψg centralizes Y.

Thus we are able to choose $\varphi \in \mathrm{Aut}(V_2)$ so that the permutation ψ $(= \alpha\varphi)$ centralizes Y. On the other hand, the stabilizer in Y of a point of Ξ does not fix any other point of Ξ. This forces ψ to stabilize all subsets of S^∞ corresponding to points of Ξ. Let $p_1 \in S^\infty$. As ψ stabilizes $\langle p_1\rangle$, we have $\psi(p_1) = \lambda_1 p_1$ for some $\lambda_1 \in \mathrm{GF}(q)\backslash\{0\}$. On the other hand, for every $\lambda \in \mathrm{GF}(q)\backslash\{0\}$ there is some element $g \in Y$ such that $g(p) = \lambda p$ for every $p \in \langle p_1\rangle$. As ψ and g commute, we have

$$\psi(\lambda p_1) = \psi(g(p_1)) = g(\psi(p_1)) = \lambda\psi(p_1) = \lambda\lambda_1 p_1 = \lambda_1 \cdot \lambda p_1$$

Consequently, the action of ψ on $\langle p_1\rangle$ is the multiplication by λ_1. We claim that λ_1 does not depend on the choice of p_1. Given another element $p_2 \in S^\infty$ with $\langle p_2\rangle$ collinear with $\langle p_1\rangle$ in Ξ, let $\lambda_2 \in \mathrm{GF}(q)\ \backslash\{0\}$ be such that $\psi(p) = \lambda_2 p$ for every $p \in \langle p_2\rangle$. Let $g \in Y$ map $\langle p_1\rangle$ onto $\langle p_2\rangle$. As ψ commutes with g, we have

$$\lambda_2 \cdot g(p_1) = \psi(g(p_1)) = g(\psi(p_1)) = g(\lambda_1 p_1) = \lambda_1 \cdot g(p_1)$$

(the last equality holds by linearity). Therefore $\lambda_1 = \lambda_2$. By the connectedness of Ξ, λ_1 does not depend on the choice of p_1, as claimed. Consequently, ψ acts by scalar multiplication on $V(m, q)$. That is, $\psi \in Z(\mathrm{GL}_m(q))$. Therefore, $\alpha = \psi\varphi^{-1} \in L_n(2)$; a contradiction. Hence $m = 1$. □

Lemma 15 finishes the proof of Theorem 6.

5. On non-canonical gluings

It is quite natural to ask how many examples exist for case (iii) of Theorem 6, for a given $n \geq 3$. (We recall that the canonical gluing is the only possibility when $n = 2$, as stated in Theorem 6). Two questions ask for an answer:

(1) Which possibilities for $X = \mathrm{Aut}(\Gamma)/(K_1 \times K_2) \leq \Gamma L_1(2^n)$ really occur?
(2) Chosen a feasible isomorphism type X for $\mathrm{Aut}(\Gamma)/(K_1 \times K_2)$, how many non-isomorphic examples exist with $\mathrm{Aut}(\Gamma)/(K_1 \times K_2) = X$?

In Section 5.1 we shall describe a family of examples with $X = \Gamma L_1(2^n)$. In Section 5.2 we shall count the number of non-isomorphic examples with $X = \Gamma L_1(2^n)$. More detailed information on the cases of $n = 3, 4, 5, 6$ will be given in Section 5.3. As a by-product, we will see that when $n = 6$ there is at least one example with $X < \Gamma L_1(2^n)$. Perhaps, the same is true whenever $2^n - 1$ and n are not relatively prime (compare Corollary 17).

5.1. *A family of examples with $X = \Gamma L_1(2^n)$*

Non-canonical gluings of two copies of $\mathrm{AS}(n, 2)$ with $X = \Gamma L_1(2^n)$ can be obtained as quotients of the elation semi-biplane associated with $\mathrm{PG}(2, 2^n)$. We shall describe these quotients in Section 5.1.2, after recalling the definition of elation semi-biplanes.

5.1.1. Elation semi-biplanes. Homology, elation and Baer semi-biplanes have been introduced by Hughes [12]. We will only consider elations semi-biplanes here.

Given a line l of $\mathrm{PG}(2,q)$ ($q = 2^n$, $n > 1$) and a point $p \in l$, let ε be an elation of $\mathrm{PG}(2,q)$ of center p and axis l. We denote by P the set of points of $\mathrm{PG}(2,q)$ not on l and by L the set of lines of $\mathrm{PG}(2,q)$ that do not pass through the point p.

Let Π_ε be the incidence structure defined as follows. The orbits of ε on P are the points of Π_ε. As blocks we take the sets $u \cup v$, with $\{u, v\}$ an orbit of ε on L. The incidence relation is defined as symmetrized inclusion. This incidence structure is a semi-biplane. It is called an *elation semi-biplane*.

It is well known that a $c.c^*$-geometry Γ can be obtained from every semi-biplane Π. The elements of Γ are the points and the blocks of Π and the unordered pairs of points of Π contained in a common block. We call these pairs of points *lines* and the blocks *planes*, to be consistent with the terminology we have chosen for $c.c^*$-geometries. According to [21], we call Γ the *enrichment* of Π.

Returning to Π_ε, let Γ_ε be its enrichment. Γ_ε is a $c.c^*$-geometry of order $q-2 = 2^n-2$. The centralizer G of ε in $P\Gamma L_3(q)$ has the following structure

$$G = H \cdot ((K_1 \times K_2) \cdot \Gamma L_1(q))$$

with H the group of elations of center p and axis l and $K_1 \cong K_2 \cong 2^n$. It is not difficult to check that G acts flag-transitively on Γ_ε with kernel $\langle\varepsilon\rangle$. Therefore $G/\langle\varepsilon\rangle$ is a flag-transitive automorphism group of Γ_ε (compare [4], Example 6).

Let us write H_ε for $H/\langle\varepsilon\rangle$ and G_ε for $G/\langle\varepsilon\rangle$, for short. When $n = 2$, a theorem of Levefre-Percsy and Van Nypelseer [18] implies that Γ_ε is isomorphic to the truncated D_4 Coxeter complex. In this case it is clear that $G_\varepsilon = \mathrm{Aut}(\Gamma_\varepsilon)$.

Assume $n > 2$. We shall prove in Section 5.1.2 that $\Gamma_\varepsilon/H_\varepsilon$ is a non-canonical gluing. Hence $G/H = \mathrm{Aut}(\Gamma_\varepsilon/H_\varepsilon)$ by Theorem 6(iii) and because H_ε is normal in G_ε. Therefore G_ε is the normalizer of H_ε in $\mathrm{Aut}(\Gamma_\varepsilon)$. On the other hand, H_ε is normal in $\mathrm{Aut}(\Gamma_\varepsilon)$, as we shall prove in a few lines. Therefore,

$$G_\varepsilon = \mathrm{Aut}(\Gamma_\varepsilon)$$

Thus, let us prove that H_ε is normal in $A = \mathrm{Aut}(\Gamma_\varepsilon)$. “Being non-collinear” is an equivalence relation on the set of points of Γ_ε with 2^n classes of size 2^{n-1}. The group H_ε acts regularly on each of these classes and the stabilizer in G_ε of a point a of Γ acts as a cyclic group on the class X_a containing a, with at least one orbit of size n. Consequently, the stabilizer A_a of a in A has at least one orbit of size $\geq n$ on X_a. On the other hand, it acts faithfully on the residue of a ([2], Lemma 2.1) and it is doubly-transitive on the set of planes incident with a. Thus, A_a is a doubly-transitive group of degree 2^n. It also has at least one orbit of size $\geq n$ on X_a. Exploiting this information and comparing the list of [8], by easy calculations one can see that A_a is almost simple only if it is placed between $L_2(3^r)$ and $P\Gamma L_2(3^r)$, for some positive integer r. If this is the case, then $1 + 3^r = 2^n$. However, $2^n \equiv 0 \pmod 8$ (because we have assumed $n > 2$), whereas $1+3^r \equiv 2$ or 4 (mod 8), according to whether r is even or odd. This contradiction forces A_a to be affine. The same for A_u, with u a plane. Hence $A_x = O_2(A_x)A_{a,u}$ and $O_2(A_x) \leq G_\varepsilon$ for $x = a$ or u.

In G_ε we see that H is the center of $\langle O_2(A_a), O_2(A_u)\rangle$. As $A_{a,u}$ normalizes both $O_2(A_a)$ and $O_2(A_u)$, H is normal both in A_a and in A_u, whence it is normal in $A = \langle A_a, A_u\rangle$.

5.1.2. A flat quotient of $\mathbf{\Gamma_\varepsilon}$. Let us keep the notation of the previous paragraph. $H/\langle\varepsilon\rangle$ defines a quotient $\bar{\Gamma}_\varepsilon$ of Γ_ε, which is flat. The group

$$G/H = (K_1 \times K_2)\Gamma L_1(q)$$

acts flag-transitively on $\bar{\Gamma}_\varepsilon$. By Theorem 6, this forces $\bar{\Gamma}_\varepsilon$ to be a gluing of two copies of $\mathrm{AS}(n, 2)$. Del Fra [11] has proved that when $n \geq 3$ this gluing is non-canonical (hence $\mathrm{Aut}(\bar{\Gamma}_\varepsilon) = G/H$, by Theorem 6).

The argument by Del Fra runs as follows. Give p the coordinates $(0, 0, 1)$ and l the Plücker coordinates $(0, 1, 0)$, and let ε be represented by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We need some notation. Denoting the additive groups of $\mathrm{GF}(q)$ and $\mathrm{GF}(2)$ by $\mathrm{GF}^+(q)$ and $\mathrm{GF}^+(2)$, we set $[\mathrm{GF}(q)]_2 = \mathrm{GF}^+(q)/\mathrm{GF}^+(2)$ and, given $x \in \mathrm{GF}(q)$, by $[x]_2$ me mean the image of x by the projection of $\mathrm{GF}^+(q)$ onto $[\mathrm{GF}(q)]_2$.

It is not difficult to see that the points and the planes of Γ_ε are represented by pairs $(x, x') \in \mathrm{GF}(q) \times [\mathrm{GF}(q)]_2$, a point (a, a') being incident with a plane (u, u') precisely when $[au]_2 + a' + u' = 0$.

An unordered pair of pairs $\{(a, a'), (b, b')\}$ with $a \neq b$ represents a pair of coplanar points of Γ_ε, namely a line of Γ_ε. The two planes on that lines are represented by the two solutions in $\mathrm{GF}(q) \times [\mathrm{GF}(q)]_2$ of the following system of equations:

$$[ax]_2 + a' + x' = 0$$
$$[bx]_2 + b' + x' = 0$$

Note that the two solutions (u, u'), (v, v') of this system satisfy the relation $(u + v)(a + b) = 1$.

The projection of Γ_ε onto $\bar{\Gamma}_\varepsilon$ maps a point (a, a') onto $a \in \mathrm{GF}(q)$ and a plane (u, u') onto $u \in \mathrm{GF}(q)$.

Let the points (a, a'), (b, b') form a line and let (u, u'), (v, v') be the two planes on that line. The image of that line in $\bar{\Gamma}_\varepsilon$ can be represented as a pair $(\{a, b\}, \{u, v\})$, where $a \neq b$, $u \neq v$ and $(a + b)(u + v) = 1$. On the other hand, every such pair represents a line of $\bar{\Gamma}_\varepsilon$.

Note that $\mathrm{GF}^+(q)$ can also be viewed as a copy of the n-dimensional vector space $V(n, 2)$ over $\mathrm{GF}(2)$. The non-zero elements of $\mathrm{GF}(q)$ are the non-zero vectors of $V(n, 2)$. Hence they correspond to the points of $\mathrm{PG}(n - 1, 2)$. Thus, the above description of $\bar{\Gamma}_\varepsilon$ amounts to the following. The vectors of $V(n, 2)$ give us both the points and the planes of $\bar{\Gamma}_\varepsilon$. The lines of $\bar{\Gamma}_\varepsilon$ are obtained by pairing two lines $e_1 = \{a, b\}$ and $e_2 = \{u, v\}$ of $\mathrm{AS}(n, 2)$, in such a way that $(a + b)(u + v) = 1$ in $\mathrm{GF}(q)$. However, $a + b$ and $u + v$ represent the points at infinity $\infty(e_1)$ and $\infty(e_2)$ of e_1 and e_2. Thus, two lines e_1, e_2 of $\mathrm{AG}(n, 2)$ are paired to form a line of $\bar{\Gamma}_\varepsilon$ whenever $\infty(e_2) = \infty(e_1)^{-1}$ in $\mathrm{GF}(q)$.

Therefore $\bar{\Gamma}_\varepsilon$ is a gluing of two copies of $\mathrm{AS}(n, 2)$ and the permutation α of $\mathrm{PG}(n-1, 2) = \mathrm{GF}(q)\backslash\{0\}$ we use for this gluing maps every element of $\mathrm{GF}(q)\backslash\{0\}$ onto its inverse in $\mathrm{GF}(q)$. When $n \geq 3$, no element of $L_n(2)$ behaves like that. Therefore, when $n \geq 3$ this gluing is non-canonical.

(The same conclusion cannot be drawn when $n = 2$, as $L_2(2) = \mathrm{Sym}(3)$. In fact, as we noticed at the beginning of Section 2.3, there is only one gluing of two copies of $\mathrm{AG}(2, 2)$, namely the canonical one.)

Remark We noticed in Section 2.3 that there is only one flag-transitive non-canonical gluing of two copies of $\mathrm{AS}(3, 2)$. That gluing is $\bar{\Gamma}_\varepsilon$. Here is another way to construct it.

Let P be the set of points of $\mathrm{PG}(2, 2)$. It is well known that $\mathrm{PG}(2, 2)$ admits a sharply flag-transitive automorphism group $F = \mathrm{Frob}(21)$ and that, for every point $p \in P$, the stabilizer of p in F has two orbits of size 3 on $P\backslash\{p\}$. One of them is a line. The other one is a non-degenerate conic, say C_p. It is not difficult to check that $(P, \{C_p\}_{p\in P})$ is a model of $\mathrm{PG}(2, 2)$. Therefore, there is a permutation β of P that maps the lines of $\mathrm{PG}(2, 2)$ onto the conics C_p, $(p \in P)$. Clearly, $L_3(2) \cap \beta L_3(2)\beta^{-1} = F$, which is transitive on P. Therefore, the gluing of $\mathrm{AS}(3, 2)$ with itself via β is flag-transitive, by Proposition 2. Clearly, $\beta \notin L_3(2)$. Hence that gluing is not the canonical one. Thus, it is isomorphic to $\bar{\Gamma}_\varepsilon$.

Let α be the permutation of P mapping every element of $P = \mathrm{GF}(8)\backslash\{0\}$ onto its inverse in $\mathrm{GF}(8)$. Then $\beta = f\alpha g$ for suitable $f, g \in L_3(2)$, by Proposition 3.

5.1.3. A conjecture. When $n = 2$ the elation semi-biplane Γ_ε is isomorphic to the truncated D_4 Coxeter complex, which is simply connected. It will turn out from the results of Section 5.3.2 that Γ_ε is simply connected when $n \leq 6$. Furthermore, the first author has obtained the following partial result: Γ_ε is simply connected when $2^n - 1$ is prime. Thus, it is quite natural to conjecture that Γ_ε is always simply connected.

5.2. *The number of examples with $X = \Gamma L_1(2^n)$*

Theorem 16 *The number of non-canonical gluings Γ as in* (iii) *of Theorem* 6 *with Aut*$(\Gamma)/(K_1 \times K_2) \cong \Gamma L_1(2^n)$ *is equal to*

$$\frac{\varphi(2^n - 1)}{n} - 1$$

with φ the Eulerian function (*i.e.*, $\varphi(2^n - 1)$ *is the number of positive integers less then* $2^n - 1$ *and relatively prime with* $2^n - 1$).

Proof: The isomorphism classes of gluings of two copies of $\mathrm{AS}(n, 2)$ bijectively correspond to the double cosets $A^\infty\alpha A^\infty$ of $A^\infty = L_n(2)$ in $\mathrm{Sym}(2^n - 1)$ (Proposition 3). Given a permutation $\alpha \in \mathrm{Sym}(2^n - 1)$, the automorphism group of the gluing obtained by α is $(K_1 \times K_2)X$ with $X = A^\infty \cap \alpha A^\infty\alpha^{-1}$ (Proposition 2).

Given any two permutations $\alpha, \beta \in \mathrm{Sym}(2^n - 1)$, if $A^\infty \alpha A^\infty = A^\infty \beta A^\infty$ then $\alpha A^\infty \alpha^{-1}$ and $\beta A^\infty \beta^{-1}$ are conjugated by an element of A^∞. On the other hand A^∞, being a copy of $L_n(2)$, is its own normalizer in $\mathrm{Sym}(2^n - 1)$. Therefore, if $a\alpha A^\infty \alpha^{-1} a^{-1} = \beta A^\infty \beta^{-1}$ for some $a \in A^\infty$, then $\beta^{-1} a\alpha \in A^\infty$, whence $A^\infty \alpha A^\infty = A^\infty \beta A^\infty$. Consequently, $A^\infty \alpha A^\infty = A^\infty \beta A^\infty$ if and only if $\alpha A^\infty \alpha^{-1}$ and $\beta A^\infty \beta^{-1}$ are conjugated by an element of A^∞.

By the above, the gluings of two copies of $\mathrm{AS}(n, 2)$ bijectively correspond to the orbits of A^∞ on the set of conjugates of A^∞ in $\mathrm{Sym}(2^n - 1)$. In particular, the orbit $O_0 = \{A^\infty\}$ corresponds to the canonical gluing, whereas the gluings with $\mathrm{Aut}(\Gamma)/(K_1 \times K_2) = \Gamma L_1(2^n)$ correspond to the orbits whose members intersect A^∞ in a subgroup isomorphic to $\Gamma L_1(2^n)$. Denoted the family of these orbits by $\mathcal{C}$, let us set $\mathcal{C}_0 = \mathcal{C} \cup \{O_0\}$.

Given a copy X of $\Gamma L_1(2^n)$ in A^∞, let S_X be its cyclic subgroup of order $2^n - 1$. The subgroup S_X is generated by a Singer cycle of $A^\infty \cong L_n(2)$ and X is its normalizer in A^∞. Hence X is its own normalizer in A^∞. Moreover, the subgroups generated by Singer cycles form one conjugacy class in $L_n(2)$. Therefore, all subgroups of A^∞ isomorphic to $\Gamma L_1(2^n)$ are conjugated with X in A^∞. Consequently, given $O \in \mathcal{C}$, some members of O intersect A^∞ in X. Let $\alpha A^\infty \alpha^{-1}$ be one of them. Then $g\alpha X \alpha^{-1} g^{-1} = X$ for some $g \in \alpha A^\infty \alpha^{-1}$. Let $f = \alpha^{-1} g \alpha$. Then $g\alpha = \alpha f$ and $X = \alpha f X f^{-1} \alpha^{-1}$. Thus, by replacing α with αf if necessary, we can assume that $\alpha X \alpha^{-1} = X$.

Assume that $a\alpha X \alpha^{-1} a^{-1}$ also intersects A^∞ in X, for some $a \in A^\infty$. Then $a\alpha A^\infty \alpha^{-1} a^{-1}$ contains both X and aXa^{-1}. On the other hand, both X and aXa^{-1} are contained in A^∞ and $a\alpha A^\infty \alpha^{-1} a^{-1} \cap A^\infty = X$. Therefore $X = aXa^{-1}$. However, X is its own normalizer in A^∞. Hence $a \in X$. Consequently, $a\alpha A^\infty \alpha^{-1} a^{-1} = \alpha A \alpha^{-1}$, because $a \in X \subseteq \alpha A^\infty \alpha^{-1}$.

Thus, there is precisely one element of O intersecting A^∞ in X and, if $\alpha A^\infty \alpha^{-1}$ is that element, we can assume that $\alpha X \alpha^{-1} = X$. The permutation α, acting by conjugation on X, determines an automorphism γ_α of X. Let β be another permutation such that $\beta A^\infty \beta^{-1} = \alpha A^\infty \alpha^{-1}$ and $\beta X \beta^{-1} = X$. Then $\beta = a\alpha$ for some $a \in A^\infty$ because A^∞ is its own normalizer in $\mathrm{Sym}(2^n - 1)$. Furthermore, $aXa^{-1} = X$ because both β and α stabilize X. Hence $a \in X$, since X is its own normalizer in A^∞. Consequently, γ_α and γ_β represent the same element of the outer automorphism group $\mathrm{Out}(X)$ of X. Let us denote that element by $\gamma(O)$. Thus, we have defined a mapping $\gamma : \mathcal{C} \to \mathrm{Out}(X)$. We extend it to $\mathcal{C}_0$ by stating that $\gamma(O_0)$ is the identity of $\mathrm{Out}(X)$.

Clearly, every automorphism of X is induced by some permutation $\alpha \in \mathrm{Sym}(2^n - 1)$ normalizing X. This implies that the above mapping γ is surjective. As $|\mathrm{Out}(X)| = \varphi(2^n - 1)/n$, in order to finish the proof we only need to prove that γ is injective.

Let $\alpha A^\infty \alpha^{-1}, \beta A^\infty \beta^{-1}$ be conjugates of A^∞ with $\alpha X \alpha^{-1} = \beta X \beta^{-1} = X$ and assume that $(\gamma_\beta)^{-1} \gamma_\alpha$ is an inner automorphism of X. Then $g^{-1} \beta^{-1} \alpha$ centralizes X, for some $g \in X$. In particular, $g^{-1} \beta^{-1} \alpha$ centralizes the cyclic subgroup S_X of X of order $2^n - 1$. Therefore, $g^{-1} \beta^{-1} \alpha \in S_X$, that is $\alpha = \beta f$ for some $f \in S_X$. Hence $\alpha A^\infty \alpha^{-1} = \beta A^\infty \beta^{-1}$. Thus, γ is injective. □

Remark The first author has proved that the $-1 + \varphi(2^n - 1)/n$ non-canonical gluings mentioned in Theorem 16 have non-isomorphic universal covers. We are not going to prove this result here.

5.3. A report on the cases of $n = 3, 4, 5, 6$

The possibilities for (iii) of Theorem 6 can be checked case-by-case by writing feasible sets of relations and computing the size of the amalgams by CAYLEY. We have done this work for $n = 3, 4, 5$ and (partially) for $n = 6$. We shall now report on the results we have obtained.

5.3.1. Preliminaries. Let $G = (K_1 \times K_1)X$ be a flag-transitive subgroup of $\mathrm{Aut}(\Gamma)$, with Γ a gluing of two copies of $\mathrm{AS}(n, 2)$, $X \leq \Gamma L_1(2^n)$, $K_1 \cong K_2 \cong V(n, 2)$ and $n \geq 3$. (Note that we are not assuming that $G = \mathrm{Aut}(\Gamma)$. In particular, if the gluing Γ is the canonical one, then G is a proper subgroup of $\mathrm{Aut}(\Gamma) = (K_1 \times K_2)L_n(2)$.)

The flag-transitivity of G amounts to the transitivity of X on the non-zero vectors of each of the two copies K_1 and K_2 of $V(n, 2)$. Thus, let X be such a subgroup of $\Gamma L_1(2^n)$ and let v, w be non-zero vectors of K_1 and K_2 respectively.

Then $\langle v, X\rangle$ is the stabilizer G_x in G of a plane x of Γ and $\langle w, X\rangle$ is the stabilizer in G of a point p incident with x. The subgroup X is the stabilizer of the flag $\{p, x\}$. If l is a line of Γ incident with p and x, its stabilizer G_l is generated by a non-zero vector $v' \in K_1$, a non-zero vector $w' \in K_2$ and a suitable subgroup Y of index $2^n - 1$ in X. We can assume to have chosen l in such a way that $v' = v$ and $w = w'$. Thus, $G_l = \langle v, w, Y\rangle$.

In order to search for examples we need to choose X and Y and to fix their actions on K_1 and K_2. We get a set of relations, we search for the group $\tilde{G}$ presented by it and, in the non-collapsing cases, we determine the geometry $\tilde{\Gamma}$ associated with $\tilde{G}$.

By Theorem 6, we have $\tilde{\Gamma} = \mathrm{Tr}(\Delta_{2^n})$ when G is a subgroup of $(K_1 \times K_2)L_n(2)$. As we saw in Section 5.1.2, flat quotients of elation semi-biplanes are non-canonical gluings and their automorphism group is $(K_1 \times K_2)\Gamma L_1(2^n)$. Thus, we also get universal covers of elation semi-biplanes for $X = \Gamma L_1(2^n)$ and for a suitable choice of its actions on K_1 and K_2. Furthermore, we also know in advance how many flat examples exist with $X = \Gamma L_1(2^n)$, by Theorem 16. Let us consider these, to begin with.

5.3.2. The case of $X = \Gamma L_1(2^n)$. Let $X = \Gamma L_1(2^n)$. This group is generated by two elements c and f of order $2^n - 1$ and n, respectively. Thus,

$$G_p = \langle w, c, f\rangle, \quad G_x = \langle v, c, f\rangle, \quad G_l = \langle v, w, f\rangle$$

and $Y = \langle f\rangle$. The generators v, w, c, f satisfy the following relations:

$$\begin{aligned} v^2 = w^2 = c^{2^n-1} = f^n &= 1 \\ \left[v, v^{c^i}\right] &= 1 \qquad (i = 1, 2, \ldots, n-1) \\ \left[w, w^{c^i}\right] &= 1 \qquad (i = 1, 2, \ldots, n-1) \\ [v, f] = [w, f] &= 1 \\ c^f &= c^2 \\ v^{p(c)} = w^{q(c)} &= 1 \\ [v, w] &= 1 \end{aligned}$$

with $p(t)$ and $q(t)$ polynomials of degree n irreducible over GF(2) and not dividing $t^a - 1$ for any proper divisor a of $2^n - 1$. As we said above, the group $\tilde{G}$ presented by these relations, if it does not collapse, defines the universal cover $\tilde{\Gamma}$ of Γ. The group $G \leq \mathrm{Aut}(\Gamma)$ which we started from is obtained from $\tilde{G}$ by factorizing over the subgroup generated by the following commutators

$$[v^{c^i}, w^{c^j}], \qquad (i, j = 1, 2, \ldots, n-1)$$

The polynomials $p(t)$ and $q(t)$ depend on the choice of c. Thus, we can fix one of them as we like, compatibly with the above conditions. Let $p(t)$ be the one we fix. Then we try all possibilities for $q(t)$. Note that when $q(t) = p(t)$ we get $G \leq (K_1 \times K_2)L_n(2)$. That is, the canonical gluing corresponds to the choice of $q(t) = p(t)$.

When $n = 3$ we can choose $p(t) = t^3 + t + 1$. Then $q(t) = t^3 + t^2 + 1$ is the only choice for $q(t) \neq p(t)$. In this case Γ is the flat quotient of the elation semi-biplane of order 6 and $\tilde{\Gamma}$ is its universal cover. Coset enumeration shows that $|\tilde{G}| = 2^8 21 = 4|G|$. Hence $\tilde{\Gamma}$ is a 4-fold cover of Γ. Thus, $\tilde{\Gamma}$ is the elation semi-biplane of order 6.

When $n = 4$ we can take $p(t) = t^4 + t + 1$. Then either $q(t) = p(t)$ or $q(t) = t^4 + t^3 + 1$. Chosen $t^4 + t^3 + 1$ as $q(t)$, the geometry Γ is the flat quotient of the elation semi-biplane of order 14 and $\tilde{\Gamma}$ is its universal cover. We now have $|\tilde{G}| = 8|G|$. Therefore $\tilde{\Gamma}$ is the elation semi-biplane of order 14 (as above).

When $n = 5$ we can take $p(t) = t^5 + t^2 + 1$. Then the following are the only choices for $q(t) \neq p(t)$:

$$\begin{array}{l} t^5 + t^3 + 1 \\ t^5 + t^4 + t^3 + t^2 + 1 \\ t^5 + t^4 + t^2 + t + 1 \\ t^5 + t^3 + t^2 + t + 1 \\ t^5 + t^4 + t^3 + t + 1 \end{array}$$

In the first case Γ is the flat quotient of the elation semi-biplane of order 30 (the action of c on K_2 is the inverse of that on K_1). Again, the elation semibiplane of order 30 is the universal cover of Γ.

In the remaining four cases $\tilde{\Gamma}$ has 2^{10} points (thus, it is a 32-fold cover of Γ). Theorem 16 says that the four flat geometries corresponding to these four cases are pairwise non-isomorphic.

Let $n = 6$. We now take $p(t) = t^6 + t + 1$ and the following are the possibilities for $q(t) \neq p(t)$:

$$\begin{array}{l} t^6 + t^5 + 1 \\ t^6 + t^4 + t^3 + t + 1 \\ t^6 + t^5 + t^4 + t + 1 \\ t^6 + t^5 + t^3 + t^2 + 1 \\ t^6 + t^5 + t^2 + t + 1 \end{array}$$

In the first case Γ is the flat quotient of the elation semi-biplane of order 62 (the actions of c on K_1 and K_2 are mutually inverse). It turns out that the elation semibiplane of order 62 is the universal cover of Γ (a 32-fold cover, in fact).

In two of the remaining four cases $\tilde{\Gamma}$ has 2^{13} points, whereas it has 2^{16} points in the other two cases. The four flat geometries corresponding to these four cases are pairwise non-isomorphic, by Theorem 16.

5.3.3. An example with $X < \Gamma L_1(2^n)$. In order to get examples of gluings different from those considered in the previous subsection we need a subgroup X of $\Gamma L_1(2^n)$ transitive on the $2^n - 1$ non-zero vectors of $V(n, 2)$ but not containing the cyclic subgroup of $\Gamma L_1(2^n)$ of order $2^n - 1$. No subgroup exists with these properties when $2^n - 1$ and n are relatively prime. Thus, by Theorem 16 we get the following.

Corollary 17 *Let $2^n - 1$ and n be relatively prime. Then*

$$\frac{\varphi(2^n - 1)}{n} - 1$$

is the total number of flag-transitive non-canonical gluings of two copies of $AS(n, 2)$. If Γ is any of them, then $Aut(\Gamma) = (K_1 \times K_2)\Gamma L_1(2^n)$.

In particular, when $n \leq 5$ no flag-transitive non-canonical gluings exist besides those considered in the previous subsection. This is no more true when $n = 6$, as we shall show now.

Let $n = 6$. Given elements c and f of $\Gamma L_1(2^6)$ of order 63 and 6 respectively and such that $c^f = c^2$, let $a = c^3$, $b = cf^2$ and $X = \langle a, b\rangle$. Then a and b have order 21 and 9 respectively, $a^b = a^4$ and $b^3 = a^7$. Thus, $c \notin X$ and $X \cong Z_{21}Z_3$.

Let $v \in K_1$ and $w \in K_2$ so that $[v, f] = [w, f] = 1$ and let $\tilde{G}$ be the group presented by the following relations:

$$\begin{aligned} v^2 = w^2 = a^{21} = b^9 &= 1 \\ \left[v, v^{a^i}\right] &= 1 \qquad (i = 1, 2, 3, 4, 5) \\ \left[w, w^{a^i}\right] &= 1 \qquad (i = 1, 2, 3, 4, 5) \\ v^{p(a)} = w^{p(a)} &= 1 \\ v^{r(a)}v^b = w^{r(a)}w^b &= 1 \\ [v, w^a] &= 1 \end{aligned}$$

with

$$\begin{aligned} p(t) &= t^6 + t^5 + t^4 + t^2 + 1 \\ r(t) &= t^4 + t^3 + 1 \end{aligned}$$

Let us consider the following subgroups of $\tilde{G}$:

$$G_x = \langle v, a, b\rangle = K_1 X, \qquad \text{where } K_1 = \langle v^X\rangle,\ X = \langle a, b\rangle,$$
$$G_p = \langle w, a, b\rangle = K_2 X, \qquad \text{where } K_2 = \langle w^X\rangle \quad \text{and}$$
$$G_l = \langle v, w^a\rangle.$$

Clearly, G_p and G_x are subgroups of $A\Gamma L_1(2^n) = 2^n \Gamma L_1(2^n)$, $G_l = 2^2$ and the coset geometry $\Gamma(\tilde{G}, (G_p, G_l, G_x)$ defines a simply connected $c.c^*$-geometry $\tilde{\Gamma}$ of order 62 with $\tilde{G}$ as a flag-transitive automorphism group.

The size of $\tilde{G}$ can be computed by coset enumeration. It turns out that $|\tilde{G}| = 2^{18}3^2 7$. Therefore $\tilde{\Gamma}$ has 2^{12} points. So in particular $\tilde{\Gamma}$ is not the D_{64}-truncation. (This can also be seen using [3], Corollary (3.5)). On the other hand, the subgroup N of $\tilde{G}$ generated by the commutators $[v^{a^i}, w^{b^j}]$ $(i, j = 1, 2, 3, 4, 5)$ is normal in $\tilde{G}$ and it has trivial intersections with each of G_p, G_x and G_l. Thus it defines a flag-transitive quotient $\Gamma = \tilde{\Gamma}$ of Γ. Furthermore, $|\tilde{G} : N| = 2^6$. Hence Γ has 2^6 points. That is, Γ is flat. By Theorem 6, Γ is a gluing of two copies of AS(6, 2).

Denoted $\tilde{G}/N$ by G, we have $\mathrm{Aut}(\Gamma) \geq G = (K_1 \times K_2)X$.

Statement 18 *We have Aut*$(\tilde{\Gamma}) = \tilde{G}$ *and Aut*$(\Gamma) = G$.

Proof: We show $\mathrm{Aut}(\tilde{\Gamma}) = \tilde{G}$. As each automorphism of Γ can be lifted to an automorphism of $\tilde{\Gamma}$, we then obtain $\mathrm{Aut}(\Gamma) = G$ as well.

Assume $A = \mathrm{Aut}(\tilde{\Gamma}) > \tilde{G}$. Since for p a point and c a plane $A_p \cong A_c$ are doubly-transitive permutation groups, we have $A_p \cong A_c \cong A_{64}, S_{64}, 2^6 L_6(2)$ or $A_p \cong A_c$ is isomorphic to a subgroup of $2^6\Gamma L_1(64)$, see [8] and [19]. As $\tilde{\Gamma}$ is not the D_{64}-truncation, $A_p \cong A_c$ are not isomorphic to A_{64} or S_{64}, see [4].

Let B be the Borel subgroup of A and let F be a flag in $\tilde{\Gamma}$. Then for the stabilizers $\tilde{G}_F$, A_F of F in $\tilde{G}$ and A, respectively, we have $A_F = \tilde{G}_F B$. Hence B normalizes K_1 and K_2, which gives $[B, v] = [B, w^a] = 1$ and $N_B(X) \neq 1$. Let $h \in N_B(X)\backslash\{1\}$. Then on one hand $h \in N_{\mathrm{Aut}(K_i)}(\langle c, f\rangle) = X$, $i = 1, 2$, and on the other hand $[v, h] = [w^a, h] = 1$. Since $C_X(\langle v\rangle) = \langle f\rangle$ and $C_X(\langle w^a\rangle) = \langle f^a\rangle$ we obtain $[K_1, h^{-1} f^i] = 1$ and $[K_2, h^{-1}(f^j)^a] = 1$ for some $i, j \in \{1, \ldots, 6\}$. The Three-subgroup Lemma, [1] (8.7), yields $[\langle c, f\rangle, h^{-1} f^i] = 1$ and $[\langle c, f\rangle, h^{-1}(f^j)^a] = 1$. So f^i and $(f^j)^a$ are inducing the same automorphism on X. Thus $f^{-i}(f^j)^a = f^{j-i}a^{-2^j+1}$ centralizes X. Since $C_{\langle c, f\rangle}(X) = \langle a^7\rangle$, we obtain $i = j = 3$ and $B \cong Z_2$.

On the other hand, $\langle X, f^3\rangle = \langle c, f\rangle$ yields $B \cong Z_6$, in contradiction to the above. Hence $A = \mathrm{Aut}(\tilde{\Gamma}) = \tilde{G}$. □

Added in proof. Conjecture 5.1.3 has been answered in the affirmative by the first author and by D. Pasechnik [22].

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