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On Distance-Regular Graphs with Height Two

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Abstract. Let Γ be a distance-regular graph with diameter at least three and height $h = 2$, where $h = \max\{i : p^d_{di} \neq 0\}$. Suppose that for every α in Γ and β in $\Gamma_d(\alpha)$, the induced subgraph on $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a clique. Then Γ is isomorphic to the Johnson graph $J(8, 3)$.

Keywords: distance-regular graph, strongly regular graph, height, clique, Johnson graph

1. Introduction

Let Γ be a connected undirected simple finite graph. We identify Γ with the set of vertices. For vertices u and v, let $\partial(u, v)$ denote the distance between u and v, i.e. the length of a shortest path from u to v in Γ. Let $d = d(\Gamma)$ denote the *diameter* of Γ, i.e. the maximal distance of two vertices in Γ. We set

$$\Gamma_i(u) = \{x \in \Gamma : \partial(u, x) = i\} \qquad (0 \leq i \leq d).$$

Γ is said to be *distance-regular* if the cardinality of the set $\Gamma_i(u) \cap \Gamma_j(v)$ depends only on the distance between u and v. In this case we write

$$p^l_{ij} = |\Gamma_i(u) \cap \Gamma_j(v)| \qquad (0 \leq i, j, l \leq d),$$

where $l = \partial(u, v)$. Let

$$k_i = p^0_{ii} = |\Gamma_i(u)| \qquad (0 \leq i \leq d).$$

In particular $k = k_1$ is the *valency* of Γ. Let

$$c_i = p^i_{1\,i-1}, \quad a_i = p^i_{1i}, \quad b_i = p^i_{1\,i+1} \qquad (0 \leq i \leq d).$$

They are called the *intersection numbers* of Γ, and

$$\iota(\Gamma) = \left\{ \begin{matrix} * & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & * \end{matrix} \right\}$$

is called the *intersection array* of Γ.

The following are basic properties of intersection numbers, which we use implicitly in this paper.

(1) $c_i + a_i + b_i = k \quad (0 \leq i \leq d)$,
(2) $1 = c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_{d-1} \leq c_d \leq k$,

(3) $k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-2} \geq b_{d-1} \geq 1$,
(4) $k_i b_i = k_{i+1} c_{i+1} \quad (0 \leq i \leq d-1)$,
(5) $k_l p^l_{ij} = k_i p^i_{lj} = k_j p^j_{li} \quad (0 \leq i, j, l \leq d)$,
(6) $p^l_{ij} \neq 0$ if $l = i + j$ or $l = |i - j|$,
(7) $c_i \leq b_j$ if $i + j \leq d$.

A graph is said to be *strongly regular* if it is distance-regular with diameter 2.

A graph is called a *clique* when any two of its vertices are adjacent. A *coclique* is a graph in which no two vertices are adjacent.

Information about the general theory of distance-regular graphs is given in [1], [3] and [5].

Let X be a finite set of cardinality v and $V = \{T \subset X : |T| = e\}$. The *Johnson graph* $J(v, e)$ is a graph whose vertex set is V and two vertices x and y are adjacent if and only if $|x \cap y| = e - 1$. It is well known that $J(v, e)$ is a distance-regular graph.

In this paper we identify a subset A of Γ with the induced subgraph on A and define the following terminology.

A subgraph A of Γ is called *geodetically closed* if for all vertices x and y in A with $\partial(x, y) = i$, $\Gamma_{i-1}(x) \cap \Gamma_1(y)$ is in A. For subsets A and B of Γ, let $\partial(A, B) = \min\{\partial(x, y) : x \in A, y \in B\}$. Let $h = \max\{i : p^d_{di} \neq 0\}$ be the *height* of Γ.

A distance-regular graph Γ is of height 0 if and only if Γ is an antipodal 2-cover, and is of height 1 if and only if $\Gamma_d(\alpha)$ is a clique for every α in Γ. So if the height of Γ is 1, Γ is the distance-2 graph of a generalized odd graph (see Proposition 4.2.10 of [5]). This paper is concerned with a distance-regular graph of height 2.

Theorem 1.1 *Let Γ be a distance-regular graph with diameter d at least 3 and height $h = 2$. Suppose that for every α in Γ and β in $\Gamma_d(\alpha)$, $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a clique. Then $d = 3$ and Γ is isomorphic to $J(8, 3)$.*

In [8] and [9] H. Suzuki showed that $d(\Gamma)$ is bounded by a function depending only on k_d if $\Gamma_d(\alpha)$ is not isomorphic to a coclique. Hence if $\Gamma_d(\alpha)$ is isomorphic to a given strongly regular graph Δ, then there are only finitely many possibilities for Γ.

On the other hand if Γ is isomorphic to Hamming graphs $H(2, q)$ $(q \geq 3)$, Johnson graphs $J(v, 2)$ $(v \geq 6)$ or $J(2d + 2, d)$ $(d \geq 2)$, then $\Gamma_d(\alpha)$ is isomorphic to a strongly regular graph.

Is it possible to characterize these distance-regular graphs by the antipodal structures $\Gamma_d(\alpha)$?

Let Δ be a graph with diameter 2. Suppose $\Gamma_d(\alpha)$ is isomorphic to Δ for every α in Γ. Then the height of Γ becomes 2. It is easy to see that in this situation Δ is distance-degree regular, i.e. $|\Delta_1(\beta)| = p^d_{d1}$, $|\Delta_2(\beta)| = p^d_{d2}$ do not depend on the choice of β in Δ.

Let Δ be a distance-degree regular graph with diameter 2 such that $\Delta_2(\beta)$ is a clique for every β in Δ. The theorem above shows that if there exists a distance-regular graph Γ of diameter d at least 3 such that $\Gamma_d(\alpha)$ is isomorphic to Δ for every α in Γ, then Γ is isomorphic to $J(8, 3)$ and Δ is isomorphic to $J(5, 2)$.

We note that there are many distance-degree regular graphs of diameter 2 such that $\Delta_2(\beta)$ is a clique for every β in Δ. The complete bipartite graphs $K_{s,s}$, the pentagon and the complements of strongly regular graphs with $a_1 = 0$ are in this class.

It is not hard to construct graphs in this class which are not strongly regular. For example, a clique extension Δ of a graph Λ in this class is also in it. By a clique extension we mean the following. Let $K^u (u \in \Lambda)$ be finite disjoint sets of the same size. Δ is a graph whose vertex set is $\cup_{u\in\Lambda} K^u$ and two distinct vertices $x \in K^u$ and $y \in K^v$ are adjacent if and only if $u = v$ or u and v are adjacent in Λ.

Corollary 1.2 *Let Γ be a distance-regular graph with diameter d at least 3, and Δ a strongly regular graph such that $\Delta_2(\beta)$ is a disjoint union of cliques for every β in Δ. If $\Gamma_d(\alpha)$ is isomorphic to Δ for every α in Γ, then $d = 3$ and Γ is isomorphic to $J(8, 3)$.*

Proof: Suppose $\Delta_2(\beta)$ is not a clique. Then it follows from Lemma 3.1 of [6] that Δ is a complete multipartite graph $K_{\tau\times s}$. Then by an unpublished work of A. Hiraki and H. Suzuki (see Appendix), we get $d \leq 2$. So we may assume that $\Delta_2(\beta)$ is a clique. Now the assertion follows from Theorem 1.1. □

2. Intersection diagram

In this section we shall introduce the intersection diagrams of rank d which we use as our main tool.

Let $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = d$. Set

$$D^i_j = D^i_j(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma_j(\beta) \qquad (0 \leq i, j \leq d).$$

It is easy to see the following.

(1) $D^i_j = \phi$ if $d > i + j$,
(2) $D^i_{d-i} \neq \phi$ if $0 \leq i \leq d$,
(3) There is no edge between D^i_j and D^f_g if $|i - f| > 1$ or $|j - g| > 1$.

An intersection diagram of rank d with respect to (α, β) is the collection $\{D^i_j\}_{i,j}$ with lines between D^i_j's and D^f_g's. We draw a line

$$D^i_j \text{---} D^f_g$$

if there is possibility of existence of edges between D^i_j and D^f_g, and we erase the line when we know there is no edge between D^i_j and D^f_g.

In the following $e(A, B)$ denotes the number of edges between subsets A and B of Γ, and $e(\{\gamma\}, A) = e(\gamma, A)$. We write $\alpha \sim \beta$, when β is in $\Gamma_1(\alpha)$, and $\alpha \not\sim \beta$, otherwise.

The following are straightforward and useful for determining the form of the intersection diagram.

For each $\gamma \in D^i_j$, we have the following.

(4) $c_i = e(\gamma, D^{i-1}_{j+1}) + e(\gamma, D^{i-1}_j) + e(\gamma, D^{i-1}_{j-1})$,
$c_j = e(\gamma, D^{i+1}_{j-1}) + e(\gamma, D^i_{j-1}) + e(\gamma, D^{i-1}_{j-1})$,
(5) $a_i = e(\gamma, D^i_{j+1}) + e(\gamma, D^i_j) + e(\gamma, D^i_{j-1})$,
$a_j = e(\gamma, D^{i+1}_j) + e(\gamma, D^i_j) + e(\gamma, D^{i-1}_j)$,
(6) $b_i = e(\gamma, D^{i+1}_{j+1}) + e(\gamma, D^{i+1}_j) + e(\gamma, D^{i+1}_{j-1})$,
$b_j = e(\gamma, D^{i+1}_{j+1}) + e(\gamma, D^i_{j+1}) + e(\gamma, D^{i-1}_{j+1})$.

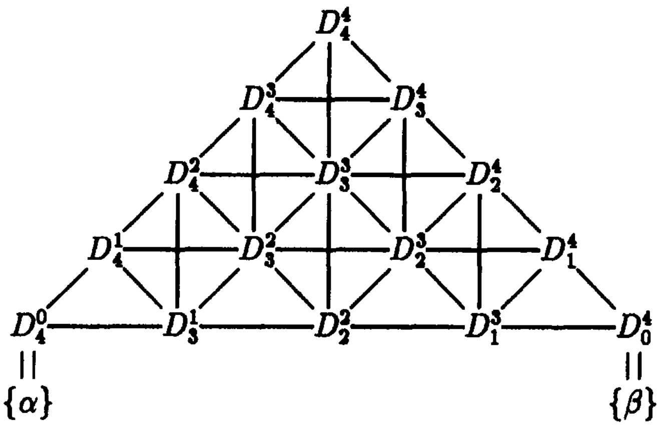

Figure 1.

Figure 1 is an example of the intersection diagram of rank $d = d(\Gamma)$ with $d = 4$.

For the properties and applications of intersection diagrams, see for example [2] and [4].

3. Preliminaries

In this section we determine the shape of the intersection diagram under the hypothesis of Theorem 1.1, and prove some basic lemmas.

Suppose there is a vertex $x \in D^i_j$, for some i, j with $i \geq 3$, $j \geq 3$, $i + j \geq d + 3$. Then there is a vertex $y \in \Gamma_d(\alpha) \cap \Gamma_{d-i}(x)$. Since β, $y \in \Gamma_d(\alpha)$ and the height $h = 2$,

$$\partial(\beta, y) \leq 2.$$

On the other hand,

$$\partial(\beta, y) \geq |\partial(\beta, x) - \partial(x, y)| = |(i + j) - d| \geq 3,$$

which is impossible. So

$$D^i_j = \emptyset \qquad \text{for } i \geq 3,\ j \geq 3,\ i + j \geq d + 3.$$

Therefore the intersection diagram becomes as in Fig. 2.

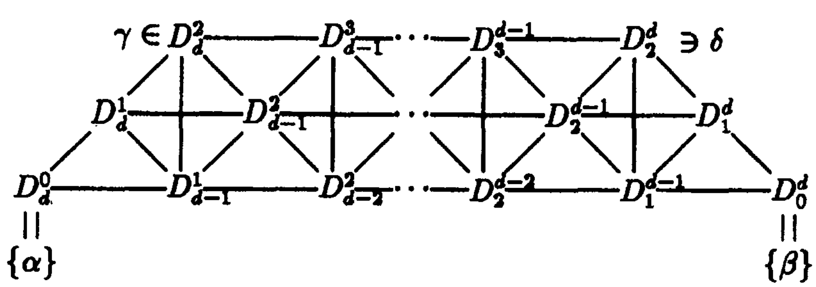

Figure 2.

Take any $\gamma \in D_d^2$, then

$$\Gamma_i(\alpha) \cap \Gamma_{i-2}(\gamma) \subseteq D_{d-i+2}^i \qquad \text{for } 2 \leq i \leq d.$$

Since $p_{i\,i-2}^2 \neq 0$, we get

$$D_{d-i+2}^i \neq \phi, \text{ i.e. } p_{i\,d-i+2}^d \neq 0 \qquad \text{for } 2 \leq i \leq d.$$

Since $k_i p_{d\,d-i+2}^i = k_d p_{i\,d-i+2}^d \neq 0$, we have

$$p_{d\,d-i+2}^i \neq 0 \quad \text{for } 2 \leq i \leq d.$$

Let $\kappa_1 = p_{d1}^d = a_d$ and $\kappa_2 = p_{d2}^d$. Then $k_d = 1 + \kappa_1 + \kappa_2$. Since D_2^d is a clique, for any $\delta \in D_2^d$, $e(\delta, D_2^d) = \kappa_2 - 1$.

Lemma 3.1 *For every α in Γ and every β, γ, δ in $\Gamma_d(\alpha)$, $\partial(\beta, \gamma) + \partial(\gamma, \delta) + \partial(\delta, \beta) \leq 5$.*

Proof: Suppose there are vertices $\beta, \gamma, \delta \in \Gamma_d(\alpha)$ such that $\partial(\beta, \gamma) + \partial(\gamma, \delta) + \partial(\delta, \beta) \geq 6$. Since the height $h = 2$,

$$\partial(\beta, \gamma) = \partial(\gamma, \delta) = \partial(\delta, \beta) = 2.$$

So $\gamma, \delta \in D_2^d$. This contradicts that D_2^d is a clique. □

Lemma 3.2 $\partial(D_{d-2}^2, D_2^d) \geq d - 1$.

Proof: Suppose there are vertices $u \in D_{d-2}^2$ and $v \in D_2^d$ such that $\partial(u, v) \leq d - 2$ (see Fig. 3).

We can take $w \in \Gamma_d(\alpha) \cap \Gamma_d(u)$ because $p_{dd}^2 \neq 0$. Since $\beta, v, w \in \Gamma_d(\alpha)$ with $\partial(\beta, v) = 2$, by Lemma 3.1, we have $\partial(w, \beta) = 1$ or $\partial(w, v) = 1$. Since $\partial(u, \beta) = d - 2$ and $\partial(u, v) \leq d - 2$, we get $\partial(u, w) \leq d - 1$. This contradicts $w \in \Gamma_d(u)$. □

Lemma 3.3 $e(D_{d-i-1}^{i+1}, D_{d-i}^{i+2}) = 0$ *for* $0 \leq i \leq d - 2$.

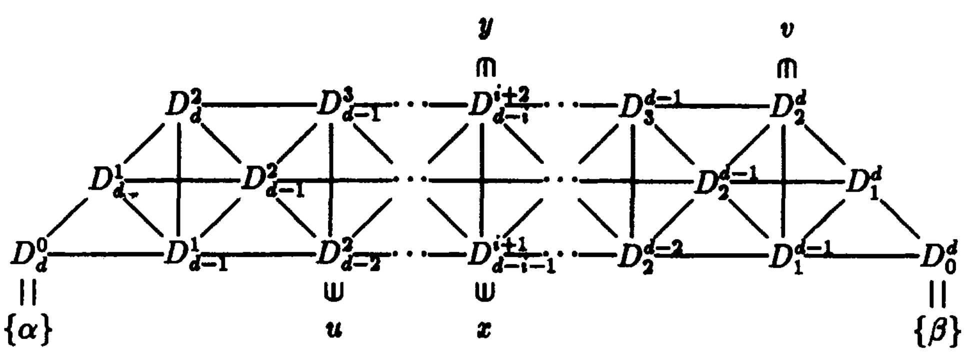

Figure 3.

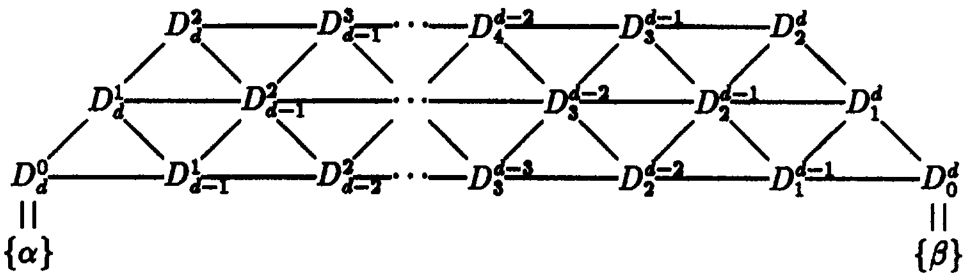

Figure 4.

Proof: Suppose not. Then there is an edge $x \sim y$ such that $x \in D^{i+1}_{d-i-1}$, $y \in D^{i+2}_{d-i}$. If $i \geq 1$, we can take $u \in D^2_{d-2}$ with $\partial(u, x) = i - 1$ and $v \in D^d_2$ with $\partial(y, v) = d - i - 2$ (see Fig. 3). We get $\partial(u, v) = d-2$, which contradicts Lemma 3.2. Since $e(D^{d-1}_1, D^d_2) = 0$, we get $e(D^1_{d-1}, D^2_d) = 0$ by symmetry. □

By Lemma 3.3, the intersection diagram becomes as in Fig. 4.

Lemma 3.4 *The following hold.*

(1) $\Gamma_d(\alpha)$ *is geodetically closed for every* α *in* Γ,
(2) $\kappa_1 \geq 2$,
(3) $c_2 = \kappa_1 - \kappa_2 + 1$.

Proof:

(1) Let $\beta, \gamma \in \Gamma_d(\alpha)$ with $\partial(\beta, \gamma) = i$. Since the height $h = 2$, we only consider the case $i = 2$. Then $\gamma \in D^d_2$. Since $e(D^{d-1}_1, D^d_2) = 0$,

$$\Gamma_1(\beta) \cap \Gamma_1(\gamma) \subseteq D^d_1 \subseteq \Gamma_d(\alpha).$$

(2) For any $\gamma \in D^d_2$, there is $\delta \in D^d_1$ such that $\gamma \sim \delta \sim \beta$. So

$$\kappa_1 = p^d_{d1} = |\Gamma_d(\alpha) \cap \Gamma_1(\delta)| \geq 2.$$

(3) Take $\gamma \in D^d_2$, then $\kappa_1 = a_d = e(\gamma, D^d_2) + e(\gamma, D^d_1) = \kappa_2 - 1 + e(\gamma, D^d_1)$. From Lemma 3.3, we get

$$c_2 = e(\gamma, D^d_1) = \kappa_1 - \kappa_2 + 1.$$ □

Lemma 3.5 $c_3 \neq 1$.

Proof: Suppose $c_3 = 1$. Then for any $x \in D^{d-1}_3$,

$$b_{d-1} = e(x, D^d_2) \leq e(x, D^d_2) + e(x, D^{d-1}_2) = c_3 = 1.$$

Hence we have

$$b_{d-1} = 1.$$

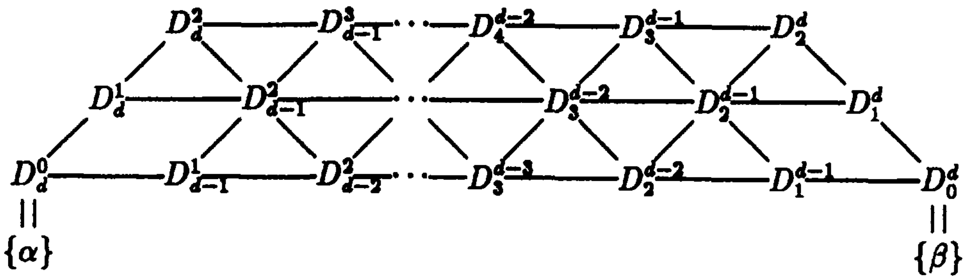

Figure 5.

For any $y \in D_1^{d-1}$,

$$1 = b_{d-1} = e(y, D_1^d) + e(y, D_0^d) = e(y, D_1^d) + 1.$$

So we get $e(y, D_1^d) = 0$. Hence we have

$$e(D_1^{d-1}, D_1^d) = 0.$$

Therefore the intersection diagram becomes as in Fig. 5.

For any $\gamma \in D_2^d$ and any $\delta \in D_1^d$, we get

$$e(\gamma, D_1^d) = c_2, \ e(\delta, D_1^d) = a_1.$$

So for any two vertices $u, v \in \Gamma_d(\alpha)$, the number of vertices which are adjacent to u and v in $\Gamma_d(\alpha)$ is c_2 if $u \not\sim v$ and a_1 if $u \sim v$. Hence $\Gamma_d(\alpha)$ becomes strongly regular.

We use bar to distinguish the parameters of $\Delta = \Gamma_d(\alpha)$ from those of Γ. Then $\bar{k} = \kappa_1$, $\bar{k_2} = \kappa_2$.

Since $c_3 = c_2 = 1$, Lemma 3.4(3) implies that $\kappa_1 = \kappa_2$. Hence the intersection array of Δ becomes

$$\iota(\Delta) = \begin{Bmatrix} * & 1 & 1 \\ 0 & \kappa_1 - 2 & \kappa_1 - 1 \\ \kappa_1 & 1 & * \end{Bmatrix}.$$

Since $\bar{b_1} = \bar{c_2} = 1$, we get $\bar{a_1} = 0$ (see Proposition 5.5.1 of [5]). So we have $\kappa_1 = 2$ and

$$k p^1_{dd} = k_d p^d_{d1} = 10,$$
$$k_2 p^2_{dd} = k_d p^d_{d2} = 10.$$

As $k < k_2$ (see Lemma 5.1.2 of [5]),

$$k = 5, \quad k_2 = 10.$$

So we have

$$b_1 = 2, \quad a_1 = 2.$$

Hence Γ is locally pentagon and we know Γ is isomorphic to the icosahedron (see Proposition 1.1.4 of [5]). This contradicts $k_2 = 10$. □

4. The case $d \geq 4$

In this section we discuss the case $d \geq 4$ and prove this case does not occur.

Lemma 4.1 *Suppose $d \geq 4$. Then the following hold.*

(1) $b_2 \geq c_{d-1}$,
(2) $b_{d-2} \geq c_3$.

Proof:

(1) Take $\gamma \in D_2^d$, then

$$b_2 = e(\gamma, D_3^{d-1}) \leq e(\gamma, D_3^{d-1}) + e(\gamma, D_2^{d-1}) = c_d.$$

Suppose $b_2 = c_d$, then $b_2 = c_d \geq c_{d-1}$. So we may assume $b_2 < c_d$. Then $e(\gamma, D_2^{d-1}) \neq 0$, so there is $\delta \in D_2^{d-1}$ such that $\gamma \sim \delta$ (see Fig. 6).

Claim $e(\delta, D_2^{d-2}) = 0$.

Suppose for some $x \in D_2^{d-2}$ such that $x \sim \delta$. Since there is $y \in D_{d-2}^2$ such that $\partial(y, x) = d - 4$, we get $\partial(y, \gamma) = d - 2$. This contradicts Lemma 3.2. Hence we get $e(\delta, D_2^{d-2}) = 0$.
By Claim, we get

$$b_2 = e(\delta, D_3^{d-2}) + e(\delta, D_3^{d-1}) \geq e(\delta, D_3^{d-2}) = c_{d-1}.$$

(2) Take $u \in D_{d-2}^4$ and argue similarly as in (1). □

Lemma 4.2 *Suppose $d \geq 4$. Then for every x in D_{d-2}^2, there are γ and δ in $\Gamma_d(x)$ such that γ in D_2^d and δ in D_4^{d-2}.*

Proof: Since $p_{dd}^2 \neq 0$, take $\gamma \in \Gamma_d(\alpha) \cap \Gamma_d(x)$. Then $\partial(\beta, \gamma) \geq \partial(x, \gamma) - \partial(x, \beta) = 2$. $\beta, \gamma \in \Gamma_d(\alpha)$ and the height $h = 2$, so $\partial(\beta, \gamma) = 2$. Hence we get

$$\gamma \in D_2^d.$$

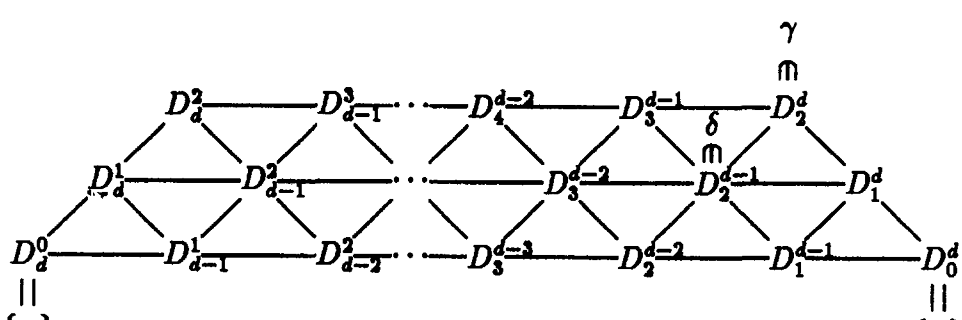

Figure 6.

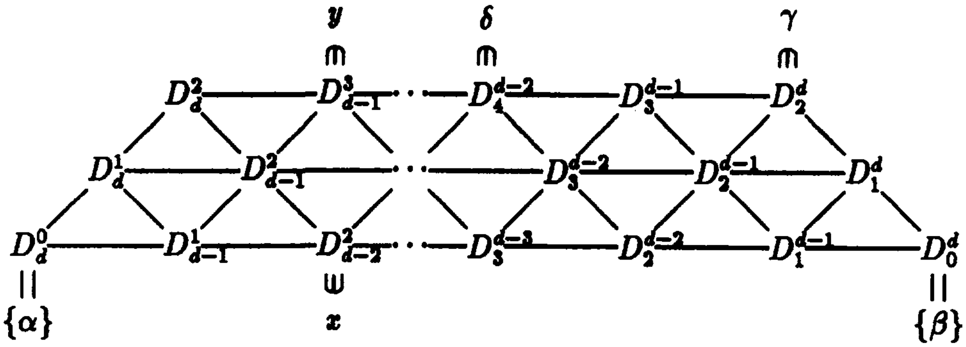

Figure 7.

Since $p^{d-2}_{d\,4} \neq 0$, take $\delta \in \Gamma_d(x) \cap \Gamma_4(\beta)$. Then $\partial(\alpha, \delta) \geq \partial(x, \delta) - \partial(\alpha, x) = d - 2$. Since $D^i_4 = \phi$ for $i \geq d - 1$, we get

$$\delta \in D^{d-2}_4. \qquad \square$$

Lemma 4.3 *Suppose $d \geq 4$. Then $\partial(D^2_{d-2}, D^3_{d-1}) \geq 3$.*

Proof: Suppose there are $x \in D^2_{d-2}$, $y \in D^3_{d-1}$ such that $\partial(x, y) = 2$. Then there is $z \in \Gamma_d(x) \cap \Gamma_d(y)$. By Lemma 4.2, there are $\gamma, \delta \in \Gamma_d(x)$ such that $\gamma \in D^d_2$, $\delta \in D^{d-2}_4$ (see Fig. 7). Since $\gamma, \delta, z \in \Gamma_d(x)$ with $\partial(\gamma, \delta) = 2$, Lemma 3.1 implies that $\partial(z, \gamma) \leq 1$ or $\partial(z, \delta) \leq 1$.

Case 1. $\partial(z, \gamma) \leq 1$.

Since there is $u \in D^d_2$ such that $\partial(y, u) = d - 3$ and D^d_2 is a clique, $\partial(y, \gamma) \leq d - 2$. So we get $\partial(y, z) \leq d - 1$, which contradicts $z \in \Gamma_d(y)$.

Case 2. $\partial(z, \delta) \leq 1$.

There is $v \in D^2_d$ such that $\partial(\delta, v) = d - 4$ and there is $w \in D^2_d$ such that $\partial(y, w) = 1$. As D^2_d is a clique, $\partial(y, z) \leq d - 1$. This is a contradiction. $\square$

Lemma 4.4 $d = 3$.

Proof: Suppose $d \geq 4$. Take $x \in D^2_{d-2}$. If $b_{d-2} > c_2$, then we can take an edge $x \sim z$ such that $z \in D^2_{d-1}$. By Lemma 4.1(1) $b_2 \geq c_{d-1}$. So

$$e(z, D^3_{d-1}) + e(z, D^3_{d-2}) \geq e(z, D^3_{d-2}) + e(z, D^2_{d-2}),$$

$$e(z, D^3_{d-1}) \geq e(z, D^2_{d-2}) \geq e(z, x) = 1.$$

Hence we can take an edge $z \sim y$ such that $y \in D^3_{d-1}$. So $\partial(x, y) = 2$, which contradicts Lemma 4.3. We may assume $b_{d-2} = c_2$. By Lemma 4.1 (2), $c_2 = b_{d-2} \geq c_3$. Therefore from Theorem 5.4.1 of [5] we get $c_3 = 1$. This contradicts Lemma 3.5. $\square$

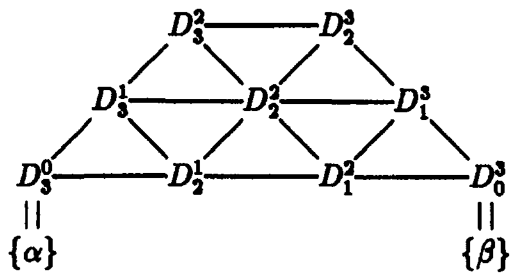

Figure 8.

5. Proof of Theorem 1.1

In the following we may assume $d = 3$. The intersection diagram becomes as in Fig. 8.

Lemma 5.1 *For every γ in D_1^2, $\Gamma_3(\alpha) \cap \Gamma_3(\gamma) \subseteq D_2^3$. In particular $p_{33}^2 \leq p_{32}^3$, and the equality holds if and only if $b_2 = c_3$.*

Proof: Take $\gamma \in D_1^2$. Since $\gamma \sim \beta$ and $D_1^3 \subseteq \Gamma_1(\beta)$, we get

$$\Gamma_3(\alpha) \cap \Gamma_3(\gamma) \subseteq D_2^3.$$

Therefore

$$p_{33}^2 = |\Gamma_3(\alpha) \cap \Gamma_3(\gamma)| \leq |D_2^3| = p_{32}^3.$$

Since $\frac{p_{32}^3}{p_{33}^2} = \frac{k_2}{k_3} = \frac{c_3}{b_2}$,

$$p_{33}^2 = p_{32}^3 \quad \text{if and only if } b_2 = c_3. \qquad \square$$

Lemma 5.2 *For every x in D_3^2, $\Gamma_3(\alpha) \cap \Gamma_1(x) = D_2^3$. In particular $b_2 = \kappa_2$.*

Proof: For any $x \in D_3^2$,

$$\Gamma_3(\alpha) \cap \Gamma_1(x) \subseteq D_2^3.$$

By way of contradiction, suppose there is $y \in D_2^3$ such that $x \not\sim y$ (see Fig. 9).

Since D_2^3 is a clique,

$$\Gamma_3(\alpha) \cap \Gamma_1(x) \subseteq \Gamma_1(y).$$

So we know

$$\partial(x, y) = 2.$$

Take $z \in \Gamma_3(x) \cap \Gamma_3(y)$. Since the height $h = 2$, $z \notin \Gamma_3(\alpha) \cup \Gamma_3(\beta)$. So $\partial(\alpha, z) = 2$ or $\partial(\beta, z) = 2$. We may assume

$$\partial(\alpha, z) = 2.$$

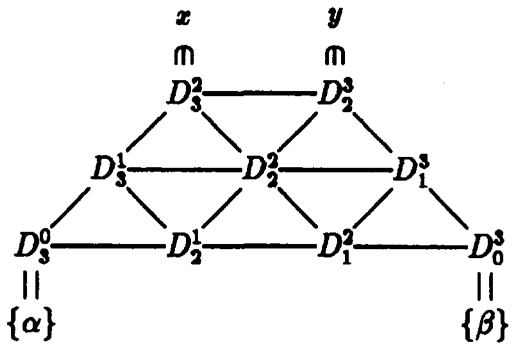

Figure 9.

From Lemma 3.4(1), $\Gamma_3(z)$ is geodetically closed. Since $x, y \in \Gamma_3(z)$ with $\partial(x, y) = 2$ and D_2^3 is a clique,

$$\begin{aligned}\Gamma_3(z) &\supseteq (\Gamma_1(x) \cap \Gamma_1(y)) \cup \{y\} \cup \{x\}\\ &\supseteq (\Gamma_3(\alpha) \cap \Gamma_1(x)) \cup \{y\}.\end{aligned}$$

So

$$\Gamma_3(\alpha) \cap \Gamma_3(z) \supseteq (\Gamma_3(\alpha) \cap \Gamma_1(x)) \cup \{y\}.$$

Claim 1 $b_2 = c_3$.

Suppose there is some $\gamma \in D_2^3$ such that $\gamma \notin \Gamma_3(\alpha) \cap \Gamma_3(z)$. Then $\partial(z, \gamma) = 2$ because D_2^3 is a clique and $\partial(z, y) = 3$. So

$$\begin{aligned}\Gamma_3(z) \cap \Gamma_1(\gamma) &\supseteq ((\Gamma_3(\alpha) \cap \Gamma_1(x)) \cup \{y\}) \cap \Gamma_1(\gamma)\\ &= (\Gamma_3(\alpha) \cap \Gamma_1(x)) \cup \{y\}.\end{aligned}$$

In this case

$$b_2 \geq b_2 + 1,$$

which is impossible. Hence we get

$$\Gamma_3(\alpha) \cap \Gamma_3(z) \supseteq D_2^3 \quad \text{i.e.} \quad p_{33}^2 \geq p_{32}^3.$$

From Lemma 5.1, we get

$$b_2 = c_3.$$

By Claim 1, for any $\delta \in D_3^2$,

$$e(\delta, D_2^3) = b_2 = c_3 = e(\delta, D_2^3) + e(\delta, D_2^2).$$

So we get $e(\delta, D_2^2) = 0$. Hence we have

$$e(D_3^2, D_2^2) = 0.$$

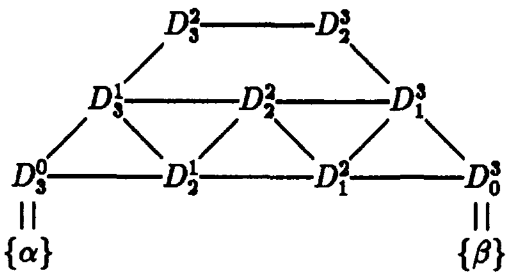

Figure 10.

Therefore the intersection diagram becomes as in Fig. 10.

Claim 2 $D^2_2 \neq \emptyset$.

Suppose $D^2_2 = \emptyset$. Then for any $u \in D^1_2$,

$$b_1 = e(u, D^2_1) = c_2.$$

By Claim 1, $c_3 = b_2 \leq b_1 = c_2$. Hence, by Theorem 5.4.1 of [5], we get $c_3 = 1$. This contradicts Lemma 3.5.

By Claim 2, take $\epsilon \in D^2_2$, then

$$c_3 = b_2 = e(\epsilon, D^3_1) \leq e(\epsilon, D^3_1) + e(\epsilon, D^2_1) = c_2.$$

Hence by Theorem 5.4.1 of [5], we get

$$c_3 = 1.$$

This contradicts Lemma 3.5. Therefore we get

$$\Gamma_3(\alpha) \cap \Gamma_1(x) = D^3_2. \qquad \square$$

Lemma 5.3 $2p^1_{33} = \kappa_1 + p^2_{33} + 1$.

Proof: Take any $x \in D^2_3$. Then by Lemma 5.2,

$$\Gamma_3(\alpha) \cap \Gamma_1(x) = D^3_2.$$

Take any $y \in D^1_3$ such that $x \sim y$ (see Fig. 11). Then

$$\Gamma_2(y) \supseteq D^3_2.$$

Claim 1 $\Gamma_3(x) \subseteq D^1_2 \cup D^2_1 \cup D^3_1 \cup D^3_0$, $\Gamma_3(y) \subseteq D^2_2 \cup D^2_1 \cup D^3_1 \cup D^3_0$.

Since $\Gamma_1(x) \supseteq D^3_2$ and the height $h = 2$, we get

$$\Gamma_3(x) \subseteq D^1_2 \cup D^2_2 \cup D^2_1 \cup D^3_1 \cup D^3_0.$$

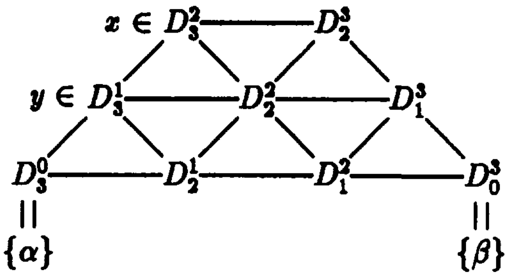

Figure 11.

So we know

$$\Gamma_3(x) \cap \Gamma_2(\beta) = \Gamma_3(x) \cap \left(D_2^1 \cup D_2^2\right),$$
$$\Gamma_3(x) \cap \Gamma_1(\alpha) = \Gamma_3(x) \cap D_2^1.$$

Since $\partial(x, \alpha) = 2$, by Lemma 5.2,

$$\kappa_2 = b_2 = |\Gamma_3(x) \cap \Gamma_1(\alpha)| = \left|\Gamma_3(x) \cap D_2^1\right|.$$

Since $\kappa_2 = |\Gamma_3(x) \cap \Gamma_2(\beta)| = |\Gamma_3(x) \cap (D_2^1 \cup D_2^2)|$, we get

$$\Gamma_3(x) \cap D_2^2 = \phi.$$

Therefore

$$\Gamma_3(x) \subseteq D_2^1 \cup D_1^2 \cup D_1^3 \cup D_0^3.$$

$\Gamma_2(y) \supseteq D_2^3$ and $y \sim \alpha$, hence we get

$$\Gamma_3(y) \subseteq D_2^2 \cup D_1^2 \cup D_1^3 \cup D_0^3.$$

Claim 2 $\Gamma_3(y) \cap D_1^2 \subseteq \Gamma_3(y) \cap \Gamma_3(x)$.

Let $\gamma \in \Gamma_3(y) \cap D_1^2$. By Lemma 5.1, there is $\delta \in \Gamma_3(\alpha) \cap \Gamma_3(\gamma)$ such that $\delta \in D_2^3$. Then $x \sim \delta$. From Lemma 3.4(1), $\Gamma_3(\gamma)$ is geodetically closed. Since $y, \delta \in \Gamma_3(\gamma)$ with $\partial(y, \delta) = 2$ and $y \sim x \sim \delta$, we get $x \in \Gamma_3(\gamma)$. Hence $\gamma \in \Gamma_3(y) \cap \Gamma_3(x)$.

Claim 3 $\Gamma_3(\alpha) \cap \Gamma_3(x) \subseteq \Gamma_3(y)$.

Take any $\epsilon \in \Gamma_3(\alpha) \cap \Gamma_3(x)$. Since $\alpha, x \in \Gamma_3(\epsilon)$ with $\partial(\alpha, x) = 2$, $\alpha \sim y \sim x$ and $\Gamma_3(\epsilon)$ is geodetically closed, we get $y \in \Gamma_3(\epsilon)$. Hence $\epsilon \in \Gamma_3(y)$.

Claim 4 $\Gamma_3(y) \cap (D_1^2 \cup D_1^3 \cup D_0^3) = \Gamma_3(y) \cap (\Gamma_3(\alpha) \cup \Gamma_3(x))$.

By Claim 2, $\Gamma_3(y) \cap D_1^2 \subseteq \Gamma_3(y) \cap \Gamma_3(x)$. Since $D_1^3 \cup D_0^3 \subseteq \Gamma_3(\alpha)$, $\Gamma_3(y) \cap (D_1^3 \cup D_0^3) \subseteq \Gamma_3(y) \cap \Gamma_3(\alpha)$. Hence

$$\Gamma_3(y) \cap \left(D_1^2 \cup D_1^3 \cup D_0^3\right) \subseteq \Gamma_3(y) \cap (\Gamma_3(\alpha) \cup \Gamma_3(x)).$$

On the other hand, take any $u \in \Gamma_3(y) \cap (\Gamma_3(\alpha) \cup \Gamma_3(x))$. If $u \in \Gamma_3(y) \cap \Gamma_3(\alpha)$, then by Claim 1, $u \in \Gamma_3(y) \cap (D_1^3 \cup D_0^3)$. If $u \in \Gamma_3(y) \cap \Gamma_3(x)$, then $u \in \Gamma_3(y) \cap (D_1^2 \cup D_1^3 \cup D_0^3)$. Therefore we get the claim.

Since $\alpha \sim y \sim x$ and $\partial(\alpha, x) = 2$, by Claim 3,

$$\begin{aligned} &|\Gamma_3(y) \cap (\Gamma_3(\alpha) \cup \Gamma_3(x))| \\ &\quad = |\Gamma_3(y) \cap \Gamma_3(\alpha)| + |\Gamma_3(y) \cap \Gamma_3(x)| - |\Gamma_3(\alpha) \cap \Gamma_3(x)| \\ &\quad = 2p_{33}^1 - p_{33}^2. \end{aligned}$$

Since $\partial(y, \beta) = 3$,

$$\begin{aligned} \kappa_1 &= |\Gamma_3(y) \cap \Gamma_1(\beta)| \\ &= \left|\Gamma_3(y) \cap \left(D_1^2 \cup D_1^3\right)\right| \\ &= \left|\Gamma_3(y) \cap \left(D_1^2 \cup D_1^3 \cup D_0^3\right) - \{\beta\}\right|. \end{aligned}$$

Hence by Claim 4, we get

$$\kappa_1 = 2p_{33}^1 - p_{33}^2 - 1.$$

□

Lemma 5.4 $p_{33}^2 = 1$.

Proof: By way of contradiction, suppose $p_{33}^2 \geq 2$. Take $x \in D_3^2$. Since $\beta \in \Gamma_3(\alpha) \cap \Gamma_3(x)$, there is $\gamma \in \Gamma_3(\alpha) \cap \Gamma_3(x) - \{\beta\}$. From Lemma 5.2, $\Gamma_1(x) \supseteq D_2^3$. Hence $\gamma \in D_1^3$ and $e(\gamma, D_2^3) = 0$.

Claim 1 $\kappa_1 \geq 2\kappa_2 - 1$.

Since $b_1 = e(\gamma, D_2^2)$, there is $\delta \in D_2^2$ such that $\gamma \sim \delta$. Suppose there is $y \in D_2^3$ such that $\delta \sim y$. As $e(\gamma, D_2^3) = 0$, $\partial(\gamma, y) = 2$. Since $\gamma, y \in \Gamma_3(\alpha)$ and $\Gamma_3(\alpha)$ is geodetically closed, we get $\delta \in \Gamma_3(\alpha)$. But this contradicts $\delta \in D_2^2$. So

$$e\left(\delta, D_2^3\right) = 0.$$

Hence

$$b_2 = e\left(\delta, D_1^3\right) \leq e\left(\delta, D_1^3\right) + e\left(\delta, D_1^2\right) = c_2.$$

From Lemma 3.4(3) and 5.2,

$$\kappa_2 = b_2 \leq c_2 = \kappa_1 - \kappa_2 + 1.$$

Since $e(\gamma, D_2^3) = 0$, we can take $\epsilon \in D_2^3$ such that $\partial(\epsilon, \gamma) = 2$ (see Fig. 12).

Claim 2 $\Gamma_3(\gamma) \subseteq D_3^0 \cup D_3^1 \cup D_3^2 \cup D_2^2 \cup D_2^1$, $\Gamma_3(\epsilon) \subseteq D_3^0 \cup D_3^1 \cup D_2^1 \cup D_1^2$.

By an argument similar to that in the Proof of Lemma 5.3, we have the claim.

Claim 3 $\Gamma_3(\gamma) \cap \Gamma_3(\epsilon) \cap D_2^1 = \emptyset$.

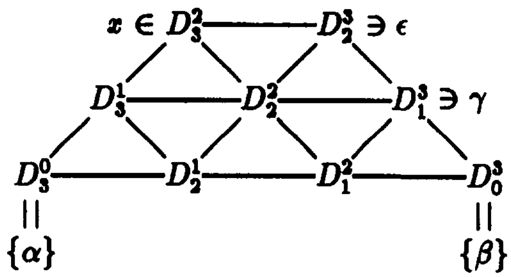

Figure 12.

Suppose there is $u \in \Gamma_3(\gamma) \cap \Gamma_3(\epsilon) \cap D_2^1$. Since $\gamma, \epsilon \in \Gamma_3(\alpha) \cap \Gamma_3(u)$ with $\partial(\epsilon, \gamma) = 2$ and $\Gamma_3(\alpha) \cap \Gamma_3(u)$ is geodetically closed,

$$(\Gamma_1(\gamma) \cap \Gamma_1(\epsilon)) \cup \{\gamma\} \cup \{\epsilon\} \subseteq \Gamma_3(\alpha) \cap \Gamma_3(u).$$

Since $e(\gamma, D_2^3) = 0$,

$$(\Gamma_1(\gamma) \cap \Gamma_1(\epsilon)) \cup \{\gamma\} \subseteq \Gamma_3(u) \cap D_1^3 \subseteq \Gamma_3(u) \cap \Gamma_1(\beta).$$

As $\partial(\gamma, \epsilon) = \partial(u, \beta) = 2$, we get

$$c_2 + 1 \leq b_2.$$

From Lemma 3.4(3) and 5.2,

$$\kappa_1 - \kappa_2 + 2 \leq \kappa_2.$$

This contradicts Claim 1.

Claim 4 $p_{33}^2 + c_2 + 1 \leq p_{33}^1$.

From Claim 2 and 3,

$$\Gamma_3(\gamma) \cap \Gamma_3(\epsilon) \subseteq \Gamma_3(\beta) \cap \Gamma_3(\gamma).$$

Since $\alpha, x \in \Gamma_3(\beta) \cap \Gamma_3(\gamma)$ with $\partial(\alpha, x) = 2$ and $\Gamma_3(\beta) \cap \Gamma_3(\gamma)$ is geodetically closed,

$$(\Gamma_1(\alpha) \cap \Gamma_1(x)) \cup \{x\} \cup \{\alpha\} \subseteq \Gamma_3(\beta) \cap \Gamma_3(\gamma).$$

As $\alpha \in \Gamma_3(\gamma) \cap \Gamma_3(\epsilon)$,

$$(\Gamma_3(\gamma) \cap \Gamma_3(\epsilon)) \cup ((\Gamma_1(\alpha) \cap \Gamma_1(x)) \cup \{x\}) \subseteq \Gamma_3(\beta) \cap \Gamma_3(\gamma).$$

By Lemma 5.2, $x \sim \epsilon$. So

$$\Gamma_3(\epsilon) \cap ((\Gamma_1(\alpha) \cap \Gamma_1(x)) \cup \{x\}) = \emptyset.$$

So we get

$$(\Gamma_3(\gamma) \cap \Gamma_3(\epsilon)) \cap ((\Gamma_1(\alpha) \cap \Gamma_1(x)) \cup \{x\}) = \emptyset.$$

Therefore as $\partial(\gamma, \epsilon) = \partial(\alpha, x) = 2$ and $\beta \sim \gamma$,

$$|(\Gamma_3(\gamma) \cap \Gamma_3(\epsilon)) \cup ((\Gamma_1(\alpha) \cap \Gamma_1(x)) \cup \{x\})| \leq |\Gamma_3(\beta) \cap \Gamma_3(\gamma)|,$$
$$p^2_{33} + c_2 + 1 \leq p^1_{33}.$$

From Claim 4 and Lemma 5.3,

$$2\big(p^2_{33} + c_2 + 1\big) \leq \kappa_1 + p^2_{33} + 1,$$
$$2\big(p^2_{33} + \kappa_1 - \kappa_2 + 2\big) \leq \kappa_1 + p^2_{33} + 1,$$
$$p^2_{33} \leq -\kappa_1 + 2\kappa_2 - 3.$$

From Claim 1, we get

$$p^2_{33} \leq -2.$$

This is impossible. Hence we get

$$p^2_{33} = 1.$$

□

Lemma 5.5 *The following hold.*
(1) $2p^1_{33} = \kappa_1 + 2$,
(2) $p^1_{33}(\kappa^2_2 + \kappa_1) = \kappa_1(1 + \kappa_1 + \kappa_2)$.

Proof:

(1) It is clear from Lemma 5.3 and 5.4.
(2) It follows from Lemma 5.4 that

$$k_2 = p^2_{33}k_2 = p^3_{32}k_3 = \kappa_2(1 + \kappa_1 + \kappa_2).$$

Since $c_3(1 + \kappa_1 + \kappa_2) = c_3k_3 = b_2k_2 = \kappa^2_2(1 + \kappa_1 + \kappa_2)$,

$$c_3 = \kappa^2_2.$$

Hence

$$k = c_3 + a_3 = \kappa^2_2 + \kappa_1.$$

Since $p^1_{33}k = p^3_{31}k_3$, we get

$$p^1_{33}\big(\kappa^2_2 + \kappa_1\big) = \kappa_1(1 + \kappa_1 + \kappa_2).$$

□

Proof of Theorem 1.1. From Lemma 5.5,

$$(\kappa_1 + 2)\big(\kappa^2_2 + \kappa_1\big) = 2\kappa_1(1 + \kappa_1 + \kappa_2).$$
$$(\kappa_1 + 2)\kappa^2_2 - 2\kappa_1\kappa_2 - \kappa^2_1 = 0.$$

Hence we get

$$\kappa_2 = \frac{\kappa_1 + \kappa_1\sqrt{\kappa_1 + 3}}{\kappa_1 + 2}.$$

Since κ_2 is a positive integer, by Lemma 3.4(2), $\sqrt{\kappa_1+3}$ is a positive integer at least 3. Let $n=\sqrt{\kappa_1+3}$. Then

$$\kappa_2 = \frac{(n^2-3)(n+1)}{n^2-1} = n+1-\frac{2}{n-1}.$$

Hence we get $n=3$ and

$$\kappa_1 = 6, \quad \kappa_2 = 3.$$

Therefore we know all the intersection numbers of Γ and they are the same as those of $J(8,3)$. By the uniqueness (see [7] and [10]), we get

$$\Gamma \simeq J(8,3). \qquad \square$$

Appendix

The next theorem was proved by A. Hiraki and H. Suzuki.

Theorem *Let Δ be a complete multipartite graph $K_{\tau\times s}$ with $\tau \geq 2$, $s \geq 2$. Then there is no distance-regular graph Γ with diameter $d \geq 3$ such that $\Gamma_d(\alpha) \simeq \Delta$ for every vertex α in Γ.*

Proof: The intersection array of Δ is as follows.

$$\iota(\Delta) = \left\{ \begin{matrix} * & 1 & (\tau-1)s \\ 0 & (\tau-2)s & 0 \\ (\tau-1)s & s-1 & * \end{matrix} \right\}.$$

Suppose there exists a graph Γ satisfying the hypothesis. Take any $\alpha, \beta \in \Gamma$ with $\partial(\alpha,\beta) = d$. Then $p^d_{d1} = |\Delta_1(\beta)| = (\tau-1)s$, $p^d_{d2} = |\Delta_2(\beta)| = s-1$ and $k_d = |\Delta| = \tau s$. By an argument similar to that in Section 3, the intersection diagram becomes as in Fig. 13.

Since $\Delta_2(\beta)$ is a coclique, D^d_2 is a coclique. For any $x \in D^d_1$ and $y \in D^d_2$, we know $e(x, D^d_1) = (\tau-2)s$ and $e(y, D^d_1) = (\tau-1)s$.

Claim 1 *For every $\alpha, \gamma \in \Gamma$ with $\partial(\alpha,\gamma) = d-1$, $\Gamma_d(\alpha) \cap \Gamma_1(\gamma)$ is a coclique.*

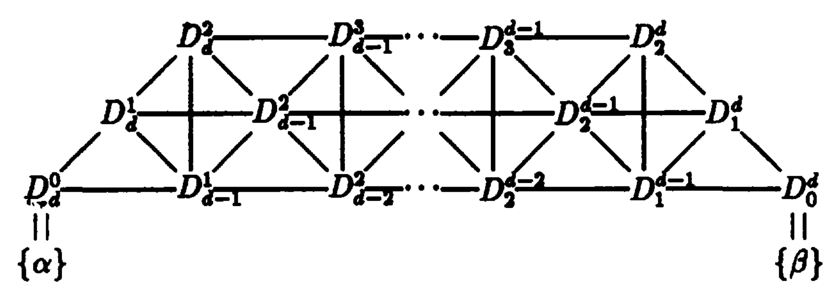

Figure 13.

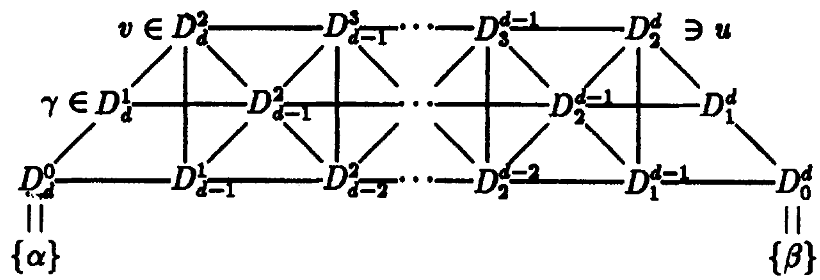

Figure 14.

Since $p^{d-1}_{d\,3} \neq 0$, we can take $\beta \in \Gamma_d(\alpha) \cap \Gamma_3(\gamma)$. Then $\gamma \in D^{d-1}_3$. As $\Gamma_d(\alpha) \cap \Gamma_1(\gamma) \subseteq D^d_2$ and D^d_2 is a coclique, we get the claim.

Claim 2 $e(D^{d-1}_1, D^d_1) = 0$, $a_1 = (\tau - 2)s$.

Suppose there is an edge $\gamma \sim \delta$ such that $\gamma \in D^{d-1}_1$, $\delta \in D^d_1$. Then $\Gamma_d(\alpha) \cap \Gamma_1(\gamma)$ contains an edge $\beta \sim \delta$, which contradicts Claim 1.

For any $x \in D^d_1$,

$$a_1 = e\big(x, D^d_1\big) = (\tau - 2)s.$$

Claim 3 *For every edge $\alpha \sim \gamma$, $\Gamma_d(\alpha) \cap \Gamma_d(\gamma)$ is a clique. $p^1_{dd} = \tau$, $k = (\tau - 1)s^2$.*

Take $\beta, \delta \in \Gamma_d(\alpha) \cap \Gamma_d(\gamma)$. Then $\gamma \in D^1_d$. For any $u \in D^d_2$, there is $v \in D^2_d$ such that $\partial(u, v) = d - 2$ (see Fig. 14).
Since $\gamma \sim v$, $\partial(\gamma, u) = d - 1$. So $\delta \in D^d_1$. Hence

$$\beta \sim \delta.$$

Therefore $\Gamma_d(\alpha) \cap \Gamma_d(\gamma)$ is a clique. Since the size of the maximal cliques of $\Gamma_d(\alpha) \simeq \Delta$ is τ,

$$p^1_{dd} \leq \tau.$$

Suppose $p^1_{dd} \leq \tau - 1$, then

$$k = \frac{k_d p^d_{d1}}{p^1_{dd}} \geq \tau s^2 > \tau s(s - 1).$$

Since $p^2_{dd} \geq 1$,

$$k_2 = \frac{k_d p^d_{d2}}{p^2_{dd}} \leq \tau s(s - 1).$$

So $k > k_2$, which is impossible (see Lemma 5.1.2 of [5]). Hence we get $p^1_{dd} = \tau$.

Claim 4 *For every $\alpha \in \Gamma$ and every edge $\beta \sim \gamma$ in $\Gamma_d(\alpha)$, $\Gamma_1(\beta) \cap \Gamma_1(\gamma) \subseteq \Gamma_d(\alpha)$.*

Since $\gamma \in D_1^d$, the claim follows from Claim 2.

Claim 5 $\tau = 2$, $p_{dd}^1 = 2$, $a_1 = 0$, $k = s^2$, $b_1 = s^2 - 1$, $k_2 \leq 2s(s-1)$.

Since $\tau \geq 2$, there is an edge $\beta \sim \delta$ in $\Gamma_d(\alpha) \cap \Gamma_d(\gamma)$ for $\alpha \sim \gamma$. From Claim 4,

$$(\Gamma_1(\beta) \cap \Gamma_1(\delta)) \cup \{\beta\} \cup \{\delta\} \subseteq \Gamma_d(\alpha) \cap \Gamma_d(\gamma).$$

In this case

$$a_1 + 2 \leq p_{dd}^1,$$
$$(\tau - 2)s + 2 \leq \tau.$$

Since $\tau \geq 2$ and $s \geq 2$, we have

$$\tau = 2.$$

So we get the claim.

Claim 6 $d = 3$.

Since $kb_1 = k_2c_2$, Claim 5 implies that

$$\frac{s^2(s^2-1)}{c_2} = k_2 \leq 2s(s-1).$$

So

$$k = s^2 < s(s+1) \leq 2c_2.$$

Since $b_2 \leq b_2 + a_2 = k - c_2 < c_2$, we get $d = 3$.

By Claim 6, the intersection diagram is as in Fig. 15.

By counting $e(D_2^3, D_1^2)$,

$$|D_1^2|(b_2 - 1) = |D_2^3|(c_2 - s),$$
$$(s^2 - s)(b_2 - 1) = (s - 1)(c_2 - s),$$
$$sb_2 = c_2.$$

Since $kb_1b_2 = k_3c_3c_2$,

$$s^2(s^2-1)b_2 = 2s(s^2 - s)c_2,$$
$$(s+1)b_2 = 2c_2.$$

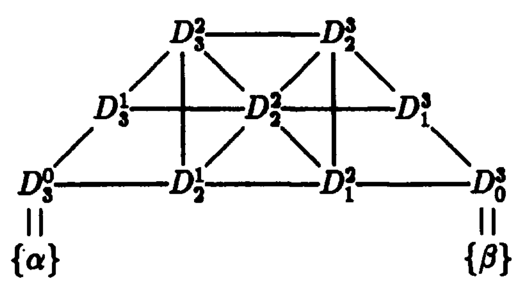

Figure 15.

Hence we get

$$(s+1)b_2 = 2sb_2.$$

Since $s \geq 2$, this is impossible. Therefore we get the assertion. □

Acknowledgment

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