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Perfect Matchings of Cellular Graphs

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Abstract. We introduce a family of graphs, called cellular, and consider the problem of enumerating their perfect matchings. We prove that the number of perfect matchings of a cellular graph equals a power of 2 times the number of perfect matchings of a certain subgraph, called the core of the graph. This yields, as a special case, a new proof of the fact that the Aztec diamond graph of order n introduced by Elkies, Kuperberg, Larsen and Propp has exactly $2^{n(n+1)/2}$ perfect matchings. As further applications, we prove a recurrence for the number of perfect matchings of certain cellular graphs indexed by partitions, and we enumerate the perfect matchings of two other families of graphs called Aztec rectangles and Aztec triangles.

Keywords: perfect matching, alternating sign pattern, Ferrers diagram

1. Introduction

Let S be the graph with vertex set consisting of the lattice points $\{(i, j) \in \mathbf{Z}^2 : i + j \text{ even}\}$, with edges between nearest neighbours. By a *cell* we mean any square formed by a 4-cycle of S whose center has even x-coordinate.

A finite subgraph G of S is called *cellular* if the edges of G can be partitioned into cells. Figure 1 shows an example of a cellular graph.

The *Aztec diamond of order n* is a cellular graph consisting of the union of the cells with centers $(2i, 2j + 1)$ for $i, j \in \{0, 1, \ldots, n - 1\}$. (Technically speaking, this is the dual of the region dubbed the Aztec diamond in [2]). Figure 2 illustrates the case $n = 4$. In [2] there are presented four proofs of the fact that the number of perfect matchings of the Aztec diamond of order n is $\mathrm{AD}(n) = 2^{n(n+1)/2}$.

The present paper was motivated by a short proof we found for this fact, using the interpretation of perfect matchings as placements of non-attacking rooks on a mutilated chessboard. The new proof of the recurrence $\mathrm{AD}(n) = 2^n \mathrm{AD}(n - 1)$ is similar to the first proof given in [2], in that both proofs make use of two formulas for $\mathrm{AD}(n)$ as a weighted sum of alternating sign matrices (see relations (1) and (2) of [2]), but the new argument is shorter and more elementary in that the notion of height functions is not needed. It turns out that a slight variation of this approach allows us to prove a more general result.

Given a cellular graph G, a *horizontal* (resp., *vertical*) *chain* is a cellular subgraph consisting of a maximal connected horizontal (resp., vertical) sequence of cells of G. Define $h(G)$, $v(G)$ to be the number of horizontal and vertical chains of G. For the graph in Figure 1, we have $h(G) = v(G) = 4$.

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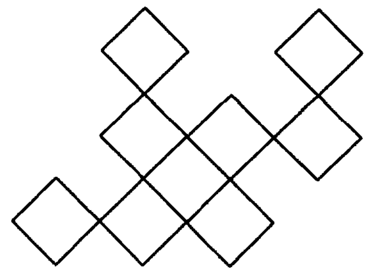

Figure 1. An example of a cellular graph.

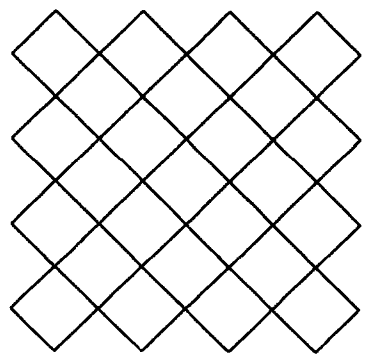

Figure 2. The Aztec diamond of order 4.

By the *endpoints* of a chain L, we mean the leftmost and rightmost vertices of that chain, if L is horizontal, and the uppermost and lowermost vertices, if L is vertical. Define the *core* of G to be the graph G' obtained from G by deleting the endpoints of all chains of G, together with the incident edges (figure 3 shows the core of the graph in figure 1; the vertices of $G \backslash G'$ are circled).

Denote by g and g' the number of perfect matchings of G and G', respectively. The main result of this paper is:

Theorem 1.1 *Let G be a cellular graph. Then $g = 0$ unless $h(G) = v(G)$. Moreover, if this condition holds, then*

$$g = 2^{h(G)} g'.$$

Since the core of the Aztec diamond of order n is the Aztec diamond of order $n - 1$, we have in particular that $\mathrm{AD}(n) = 2^n \mathrm{AD}(n-1)$.

In the following section we prove Theorem 1.1. A key ingredient of the proof is the concept of alternating sign pattern, a natural generalization of the alternating sign matrices introduced by Mills, Robbins and Rumsey in [4].

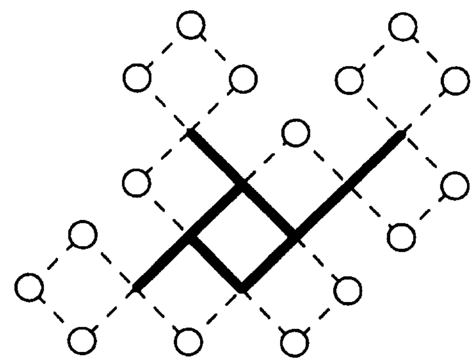

Figure 3. A cellular graph and its core.

By a theorem of Kasteleyn [3], we know that the graph $D_{2m,2n}$ having vertex set $\{1, 2, \dots, 2m\} \times \{1, 2, \dots, 2n\}$ and edges between nearest neighbours has $2^{2mn} \prod_{i=1}^{m} \prod_{j=1}^{n} (\cos^2 \frac{\pi i}{2m+1} + \cos^2 \frac{\pi j}{2n+1})$ perfect matchings. We introduce a family of cellular graphs, called Aztec rectangles, and enumerate their perfect matchings using Theorem 1.1 and Kasteleyn's result. In Section 3 we introduce the cellular Ferrers graphs, which are indexed by partitions, and we derive a recurrence for the number of their perfect matchings. In Section 4 we define a family of graphs called Aztec triangles and we determine the generating function of their perfect matchings. We conclude by giving a 3-dimensional version of Theorem 1.1.

2. The Proof of Theorem 1.1

Let x be any vertex of G. Consider lines through x parallel to the coordinate axes. Exactly one of these lines has the property that it passes through the center of some cells of G. Call x a *row-vertex* if this line is horizontal, and a *column-vertex* if the line is vertical. Let A and B be the sets of row and column vertices of G, respectively. Then any edge of G has one of its endpoints in A and the other in B. So, if G has a perfect matching, then $|A| = |B|$.

For a cellular graph F, denote by $c(F)$ the number of cells of F. Then we have

$$|A| = \sum_{L} (c(L) + 1) = c(G) + h(G),$$

where L ranges over all horizontal chains of G. Similarly, $|B| = c(G) + v(G)$. This proves that G has no perfect matching unless $h(G) = v(G)$.

Our proof of the second part of Theorem 1.1 requires a number of preliminary lemmas.

Given a perfect matching of G, associate to each cell c of G one of the numbers $+1$, 0 or -1 corresponding to the instances when the given perfect matching contains 2, 1 or 0 edges that lie in c. Figure 4 shows a perfect matching of a cellular graph, with the corresponding numbers in the cells.

We notice that in this example the entries form an alternating sign pattern (ASP), i.e., in each horizontal or vertical chain of G, the $+1$'s and -1's alternate, and all chain sums

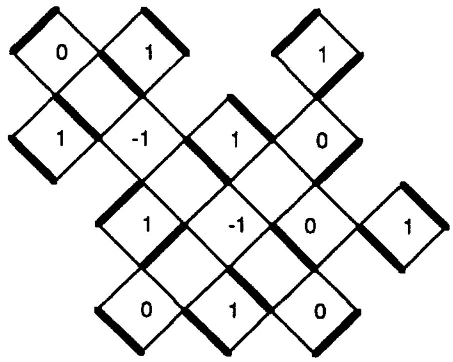

Figure 4. The ASP corresponding to a perfect matching.

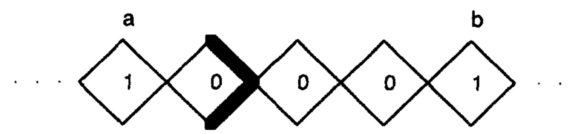

Figure 5. A portion of the chain L.

equal $+1$. It turns out that this is always the case. Denote by ASP(G) the set of alternating sign patterns of shape G.

Lemma 2.1 *The labeling of the cells of G described above is an alternating sign pattern.*

Proof: Fix a chain L of G (horizontal or vertical). We prove first that between any 2 consecutive $+1$'s in L there is a -1. Suppose this is not the case. Then there are 2 $(+1)$-cells a, b of L having only 0-cells between them (see figure 5).

Consider the 0-cell immediately to the right of a. Its unique edge that is part of the perfect matching must be contained in the "eastern hook" of this cell, i.e., in the union of the two edges that face NE and SE. But then the leftmost vertex of the next 0-cell is matched to a vertex external to that cell, so again the unique edge of this cell participating in the perfect matching must be in the eastern hook. Continuing in this manner, we get that the 0-cell immediately to the left of b contributes to the perfect matching with an edge of its eastern hook, a contradiction. So there is a -1 between any two consecutive $+1$'s of L. This implies that every chain sum in the described pattern is at most 1.

Let now x, y and z denote the number of cells of G that are labelled 1, 0 and -1, respectively. Let $V(G)$ be the number of vertices of G. We then have that $x + y + z = c(G)$, and $2x + y = 1/2V(G)$. In particular, if $|L|$ is the sum of the entries in the cells of L,

$$\sum_{L \text{ horizontal}} |L| = x - z = 1/2V(G) - c(G).$$

But since the number of vertices of G is the sum of the number of row-vertices and column-vertices, we have

$$V(G) = \sum_{L \text{ horizontal}} (c(L)+1) + \sum_{L \text{vertical}} (c(L)+1)$$
$$= 2c(G) + h(G) + v(G) = 2(c(G) + h(G)). \tag{1}$$

Therefore,

$$\sum_{L \text{ horizontal}} |L| = h(G), \tag{2}$$

which together with our previous remark that $|L| \leq 1$ for all L shows that the sum along every horizontal (and similarly vertical) chain equals 1. □

Lemma 2.2 *Let A be an alternating sign pattern of shape G. Then the number of perfect matchings of G having sign pattern A is $2^{N_+(A)}$, where $N_+(A)$ is the number of $+1$'s in A.*

Proof: In order to produce a perfect matching of G with associated ASP A, we have to choose two opposite edges from each $(+1)$-cell, one edge from each 0-cell and no edge from the (-1)-cells. There are $2^{N+(A)}$ ways to make this choice for the union of $(+1)$-cells and (-1)-cells. To prove the Lemma, we show that each of these $2^{N+(A)}$ collections of edges can be uniquely completed to a perfect matching by choosing exactly one edge from each of the remaining cells, which are the 0-cells.

By a hook we mean the union of 2 incident edges in a cell. So we have 4 hooks in each cell. According to the direction they point to, we call them western, eastern, northern and southern (abbreviated by W, E, N and S, respectively).

Let us consider a chain L of cells; say L is horizontal (see figure 6). Place a hook on each 0-cell pointing away from the 1-cells. Since A is an ASP, any string of zeroes lies either between a $+1$ and a -1, or between an endpoint of L and a $+1$. This ensures that the hooks we defined and the $(+1)$-cells are disjoint. By an argument similar to that in the proof of Lemma 2.1, the edge we need to pick from each 0-cell in order to complete our perfect matching must lie in the hook placed on that cell. So, considering the horizontal chain through a fixed 0-cell c_0, we get that the edge of this 0-cell participating in the perfect matching must lie in a specific hook of type W or E.

By similar consideration of the vertical chain through c_0, we see that the edge of c_0 participating in the perfect matching must also lie in a specific hook of type N or S.

The key fact is that if α is either the W or E hook of c_0, and β is either the N or S hook of c_0, then $\alpha \cap \beta$ is always a single edge of c_0. This proves that the choice we are allowed to make from the edges of each 0-cell is unique.

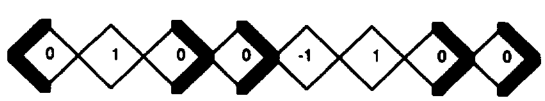

Figure 6. A chain with hooks on the 0-cells.

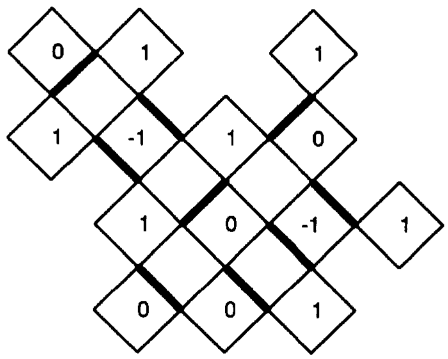

Figure 7. The ASP of shape G induced by a perfect matching of G'.

Also, since both horizontally and vertically the hooks and the $(+1)$-cells are disjoint, all the edges selected in the above way are disjoint. Since we have selected the prescribed number of edges from each cell of G, we have produced a perfect matching. □

We consider now a second labeling of the cells of G, this one induced by a perfect matching of the core graph G'.

In figure 7 G is the same as in figure 4, and we have indicated a perfect matching of G'. The rule for labeling the cells of G is now the following: write -1, 0 or 1 in each cell, corresponding to the cases that the cell contains 2, 1 or 0 edges of the perfect matching of G'.

Lemma 2.3 *The labels described above form an* ASP *of shape* G.

Proof: The proof is similar to that of Lemma 2.1. Let L be a chain in G. One can prove that between any two consecutive $+1$'s in L there is a -1 in a way that is perfectly analogous to the one we used in the proof of Lemma 2.1. Let x, y and z be the number of $(+1)$-, 0- and (-1)-cells in G, respectively. Then we have $x + y + z =$ (# cells of G) and $y + 2z = 1/2$ (# vertices of G'). Therefore,

$$\sum_{L \text{ horizontal}} |L| = x - z = c(G) - 1/2V(G').$$

But

$$V(G') = V(G) - 2h(G) - 2v(G) = V(G) - 4h(G)$$

Therefore, using relation (1), we have

$$\sum_{L \text{ horizontal}} |L| = c(G) - 1/2V(G) + 2h(G) = -h(G) + 2h(G) = h(G).$$

By the argument at the end of the proof of Lemma 2.1, it again follows that we have an ASP. □

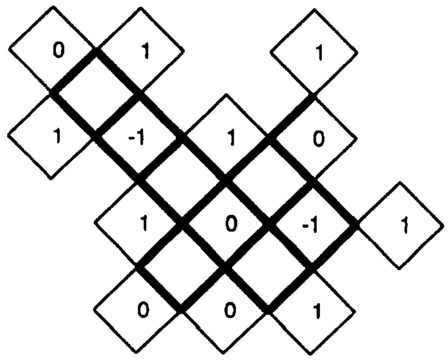

Figure 8. A cellular graph, its core and an ASP.

So we have ASP's of shape G associated to perfect matchings of both G and G'.

Lemma 2.4 *Let A be an* ASP *of shape G. Then the number of perfect matchings of G' with associated* ASP *A is $2^{N_-(A)}$, where $N_-(A)$ is the number of -1's in A.*

Proof: To produce a perfect matching of G' with associated ASP A, we need to pick two edges of G' from each (-1)-cell, one edge of G' from each 0-cell and no edge from the $(+1)$-cells. Therefore we can make this choice in $2^{N_-(A)}$ ways for the union of the $(+1)$-cells and (-1)-cells. The lemma will be proved once we show that each of these collections of edges of G' can be uniquely completed to a perfect matching of G' by selecting one more edge of G' from each 0-cell of G.

This can be proved by nearly the same argument as in the proof of Lemma 2.2. The only difference is that G' omits some edges of the cells of G (see figure 8). So we have to ensure that the unique candidate from each 0-cell for completion of a perfect matching of G' does not lie in the missing hooks. But this is a consequence of the fact that the $(+1)$ and (-1)-cells have now the opposite significance to the one they had in Lemma 2.2, and therefore the hooks in the 0-cells along each chain L are pointing now *towards* the 1's. So in particular if a 0-cell is at one end of L, the corresponding hook will point inside L, thus avoiding the hook that G' omits. □

Proof of Theorem 1.1: By relation (2), we have $N_+(A) - N_-(A) = h(G)$. Therefore, by Lemmas 2.1–2.4 we obtain

$$g = \sum_{A \in \mathrm{ASP}(G)} 2^{N_+(A)} = 2^{h(G)} \sum_{A \in \mathrm{ASP}(G)} 2^{N_-(A)} = 2^{h(G)} g'. \qquad \square$$

Let us now rotate the Aztec diamond of order n by $\pi/4$ and take the origin to be in its center. Let $k, l \geq 1$ be such that $k + l = n$. Define the $k \times l$ *Aztec rectangle* to be the cellular graph consisting of the cells of the Aztec diamond having centers of coordinates (x, y) satisfying $|x| \leq k$ and $|y| \leq l$. Figure 9 shows the case $k = 3, l = 2$.

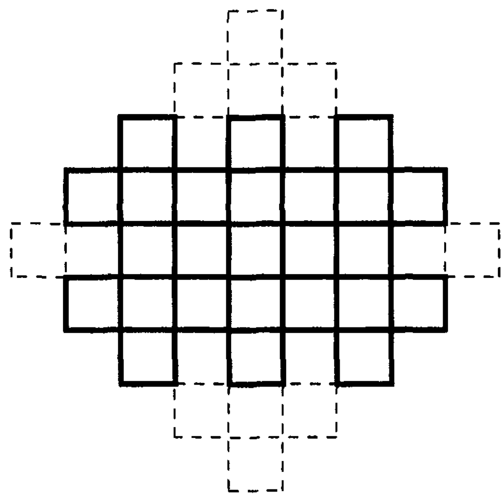

Figure 9. The 3 × 2 Aztec rectangle.

Corollary 2.5 *Let $AR(k, l)$ be the number of perfect matchings of the $k \times l$ Aztec rectangle. Then,*

$$AR(k, l) = 2^{k+l} d_{2k,2l},$$

where $d_{2k,2l}$ is the number of perfect matchings of Kasteleyn's grid graph $D_{2k,2l}$ mentioned in the introduction.

Proof: We apply Theorem 1.1 with G the $k \times l$ Aztec rectangle. Then the core of G is the $2k \times 2l$ grid graph and $h(G) = k + l$. □

Remark The above theorem remains true in a more general setting. Let R be any equivalence relation on the set of vertices of a cellular graph G such that each equivalence class contains at most 2 vertices, and if it does contain two, both are endpoints of chains of the same kind (i.e., both horizontal or both vertical). Let $\tilde{G} = G/R$ be the quotient graph. Then the horizontal (and similarly the vertical) chains of $\tilde{G}$ consist of sequences of horizontal chains of G, and it may happen that we have no endpoints for a chain of $\tilde{G}$. If this is the case, we say that we have a *cycle*, rather than a chain; otherwise we shall still use the term "chain".

With these conventions, let $h(\tilde{G})$ and $v(\tilde{G})$ have the same meaning as before, namely the number of horizontal and vertical chains, respectively. Let $\tilde{G}'$ be obtained from $\tilde{G}$ by removing the endpoints of the chains of $\tilde{G}$.

Then Theorem 1.1 remains true with $\tilde{G}$ in place of G. The proof goes through in the same way. The only difference is that now an ASP A of shape $\tilde{G}$ is an arrangement of the numbers 1, 0 and −1 in the cells of $\tilde{G}$ such that the 1's and −1's alternate along each chain and cycle, and all chain sums are $+1$ (clearly the cycle sums are then 0). Moreover, if a cycle of the ASP contains only zeroes, we have no hooks to place on the cells of that cycle (recall that the hooks were determined by the $+1$'s). Along each such cycle, there are 2 ways of placing disjoint hooks on the 0-cells so that they point along the cycle. Therefore, there are $2^{z(A)}$ ways of completing a perfect matching of the subgraph consisting of the union of the $+1$ and (-1)-cells to a perfect matching of $\tilde{G}$, where $z(A)$ is the number of cycles of A consisting entirely of zeroes. Therefore,

$$\tilde{g} = \sum_{A \in \mathrm{ASP}(\tilde{G})} 2^{N_+(A)} 2^{z(A)}.$$

Similarly,

$$\tilde{g}' = \sum_{A \in \mathrm{ASP}(\tilde{G})} 2^{N_-(A)} 2^{z(A)},$$

and thus the statement of Theorem 1.1 holds for $\tilde{G}$.

Note. As one of the referees has pointed out, the hooks used in the above proofs are related to the "arrows" used in [6] to obtain formulas for the determinant of a matrix in terms of connected minors.

3. Cellular Ferrers graphs

Let us consider now the following special case. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition, denote by A_λ the cellular graph consisting of cells arranged in the shape of the Ferrers diagram of λ, i.e., λ_1 cells in its first row, λ_2 cells in the second row, etc., the rows being left-justified. Figure 10 shows A_λ for $\lambda = (4, 4, 3, 1)$. We will say, for simplicity, that an ASP associated to A_λ has shape λ.

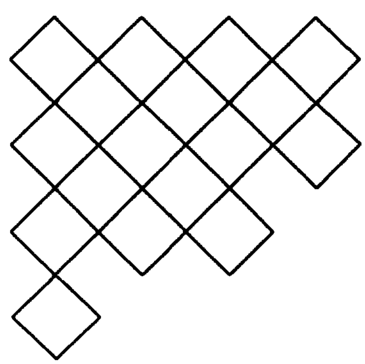

Figure 10. $A_{(4,4,3,1)}$.

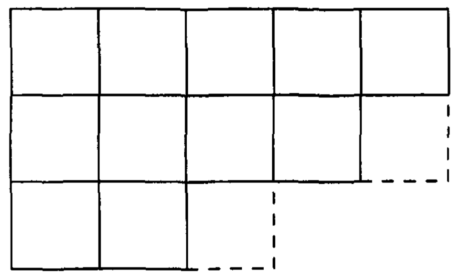

Figure 11. $C(\lambda)$ for $\lambda = (5, 4, 2)$.

Let D_λ denote the Young diagram of λ. The corner cells of D_λ are the unit squares x outside D_λ with the property that $D_\lambda \cup \{x\}$ is a partition diagram in which x does not lie in the first row or in the first column. Let $C(\lambda)$ denote the set of corner cells (corners) of D_λ. Figure 11 indicates this set for $\lambda = (5, 4, 2)$. Then for any subset S of $C(\lambda)$, $D_\lambda \cup S$ is the diagram of a partition, which we will denote by $\lambda \cup S$. Let λ' denote the conjugate partition of λ.

Lemma 3.1 *If λ is a partition, then A_λ has no perfect matching unless $\lambda_1 = \lambda'_1 = n$ for some n, and the diagram of λ contains the staircase shape of size n (i.e., the diagram of the partition $\delta_n = (n, n-1, \ldots, 1)$).*

Proof: We have $\lambda_1 = \lambda'_1$ by Theorem 1.1. Suppose the diagram of λ does not contain the staircase shape. Then there exist $k, l \geq 1$ with $k + l > n$ such that the $k \times l$ rectangle sitting in the lower-right corner of the $n \times n$ square containing D_λ is disjoint from D_λ. Suppose that A_λ has a perfect matching. Consider the ASP of shape λ it induces. By assigning 0's to the unit cells of the $n \times n$ square outside D_λ, we obtain an $n \times n$ matrix M. By Lemma 2.1, M is an alternating sign matrix. Following [4], let $\tilde{M}$ be the $n \times n$ matrix where columns represent the partial sums of the columns of M, starting at the top. Since M is sign-alternating, $\tilde{M}$ contains exactly i 1's in the ith row. On the other hand, since the $k \times l$ lower-right block of M consists of 0's, the $(n-k)$th row of $\tilde{M}$ has 1's in the last l positions. Therefore $l \leq n - k$, so $k + l \leq n$, a contradiction. □

Let λ be a partition with $\lambda_1 = \lambda'_1$ and with diagram containing the staircase shape. Then $\lambda_1 - \lambda_2 \leq 1$ and $\lambda'_1 - \lambda'_2 \leq 1$. Suppose $\lambda_1 - \lambda_2 = 1$. Then any ASP of shape λ must have a 1 in the last cell of the first row. Thus we have (writing a_λ for the number of perfect matchings of A_λ)

$$a_\lambda = \sum_{A \in \mathrm{ASP}(\lambda)} 2^{N_+(A)} = 2 \sum_{A \in \mathrm{ASP}(\mu)} 2^{N_+(A)} = 2\, a_\mu$$

where μ is the partition whose diagram is obtained from D_λ by deleting its first row. A similar statement is true if $\lambda'_1 - \lambda'_2 = 1$.

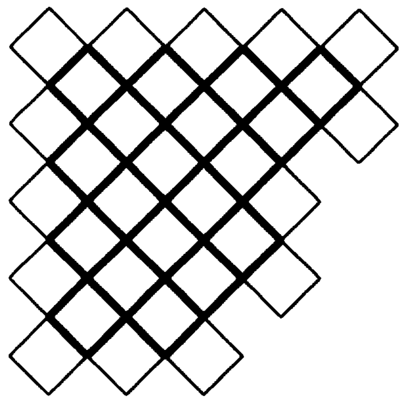

Figure 12. A_λ and its core for $\lambda = (5, 5, 4, 4, 3)$.

Therefore, for the purpose of determining the a_λ's recursively, the only interesting case is when $\lambda_1 = \lambda_2 = \lambda'_1 = \lambda'_2$.

Let λ be such a partition. Let μ be the partition whose diagram is obtained from D_λ by cutting off the first row and column. Denote by $\bar{A}_\mu$ the graph obtained from A_μ by adding $|C(\mu)|$ more edges in the positions corresponding to the corners of μ. Notice that the core of A_λ is isomorphic to $\bar{A}_\mu$. Figure 12 illustrates this for $\lambda = (5, 5, 4, 4, 3)$. Let $\bar{a}_\mu$ be the number of perfect matchings of $\bar{A}_\mu$.

Theorem 3.2 *Let λ, μ, $\bar{A}_\mu$ be as above and let $n = \lambda_1 = \lambda_2 = \lambda'_1 = \lambda'_2$. Then*

$$a_\lambda = 2^n\, \bar{a}_\mu = 2^{n-|C(\mu)|} \sum_{S \subseteq C(\mu)} a_{\mu \cup S}.$$

Proof: Apply Theorem 1.1 with G taken to be A_λ. Since G' is precisely $\bar{A}_\mu$, the first equality follows. The second equality follows by applying Lemma 3.3 to each of the $|C(\mu)|$ edges of $\bar{A}_\mu - A_\mu$. □

Lemma 3.3 *Let e be an edge with endpoints a and b in the graph G. Let G^+ be the graph obtained from G by adding two new vertices u and v and three new edges $\{a, u\}$, $\{u, v\}$ and $\{v, b\}$. Let g^+, g^- be the number of perfect matchings of G^+ and $G \backslash \{e\}$, respectively. Then*

$$g = 1/2(g^+ + g^-).$$

Proof: Let m be a perfect matching of G^+. Then either $\{u, v\} \in m$, or $\{a, u\}, \{b, v\} \in m$ and the two cases are mutually exclusive. The perfect matchings m of the first case are in bijection with the perfect matchings of G, while those for which the second is true are in bijection with the perfect matchings of G in which $\{a, b\}$ appears. Denoting the number

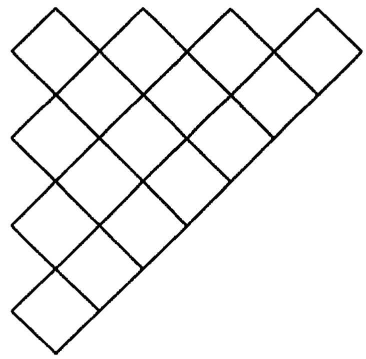

Figure 13. The Aztec triangle of size 4.

of these last ones by x, we get $g = g^+ - x$. But clearly we also have that $g = g^- + x$, implying the statement of the lemma. □

4. The Aztec triangle

Define the *Aztec triangle* of size n to be the graph $\bar{A}_\delta$, for $\delta = (n, n-1, \ldots, 1)$ (figure 13 shows this for $n = 4$).
Let T_n denote the number of its perfect matchings and define $T_0 = 1$.

Theorem 4.1 *We have*

$$\sum_{n \geq 0} T_n x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} = 1 + 2x + 6x^2 + 22x^3 + 90x^4 + \cdots.$$

To give the proof, we need the following result. Let λ be a partition containing the staircase shape $(n, n-1, \ldots, 1)$ such that $\lambda_1 = \lambda'_1 = n$. The diagram of λ is contained in a $\lambda_1 \times \lambda_1$ square S. A critical point of D_λ is a vertex of a unit box of D_λ which lies both on the boundary of D_λ and on the SW-NE diagonal of S. The critical points of D_λ determine critical blocks: for A, B two consecutive critical points, the critical block determined by them is the partition diagram obtained by intersecting D_λ with the square having AB as its SW-NE diagonal. (Figure 14 illustrates this for $\lambda = (6, 5, 5, 4, 2, 2)$).

Lemma 4.2 *Let λ be a partition as above. Then $a_\lambda = \prod_\mu a_\mu$, where the product ranges over the critical blocks μ of λ.*

Proof: It is enough to show that if x is a critical point, then $a_\lambda = a_{\mu_1} a_{\mu_2}$ where μ_1 and μ_2 are the regions of D_λ lying below the horizontal line through x and to the right of the vertical line through x, respectively (these are clearly partition diagrams; see figure 15).

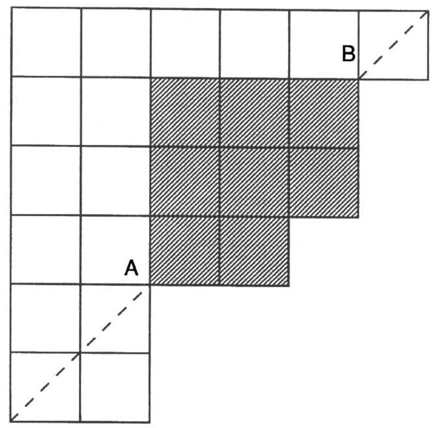

Figure 14. The critical block determined by two critical points.

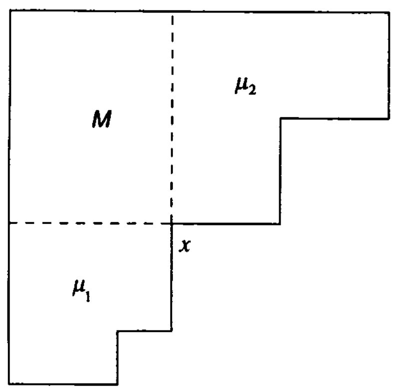

Figure 15. The partition diagrams determined by a critical point of λ.

Let A be the ASP associated to some perfect matching of A_λ.

Let M be the rectangle for which $D_\lambda = D_{\mu_1} \cup D_{\mu_2} \cup M$, and let A_0 be the part of A inside M. We claim that A_0 consists only of zeroes. This implies

$$a_\lambda = \sum_{A \in \mathrm{ASP}(\lambda)} 2^{N_+(A)} = \sum_{A_1 \in \mathrm{ASP}(\mu_1), A_2 \in \mathrm{ASP}(\mu_2)} 2^{N_+(A_1)} 2^{N_+(A_2)} = a_{\mu_1} a_{\mu_2},$$

by Lemmas 2.1 and 2.2.

To prove the claim let A_1 be the portion of A lying inside μ_1 and note that all row-sums of A_1 are 1. Since A_1 has the same number of rows and columns and since the column sums of A_1 are ≤ 1 (A is sign-alternating) these column sums must equal 1. Therefore the column sums of A_0 are all 0. But there can be no -1 in the first column of A_0, so all entries

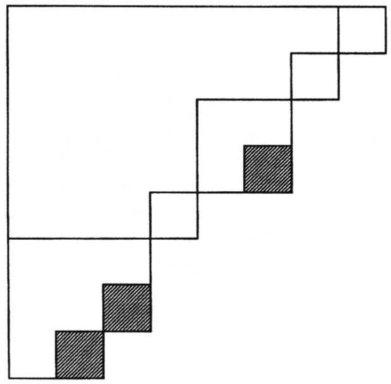

Figure 16. The case $n = 8$, $S = \{1, 2, 5\}$.

in this column must be zeroes. Consequently, neither can the second column of A_0 contain any -1, so all its entries are again zeroes. Iterating this proves the claim. □

Proof of Theorem 4.1: Applying Lemma 3.3 to each of the $n-1$ edges of $\bar{A}_\delta - A_\delta$, we obtain

$$T_n = \frac{1}{2^{n-1}} \sum_{S \subseteq C(\delta)} a_{\delta \cup S}. \tag{3}$$

Identify $C(\delta)$ with the set $\{1, 2, \dots, n-1\}$ and fix a subset S of it. By a connected component of S we mean a maximal string of consecutive integers in S. Assume that the connected components of S have lengths $i_1, i_2, \dots, i_k$, respectively. Then, as $a_{(i)} = 2$, Lemma 4.2 yields

$$a_{\delta \cup S} = 2^{n-k-(i_1+\cdots+i_k)} a_{\alpha_{i_1}} a_{\alpha_{i_2}} \cdots a_{\alpha_{i_k}}, \tag{4}$$

where $\alpha_j = (j+1, j+1, j, j-1, \dots, 3, 2)$ for $j \geq 1$. Figure 16 illustrates this for $n = 8$, $S = \{1, 2, 5\}$.

By Theorem 1.1, $a_{\alpha_j} = 2^{j+1} T_j$. Therefore, (4) gives $a_{\delta \cup S} = 2^n\, T_{i_1} \cdots T_{i_k}$, and then we have by (3) that

$$T_n = 2 \sum_{S \subseteq \{1,2,\dots,n-1\}} T_{i_1} \cdots T_{i_k}, \tag{5}$$

where S consists of strings of consecutive integers of lengths $i_1, \dots, i_k$. Let $F_n = T_n/2$. Then, splitting the sum in (5) according to the length of the string of S containing 1, we have

$$F_n = F_{n-1} + T_1 F_{n-2} + T_2 F_{n-3} + \cdots + T_{n-3} F_2 + T_{n-2} F_1 + T_{n-1},$$

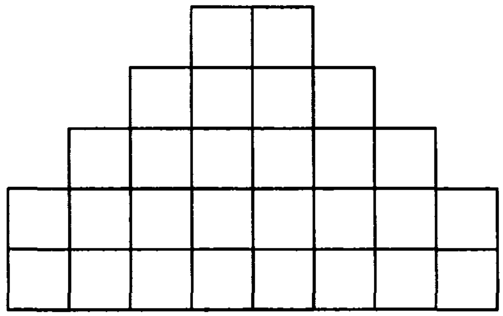

Figure 17. A realisation of the Aztec triangle.

or equivalently,

$$T_n = T_{n-1} + \sum_{k=0}^{n-1} T_k T_{n-1-k},$$

since $T_0 = 1$. Setting $T(x) = \sum_{n\geq 0} T_n x^n$, we obtain $T(x) - 1 = xT(x) + xT^2(x)$, which gives the desired generating function. □

Remark The generating function we obtained is the generating function of the Schröder numbers r_n, which have a great number of combinatorial interpretations (see [8] and [9]). One of them describes r_n as being the number of lattice paths from (0, 0) to (n, n) using steps (1, 0), (0, 1) and (1, 1) and never rising above the diagonal. It turns out that we can define an explicit bijection between perfect matchings of the Aztec triangle of size n and the above set of paths, using a method previously employed by Sachs and Zernitz [7] (also discovered independently by Randall [5]) for domino tilings of the diamond-shaped planar region having rows consisting of $2, 4, 6, \ldots, 2n-2, 2n, 2n, 2n, 2n-2, \ldots, 6, 4, 2$ unit boxes.

Consider the planar region T shown in figure 17, having rows consisting of $2, 4, 6, \ldots,$ $2n, 2n$ unit squares. The Aztec triangle of order n is isomorphic to the graph that has vertices corresponding to the squares of T and edges corresponding to all pairs of adjacent squares. A perfect matching corresponds to a tiling by 2×1 or 1×2 dominoes. Suppose that each domino can be marked in one of the four ways indicated in figure 18.

Consider a tiling of T by dominoes and mark the middle of the leftmost and rightmost vertical segments of the last row of T. Then there is a unique way of marking the dominoes such that the marked points on their boundaries agree among each other and with the two points we just marked and there are no marked points on the boundary of T other than these two. This creates a path in the (rotated) lattice $(\sqrt{2}\mathbf{Z}) \times (\sqrt{2}\mathbf{Z})$ that has the properties described above. Conversely, given such a lattice path, place a marked domino on each of its steps. Then it can be shown that the remaining region of T can be uniquely tiled by dominoes.

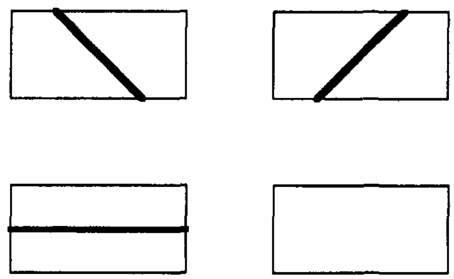

Figure 18. Four ways of marking a domino.

5. Cellular hypergraphs

It is interesting that Theorem 1.1 admits a 3-dimensional version. To describe it, consider the tesselation of R^3 by regular octahedra of side $\sqrt{2}$ and polyhedra obtained from a cube of side 2 by chopping off from each vertex the tetrahedron determined by that vertex and the midpoints of the 3 edges incident to the vertex (this describes a semi-regular honeycomb; see [1]). Let a *cell* be the hypergraph whose vertex set is the vertex set of an octahedron with the hyperedges being the faces of the octahedron. Regarding the tesselation as a hypergraph in this manner, a *cellular hypergraph* is any sub-hypergraph obtained from the union of cells. A *chain* will be now a sub-hypergraph consisting of a maximal connected sequence of cells in any of the 3 directions of the coordinate axes. Let $x(G)$, $y(G)$ and $z(G)$ denote the number of chains parellel to the x, y and z axis, respectively.

Let G be a cellular hypergraph. We define G' to be the hypergraph obtained from G by removing the vertices that are endpoints of chains of G, together with the hyperedges containing them. A perfect hypermatching of G is a collection of hyperedges of G with the property that every vertex is incident to exactly one hyperedge from this collection.

We have the following

Theorem 5.1 *Let G be a cellular hypergraph. Then G has no perfect hypermatching unless* $x(G) = y(G) = z(G)$*. Moreover, if this condition holds, then*

$$g = 4^{x(G)} g',$$

where g and g′ represent the number of perfect hypermatchings of G and G′, respectively.

The proof is completely analogous to the 2-dimensional case. The 4 on the right hand side represents the number of perfect hypermatchings of a cell.

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