

Plactification*

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Abstract. We study a map called *plactification* from reduced words to words. This map takes Coxeter-Knuth equivalence to Knuth equivalence, and has applications to the enumeration of reduced words, Schubert polynomials and certain Specht modules.

Keywords: reduced words, Knuth equivalence, Specht module, Schubert polynomial

1. Introduction

The problem of counting the reduced words of a given permutation has received a great deal of attention since the early 1980's [19]. A fundamental tool in this subject is the mysterious equivalence relation on reduced words known as *nilplactic equivalence* [9] or *Coxeter-Knuth equivalence* [2], which bears a striking resemblance to the better understood equivalence relation on words known as *plactic equivalence* [10] or *Knuth equivalence*.

This paper describes a map (considered earlier in a different form by Lascoux and Schützenberger [8]) called *plactification*. This map takes reduced words to words and maps nilplactic equivalence to plactic equivalence, substantiating the “striking resemblance” alluded to above. The map has other pleasant properties, giving rise to applications to the enumeration of reduced words, the theory of *Schubert polynomials*, and decompositions of certain *Specht modules*.

The paper is organized as follows. Section 2 establishes terminology. Section 3 defines the plactification map and proves its main properties. Section 4 discusses applications.

2. Definitions

It will be assumed that the reader has some familiarity with the notions of *partitions* λ and their *Ferrers diagrams*, *standard Young tableaux*, *column strict tableaux*, and *skew column strict tableaux* [18]. All tableaux will be assumed to be of Ferrers shape unless they are specifically referred to as skew. It is also assumed that the reader is familiar with the notions of *plactic (Knuth) equivalence*, the *Robinson-Schensted-Knuth correspondence*, and *jeu-de-taquin*, which are briefly reviewed here (for more thorough discussions, see [2, 6, 18]). The *plactic or Knuth equivalence* on words is the transitive closure of the relations

$$\dots ikj \dots \underset{k}{\sim} \dots kij \dots$$

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if $i \leq j < k$, and

$$\dots jik \dots \underset{K}{\sim} \dots jki \dots$$

if $i < j \leq k$. The *column-reading word* of the skew column strict tableau T , denoted $word(T)$, is defined to be the concatenation $v^1v^2v^3 \dots$, where v^r is the strictly decreasing word comprising the r th column of T . By abuse of language, we will occasionally refer to the skew tableau T when we really mean the column-reading word of T . For example, the Knuth equivalence class of T is the Knuth class of the column-reading word of T . The *row-reading word* of a skew tableau T is defined to be the concatenation $\dots u^3u^2u^1$, where u^r is the weakly increasing word comprising the r th row of T . It is well-known that the row-reading and column-reading word of T are Knuth equivalent, so the Knuth class of T is given by *either* reading word. Every word \underline{b} is Knuth equivalent to a unique column strict tableau which will be denoted $P(\underline{b})$. The tableau $P(\underline{b})$ can be computed using the algorithm known as *Robinson-Schensted-Knuth row insertion*. $P(\underline{b})$ is often called the *insertion tableau* of \underline{b} . Let $P_r(\underline{b}) = P(b_1b_2 \dots b_r)$. The *recording tableau* $Q(\underline{b})$ for the row insertion of the word \underline{b} is defined to be the standard tableau having the same shape as $P(\underline{b})$ in which the entry r is located in the cell of $P_r(\underline{b})$ which is not in $P_{r-1}(\underline{b})$. Row insertion gives a bijection between words \underline{b} and pairs of tableaux $(P(\underline{b}), Q(\underline{b}))$ of the same shape where P is column strict and Q is standard. This correspondence is written as

$$(\emptyset \leftarrow \underline{b}) = (P(\underline{b}), Q(\underline{b}))$$

Given a skew tableau, one can also obtain the insertion tableau for its reading word by sliding it into the northwest corner using the *jeu-de-taquin* (see [18]).

Example 1 Let $\underline{b} = 645436546$. Then its Robinson-Schensted-Knuth row insertion is given by

$$\begin{array}{ccccccc}
 P_1(\underline{b}) = 6, & P_2(\underline{b}) = \begin{array}{c} 4 \\ 6 \end{array}, & P_3(\underline{b}) = \begin{array}{cc} 4 & 5 \\ 6 & \end{array}, & P_4(\underline{b}) = \begin{array}{cc} 4 & 4 \\ 5 & 6 \end{array}, & P_5(\underline{b}) = \begin{array}{cc} 3 & 4 \\ 4 & 5 \\ 6 & \end{array} \\
 P_6(\underline{b}) = \begin{array}{ccc} 3 & 4 & 6 \\ 4 & & \\ 5 & & \\ 6 & & \end{array}, & P_7(\underline{b}) = \begin{array}{ccc} 3 & 4 & 5 \\ 4 & 6 & \\ 5 & & \\ 6 & & \end{array}, & P_8(\underline{b}) = \begin{array}{ccc} 3 & 4 & 4 \\ 4 & 5 & \\ 5 & 6 & \\ 6 & & \end{array}, & P_9(\underline{b}) = \begin{array}{cccc} 3 & 4 & 4 & 6 \\ 4 & 5 & & \\ 5 & 6 & & \\ 6 & & & \end{array} \\
 & & & & & & = P(\underline{b})
 \end{array}$$

and hence

$$(\emptyset \leftarrow \underline{b}) = (P(\underline{b}), Q(\underline{b})) = \left(\begin{array}{cccc|cccc}
 3 & 4 & 4 & 6 & 1 & 3 & 6 & 9 \\
 4 & 5 & & & 2 & 7 & & \\
 5 & 6 & & & 4 & 8 & & \\
 6 & & & & 5 & & &
 \end{array} \right).$$

Since $\underline{b} = \text{word}(T)$ for the skew tableau

$$T = \begin{array}{ccc} & 3 & 4 & 6 \\ & & 4 & 5 \\ 4 & 5 & 6 & \\ & & & 6 \end{array}$$

one can also obtain $P(\underline{b})$ using the jeu-de-taquin:

$$\begin{array}{ccc} \begin{array}{ccc} 3 & 4 & 6 \\ 4 & 5 & \\ 4 & 5 & 6 \\ 6 & & \end{array} & \mapsto & \begin{array}{ccc} 3 & 4 & 6 \\ 4 & 4 & 5 \\ 5 & 6 & \\ 6 & & \end{array} & \mapsto & \begin{array}{cccc} 3 & 4 & 4 & 6 \\ 4 & 5 & & \\ 5 & 6 & & \\ 6 & & & \end{array} \end{array}$$

Now the definitions regarding *reduced words* will be reviewed. Given a permutation w in the symmetric group S_n , a reduced word \underline{a} is a sequence $a_1 a_2 \cdots a_l$ of minimal length such that $w = s_{a_1} s_{a_2} \cdots s_{a_l}$, where s_i is the adjacent transposition $(i \ i + 1)$. Here l is called the *length* $l(w)$ of w . Let $Red(w)$ denote the set of reduced words for w . Since the adjacent transpositions s_i obey the *braid relations*

$$\begin{aligned} s_i s_j &= s_j s_i & \text{if } |i - j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

one can define the *nilplactic* or *Coxeter-Knuth* equivalence relation on $Red(w)$ to be the transitive closure of the relations

$$\begin{aligned} \dots ikj \dots &\underset{CK}{\sim} \dots kij \dots \\ \dots jik \dots &\underset{CK}{\sim} \dots jki \dots \\ \dots i i + 1 i \dots &\underset{CK}{\sim} \dots i + 1 i i + 1 \dots \end{aligned}$$

where $i < j < k$. The papers [2] and [9] show that the above constructions involving Knuth equivalence carry over to Coxeter-Knuth equivalence. For any reduced word \underline{a} , there is a unique column strict tableau $\tilde{P}(\underline{a})$ whose reading word is Coxeter-Knuth equivalent to \underline{a} . *Coxeter-Knuth insertion* [2] is an algorithm for computing $\tilde{P}(\underline{a})$. As in the definition of the Robinson-Schensted-Knuth correspondence, let $\tilde{P}_r(\underline{a}) = \tilde{P}(a_1 a_2 \cdots a_r)$ and let $\tilde{Q}(\underline{a})$ be the standard tableau of the same shape as $\tilde{P}(\underline{a})$ in which the letter r is located in the cell of $\tilde{P}_r(\underline{a})$ which is not in $\tilde{P}_{r-1}(\underline{a})$. The following notation will be used for the Coxeter-Knuth row insertion of the reduced word \underline{a} .

$$(\emptyset \xleftarrow{\circ} \underline{a}) = (\tilde{P}(\underline{a}), \tilde{Q}(\underline{a}))$$

Example 2 $\underline{a} = 645437658$ is a reduced word for the permutation $w = 127385496$, where the one-line notation for w means $w(1) = 1, w(2) = 2, w(3) = 7$, etc. Then the

Coxeter-Knuth insertion of the reduced word \underline{a} is given by

$$\begin{aligned} \tilde{P}_1(\underline{a}) &= 6, \tilde{P}_2(\underline{a}) = \begin{matrix} 4 \\ 6 \end{matrix}, \tilde{P}_3(\underline{a}) = \begin{matrix} 4 & 5 \\ 6 \end{matrix}, \tilde{P}_4(\underline{a}) = \begin{matrix} 4 & 5 \\ 5 & 6 \end{matrix}, \tilde{P}_5(\underline{a}) = \begin{matrix} 3 & 5 \\ 4 & 5 \\ 6 \end{matrix} \\ \tilde{P}_6(\underline{a}) &= \begin{matrix} 3 & 5 & 7 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 \end{matrix}, \tilde{P}_7(\underline{a}) = \begin{matrix} 3 & 5 & 6 \\ 4 & 7 \\ 5 & 6 \end{matrix}, \tilde{P}_8(\underline{a}) = \begin{matrix} 3 & 5 & 6 \\ 4 & 6 \\ 5 & 7 \\ 6 \end{matrix}, \tilde{P}_9(\underline{a}) = \begin{matrix} 3 & 5 & 6 & 8 \\ 4 & 6 \\ 5 & 7 \\ 6 \end{matrix} \\ &= \tilde{P}(\underline{a}) \end{aligned}$$

and hence

$$((\emptyset \xleftarrow{\circ} \underline{a}) = (\tilde{P}(\underline{a}), \tilde{Q}(\underline{a})) = \left(\begin{matrix} 3 & 5 & 6 & 8 & 1 & 3 & 6 & 9 \\ 4 & 6 & & & 2 & 7 & & \\ 5 & 7 & & & 4 & 8 & & \\ 6 & & & & 5 & & & \end{matrix} \right).$$

One more tool is required in order to define the placticification map, namely, the *plactic action* of the symmetric group on words [10]. Given a word \underline{b} and positive integer r , the r -parenthesization of \underline{b} consists of replacing each occurrence of $r + 1$ by a left parenthesis “(” and each occurrence of r by a right parenthesis “)”. Say that an occurrence of $r + 1$ and an occurrence of r are r -paired if they are replaced by a pair of parentheses which close each other under the usual rules of parenthesization. A letter $r + 1$ or r which corresponds to a parenthesis which is unclosed will be called r -unpaired. It follows that the subsequence of r -unpaired letters in \underline{b} must be of the form $r^s (r + 1)^t$, where x^m denotes the word consisting of m copies of the letter x . The plactic action of the adjacent transposition $s_r = (r r + 1)$ on \underline{b} leaves all numbers other than r or $r + 1$ fixed and replaces the subword $r^s (r + 1)^t$ by $r^t (r + 1)^s$ to form a new word denoted $\sigma_r(\underline{b})$.

Example 3 If $\underline{b} = 133431312213432$ then $\sigma_2(\underline{b}) = 122431312212432$.

Remark 4 It follows immediately from the definitions that

$$\sigma_r(q\underline{b}) = q\sigma_r(\underline{b})$$

for any word \underline{b} and letter $q \notin \{r, r + 1\}$.

The following properties of r -pairing and σ_r are not hard to check directly [10]:

Proposition 5 The word \underline{b} is the column-reading word of a column strict tableau T of the skew shape λ/μ if and only if $\sigma_r(\underline{b})$ is the column-reading word of a skew column strict tableau of the same shape λ/μ . In this case denote the latter skew tableau by $\sigma_r(T)$.

Proposition 6 The number of r -paired r 's and $(r + 1)$'s is invariant under Knuth transformations, and hence constant on Knuth equivalence classes.

Proposition 7 σ_r commutes with Knuth equivalence, i.e. if $\underline{b} \sim_K \underline{b}'$ then $\sigma_r(\underline{b}) \sim_K \sigma_r(\underline{b}')$. In terms of the P symbol, $\sigma_r(P(\underline{b})) = P(\sigma_r(\underline{b}))$.

Proposition 8 $Q(\sigma_r(\underline{b})) = Q(\underline{b})$.

Proposition 9

- (1) $\sigma_i^2 = id$;
- (2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$;
- (3) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Remark 10 Although it will not be used here, Proposition 9 allowed Lascoux and Schützenberger to define the *plactic action* of any permutation w on a word \underline{b} by

$$\sigma_{i_1} \cdots \sigma_{i_l}(\underline{b})$$

for any decomposition $w = s_{i_1} \cdots s_{i_l}$, since it is well-known that the symmetric group has the presentation given by the generators $\{s_i\}$ and relations (1)–(3).

3. The plactification map

We now define the plactification map ϕ from reduced words to words. Let $\phi(\emptyset) = \emptyset$ where \emptyset is the empty word. Given a nonempty reduced word \underline{a} , the word $\phi(\underline{a})$ is defined by the recurrence

$$\phi(\underline{a}) = r\sigma_r(\phi(\hat{\underline{a}}))$$

where $\hat{\underline{a}}$ is the word obtained from \underline{a} by removing its first letter r . Thus the plactification of the reduced word $\underline{a} = a_1 \cdots a_l$ is given by

$$\phi(\underline{a}) = a_1 \sigma_{a_1} (a_2 \sigma_{a_2} (\cdots a_{l-1} \sigma_{a_{l-1}} (a_l) \cdots))$$

Example 11

$$\begin{aligned} \phi(645437658) &= 6\sigma_6(4\sigma_4(5\sigma_5(4\sigma_4(3\sigma_3(7\sigma_7(6\sigma_6(5\sigma_5(8)))))))) \\ &= 6\sigma_6(4\sigma_4(5\sigma_5(4\sigma_4(3\sigma_3(7\sigma_7(6\sigma_6(58))))))) \\ &= 6\sigma_6(4\sigma_4(5\sigma_5(4\sigma_4(3\sigma_3(7\sigma_7(658)))))) \\ &= 6\sigma_6(4\sigma_4(5\sigma_5(4\sigma_4(3\sigma_3(7657)))) \\ &= 6\sigma_6(4\sigma_4(5\sigma_5(4\sigma_4(37657)))) \\ &= 6\sigma_6(4\sigma_4(5\sigma_5(437647))) \\ &= 6\sigma_6(4\sigma_4(5437547)) \\ &= 6\sigma_6(45437547) \\ &= 645436546 \end{aligned}$$

Remark 12 The plactification map is injective. The recurrence

$$\phi^{-1}(r\underline{b}) = r\phi^{-1}(\sigma_r(\underline{b}))$$

defines the inverse of ϕ for words \underline{b} in the image of ϕ .

ϕ also preserves the property that a word is the reading word of a skew tableau.

Proposition 13

- (1) Let T be a skew tableau whose column-reading word $\underline{a} = \text{word}(T)$ is reduced. Then $\phi(\underline{a})$ is the column-reading word of a skew tableau of the same shape, which will be denoted $\phi(T)$. Furthermore, the first columns of T and $\phi(T)$ agree.
- (2) If \underline{a} is a reduced word such that $\phi(\underline{a})$ is the column-reading word of a skew tableau S , then \underline{a} is the column-reading word of a skew tableau T such that $\phi(T) = S$.

Proof: Only (1) is proven here since (2) can be established similarly. The proof proceeds by induction on the number of columns of T . If T has one column with the decreasing column-reading word $a_1 \cdots a_m$ then

$$\begin{aligned} \phi(\text{word}(T)) &= a_1 \sigma_{a_1} \cdots a_m \sigma_{a_m}(\emptyset) \\ &= a_1 a_2 \cdots a_m \sigma_{a_1} \cdots \sigma_{a_m}(\emptyset) = a_1 \cdots a_m = \text{word}(T) \end{aligned}$$

since σ_{a_i} does not affect any of the letters comprising the subword $a_{i+1} \cdots a_m$ (which are all smaller than a_i). So $\phi(T) = T$ and the result is trivially true in this case. Now suppose T has more than one column. Let $a_1 \cdots a_m$ be the decreasing word comprising the first column of T and let \hat{T} be the remainder of T after its first column is removed. Of course $\text{word}(T) = a_1 \cdots a_m \text{word}(\hat{T})$. By induction $\phi(\hat{T})$ is a column strict tableau having the same shape and first column as \hat{T} . From the following computation of $\phi(\text{word}(T))$,

$$\phi(\text{word}(T)) = a_1 \sigma_{a_1} \cdots a_m \sigma_{a_m} \phi(\text{word}(\hat{T})) = a_1 \cdots a_m \sigma_{a_1} \cdots \sigma_{a_m} \phi(\text{word}(\hat{T}))$$

it is clear that $\text{word}(T)$ and $\phi(\text{word}(T))$ both start with the decreasing sequence $a_1 \cdots a_m$. By Proposition 5 the word $\sigma_{a_1} \cdots \sigma_{a_m} \phi(\text{word}(\hat{T}))$ is the column-reading word of a column strict tableau of the same shape as $\phi(\hat{T})$ (and \hat{T}). The only thing remaining to be shown is that the “first column” $a_1 \cdots a_m$ and the skew tableau $\sigma_{a_1} \cdots \sigma_{a_m} \phi(\hat{T})$ fit together to form a column strict tableau of the same shape as T . Let T_i be the tableau (not necessarily column strict a priori) of the same shape as T whose column-reading word is $a_1 \cdots a_m \sigma_{a_{i+1}} \cdots \sigma_{a_m} \phi(\text{word}(\hat{T}))$. It is enough to show that T_i is a column strict tableau for all i such that $0 \leq i \leq m$, since $\text{word}(T_0) = \phi(\text{word}(T))$. T_m , whose column-reading word is $a_1 \cdots a_m \phi(\text{word}(\hat{T}))$, is a column strict tableau, since the first columns of $\phi(\hat{T})$ and \hat{T} agree and the column $a_1 \cdots a_m$ fits together with \hat{T} to form the column strict tableau T . Suppose that i is maximal such that $1 \leq i \leq m$ and T_i is column strict but T_{i-1} is not. The only way this could happen is if in passing from T_i to T_{i-1} , the operator σ_r (where $r = a_i$) changed a letter $r + 1$ in the second column of T_i into the letter r and this letter $r + 1$ was located immediately to the right of a letter $r + 1$ in the first column of T_i . So the first two columns of T_i contain the 2×2 subtableau

$$\begin{array}{cc} a_i & = r & & y \\ a_{i-1} & = r + 1 & r + 1 & \end{array}$$

Since T_i is column strict, the letter y must be equal to r . But then the letter $r + 1$ in the second column of T_i is paired with the letter $y = r$ just above it, which contradicts the assumption that this letter $r + 1$ was changed by the operator σ_r in passing from T_i to T_{i-1} . Therefore T_{i-1} is column strict. By induction on i , (1) follows. □

Remark 14 Proposition 13 still holds if the row-reading word is used, and the tableaux resulting from plactifying the row- and column-reading words agree. This follows from

Theorem 19 below, combined with the fact that the row- and column-reading words of T are Coxeter-Knuth equivalent.

Our next goal is to characterize the image $\phi(\text{Red}(w))$. To this end, recall [14] that the *Rothe diagram* $D(w)$ is the set of pairs of positive integers

$$\{(i, j): i < (w^{-1})_j \text{ and } j < w_i\}.$$

Given any *diagram* D , that is, finite set of *cells* (pairs of positive integers, indexed matrix-style), the *row-filled diagram* $F(D)$ is the filling of D in which every cell in the r -th row is assigned the entry r . The tableau T said to be *D-peelable* if the first column C of $F(D)$ is contained in the first column of Q as an initial segment V , and the tableau \hat{T} is $\hat{D} = D/C$ -peelable, where $\hat{T} = P(\text{word}(T/V))$ is the tableau obtained from T by removing the subtableau V and sliding the skew tableau T/V into the vacated cells by the jeu-de-taquin.

Example 15 For the permutation $w = 341286597$, the Rothe diagram $D(w)$, the row-filled diagram $F(D)$, and an interesting $D(w)$ -peelable tableau T are given below.

$$\begin{array}{ccc}
 \begin{array}{cc} \times & \times \\ \times & \times \end{array} & & \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \\
 \\
 D(w) = & \begin{array}{ccc} & \times & \times & \times \\ & \times & & \\ & & & \times \end{array} & F(D(w)) = & \begin{array}{ccc} 5 & 5 & 5 \\ 6 & & \\ & & 8 \end{array} \\
 \\
 T = & \begin{array}{ccc} 1 & 1 & 5 \\ 2 & 2 & 6 \\ 5 & 5 & 8 \end{array}
 \end{array}$$

Below we exhibit the “ $D(w)$ -peeling” of T , that is, the sequence of tableaux obtained by iterating the $\hat{\cdot}$ operation on the pair $(D(w), T)$. The letters of the subtableaux V are underlined.

$$\begin{array}{ccccccc}
 \underline{1} & 1 & 5 & \underline{1} & 5 & 5 & \underline{5} & 5 & 5 \\
 \underline{2} & 2 & 6 & \underline{2} & 6 & & \underline{6} & & \\
 5 & 5 & 8 & 5 & 8 & & 8 & &
 \end{array}
 \mapsto
 \begin{array}{ccc}
 \underline{5} & 5 & 5 \\
 \underline{6} & & \\
 8 & &
 \end{array}
 \mapsto
 \begin{array}{ccc}
 \underline{5} & 5 & 5 \\
 \underline{8} & & \\
 & &
 \end{array}
 \mapsto \emptyset$$

Remark 16 The natural class of diagrams D for which the notion of D -peelability makes the most sense is the class of diagrams with the *northwest property*, that is, $(i, j) \in D$ and $(i', j') \in D$ imply $(\min(i, i'), \min(j, j')) \in D$. Rothe diagrams have the northwest property. The first column of a diagram with the northwest property will always define an initial segment in the set of indices of all its nonempty rows, and the removal of this first column also yields a diagram with the northwest property.

It will be shown in Theorem 22 that $\phi(\text{Red}(w^{-1}))$ is exactly the set of $D(w)$ -peelable words \underline{b} . But first it is convenient to take note of a few consequences of $D(w)$ -peelability. For any word \underline{b} , the *content* of \underline{b} is the sequence (c_1, c_2, \dots) , where c_i is the number of

occurrences of the letter i in \underline{b} . For a permutation w , the (*Lehmer*) *code* of w is the sequence $(c_1(w), c_2(w), \dots)$ where

$$c_i(w) = \#\{(i, j): i < j, w_i > w_j\}.$$

The following proposition is an immediate consequence of the definitions, since the i -th part $c_i(w)$ of the code of w is equal to the number of cells in the i -th row of $D(w)$.

Proposition 17 *If \underline{b} is $D(w)$ -peelable, then the content of \underline{b} equals the code of w .*

Peelable words for Rothe diagrams always satisfy the following property involving the r -pairing.

Proposition 18 *If \underline{b} is $D(w)$ -peelable, then*

- (1) *all r 's in \underline{b} are r -paired if $w_r < w_{r+1}$,*
- (2) *all $(r + 1)$'s in \underline{b} are r -paired if $w_r > w_{r+1}$.*

Proof: In light of Proposition 6, assume without loss that \underline{b} is the column-reading word of a tableau P (namely $P(\underline{b})$). Only the proof of (1) will be given here (the other case is similar). So let $w_r < w_{r+1}$. Then every cell in row r of $D(w)$ has a cell in row $r + 1$ directly beneath it, so $D(w)$ is a diagram D with the following property (*): every column of $F(D)$ containing an r also contains an $r + 1$.

It suffices to prove the following general fact about peelables: If a diagram D has property (*) and P is D -peelable, then every r in P is r -paired. The proof proceeds by induction on the number of columns of D . Let C, V, \hat{D} and \hat{P} be as in the definition of the D -peelability of the tableau P . Since \hat{D} also has property (*), every r in \hat{P} is r -paired by induction. Since the reading words of \hat{P} and the skew tableau P/V are Knuth-equivalent, by Proposition 6 every r in P/V is r -paired. Since P is obtained from P/V by adding in the decreasing subsequence V , which never contains an r without an $r + 1$ preceding it, every r in P is r -paired. \square

We now prove the two fundamental properties of plactification:

Theorem 19

(A) *If \underline{a} and \underline{a}' in $\text{Red}(w)$ satisfy $\underline{a} \underset{CK}{\sim} \underline{a}'$ then $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$. That is,*

$$\phi(\tilde{P}(\underline{a})) = P(\phi(\underline{a}))$$

(B) *For \underline{a} in $\text{Red}(w^{-1})$, $\phi(\underline{a})$ is $D(w)$ -peelable.*

Proof: Both assertions are proven simultaneously by induction on $l(w)$. Let A_l and B_l be the assertions A and B respectively, for all permutations w with $l(w) = l$. The theorem immediately follows from the next two lemmas. \square

Lemma 20 *B_k for $k < l$ implies A_l .*

Lemma 21 *A_k for $k \leq l$ and B_k for $k < l$ imply B_l .*

Proof of Lemma 20: Let \underline{a} and \underline{a}' in $Red(w)$ satisfy $\underline{a} \underset{CK}{\sim} \underline{a}'$ with $l(w) = l$. Assume without loss of generality that \underline{a} and \underline{a}' have the form

$$\begin{aligned} \underline{a} &= \hat{a}a_i a_{i+1} a_{i+2} \hat{a} \\ \underline{a}' &= \hat{a}a'_i a'_{i+1} a'_{i+2} \hat{a} \end{aligned}$$

where

$$a_i a_{i+1} a_{i+2} \underset{CK}{\sim} a'_i a'_{i+1} a'_{i+2}$$

is a Coxeter-Knuth transformation. By Proposition 7, it suffices to show

$$\phi(a_i a_{i+1} a_{i+2} \hat{a}) \underset{K}{\sim} \phi(a'_i a'_{i+1} a'_{i+2} \hat{a})$$

since $\phi(\underline{a})$ and $\phi(\underline{a}')$ are obtained from these two words by applying the same sequence of $r\sigma_r$'s. To establish the Knuth-equivalence of these words, there are a number of cases depending upon the nature of the Coxeter-Knuth transformation involved.

Case 1. $\underline{a} = rr + 1r\hat{a}$ and $\underline{a}' = r + 1rr + 1\hat{a}$. We have

$$\begin{aligned} \phi(\underline{a}) &= r\sigma_r(r + 1\sigma_{r+1}(r\sigma_r(\phi(\hat{a})))) = r\sigma_r(r + 1r\sigma_{r+1}\sigma_r(\phi(\hat{a}))) \\ &= rr + 1r\sigma_r\sigma_{r+1}\sigma_r(\phi(\hat{a})) = rr + 1r\sigma_{r+1}\sigma_r\sigma_{r+1}(\phi(\hat{a})) \end{aligned}$$

The second equality holds by Remark 4, the third equality follows from the fact that any pair of adjacent letters of the form $r + 1r$ will be r -paired, and the fourth equality is furnished by Proposition 9. On the other hand,

$$\begin{aligned} \phi(\underline{a}') &= r + 1\sigma_{r+1}(r\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a})))) = r + 1r\sigma_{r+1}\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a}))) \\ &= r + 1r\sigma_{r+1}r\sigma_r\sigma_{r+1}(\phi(\hat{a})) = r + 1rr\sigma_{r+1}\sigma_r\sigma_{r+1}(\phi(\hat{a})) \end{aligned}$$

The second and fourth equalities follow from Remark 4. The only application of the hypothesis B_k for $k < l$ is to establish the third equality. Since \hat{a} is in $Red(\hat{w})$ for some \hat{w} with $\hat{w}_r < \hat{w}_{r+1} < \hat{w}_{r+2}$, it follows by induction that $\phi(\hat{a})$ is $D(\hat{w})$ -peelable, and hence by Proposition 18, that all r 's are r -paired in $\phi(\hat{a})$ and all $r + 1$'s are $(r + 1)$ -paired in $\phi(\hat{a})$. In passing from $\phi(\hat{a})$ to $\sigma_{r+1}(\phi(\hat{a}))$, some $r + 2$'s are replaced by $r + 1$'s, but this does not destroy any r -pairs. This means that all of the r 's in $\sigma_{r+1}(\phi(\hat{a}))$ are still r -paired, so that the leftmost $r + 1$ is r -unpaired in $r + 1\sigma_{r+1}(\phi(\hat{a}))$. Therefore $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$, since $rr + 1r \underset{K}{\sim} r + 1rr$.

Case 2. $\underline{a} = ikj\hat{a}$ and $\underline{a}' = kij\hat{a}$ where $i + 1 < j < k - 1$. We have

$$\begin{aligned} \phi(\underline{a}) &= i\sigma_i(k\sigma_k(j\sigma_j(\phi(\hat{a})))) = ikj\sigma_i\sigma_k\sigma_j(\phi(\hat{a})) = ikj\sigma_k\sigma_i\sigma_j(\phi(\hat{a})), \\ \phi(\underline{a}') &= k\sigma_k(i\sigma_i(j\sigma_j(\phi(\hat{a})))) = kij\sigma_k\sigma_i\sigma_j(\phi(\hat{a})) \end{aligned}$$

Since $ikj \underset{K}{\sim} kij$, it follows that $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$. The case where $\underline{a} = jik\hat{a}$ and $\underline{a}' = jki\hat{a}$ with $i + 1 < j < k - 1$ is similar.

Case 3. $\underline{a} = rqr + 1\hat{a}$ and $\underline{a}' = qrr + 1\hat{a}$ where $r + 1 < q$. We have

$$\begin{aligned} \phi(\underline{a}) &= r\sigma_r(q\sigma_q(r + 1\sigma_{r+1}(\phi(\hat{a})))) = rq\sigma_r\sigma_q(r + 1\sigma_{r+1}(\phi(\hat{a}))) \\ &= rq\sigma_q\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a}))), \\ \phi(\underline{a}') &= q\sigma_q(r\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a})))) = qr\sigma_q\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a}))) \end{aligned}$$

The word $\sigma_q\sigma_r(r + 1\sigma_{r+1}(\phi(\hat{a})))$ either begins with $x = r$ or $x = r + 1$, but in either case $rqx \underset{K}{\sim} qrx$. So $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$. The case where $\underline{a} = qr + 1r\hat{a}$ and $\underline{a}' = r + 1qr\hat{a}$ with $q < r$ is similar.

Case 4. $\underline{a} = r + 1qr\hat{a}$ and $\underline{a}' = r + 1rq\hat{a}$ where $r + 1 < q$. We have

$$\begin{aligned} \phi(\underline{a}) &= r + 1\sigma_{r+1}(q\sigma_q(r\sigma_r(\phi(\hat{a})))) = r + 1\sigma_{r+1}(qr\sigma_q\sigma_r(\phi(\hat{a}))) \\ &= r + 1\sigma_{r+1}(qr\sigma_r\sigma_q(\phi(\hat{a}))), \\ \phi(\underline{a}') &= r + 1\sigma_{r+1}(r\sigma_r(q\sigma_q(\phi(\hat{a})))) = r + 1\sigma_{r+1}(rq\sigma_r\sigma_q(\phi(\hat{a}))) \end{aligned}$$

The words $\sigma_{r+1}(qr\sigma_r\sigma_q(\phi(\hat{a})))$ and $\sigma_{r+1}(rq\sigma_r\sigma_q(\phi(\hat{a})))$ are of the form $xr \dots$ and $rx \dots$, where $x = q$ or $x = q - 1$, but in either case $r + 1xr \underset{K}{\sim} r + 1rx$. Again $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$. The case where $\underline{a} = rr + 1q\hat{a}$ and $\underline{a}' = rqr + 1\hat{a}$ with $q < r$ is similar.

This exhausts all the cases, and the lemma is proven. □

Proof of Lemma 21: Assume that A_k holds for $k \leq l$ and B_k for $k < l$. To show B_k , let \underline{a} be in $Red(w^{-1})$ with $l(w) = l$, and let $\tilde{P}(\underline{a})$ be its Coxeter-Knuth insertion tableau. By Proposition 13, $\phi(\tilde{P}(\underline{a}))$ is a column strict tableau, which is Knuth equivalent to $\phi(\underline{a})$ by A_l . Thus

$$\phi(\tilde{P}(\underline{a})) = P(\phi(\underline{a}))$$

since there is a unique column strict tableau which is Knuth equivalent to a given word. Let $\tilde{P} = \tilde{P}(\underline{a})$ and $P = \phi(\tilde{P}(\underline{a})) = P(\phi(\underline{a}))$. It is enough to show that P is $D(w)$ -peelable.

Let $k = (w^{-1})_1 - 1$ and $\hat{w} = ws_k s_{k-1} \dots s_1$. One can easily check that $D(\hat{w})$ is obtained from $D(w)$ by removing the first column and moving all remaining cells in rows $1, 2, \dots, k$ down one row, and that $F(D(\hat{w}))$ is obtained from $F(D)$ by removing the first column C and adding 1 to each remaining entry which is $\leq k$.

Observe that the column-reading word $word(\tilde{P})$ of \tilde{P} is a reduced word for w^{-1} . In order to have $(w^{-1})_1 = k + 1$, it must be of the form

$$word(\tilde{P}) = a_1 a_2 \dots a_m k k - 1 \dots 2 1 \hat{a}$$

where $a_1 > a_2 > \dots > a_m \geq k + 2$. This shows that C is contained in the first column of \tilde{P} as an initial segment V . Since \tilde{P} and P have the same first column by Proposition 13, P contains V as well.

It now remains to show that the skew tableau P/V is \hat{D} -peelable, where \hat{D} is the diagram obtained from $D(w)$ by removing its first column.

It is now shown that the skew tableau P/V is obtained from $\phi(\tilde{P}/V)$ by a trivial relabelling, namely, by subtracting 1 to each entry which is $\leq k + 1$. Using the fact that

$a_1 > \dots > a_m > k + 1$, one can reorder the operators in the calculation of $\phi(\underline{a})$:

$$\phi(\underline{a}) = a_1 \cdots a_m k k - 1 \cdots 2 1 \sigma_k \sigma_{k-1} \cdots \sigma_1 \sigma_{a_1} \cdots \sigma_{a_m} \phi(\hat{\underline{a}})$$

This allows the following description of the relationship between P/V and \tilde{P}/V :

$$\begin{aligned} \text{word}(P/V) &= a_1 \cdots a_m \sigma_k \sigma_{k-1} \cdots \sigma_1 \sigma_{a_1} \cdots \sigma_{a_m} \phi(\hat{\underline{a}}) \\ &= \sigma_k \sigma_{k-1} \cdots \sigma_1 a_1 \cdots a_m \sigma_{a_1} \cdots \sigma_{a_m} \phi(\hat{\underline{a}}) = \sigma_k \sigma_{k-1} \cdots \sigma_1 \phi(\text{word}(\tilde{P}/V)) \end{aligned}$$

It must now be shown that the successive applications on $\phi(\tilde{P}/V)$ of σ_i for i going from 1 to k have the effect of replacing the 2's in $\phi(\tilde{P}/V)$ by 1's, the 3's by 2's, etc. The word $\hat{\underline{a}}$ is a reduced word for a permutation v which fixes 1. By induction on the length of the reduced word, $\phi(\hat{\underline{a}})$ is $D(v)$ -peelable, which by Proposition 17 implies that $\phi(\hat{\underline{a}})$ has no 1's. Since $a_1 > \dots > a_m > k + 1$, it is clear that $\sigma_{a_1} \cdots \sigma_{a_m} \phi(\hat{\underline{a}}) = \phi(\text{word}(\tilde{P}/V))$ also contains no 1's. So $\phi(\tilde{P}/V)$ contains no 1's. In applying σ_1 to $\phi(\tilde{P}/V)$, all of the 2's are 1-unpaired and are replaced by 1's. This means that $\sigma_1 \phi(\tilde{P}/V)$ contains no 2's. Then the application of σ_2 changes all the 3's to 2's, etc. Thus P/V is obtained from $\phi(\tilde{P}/V)$ by replacing the $i + 1$'s by i 's for $1 \leq i \leq k$.

By induction, $\phi(\tilde{P}/V)$ is $D(\hat{w})$ -peelable. If \hat{D} is the diagram obtained from $D(w)$ by removing its first column, then the relabelling of entries indicated above shows that P/V is \hat{D} -peelable, and hence that P is D -peelable.

This completes the proof of both lemmas. Theorem 19 follows. □

Finally the image of $\phi(\text{Red}(w))$ is characterized.

Theorem 22 *The word \underline{b} is the plactification of a reduced word \underline{a} for the permutation w^{-1} if and only if \underline{b} is a $D(w)$ -peelable word.*

Proof: Assertion *B* of Theorem 19 gives the forward implication, so it remains to show the converse: if \underline{b} is $D(w)$ -peelable, then $\underline{b} = \phi(\underline{a})$ for some $\underline{a} \in \text{Red}(w^{-1})$. The proof proceeds by induction on $l(w)$. Let $P = P(\underline{b})$. It must first be shown that P is the plactification of a tableau \tilde{P} whose column-reading word is a reduced word for w^{-1} . The same notation as in the proof of Lemma 21 will be used here. Let $k = (w^{-1})_1 - 1$. Since P is $D(w)$ -peelable its first column contains the first column C of $F(D(w))$, which consists of the numbers $1, 2, \dots, k$, as an initial segment. By definition $\text{word}(P/V)$ is $\widehat{D}(w)$ -peelable. The skew tableau S obtained from P/V by relabelling each letter i by $i + 1$ for $1 \leq i \leq k$ is $D(\hat{w})$ -peelable, where as before $\hat{w} = w s_k s_{k-1} \cdots s_1$. By induction on $l(w)$ and Proposition 13 there is a skew tableau \tilde{S} of the same shape as S such that $\phi(\tilde{S}) = S$ and $\text{word}(\tilde{S}) \in \text{Red}(\hat{w}^{-1})$. Let \tilde{P} be obtained from \tilde{S} by adding V into the empty vertical segment in the first column. It is clear that \tilde{P} is a column strict tableau of the same shape as P , such that $\text{word}(\tilde{P}) \in \text{Red}(w^{-1})$. It follows from the proof of Lemma 21 that $\phi(\tilde{P}) = P$. Let \underline{a} be the unique reduced word of w^{-1} satisfying $(\emptyset \overset{\circ}{\leftarrow} \underline{a} = (\tilde{P}, Q(\underline{b})))$. Proposition 23 below shows that $\phi(\underline{a}) = \underline{b}$. □

Interestingly, the plactification map not only takes Coxeter-Knuth to Knuth equivalence, but also preserves recording tableaux:

Proposition 23

$$Q(\phi(\underline{a})) = \tilde{Q}(\underline{a})$$

for all reduced words \underline{a} .

Proof: Let \underline{a} be in $Red(w)$. The proof proceeds by induction on $l(w)$. Write $\underline{a} = r\hat{a}$, so that $\phi(\underline{a}) = r\sigma_r\phi(\hat{a})$. It is convenient to make use of the *column insertion* versions of the Coxeter-Knuth and Robinson-Schensted-Knuth algorithms, which will be written

$$\begin{aligned} (\underline{b} \rightarrow \emptyset) &= (P, Q) \\ (\underline{a} \overset{\circ}{\rightarrow} \emptyset) &= (\tilde{P}, \tilde{Q}) \end{aligned}$$

The Robinson-Schensted-Knuth column insertion of the word \underline{b} is characterized by the property that it shares the same insertion tableau $P(\underline{b})$ as the row insertion of \underline{b} , but its recording tableau is the standard tableau in which the letter $l + 1 - r$ occupies the cell of $P(b_r b_{r+1} \cdots b_l)$ which is not in $P(b_{r+1} \cdots b_l)$ where l is the length of the word \underline{b} . Let

$$\begin{aligned} (\underline{a} \overset{\circ}{\rightarrow} \emptyset) &= (\tilde{P}, \tilde{Q}) \\ (\hat{a} \overset{\circ}{\rightarrow} \emptyset) &= (\tilde{P}', \tilde{Q}') \end{aligned}$$

so that \tilde{Q} is obtained from \tilde{Q}' by adjoining the letter l at some corner cell. Induction and assertion A of Theorem 19 yield

$$(\phi(\hat{a}) \rightarrow \emptyset) = (\phi(\tilde{P}'), \tilde{Q}').$$

Proposition 7 implies that

$$(\sigma_r\phi(\hat{a}) \rightarrow \emptyset) = (\sigma_r\phi(\tilde{P}'), \tilde{Q}')$$

so that

$$(r\sigma_r\phi(\hat{a}) \rightarrow \emptyset) = (\phi(\tilde{P}), Q)$$

where Q is obtained from \tilde{Q}' by adjoining the letter l at some corner cell. But

$$shape(Q) = shape(\phi(\tilde{P})) = shape(\tilde{P}) = shape(\tilde{Q})$$

so $\tilde{Q} = Q$, and hence

$$Q(\phi(\underline{a})) = Q^{evac} = \tilde{Q}^{evac} = \tilde{Q}(\underline{a})$$

since the respective recording tableaux for the row and column insertion of a word are related by Schützenberger's *evacuation* involution $Q \mapsto Q^{evac}$ (see [18]). \square

4. Applications

The first application gives a new way of counting the number of reduced words of a permutation.

It is easy to show that the only \hat{D} -peelable tableau is

$$\begin{array}{cc} 2 & 2 \\ 4 & 4 \\ 5 & \end{array}$$

and there are five ways to place a vertical 2-strip on the frontier:

$$\left\{ \begin{array}{ccc} \begin{array}{ccc} 2 & 2 & . \\ 4 & 4 & . \\ 5 & & . \end{array} & , & \begin{array}{ccc} 2 & 2 & . \\ 4 & 4 & . \\ 5 & & . \end{array} & , & \begin{array}{ccc} 2 & 2 & . \\ 4 & 4 & . \\ 5 & & . \end{array} & , & \begin{array}{ccc} 2 & 2 & . \\ 4 & 4 & . \\ 5 & & . \end{array} & , & \begin{array}{ccc} 2 & 2 & . \\ 4 & 4 & . \\ 5 & & . \end{array} \end{array} \right\}$$

Sliding into the cells of these vertical strips in order from top to bottom yields the skew tableaux

$$\left\{ \begin{array}{ccc} \begin{array}{ccc} . & 2 & 2 \\ . & 4 & 4 \\ 5 & & . \end{array} & , & \begin{array}{ccc} . & 2 & 2 \\ . & 4 & 4 \\ 4 & 5 & . \end{array} & , & \begin{array}{ccc} . & 2 & 2 \\ . & 4 & 4 \\ 4 & & 5 \end{array} & , & \begin{array}{ccc} . & 2 & . \\ . & 4 & . \\ 2 & 5 & . \end{array} & , & \begin{array}{ccc} . & 2 & . \\ . & 4 & . \\ 4 & & 5 \end{array} \end{array} \right\}$$

Filling the vacancies in the first column with the letters 1, 2 gives

$$\left\{ \begin{array}{ccc} \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 5 & & . \end{array} & , & \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 4 & 5 & . \end{array} & , & \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 4 & & 5 \end{array} & , & \begin{array}{ccc} 1 & 2 & . \\ 2 & 4 & . \\ 2 & 5 & . \end{array} & , & \begin{array}{ccc} 1 & 2 & . \\ 2 & 4 & . \\ 4 & & 5 \end{array} \end{array} \right\}$$

The last two of these are not column strict tableaux and are discarded. The first three tableaux form the set of $D = D(w)$ -peelable tableaux.

The next application of plactification requires a discussion of the notion of keys [12]. A column strict tableau is called a *key tableau* if the i th column contains the j th for all $i < j$. It is easy to see that there is a unique key tableau $key(\alpha)$ of content α for each i composition α (sequence of nonnegative integers, almost all zero).

Example 27 If $\alpha = 104252$ then $key(\alpha)$ is

$$\begin{array}{cccccc} 1 & 3 & 3 & 3 & 5 & \\ 3 & 4 & 5 & 5 & & \\ 4 & 5 & & & & \\ 5 & 6 & & & & \\ 6 & & & & & \end{array}$$

For each column strict tableau P there is an associated key tableau $K_-(P)$ of the same shape called the *left key* of P . The j th column of $K_-(P)$ is given as follows. Let α be any

composition whose nonzero parts give the lengths of the columns of P and whose first part α_1 equals the length l of the j th column of P . Let \underline{b} be the word satisfying

$$(\emptyset \leftarrow \underline{b}) = (P, \text{std}(\text{key}(\alpha))^t)$$

where T^t means the transpose of the tableau T and $\text{std}(T)$ is the *standardization* of the column strict tableau T , defined by replacing the 1's in T by $1, 2, \dots, s$ from left to right, replacing the 2's by $s + 1, s + 2, \dots$, etc. Then $b_1 b_2 \cdots b_l$ form the strictly decreasing word which comprises the j th column of $K_-(P)$.

Example

$$P = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & 5 & \end{array}$$

has columns of lengths 3,3,1. To obtain the columns of length 3 in $K_-(P)$, let $\alpha = 331$ so that

$$\text{std}(\text{key}(\alpha))^t = \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & \\ 3 & 6 & \end{array}$$

and $\underline{b} = 421\ 542$. The first and second columns of $K_-(P)$ are given by the word 421. To obtain the column of length 1 in $K_-(P)$, let $\alpha = 133$ so that

$$\text{std}(\text{key}(\alpha))^t = \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & 7 & \end{array}$$

and $\underline{b} = 2\ 421\ 542$. The third column of $K_-(P)$ is given by the word 2. Therefore,

$$K_-(P) = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \\ 4 & 4 & \end{array}$$

Similarly if \tilde{P} is a column strict tableau whose reading word is reduced, one can define its *left nil key* $K_-^0(\tilde{P})$ by replacing Robinson-Schensted-Knuth insertion by Coxeter-Knuth insertion in the above definition.

Example

$$\tilde{P} = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \\ 4 & 5 & \end{array}$$

has columns of length 3,3,1. To obtain the columns of length 3 in $K_-^0(\tilde{P})$, let $\alpha = 331$. Then

$$\text{std}(\text{key}(\alpha))^t = \begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & \\ 3 & 6 & \end{array}$$

and $\underline{a} = 421\ 543\ 4$. To obtain the column of length 1 in $K_-^0(\tilde{P})$, let $\alpha = 133$. Then

$$\text{std}(\text{key}(\alpha))^t = \begin{matrix} & 1 & 2 & 5 \\ 3 & & & \\ 4 & & & 7 \end{matrix}$$

and $\underline{a} = 2\ 431\ 543$. Therefore

$$K_-^0(\tilde{P}) = \begin{matrix} & 1 & 1 & 2 \\ 2 & & & \\ 4 & & & \end{matrix}$$

Notice that $\phi(\tilde{P}) = P$ and $K_-(\phi(\tilde{P})) = K_-^0(\tilde{P})$, where P is from the previous example.

Proposition 28 *If \tilde{P} is a tableau whose reading word is reduced, then*

$$K_-(\phi(\tilde{P})) = K_-^0(\tilde{P})$$

Proof: Let $\phi(\tilde{P}) = P$. To obtain the j th columns of $K_-^0(\tilde{P})$ and $K_-(P)$, let α be any composition as in the definitions above, and let $\underline{a}, \underline{b}$ satisfy

$$\begin{aligned} (\emptyset \xleftarrow{\circ} \underline{a}) &= (\tilde{P}, \text{std}(\text{key}(\alpha))^t) \\ (\emptyset \xleftarrow{\leftarrow} \underline{b}) &= (P, \text{std}(\text{key}(\alpha))^t) \end{aligned}$$

The j th columns of $K_-^0(\tilde{P})$ and $K_-(P)$ are given by the decreasing subwords

$$\begin{aligned} a_1 &> a_2 > \cdots > a_l \\ b_1 &> b_2 > \cdots > b_l \end{aligned}$$

respectively. Then $\phi(\underline{a}) = \underline{b}$ by Theorem 19, Proposition 23, and the bijectivity of row insertion. Since the sequence $a_1 a_2 \cdots a_l$ comes at the left end of the word \underline{a} and is decreasing, it will be unchanged during the plactification (see the proof of Proposition 13) and hence is identical to $b_1 b_2 \cdots b_l$. Therefore $K_-^0(\tilde{P}), K_-(P)$ agree in every column. \square

Theorem 29 *For any permutation w ,*

$$\mathfrak{S}_w = \sum_P \kappa_{\text{content}(K_-(P))}$$

where P runs over the set of $D(w)$ -peelable tableaux, \mathfrak{S}_w is the Schubert polynomial [11] and κ_α is the key polynomial [12]

Proof: In [13], [16] it is shown that

$$\mathfrak{S}_w = \sum_{\tilde{P}} \kappa_{\text{content}(K_-^0(\tilde{P}))}$$

where \tilde{P} runs over all tableaux whose reading words are reduced words for w^{-1} . Proposition 28 completes the proof. \square

Remark 30 There is a refinement of the previous result which answers a question raised in [1]. In that paper, it is proven that

$$\mathfrak{S}_w = \sum_{(\underline{a}, \underline{i})} x_{i_1} x_{i_2} \cdots$$

where \underline{a} is a reduced word for w and \underline{i} is a *compatible sequence* for \underline{a} , that is,

- (1) \underline{i} is weakly increasing;
- (2) $i_k < i_{k+1}$ if $a_k < a_{k+1}$ for each k ;
- (3) $i_k \leq a_k$ for each k .

It is also observed that for each such pair $(\underline{a}, \underline{i})$, one can produce a column strict tableau T as follows. Consider the recording tableau \tilde{Q} resulting from the Coxeter-Knuth row insertion of the reduced word \underline{a} :

$$(\emptyset \xrightarrow{\circ} \underline{a}) = (\tilde{P}, \tilde{Q})$$

Let T be the tableau obtained from \tilde{Q}' by replacing the letter k by the letter i_k for all k . Equivalently, T is the tableau obtained from the recording tableau of the Coxeter-Knuth column insertion of the reverse $rev(\underline{a})$ of \underline{a} by replacing the letter k by i_k for all k . It is easy to see that T is a column strict tableau such that $content(T) = content(\underline{i})$. Let \mathcal{M}_w be the *multiset* of column strict tableaux T obtained in this manner. It follows that

$$\mathfrak{S}_w = \sum_{T \in \mathcal{M}_w} x^T$$

where $x^T = x_1^{c_1} x_2^{c_2} \cdots$ and $content(T) = (c_1, c_2, \dots)$. [1] asks the question of whether \mathcal{M}_w has “a simple direct description avoiding the use of [Coxeter-Knuth insertion].”

Theorem 31 *Let w be a permutation.*

(1)

$$\mathfrak{S}_w = \sum_{(\underline{b}, \underline{i})} x_{i_1} x_{i_2} \cdots$$

where $(\underline{b}, \underline{i})$ runs over all pairs where $rev(\underline{b})$ is a $D(w)$ -peelable word and \underline{i} is a compatible sequence for \underline{b} (the definition of compatible sequence makes sense for any word).

(2) \mathcal{M}_w is the multiset given by

$$\mathcal{M}_w = \bigcup_P \{T : K_+(T) \leq K_-(P)\}$$

where P runs over the set of $D(w)$ -peelable tableaux, $K_+(T)$ is the right key of T (which is defined below), and $A \leq B$ for two tableaux of the same shape means A is entrywise dominated by B .

$K_+(T)$ is defined similarly to $K_-(T)$, except that the *last* nonzero part of α must be equal to the length l of the j th column of T , and the *last* l letters of the word \underline{b} comprise the j th column of $K_+(T)$.

Proof of Theorem 31: (1) follows immediately from Theorem 29 and [16, Theorem 5]. (2) may be deduced using Proposition 28 and the results and techniques of [16]. \square

Example 32 Let $w^{-1} = w = 21543$. The $D(w)$ -peelable tableaux are

$$\left\{ \begin{array}{c} 1 \ 3 \\ 3 \quad \\ 4 \end{array}, \begin{array}{c} 1 \ 3 \\ 3 \ 4' \end{array}, \begin{array}{c} 1 \ 3 \ 3 \\ 4 \end{array} \right\}$$

which have corresponding left keys

$$\left\{ \begin{array}{c} 1 \ 3 \\ 3 \quad \\ 4 \end{array}, \begin{array}{c} 1 \ 1 \\ 3 \ 3' \end{array}, \begin{array}{c} 1 \ 1 \ 1 \\ 4 \end{array} \right\}.$$

Therefore Theorem 29 says

$$\mathfrak{S}_{21543} = \kappa_{1021} + \kappa_{202} + \kappa_{3001}.$$

The reduced words \underline{a} of w are listed horizontally, and below each word \underline{a} is a list of its

$$Red(w) = \{ 4314 \ 4341 \ 3431 \ 3413 \ 3143 \ 4134 \ 1434 \ 1343 \}$$

$$\begin{array}{cccccc} 1112 & 1122 & 1123 & 1223 & 1233 & \\ 1113 & 1123 & 1124 & 1224 & & \\ 1114 & 1133 & 1134 & 1234 & & \\ & & & 1334 & & \end{array}$$

The multiset \mathcal{M}_w is given by

$$\left\{ \begin{array}{c} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 1 \ 2 \ 1 \ 3 \ 1 \ 3 \ 1 \ 3 \\ 2 \quad \quad \quad 2 \quad \quad \quad 3 \quad \quad \quad 2 \quad \quad \quad 2 \quad \quad \quad 2 \quad \quad \quad 3 \quad \quad \quad 2 \\ 3 \quad \quad \quad 4 \quad \quad \quad 4 \quad \quad \quad 3 \quad \quad \quad 4 \quad \quad \quad 4 \quad \quad \quad 4 \quad \quad \quad 3 \end{array}, \begin{array}{c} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 2 \ 2' \ 2 \ 3' \ 3 \ 3' \end{array}, \begin{array}{c} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 2 \quad \quad \quad 3 \quad \quad \quad 4 \end{array} \right\}$$

The above tableaux T are grouped according to their associated $D(w)$ -peelable P which satisfies $K_+(T) \leq K_-(P)$. For example, the first eight tableaux T all have right keys which are entrywise dominated by the left key of the first $D(w)$ -peelable tableau P listed earlier.

The last application of plactification is the decomposition of *Specht modules* of the symmetric group over \mathbb{C} into irreducible representations. Given a diagram D of cardinality n and a field \mathbb{F} , the *Specht module* S^D is a representation of the symmetric group S_n over the field \mathbb{F} defined using the *Young-symmetrizer construction* (see [5] for a definition). When D is a Ferrers diagram λ and $char(\mathbb{F}) = 0$, S^λ is irreducible and such modules yield all isomorphism classes of irreducible modules. If $char(\mathbb{F}) > 0$, the irreducible modules can

still be constructed using the Specht modules S^λ . When D is a skew partition diagram, the *Littlewood-Richardson rule* gives the decomposition of S^D into irreducibles [5]. It is a natural question to ask about such decompositions for more general diagrams D .

It was known (although never published) that using results of Kraskiewicz and Pragacz [7] and [9], [2] along with *Schur-Weyl duality*, one can prove the following decomposition formula for the Specht module of the Rothe diagram of a permutation.

Theorem 33 *Let \mathbb{F} be a field of characteristic zero. Then for any permutation w*

$$S^{D(w)} = \bigoplus_{\tilde{P}} S^{\text{shape}(\tilde{P})}$$

where \tilde{P} runs over the set of tableaux whose reading words are reduced words for w^{-1} .

Plactification shows that the $D(w)$ -peelable tableaux may be used to index the decomposition of the Specht module $S^{D(w)}$.

Corollary 34 *If \mathbb{F} is a field of characteristic zero and w is any permutation, then*

$$S^{D(w)} = \bigoplus_P S^{\text{shape}(P)}$$

where P runs over the set of $D(w)$ -peelable tableaux.

In [17], it is shown that there is *another* class of diagrams D for which the decomposition of S^D into irreducibles over \mathbb{C} is given by the shapes of the D -peelable tableaux, namely the class of *column convex* diagrams D , that is, those D for which each column has no gaps between its cells. Both Rothe diagrams and column convex diagrams have the property that one can rearrange their columns (this does not change the isomorphism class of the associated Specht module) to obtain a diagram which possesses the *northwest property* (see the definition of peelable tableau). Rothe diagrams and column-convex diagrams both generalize the notion of a skew diagram (see [1]). These facts suggest the following conjecture.

Conjecture 35 *If \mathbb{F} is a field of characteristic zero and D has the northwest property, then*

$$S^D = \bigoplus_P S^{\text{shape}(P)}$$

as P runs over all D -peelable tableaux.

Conjecture 35 has been verified for all diagrams with at most 8 cells. The following lemmas (whose proofs we omit) provide further evidence for this conjecture. Lemma 36 shows that the conjecture agrees with the fact that switching rows or columns in a diagram does not change the isomorphism class of the associated Specht module.

Lemma 36 *Let D and D' be diagrams which possess the northwest property and differ by exchanging their r th and $(r + 1)$ st*

(1) *columns. Then the diagrams D and D' have the same peelable tableaux.*

(2) rows. Then the plactic transposition σ_r gives a shape-preserving bijection from the D -peelable tableaux to the D' -peelable tableaux.

The next lemma shows that Conjecture 35 agrees with the well-known fact that the multiplicity of the irreducible S^λ in S^D is the same as the multiplicity of the transpose irreducible $S^{\lambda'}$ in $S^{D'}$.

Lemma 37 *Let D be a diagram having the northwest property. There is a shape-transposing bijection from the set of D -peelable tableaux to the set of D' -peelable tableaux.*

If D is a skew diagram, this bijection agrees with a simple variant of the bijection of [3] and [20]. If $D = D(w)$, this bijection agrees with the map $P \mapsto \phi((\phi^{-1}(P))')$ from the set of $D(w)$ -peelable tableaux to the $(D(w))' = D(w^{-1})$ -peelable tableaux, where $'$ means transpose.

Further evidence for this conjecture is provided by recent results of Magyar [15], who gives a geometric construction for Schur modules (which are closely related to Specht modules by Schur-Weyl duality). More precisely, he constructs a representation of $GL_N(\mathbb{C})$ on the space of global sections of a line bundle \mathcal{L}_D over a projective variety X_D associated to any diagram D , and then shows that this representation coincides with the classical Schur module construction in the case where D has the northwest property. He also shows that several nice things happen when D has the northwest property:

- (1) The line bundle \mathcal{L}_D has no higher cohomology.
- (2) The variety X_D has only rational singularities, and can be “blown-up” to a smooth variety $X_{D'}$, where D' is another diagram with the northwest property.
- (3) Using the last two properties, one can apply the Atiyah-Bott fixed point formula to obtain a character formula for the Schur module of D .

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Note added in proof: We can now prove Conjecture 35 using a result of Magyar.

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