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Two Variable Pfaffian Identities and Symmetric Functions*

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Abstract. We give sign-reversing involution proofs of a pair of two variable Pfaffian identities. Applications to symmetric function theory are given, including identities relating Pfaffians and Schur functions. As a corollary we are able to compute the plethysm $p_2 \circ s_{k^n}$.

Keywords: Pfaffian, involution, Schur function, plethysm, root system

1. Introduction

Our main result (Theorem 2.1) is a two-variable generalization of the following pair of identities:

$$\mathrm{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right) = \prod_{1 \le i < j \le 2n} \left(\frac{x_i - x_j}{x_i + x_j}\right) \tag{1.1}$$

$$\mathrm{Pf}\left(\frac{x_i - x_j}{1 + x_i x_j}\right) = \prod_{1 \le i < j \le 2n} \left(\frac{x_i - x_j}{1 + x_i x_j}\right) \tag{1.2}$$

These identities are interesting in that they are related to the Weyl identities for the classical root systems. In the proofs (Section 3) we will see how an identity of Littlewood corresponding to the root system of type D_n plays a role in (1.2), and in Section 6 we generalize this connection to types B_n and C_n.

In Sections 4 and 5 we give some applications to symmetric function theory, including several identities which express Schur functions in terms of Pfaffians. In particular, we obtain a Pfaffian expression for the plethysm $p_2 \circ s_{k^n}$ (Corollary 4.1) for which we are able to give an explicit expansion into Schur functions (Theorem 5.3).

2. Two Pfaffian identities

In this section we state two-variable generalizations of (1.1) and (1.2) (Theorem 2.1). The proofs are in Section 3. Many of the following definitions are taken from [7].

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Definitions For n an integer let $[2n] = \{1, 2, \ldots, 2n\}$. Let K_{2n} denote the complete graph on the vertex set $[2n]$ (no loops or multiple edges.) We represent edges in K_{2n} as ordered pairs; by convention the first element of the pair is the smaller vertex. A perfect matching (henceforth called a matching) is a set of edges

$$\pi = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\} \quad \text{with} \quad [2n] = \{i_1, j_1\} \cup \{i_2, j_2\} \cup \cdots \cup \{i_n, j_n\}.$$

Let $\mathcal{F}_{2n}$ denote the set of perfect matchings on $[2n]$. The following two matchings in $\mathcal{F}_{2n}$ will be used in Sections 3 and 4:

$$\begin{aligned} \pi_0 &= \{(1, 2), (3, 4), \ldots, (2n-1, 2n)\} \\ \pi_1 &= \{(1, 2n), (2, 2n-1), \ldots, (n, n+1)\}. \end{aligned}$$

Given $\pi \in \mathcal{F}_{2n}$ let $\epsilon(\pi) = (-1)^{\text{cross}(\pi)}$, where $\text{cross}(\pi)$ is the crossing number of π, which we can take to be the number of intersections when edges of π are drawn in the upper half-plane as semicircular arcs between integer points of the x-axis.

If $A = (a_{i,j})$ is a $2n \times 2n$ skew-symmetric matrix then we define the Pfaffian of A to be

$$\text{Pf}(A) = \sum_{\pi \in \mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j) \in \pi} a_{i,j}.$$

In this way we view the Pfaffian as a weighted generating function for matchings. We often write $\text{Pf}(a_{i,j})$ for $\text{Pf}(A)$.

In our main result, we express Pfaffians in terms of skew-symmetrizations of certain monomials. Let $x = \{x_1, x_2, \ldots, x_{2n}\}$ and $y = \{y_1, y_2, \ldots, y_{2n}\}$ be two sets of variables and let S_{2n} act on each by permuting indices. For α and β compositions of length n define

$$a_{\alpha,\beta}(x, y) = \sum_{\sigma \in S_{2n}} \epsilon(\sigma)\, \sigma\big(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n\, x_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}\big).$$

For example, $a_{\delta_n,\delta_n}(x, x^n) = \det(x_i^{2n-j})$ is the usual alternant in x, where $\delta_n = (n-1, \ldots, 1, 0)$.

We are now ready to state the main result; (2.1) is also due to Proctor.

Theorem 2.1

$$\text{Pf}\left(\frac{y_i - y_j}{x_i + x_j}\right) \prod_{i<j}^{2n} (x_i + x_j) = (-1)^{\binom{n}{2}}\, a_{2\delta_n, 2\delta_n}(x, y) \tag{2.1}$$

$$\text{Pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right) \prod_{i<j}^{2n} (1 + x_i x_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n, \mu+\delta_n}(x, y) \tag{2.2}$$

where the sum is over pairs of partitions $\lambda = (\alpha_1, \ldots, \alpha_p \mid \alpha_1 + 1, \ldots, \alpha_p + 1)$ *and* $\mu = (\beta_1, \ldots, \beta_q \mid \beta_1 + 1, \ldots, \beta_q + 1)$ *in Frobenius notation, with* $\alpha_1, \beta_1 < n - 1$.

Remark 2.1 Equations (2.1) and (2.2) are generalizations of (1.1) and (1.2) respectively. Setting $y = x$ in (2.1) gives

$$\mathrm{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right)\prod_{i<j}(x_i + x_j) = (-1)^{\binom{n}{2}}\sum_{\sigma}\epsilon(\sigma)\,\sigma\left(x_1^{2n-1}\cdots x_n^1\, x_{n+1}^{2n-2}\cdots x_{2n}^0\right)$$
$$= \prod_{i<j}(x_i - x_j).$$

Setting $y = x$ in (2.2) we see that

$$\mathrm{Pf}\left(\frac{x_i - x_j}{1 + x_i x_j}\right)\prod_{i<j}(1 + x_i x_j) = \sum_{\lambda,\mu}\sum_{\sigma}\epsilon(\sigma)\,\sigma\left(x_1^{\lambda_1+n}\cdots x_n^{\lambda_n+1}\, x_{n+1}^{\mu_1+n-1}\cdots x_{2n}^{\mu_n}\right).$$

The inner sum vanishes unless the exponents $\lambda_1+n, \ldots, \mu_n$ are distinct. Since $\lambda_1, \mu_1 < n$, the exponents must be a permutation of $\{0, 1, \ldots, 2n-1\}$. We see immediately that $\lambda_1 = n-1$ and $\mu_n = 0$. By induction we get $\lambda = (n-1)^n$ and $\mu = \emptyset$, as desired.

Remark 2.2 The shapes λ and μ which occur in the last sum are those which occur in the expansion of $\prod(1 + x_i x_j)$ into Schur functions. In fact (see [4, pp. 46, 47])

$$\prod_{i<j}^{n}(1 + x_i x_j) = \sum_{\lambda} s_\lambda(x)$$

where the sum is over all partitions $\lambda = (\alpha_1, \ldots, \alpha_p \mid \alpha_1 + 1, \ldots, \alpha_p + 1)$ with $\alpha_1 < n-1$. Moreover, the right hand side of (2.1) has a similar interpretation as a sum of terms $a_{\lambda+\delta,\mu+\delta}(x, y)$ where λ and μ range over all shapes in the expansion of $\prod(x_i + x_j)$, namely the single shape δ. The reason for this will become apparent in the proofs.

3. Proofs

In our proofs we use sign-reversing involutions similar to those found in [1] and [3]. Our involutions are defined on sets of matchings and *tournaments*. A tournament on $[2n]$ is an assignment of a direction ($i \to j$ or $j \to i$) to each edge $(i, j) \in K_{2n}$; let $\mathcal{T}_{2n}$ denote the set of tournaments on $[2n]$. We represent tournaments as sets of ordered pairs; the pair (i, j) corresponds to the directed edge $i \to j$. Pairs $(j, i) \in T$ with $i < j$ are called upsets in T, and for $T \in \mathcal{T}_{2n}$ the sign of T is $\epsilon(T) = (-1)^{\mathrm{up}(T)}$ where $\mathrm{up}(T)$ is the number of upsets in T. Given $T \in \mathcal{T}_{2n}$, the degree (or more specifically, the outdegree) of the vertex k is

$$\deg(k) = \#\{j \in [2n] \mid (k, j) \in T\},$$

that is, the number of edges out of vertex k. In the proofs of (2.1) and (2.2) we make use of the fact that a product of $\binom{2n}{2}$ binomials can be written as a weighted sum over $\mathcal{T}_{2n}$:

$$\prod_{i<j}^{2n}(x_i - x_j) = \sum_{T\in\mathcal{T}_{2n}}\epsilon(T)\, x_1^{\deg(1)}\cdots x_{2n}^{\deg(2n)}. \tag{3.1}$$

Given a tournament T and a matching π, we refer to the undirected edges of T as either matched or unmatched, according to whether or not they are contained in π. We will use the following lemma to show that our involutions are sign-reversing.

Sign Lemma *Let π be a matching and T a tournament. Let $mup(\pi, T)$ denote the number of edges in π which are upsets in T (matched upsets). Suppose i and j are matched (in π) to k and l respectively. Let π' be the matching obtained by interchanging i and j, and let T' be the tournament obtained by interchanging i and j in the edges which contain k or l. Then*

$$\epsilon(\pi)(-1)^{mup(\pi,T)} = -\epsilon(\pi')(-1)^{mup(\pi',T')}$$

Proof: The Sign Lemma is closely related to a lemma of Stembridge [7, Lemma 2.1]. Using his argument we can reduce to the case where there are only four vertices, $\{i, j, k, l\}$, and so we have only to check a small number of cases. If neither k nor l is between i and j then we have $\epsilon(\pi) = -\epsilon(\pi')$ and it is easy to check that $\text{mup}(\pi, T) = \text{mup}(\pi', T')$. If both k and l are between i and j then $\epsilon(\pi) = -\epsilon(\pi)$ and in that case it is easy to see that two matched upsets are changed, so again $(-1)^{\text{mup}(\pi,T)} = (-1)^{\text{mup}(\pi',T')}$. Finally, if exactly one of k and l is between i and j, then $\epsilon(\pi) = \epsilon(\pi)$, and exactly one matched upset is changed. □

Remark The interchange mentioned in the Sign Lemma occurs in our involutions and will be called a *simple i-j interchange.*

To prove (2.1) we expand the left hand side as a sum over matchings and tournaments.

$$\begin{aligned}
\text{LHS} &= \text{Pf}\left(\frac{y_i - y_j}{x_i + x_j}\right) \prod_{i<j}(x_i + x_j) \\
&= \sum_{\pi \in \mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j)\in\pi} (y_i - y_j) \prod_{(i,j)\notin\pi} (x_i + x_j) \\
&= \sum_{\substack{\pi\in\mathcal{F}_{2n} \\ T\in\mathcal{T}_{2n}}} \epsilon(\pi)(-1)^{\text{mup}(\pi,T)}\, x_1^{\text{udeg}(1)} \cdots x_{2n}^{\text{udeg}(2n)}\, y_1^{\text{mdeg}(1)} \cdots y_{2n}^{\text{mdeg}(2n)},
\end{aligned} \tag{3.2}$$

where $\text{udeg}(i)$ is the number of unmatched edges out of i, and $\text{mdeg}(i)$ is the number of matched edges out of i.

Given $(\pi, T) \in \mathcal{F}_{2n} \times \mathcal{T}_{2n}$, let V_1 be the set of vertices with mdeg $= 1$ and V_0 the set of vertices with mdeg $= 0$. Let E_1 be the set of unmatched edges between vertices of V_1 and vertices of V_0, and let E_2 be the remaining unmatched edges. We now describe an involution on $\mathcal{F}_{2n} \times \mathcal{T}_{2n}$ by considering the subgraph on edges E_1.

Description of ϕ. Given (π, T) find the lexicographically smallest pair of integers (i, j) with $\text{mdeg}(i) = \text{mdeg}(j)$ and $\deg_{E_1}(i) = \deg_{E_1}(j)$; i.e., we seek a pair of vertices in V_1 or in V_0 with the same degree in the subgraph induced by the edges E_1. If no such pair exists then (π, T) is a fixed point of ϕ. Otherwise, suppose i is matched to k and j is matched to l. We define a new pair $(\pi', T') = \phi(\pi, T)$ by making the following changes:

- Make a simple i-j interchange.
- In T replace every pair of E_1 edges of the form $(i, s), (s, j)$ with $(j, s), (s, i)$, and conversely.

In almost all cases it is immediately clear that mdeg and udeg are preserved by these operations, the only difficult cases being udeg(i) and udeg(j). Call a vertex type 1 if it contributes to $\deg_{E_1}(i)$ but not to $\deg_{E_1}(j)$ and type 2 if it contributes to $\deg_{E_1}(j)$ but not to $\deg_{E_1}(i)$. By the choice of i and j we see that there are equal numbers of type 1 and type 2 vertices. After the interchange, the roles of type 1 and type 2 have switched, so udeg(i) and udeg(j) are preserved as well. If we apply ϕ to the pair (π', T') we see that the same pair (i, j) is selected (since the choice depends only on degrees) and repeating the switches returns the original pair (π, T). This shows that ϕ is a weight-preserving involution, and the Sign Lemma shows that ϕ is sign-reversing.

If (π, T) is a fixed point of ϕ, the vertices of V_1 and V_0 must have distinct E_1-degrees. Since $0 \leq \deg_{E_1} \leq n-1$, the E_1-degrees must be $0, 1, \ldots, n-1$ for each set. Note: if $i \in V_1$ is matched to $j \in V_0$ and $\deg_{E_1}(i) = n-1$ then all vertices in V_0 other than j have $\deg_{E_1} < n-1$, so $\deg_{E_1}(j) = n-1$ as well. By induction we see that $\deg_{E_1}(i) = \deg_{E_1}(j)$ whenever $(i, j) \in \pi$.

Thus each fixed point of ϕ determines a permutation $\sigma \in S_{2n}$ by the following equations:

$$\begin{array}{lll} \operatorname{udeg}(\sigma_i) = n - i & \operatorname{mdeg}(\sigma_i) = 1 & i = 1, \ldots, n \\ \operatorname{udeg}(\sigma_{n+i}) = i - 1 & \operatorname{mdeg}(\sigma_{n+i}) = 0 & i = 1, \ldots, n, \end{array}$$

that is, the match winners in E_1 order determine the first half of σ while the match losers in reverse E_1 order determine the second half of σ. Let $[\sigma]$ be the equivalence class of fixed points (π, T) which correspond to σ. Note that equivalence classes are exactly determined by matched edges together with edges in E_1, and so all pairs in $[\sigma]$ have the same sign. Thus

$$\sum_{(\pi,T)\in[\sigma]} x^{\mathrm{udeg}} y^{\mathrm{mdeg}} = y_{\sigma_1} \cdots y_{\sigma_n} x_{\sigma_1}^{n-1} \cdots x_{\sigma_n}^{0} x_{\sigma_{n+1}}^{0} \cdots x_{\sigma_{2n}}^{n-1} \times \prod_{1\leq i<j\leq n} (x_{\sigma_i} + x_{\sigma_j}) \prod_{n+1\leq i<j\leq 2n} (x_{\sigma_i} + x_{\sigma_j}),$$

the products arising from all possible choices of edge sets E_2.

We claim that if $(\pi, T) \in [\sigma]$ then $\epsilon(\sigma) = \epsilon(\pi)(-1)^{\mathrm{mup}(\pi,T)}$. This is clear if σ is the identity since then pairs must be of the form (π_1, T_0) where π_1 is defined in Section 2 and T_0 satisfies $\mathrm{mup}(\pi_1, T_0) = 0$. Any other equivalence class can be obtained by making simple interchanges or by interchanging the elements in a matched edge, both of which change the sign of pairs in the class, and both of which correspond to acting on σ by a transposition.

Thus by induction we see that the sign is correct for all classes of fixed points. Now we return to (3.2):

$$\text{LHS} = \sum_{\pi,T} \epsilon(\pi)(-1)^{\text{mup}(\pi,T)}\, x^{\text{udeg}(1)}\, y^{\text{mdeg}(1)} \cdots x_{2n}^{\text{udeg}(2n)}\, y_{2n}^{\text{mdeg}(2n)}$$

$$= \sum_{\sigma\in S_{2n}} \epsilon(\sigma)\, y_{\sigma_1}\cdots y_{\sigma_n}\, x_{\sigma_1}^{n-1}\cdots x_{\sigma_n}^{0}\, x_{\sigma_{n+1}}^{0}\cdots x_{\sigma_{2n}}^{n-1} \prod_{1}^{n}\left(x_{\sigma_i}+x_{\sigma_j}\right)\prod_{n+1}^{2n}\left(x_{\sigma_i}+x_{\sigma_j}\right)$$

Now write this last sum as a sum over triples (A, ρ, τ) where A $(= \{\sigma_1, \ldots, \sigma_n\})$ is an n-element subset of $[2n]$ and ρ and τ are permutations on A and $\bar{A}$. Let $\text{inv}(A)$ be the number of $i < j$ with $i \in \bar{A}$ and $j \in A$, and let $V(x_A) = \prod(x_i + x_j)$ over $i < j$ in A. Then

$$\text{LHS} = \sum_{A}(-1)^{\text{inv}(A)} \sum_{\rho} \epsilon(\rho)\, y_A\, \rho\left(x_A^{\delta}\right)\, V(x_A) \sum_{\tau} \epsilon(\tau)(-1)^{\binom{n}{2}}\, \tau\left(x_{\bar{A}}^{\delta}\right)\, V(x_{\bar{A}})$$

$$= (-1)^{\binom{n}{2}} \sum_{A}(-1)^{\text{inv}(A)} y_A \sum_{\rho} \epsilon(\rho)\, \rho\left(x_A^{2\delta}\right) \sum_{\tau} \epsilon(\tau)\, \tau\left(x_{\bar{A}}^{2\delta}\right)$$

$$= (-1)^{\binom{n}{2}} \sum_{\sigma} \epsilon(\sigma) y_{\sigma_1}\cdots y_{\sigma_n}\, x_{\sigma_1}^{2(n-1)}\cdots x_{\sigma_n}^{0}\, x_{\sigma_{n+1}}^{2(n-1)}\cdots x_{\sigma_{2n}}^{0}$$

$$= (-1)^{\binom{n}{2}} a_{2\delta,2\delta}(x, y),$$

which completes the proof of (2.1).

The proof of (2.2) is entirely analogous; this time we use an involution on $\mathcal{F}_{2n} \times \mathcal{T}_{2n}$ which preserves a pair of statistics to reduce to equivalence classes of fixed points which have generating functions of the form $x_A^{\delta} x_{\bar{A}}^{\delta} y_A \prod(1 + x_i x_j)$. We begin as before by expanding the left hand side over matchings and using a modification of (3.1) to expand products over tournaments:

$$\text{LHS} = \text{Pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right) \prod_{i<j}(1 + x_i x_j)$$

$$= \sum_{\pi\in\mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j)\in\pi} (y_i - y_j) \prod_{(i,j)\notin\pi} (1 + x_i x_j)$$

$$= \sum_{\substack{\pi\in\mathcal{F}_{2n}\\ T\in\mathcal{T}_{2n}}} \epsilon(\pi)(-1)^{\text{mup}(\pi,T)}\, x_1^{\text{uup}(1)} \cdots x_{2n}^{\text{uup}(2n)}\, y_1^{\text{mdeg}(1)} \cdots y_{2n}^{\text{mdeg}(2n)},$$

where $\text{uup}(i)$ is the number of unmatched upsets that i is contained in. As before define vertex sets V_0 and V_1 and edge sets E_1 and E_2. We now describe an involution $\bar{\phi}$ on $\mathcal{F}_{2n} \times \mathcal{T}_{2n}$ which preserves mdeg and uup.

Description of $\bar{\phi}$. Given (π, T) find the lexicographically smallest pair of integers (i, j) with $\text{mdeg}(i) = \text{mdeg}(j)$ and $\text{uup}_{E_1}(i) = \text{uup}_{E_1}(j)$; i.e., we seek a pair of vertices in V_1 or in V_0 having the same number of E_1-upsets. If no such pair exists then (π, T) is a fixed point of $\bar{\phi}$. Otherwise, suppose i is matched to k and j is matched to l. We define a new pair $(\pi', T') = \phi(\pi, T)$ by making the following changes:

- Make a *dual* i-j interchange.
- In T, whenever there is an E_1 upset involving i and some vertex s together with an E_1 non-upset involving s and j, reverse both edges so that the upset status of each changes. Similarly, reverse pairs with an E_1 non-upset involving i and s and an E_1 upset involving s and j.

A dual i-j interchange consists of a simple i-j interchange with the extra condition that after the interchange, edges in T are reversed if their upset status was altered by the interchange; i.e., the interchange does not effect the number of E_1 upsets containing k or l. The proof that $\bar{\phi}$ is a sign-reversing involution is the same as the proof that ϕ is a sign-reversing involution.

If (π, T) is a fixed point of $\bar{\phi}$, the statistic uup_{E_1} must be distinct on V_1 and V_0. Since $0 \le \mathrm{uup}_{E_1} \le n-1$, the values must be $0, 1, \ldots, n-1$ for each set. Note: if $i \in V_1$ is matched to $j \in V_0$ and $\mathrm{uup}_{E_1}(i) = n-1$ then all vertices in V_0 other than j have $\mathrm{uup}_{E_1} > 0$, so we must have $\mathrm{uup}_{E_1}(j) = 0$. By induction we see that $\mathrm{uup}_{E_1}(i) + \mathrm{uup}_{E_1}(j) = n-1$ whenever $(i, j) \in \pi$. Thus equivalence classes of fixed points correspond to permutations $\sigma \in S_{2n}$ and the weight of an equivalence class is

$$\sum_{(\pi,T)\in[\sigma]} x^{\mathrm{uup}} y^{\mathrm{mdeg}} = y_{\sigma_1} \cdots y_{\sigma_n} x_{\sigma_1}^{n-1} \cdots x_{\sigma_n}^{0} x_{\sigma_{n+1}}^{n-1} \cdots x_{\sigma_{2n}}^{0} \times \prod_{1\le i<j\le n} \left(1 + x_{\sigma_i} x_{\sigma_j}\right) \prod_{n+1\le i<j\le 2n} \left(1 + x_{\sigma_i} x_{\sigma_j}\right).$$

Now we proceed with a computation similar to the one at the end of the previous proof, the crucial difference being that the identity

$$\sum_{\rho} \epsilon(\rho)\, \rho(x^{\delta}) \prod_{i<j} (x_i + x_j) = \sum_{\rho} \epsilon(\rho)\, \rho(x^{\delta+\delta})$$

gets replaced by the corresponding D_n identity

$$\sum_{\rho} \epsilon(\rho)\, \rho(x^{\delta}) \prod_{i<j} (1 + x_i x_j) = \sum_{\lambda} \sum_{\rho} \epsilon(\rho)\, \rho(x^{\lambda+\delta}),$$

where the last sum is over all $\lambda = (\alpha_1, \ldots, \alpha_p \mid \alpha_1 + 1, \ldots, \alpha_p + 1)$ with $\alpha_1 < n-1$. Setting $W(x_A) = \prod (1 + x_i x_j)$ over $i < j$ in A we have the computation

$$\begin{aligned} \mathrm{LHS} &= \mathrm{Pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right) \prod_{i<j} (1 + x_i x_j) \\ &= \sum_{(\pi,T)} \epsilon(\pi) (-1)^{\mathrm{mup}(\pi,T)}\, x^{\mathrm{uup}} y^{\mathrm{mdeg}} \end{aligned}$$

Now we use the sign-reversing involution $\bar{\phi}$ to cancel terms leaving behind equivalence classes of fixed points which correspond to permutations $\sigma \in S_{2n}$. This gives:

$$\begin{aligned}
\text{LHS} &= \sum_{\sigma \in S_{2n}} \epsilon(\sigma)\; y_{\sigma_1} \cdots y_{\sigma_n}\, x_{\sigma_1}^{n-1} \cdots x_{\sigma_n}^{0}\, x_{\sigma_{n+1}}^{n-1} \cdots x_{\sigma_{2n}}^{0}\;\; W(x_A)W(x_{\bar{A}}) \\
&= \sum_{(A,\rho,\tau)} (-1)^{\mathrm{inv}(A)} \epsilon(\rho)\epsilon(\tau)\, y_A\; \rho\big(x_A^{\delta}\big)\; W(x_A)\; \tau\big(x_{\bar{A}}^{\delta}\big)\; W(x_{\bar{A}}) \\
&= \sum_{A} (-1)^{\mathrm{inv}(A)} y_A \sum_{\rho} \epsilon(\rho)\rho\big(x_A^{\delta}\big)\; W(x_A) \sum_{\tau} \epsilon(\tau)\tau\big(x_{\bar{A}}^{\delta}\big)\; W(x_{\bar{A}}) \\
&= \sum_{A} (-1)^{\mathrm{inv}(A)} y_A \sum_{\lambda}\sum_{\rho} \epsilon(\rho)\rho\big(x_A^{\lambda+\delta}\big) \sum_{\mu}\sum_{\tau} \epsilon(\tau)\tau\big(x_{\bar{A}}^{\mu+\delta}\big) \\
&= \sum_{\lambda,\mu}\sum_{\sigma} \epsilon(\sigma)\; y_{\sigma_1} \cdots y_{\sigma_n}\, x_{\sigma_1}^{\lambda_1+n-1} \cdots x_{\sigma_n}^{\lambda_n}\, x_{\sigma_{n+1}}^{\mu_1+n-1} \cdots x_{\sigma_{2n}}^{\mu_n} \\
&= \sum_{\lambda,\mu} a_{\lambda+\delta,\mu+\delta}(x, y).
\end{aligned}$$

This completes the proof of (2.2).

4. Pfaffians and Schur functions

In this section we obtain identities expressing Schur functions in terms of certain Pfaffians.

For α a composition of length $2n$ let

$$a_\alpha(x) = \sum_{\sigma \in S_{2n}} \epsilon(\sigma)\, \sigma\big(x_1^{\alpha_1} \cdots x_{2n}^{\alpha_{2n}}\big).$$

The Schur function of shape λ is $s_\lambda(x) = a_{\lambda+\delta}(x)/a_\delta(x)$. In (2.1) or (2.2), if we replace x and y by powers of x and divide both sides by $a_\delta(x)$ the right hand side is easily expressed in terms of non-standard Schur functions. One case of interest is

Proposition 4.1

$$\frac{1}{a_\delta(x)} \operatorname{Pf}\left(\frac{x_i^N - x_j^N}{x_i^M + x_j^M}\right) \prod_{i<j}^{2n} \big(x_i^M + x_j^M\big) = \pm s_{\Lambda_N^M}(x_1, \ldots, x_{2n}),$$

$$\Lambda_N^M = D\,(2M\delta_n + N, 2M\delta_n) - \delta_{2n},$$

where $D(2M\delta_n + N, 2M\delta_n)$ is the decreasing rearrangement of $(2M\delta_n + N, 2M\delta_n)$, and the sign is $(-1)^{\binom{n}{2}}$ times the sign of the permutation in S_{2n} which rearranges $(2M\delta_n + N, 2M\delta_n)$ into $D(2M\delta_n + N, 2M\delta_n)$.

Proof: In (2.1) replace x by x^M, replace y by x^N, and divide both sides by $a_\delta(x)$. □

The next corollary, originally due to Proctor [6], expresses the plethysm $p_2 \circ s_{k^n}$ in terms of a Pfaffian.

Corollary 4.1

$$\frac{1}{a_\delta(x)} \mathrm{Pf}\left(\frac{x_i^{2(n+k)} - x_j^{2(n+k)}}{x_i + x_j} \right) = (-1)^{\binom{n}{2}} s_{k^n}(x_1^2, \dots, x_{2n}^2).$$

Proof: Set $N = (n+k)$ and $M = 1/2$ in Proposition 4.1 and replace x by x^2. Then $\Lambda_{(n+k)}^{1/2} = k^n$ and the shuffle permutation is the identity. On the left hand side, the factors $\prod(x_i^M + x_j^M)/a_\delta(x)$ become $1/a_\delta(x)$ as desired. □

There is another way to express Schur functions in terms of Pfaffians. More generally any determinant can be written as a Pfaffian [2, 5]. Given an even order matrix A, choose J skew symmetric with determinant 1 and set $B = AJA^t$. Then

$$\mathrm{Pf}^2(B) = \det(B) = \det{}^2(A) \quad \text{so} \quad \mathrm{Pf}(B) = \det(A). \tag{4.1}$$

For A of odd order let $\bar{A} = A \oplus (1)$ (matrix direct sum) so that $\bar{A}$ has even order and $\det(A) = \det(\bar{A})$.

Thus any Schur function can be written as a quotient of a Pfaffian by $a_\delta(x)$ in many ways. One such way is given by the following proposition, which is a special case of Theorem 5.1:

Proposition 4.2

$$s_\lambda(x) = \frac{1}{a_\delta(x)} \mathrm{Pf}(f_\lambda(x_i, x_j)),$$

where

$$f_\lambda(x, y) = \sum_{i=0}^{2n-1} (-1)^i x^{(\delta+\lambda)_{i+1}} y^{(\delta+\lambda)_{2n-i}}.$$

Proof: Apply (4.1) to $a_{\lambda+\delta}(x)$ where J has entries $(-1)^{i+1}$ on the antidiagonal. □

Remark 4.1 We can express a Schur function as a single Pfaffian by applying our method to the Jacobi-Trudi identity. Let λ be a partition of length at most $2n$. For $1 \le i < j \le 2n$ let

$$s_{i,j} = s_{\lambda_i - i+1, \lambda_j - j+2} + s_{\lambda_i - i+3, \lambda_j - j+4} + \cdots + s_{\lambda_i - i+2n-1, \lambda_j - j+2n},$$

(sums of Schur functions with two parts), and for $i > j$ let $s_{i,j} = -s_{j,i}$. Then

$$s_\lambda(x) = \mathrm{Pf}(s_{i,j}).$$

Actually, this is a special case of a theorem of Stembridge [7, Theorem 3.1], but we can also obtain it by applying (4.1) to the Jacobi-Trudi identity with J equal to a block diagonal matrix with blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Remark 4.2 Using the matrix J with entries -1 on the upper half of the antidiagonal and 1 on the lower half of the antidiagonal we can show

$$a_{2\delta_n,2\delta_n}(x, y) = \mathrm{Pf}\left(\frac{(y_i - y_j)(x_i^{2n} - x_j^{2n})}{x_i^2 - x_j^2}\right). \tag{4.2}$$

Then we have

$$\mathrm{Pf}\left(\frac{y_i - y_j}{x_i + x_j}\right)\mathrm{Pf}\left(\frac{x_i^{4n} - x_j^{4n}}{x_i^2 + x_j^2}\right) = (-1)^{\binom{n}{2}}\mathrm{Pf}\left(\frac{x_i^{2n} - x_j^{2n}}{x_i + x_j}\right)\mathrm{Pf}\left(\frac{(y_i - y_j)(x_i^{2n} - x_j^{2n})}{x_i^2 - x_j^2}\right).$$

To obtain this, multiply both sides of (2.1) by $a_{\delta_{2n}}(x) = (-1)^{\binom{n}{2}}a_{2\delta_n,2\delta_n}(x, x)$. Then use (4.2) to convert determinants to Pfaffians.

5. Symmetric function expansions

In this section we study how Pfaffians give rise to alternating functions and give a technique for expanding such Pfaffians.

We say that the formal power series $f(x_1, \ldots, x_{2n})$ is *alternating* if $\sigma\, f(x) = \epsilon(\sigma)\, f(x)$ for all permutations $\sigma \in S_{2n}$. We say that the formal power series $f(u, v)$ (in two variables) is *skew symmetric* if $f(u, v) = -f(v, u)$.

Lemma 5.1 *Let $f(u, v)$ be skew symmetric and define $a_{i,j} = f(x_i, x_j)$. Then $Pf(a_{i,j})$ is alternating.*

Proof: Given $\sigma \in S_{2n}$ let P be the permutation matrix corresponding to σ. Then $\sigma\,(a_{i,j}) = P^t(a_{i,j})P$. Hence

$$\sigma(\mathrm{Pf}(a_{i,j})) = \mathrm{Pf}(P^t(a_{i,j})P) = \det(P)\mathrm{Pf}(a_{i,j}). \qquad \square$$

Consequently we have

Theorem 5.1 *Let f be a skew symmetric formal power series in two variables, $f(u, v) = \sum_{r,s} c_{r,s}\, x^r y^s$. Then*

$$\frac{1}{a_\delta(x)} Pf(f(x_i, x_j)) = \sum_\lambda Pf(C_{\lambda+\delta})\, s_\lambda(x),$$

where C_μ is the skew symmetric matrix with entries c_{μ_i,μ_j}.

Proof: By Lemma 5.1 we know that the left hand side is a symmetric function so it suffices to show that the coefficient of x^μ in $\mathrm{Pf}(f(x_i, x_j))$ is $\mathrm{Pf}(C_\mu)$ for any shape $\mu = \lambda + \delta$. Let $\langle x^\mu \rangle\, f$ denote the coefficient of x^μ in f. Then

$$\begin{aligned}\langle x^\mu\rangle \,\mathrm{Pf}(f(x_i, x_j)) &= \langle x^\mu\rangle \sum_{\pi\in\mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j)\in\pi} f(x_i, x_j)\\ &= \sum_{\pi\in\mathcal{F}_{2n}} \epsilon(\pi)\,\langle x^\mu\rangle \prod_{(i,j)\in\pi} f(x_i, x_j)\\ &= \sum_{\pi\in\mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j)\in\pi} c_{\mu_i,\mu_j}\\ &= \mathrm{Pf}(C_\mu).\end{aligned}$$

□

We give two applications of Theorem 5.1.

Theorem 5.2

$$\frac{1}{a_\delta(x)} Pf\left(\frac{x_i - x_j}{1 - x_i^2 x_j^2}\right) = \sum_\lambda s_\lambda(x),$$

where the sum is over all shapes with even length rows and even length columns.

Proof: To apply Theorem 5.1 we must expand $(u - v)/(1 - u^2 v^2) = \sum_{r,s} c_{r,s}\, u^r\, v^s$. Evidently

$$c_{r,s} = \begin{cases} 1 & r = s+1,\ s \text{ even}\\ -1 & s = r+1,\ r \text{ even}\\ 0 & \text{otherwise.}\end{cases}$$

Suppose $\mathrm{Pf}(C_\mu) \neq 0$ for some $\mu = \lambda + \delta$. Let $\pi \neq \pi_0$ be a matching that makes a non-zero contribution to $\mathrm{Pf}(C_\mu)$. There must be an edge $(i, j) \in \pi$ with $j > i + 1$. Since $c_{\mu_i,\mu_j} \neq 0$ and $\mu_i > \mu_j$, we must have $\mu_i = \mu_j + 1$. But $\mu_i > \mu_{i+1} > \mu_j$ which is impossible. Hence $\mathrm{Pf}(C_\mu) = c_{\mu_1,\mu_2} \cdots c_{\mu_{2n-1},\mu_{2n}} \neq 0$. This forces $c_{\mu_{2i-1},\mu_{2i}} = 1$ for all i, so $\mu_{2i-1} = \mu_{2i} + 1$ and μ_{2i} must be even. But this is equivalent to λ having even rows and columns. □

Remark 5.1 We can use (2.2) to get a different expansion of the previous Pfaffian involving plethysms.

$$\sum_\pi s_\pi(x) = \sum_\nu (-1)^{|\nu|/2} s_\nu(x^2) \sum_{\lambda,\mu} \epsilon(\lambda, \mu)(-1)^{\binom{n}{2}+|\lambda|/2+|\mu|/2} s_{\Lambda(\lambda,\mu)}(x),$$

where π has even rows and columns, ν has Frobenius type $(\alpha_1, \ldots \,|\, \alpha_1 + 1 \cdots)$ with $\alpha_1 < 2n - 1$, λ and μ have Frobenius type $(\alpha_1, \ldots \,|\, \alpha_1 + 1 \cdots)$ with $\alpha_1 < n - 1$, $\Lambda(\lambda, \mu) = D(\lambda + 2\delta_n + 1, \mu + 2\delta_n) - \delta_{2n}$, and $\epsilon(\lambda, \mu)$ is the sign of the shuffle that rearranges $(\lambda + 2\delta_n + 1, \mu + 2\delta_n)$.

Proof: In (2.2), move $\prod(1 + x_i x_j)$ to the denominator of the right hand side, make the change of variables $x \mapsto ix^2$ and $y \mapsto x$, and then divide through by $a_\delta(x)$. □

As a second application, we can expand the Pfaffian in Corollary 4.1 to get an explicit expansion of the plethysm $p_2 \circ s_{k^n}$ into Schur functions. This is also in [6].

Theorem 5.3

$$s_{k^n}\left(x_1^2, \ldots, x_{2n}^2\right) = \sum_{\lambda} (-1)^{\lambda_1 + \cdots + \lambda_n} s_\lambda(x_1, \ldots, x_{2n}),$$

where the sum is over all self-complementary partitions inside the $2n \times 2k$ *rectangle, i.e., partitions satisfying* $\lambda_i + \lambda_{2n+1-1} = 2k$ *for* $i = 1, \ldots, n$.

Proof: We apply Theorem 5.1 to Corollary 4.1. First we expand the formal power series $f(u, v) = (u^{2(n+k)} - v^{2(n+k)})/(u + v) = \sum_{r,s} c_{r,s}\, u^r v^s$.

$$c_{r,s} = \begin{cases} 1 & r + s = 2(n+k) - 1, \ s \text{ even} \\ -1 & r + s = 2(n+k) - 1, \ s \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\mathrm{Pf}(C_\mu) \neq 0$ for some $\mu = \delta + \lambda$. Then only π_1 can contribute to $\mathrm{Pf}(C_\mu)$ since (C_μ) has all its non-zero entries on the antidiagonal. Then

$$\mathrm{Pf}(C_\mu) = c_{\mu_1, \mu_{2n}} \cdots c_{\mu_n, \mu_{n+1}} = (-1)^{\mu_{2n} + \cdots + \mu_{n+1}} = (-1)^{\binom{n}{2} + \lambda_{n+1} + \cdots + \lambda_{2n}},$$

and $\mu_i + \mu_{2n+1-i} = 2(n + k) - 1$ for $i = 1, \ldots, n$. This last condition is equivalent to $\lambda_i + \lambda_{2n+1-i} = 2k$, and so also

$$(-1)^{\lambda_{n+1} + \cdots + \lambda_{2n}} = (-1)^{\lambda_1 + \cdots + \lambda_n}.$$ □

6. Remarks

Remark 6.1 Identity (2.2) corresponds to the root system D_n in the sense that the shapes which occur in the expansion on the right hand side are those which appear in $\prod(1 - x_i x_j)$, the product half of Weyl's identity for the root system D_n [4, p. 46]. Other identities corresponding to root systems B_n and C_n can easily be developed. More generally we have

$$\mathrm{Pf}\left(\frac{(y_i - y_j)\left(1 - x_i^p\right)\left(1 - x_j^p\right)}{1 - x_i x_j}\right) \prod_{i<j}^{2n} (1 - x_i x_j) = \sum_{\lambda, \mu} c_\lambda^{(p)} c_\mu^{(p)}\, a_{\lambda+\delta, \mu+\delta}(x, y) \quad (6.1)$$

where the coefficients $c_\lambda^{(p)}$ are determined by

$$\prod_{i<j}^{n}(1-x_ix_j)\prod_{i=1}^{n}\left(1-x_i^p\right)=\sum_\lambda c_\lambda^{(p)}s_\lambda(x).$$

The cases $p=1$, $p=2$, and $p=\infty$ correspond to root systems B_n, C_n, and D_n respectively.

The factors $(1-x_i^p)$ and $(1-x_j^p)$ can be factored out of the Pfaffian in (6.1) as $\prod(1-x_i^p)$. This leads to the identity

$$\prod_{i=1}^{2n}\left(1-x_i^p\right)\sum_{\lambda,\mu}c_\lambda^{(\infty)}c_\mu^{(\infty)}a_{\lambda+\delta,\mu+\delta}(x,y)=\sum_{\lambda,\mu}c_\lambda^{(p)}c_\mu^{(p)}a_{\lambda+\delta,\mu+\delta}(x,y). \tag{6.2}$$

We can similarly modify (2.1) to get

$$\mathrm{Pf}\left(\frac{(y_i-y_j)\left(1-x_i^p\right)\left(1-x_j^p\right)}{x_i+x_j}\right)\prod_{i<j}^{2n}(x_i+x_j)=\sum_{\lambda,\mu}d_\lambda^{(p)}d_\mu^{(p)}a_{\lambda+\delta,\mu+\delta}(x,y), \tag{6.3}$$

where the coefficients $d_\lambda^{(p)}$ are determined by

$$\prod_{i<j}^{n}(x_i+x_j)\prod_{i=1}^{n}\left(1-x_i^p\right)=\sum_\lambda d_\lambda^{(p)}s_\lambda(x).$$

Remark 6.2 It is easy to modify Theorem 5.2 to obtain other symmetric function expansions. For example, it is known how to find the coefficient of $s_\lambda(x)$ in

$$\frac{1}{a_\delta(x)}\mathrm{Pf}\left(\frac{x_i^N-x_j^N}{1-x_i^Mx_j^M}\right),$$

and all coefficients are -1, 0, or 1 (see [8]). In some cases, the Pfaffian can be computed from (1.2), resulting in a Littlewood formula.

Remark 6.3 The two-variable identities may have three-variable generalizations. For instance, it is known that

$$\mathrm{Pf}\left(\frac{(y_i-y_j)(z_i-z_j)}{x_i-x_j}\right)\prod_{i<j}^{2n}(x_i-x_j)=a_{\delta,\delta}(x,y)\,a_{\delta,\delta}(x,z).$$

This generalizes (2.1) since the change of variables $x\mapsto x^2$, $z\mapsto x$ yields (2.1). There is also a conjecture for a three-variable version of (2.2). These and other generalizations will be presented in a following paper.

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