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Splitting the Square of a Schur Function into its Symmetric and Antisymmetric Parts

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Abstract. We propose a new combinatorial description of the product of two Schur functions. In the particular case of the square of a Schur function S_I, it allows to discriminate in a very natural way between the symmetric and antisymmetric parts of the square. In other words, it describes at the same time the expansion on the basis of Schur functions of the plethysms $S^2(S_I)$ and $\Lambda^2(S_I)$. More generally our combinatorial interpretation of the multiplicities $c_{IJ}^K = (S_I S_J, S_K)$ leads to interesting q-analogues $c_{IJ}^K(q)$ of these multiplicities. The combinatorial objects that we use are domino tableaux, namely tableaux made up of 1×2 rectangular boxes filled with integers weakly increasing along the rows and strictly increasing along the columns. Standard domino tableaux have already been considered by many authors [33], [6], [34], [8], [1], but, to the best of our knowledge, the expression of the Littlewood-Richardson coefficients in terms of Yamanouchi domino tableaux is new, as well as the bijection described in Section 7, and the notion of the diagonal class of a domino tableau, defined in Section 8. This construction leads to the definition of a new family of symmetric functions (H-functions), whose relevant properties are summarized in Section 9.

Keywords: symmetric function, domino tableaux, plethysm, Littlewood-Richardson rule

1. Introduction

The problem addressed in the title of this paper may be formulated in various ways. Recall that a tensor of rank 2 separates into a symmetric and an antisymmetric part

$$x \otimes y = \tfrac{1}{2}(x \otimes y + y \otimes x) + \tfrac{1}{2}(x \otimes y - y \otimes x).$$

In other words, if V denotes a finite-dimensional vector space over the field of complex numbers, one has

$$V \otimes V = S^2(V) \oplus \Lambda^2(V),$$

which may be seen as the decomposition of the representation $V \otimes V$ of $GL(V)$ into its irreducible components $S^2(V)$ and $\Lambda^2(V)$. Now suppose that $V = S_I(W)$ is itself a model of the irreducible representation of $GL(W)$ indexed by the partition I. One has again

$$S_I(W) \otimes S_I(W) = S^2(S_I(W)) \oplus \Lambda^2(S_I(W)), \tag{1}$$

but the spaces $S^2(S_I(W))$ and $\Lambda^2(S_I(W))$ are no longer irreducible under the action of $GL(W)$. The problem is to decompose these spaces into their irreducible components.

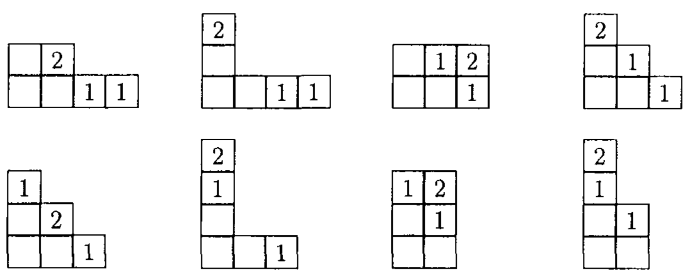
Considering the characters of these representations, one can give an equivalent formulation in terms of symmetric functions. Denoting by S_I the Schur function indexed by the partition I, formula (1) is equivalent to

$$S_I S_I = S^2(S_I) + \Lambda^2(S_I),$$

where the symmetric functions $S^2(S_I)$ and $\Lambda^2(S_I)$ are special cases of plethysms of Schur functions, as defined by Littlewood in [20] (see also [21], [22] and [29]). The problem is now to decompose these plethysms on the basis of Schur functions.

As shown by Littlewood, this question admits also an interpretation in classical invariant theory (actually, invariant theory was his motivation for defining the plethysm of Schur functions). Indeed, the coefficient α_{IJ} in the expansion $S^2(S_I) = \sum_J \alpha_{IJ} S_J$ is equal to the number of concomitants of type J and degree 2 in the coefficients of a ground form of type I. We refer the reader to [7, 9] for a modern formulation of the general plethysm problem in invariant theory.

Now, following Littlewood, we present the corresponding combinatorial problem. Since the decomposition of the tensor square $S_I(W) \otimes S_I(W)$ is given by the well-known Littlewood-Richardson rule, the problem is in fact to discriminate between the Young tableaux coming from the symmetric part of this square, and those coming from its anti-symmetric part. For example, in order to compute the square of S_{12} the following eight tableaux are constructed



and by reading their shapes it is found that

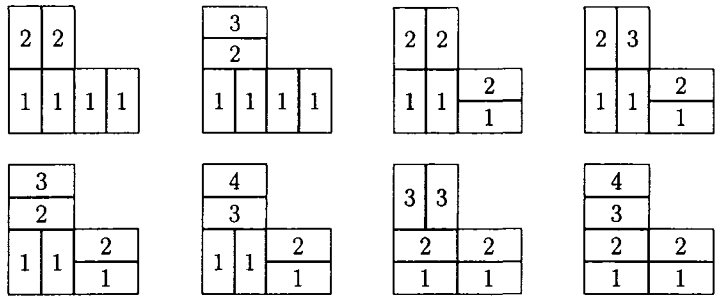
$$S_{12}S_{12} = S_{24} + S_{114} + S_{33} + 2S_{123} + S_{1113} + S_{222} + S_{1122}.$$

This square splits into

$$\begin{cases} S^2(S_{12}) = S_{24} + S_{123} + S_{1113} + S_{222}, \\ \Lambda^2(S_{12}) = S_{114} + S_{33} + S_{123} + S_{1122}. \end{cases}$$

But Littlewood could not find a general simple method of discriminating the tableaux pertaining to $S^2(S_I)$.

In Section 4 we shall explain a different combinatorial description of the product of two Schur functions, in terms of domino tableaux. Using this new rule, the previous example would correspond to the following tableaux



All these tableaux have the same shape 2244, and the result of the multiplication is obtained by reading their evaluation, namely the number of dominoes labelled 1, labelled 2, and so on. Now the splitting becomes obvious. Count the number of horizontal dominoes, which is always even, and divide it by two. If the result is even the corresponding Schur function comes from the symmetric part $S^2(S_{12})$, if it is odd it comes from the antisymmetric part $\Lambda^2(S_{12})$.

This observation suggests that the number of horizontal (or vertical) dominoes is an interesting statistic on domino tableaux. We call it the spin (see Section 3 for a precise definition). Taking into account the spin of the domino tableaux which correspond to a given multiplicity $c_{IJ}^K = (S_I S_J, S_K)$, we are led to a q-analogue $c_{IJ}^K(q)$ of this multiplicity. In Section 8 we study the distribution of this statistic on the set of domino tableaux of given shape and evaluation, and show how this set may be partitioned into simple classes whose spin polynomial is of the type $q^a(1+q)^b$. This yields a decomposition of the polynomials $c_{IJ}^K(q)$ into elementary blocks. For example the multiplicity $c_{1234,1234}^{123356} = 18$ gives rise to the q-analogue

$$c_{1234,1234}^{123356}(q) = q^3+5q^4+8q^5+4q^6 = q^4(1+q)^2+q^4(1+q)^2+q^3(1+q)^3+q^5(1+q).$$

The paper is organized as follows. In Section 2 we review the necessary background in the theory of symmetric functions, including plethysms, adjoint and differential operators, and the concepts of 2-quotient and 2-sign of a partition. In Section 3 we describe the combinatorial objects used in the sequel, viz. domino tableaux, Yamanouchi domino tableaux, and their associated spin and diagonals. In Section 4 we state the rule for computing the scalar product $(S_I\,\psi^2(S_J), S_K)$ as a number of Yamanouchi domino tableaux (Theorem 4.1), thus providing a combinatorial description of the expansion on the basis of Schur functions of various symmetric functions, including the product $S_I S_J$, the plethysm $\psi^2(S_I)$ and the derivative $D_{\psi^2(S_I)}S_J$. The connection with a recent formula [24] for expressing a P-Schur function as a quadratic form of S-Schur functions is also mentioned. In Section 5 we explain how to split the square of a Schur function (Corollary 5.5), using the spin of the Yamanouchi domino tableaux of Theorem 4.1, and we introduce the q-analogues $c_{IJ}^K(q)$. In Section 6 we study a bijection due to Stanton and White [33] between domino tableaux and pairs of ordinary tableaux. We give a description of it explained to us by Schützenberger which is more simple and stresses the role played by the diagonals of the tableaux. This method is equivalent to the approach of Fomin and Stanton [6]. We also fulfil the program suggested by Stanton and White of extending the plactic monoid of Lascoux and Schützenberger to dominoes. Indeed we show that domino tableaux may be seen as the elements of a monoid, which we call the super plactic monoid. It is isomorphic to the direct product of two plactic monoids. In Section 7 we describe a different algorithm associating to a given domino tableau a pair of tableaux, the first being a Yamanouchi domino tableau and the second an ordinary tableau. This algorithm, which is the analogue for domino tableaux of the classical algorithm used by Robinson [31], Littlewood [22] or Macdonald [29] for proving the rule of multiplication of Schur functions, furnishes the proof of Theorem 4.1. It also enables an action of the symmetric group on domino tableaux to be defined, which permutes the evaluation without changing the spin. In Section 8 we analyse the distribution of spin on the set of domino tableaux of given shape, evaluation and diagonals, and deduce from this study the proof of Theorem 5.3. In Section 9 we show that our construction leads to the definition of a new family of symmetric functions (H-functions), and we sum up their relevant properties. Finally, we give in Section 10 the proof of a series of lemmas stated and used in Section 8.

2. Symmetric functions and plethysms

Our notations for symmetric functions are as in [26]. In particular a partition $I = (i_1, i_2, \ldots, i_n)$ is a weakly *increasing* sequence of nonnegative integers. The integer n is called the length of I. When there is no risk of confusion, we sometimes write for short $I = i_1 i_2 \cdots i_n$. Schur functions and monomial functions are denoted respectively by S_I and ψ_I. When $I = i$ is reduced to one part, one obtains the complete symmetric function S_i also denoted by S^i, and the power sum ψ_i also denoted by ψ^i. The products of these functions are denoted by $S^I := S^{i_1} S^{i_2} \cdots S^{i_n}$ and $\psi^I := \psi^{i_1} \psi^{i_2} \cdots \psi^{i_n}$. We write $I^\sim$ for the conjugate partition of I (i.e. the partition whose diagram is obtained by interchanging the rows and columns of the diagram of I), and we set $\Lambda_I = S_{I^\sim}$. In particular, $\Lambda_i = \Lambda^i$ denotes the i-th elementary symmetric function.

The algebra of symmetric functions is endowed with a scalar product denoted by $(\, .\, ,.)$, defined by the requirement that the Schur functions form an orthonormal basis. Given a symmetric function F, one defines the *differential operator* D_F as the adjoint operator of the multiplication by F for this scalar product. In other words, for any symmetric functions G, H there holds

$$(FG, H) = (G, D_F H).$$

As an example one has $D_{S_I} S_J = S_{J/I}$, the skew Schur function associated with the skew diagram J/I.

Usually, we omit to specify the set of variables on which depend the symmetric functions we are dealing with. When it proves to be necessary, this set of variables, or *alphabet* is denoted by $A = \{a_1 < a_2 < \cdots\}$. Let $a^J = a_1^{j_1} a_2^{j_2} \ldots$ denote the monomial with multi-degree J in the polynomial algebra $Z[A]$, and suppose that the symmetric function F admits the expansion $F = \sum_J \lambda_J a^J$, where the λ_J are integers. The *plethysms* $S^k(F)$ are defined by means of the generating series

$$\sum_{k \geq 0} z^k S^k(F) = \prod_J \frac{1}{(1 - z a^J)^{\lambda_J}}.$$

More generally if G is another symmetric function, we express $G = g(S^1, S^2, \ldots)$ as a polynomial in the S^k and we define the *plethysm* $G(F)$ by $G(F) := g(S^1(F), S^2(F), \ldots)$.

The plethysms that will be dealt with in the sequel are $S^2(S_I)$, $\Lambda^2(S_I)$ and $\psi^2(S_I) = S_I(\psi^2) = S_I(x_1^2, x_2^2, \ldots)$. We note the following simple but important relations

$$S^2(S_I) + \Lambda^2(S_I) = (S_I)^2; \qquad S^2(S_I) - \Lambda^2(S_I) = \psi^2(S_I).$$

We recall that for any k, $F \to \psi^k(F)$ is a linear operator on the algebra of symmetric function (the so-called Adams operation). Its *adjoint operator* with regard to the scalar product defined above is denoted by ϕ^k. In this paper we are primarily concerned with a combinatorial expression of

$$\left(S_I\psi^2(S_J), S_K\right) = \left(\psi^2(S_J), S_{K/I}\right) = \left(S_J, \phi^2(S_{K/I})\right) = (S_I, D_{\psi^2(S_J)}S_K).$$

The particular case when $I = \emptyset$ gives a combinatorial description of the expansion on the basis of Schur functions of $\psi^2(S_K)$ and $\phi^2(S_K)$. Now, as shown by Littlewood [23], $\phi^2(S_K)$ is either zero or equal up to sign to the product of two Schur functions $S_{K_0}S_{K_1}$.

Here is how to obtain the ordered pair (K_0, K_1), called the *2-quotient* of K. Make K into a partition of even length $2n$ by adding if necessary a zero part. Add to K the staircase partition $\rho_{2n} = (0, 1, 2, \ldots, 2n-1)$. Reduce modulo 2 the successive parts of $L = K + \rho_{2n}$ without using two times the same representative. This gives a sequence M, which is put in increasing order by means of a permutation σ. Then, if $\sigma(M) \neq \rho_{2n}$, set $\epsilon_2(K) = 0$, otherwise $\epsilon_2(K) = sign(\sigma)$. This is the *2-sign* of K. Finally if $\epsilon_2(K) \neq 0$, subtract from the even parts of L the corresponding residues in M and divide by 2 to obtain K_0. The same procedure applied to the odd parts gives the second partition K_1.

Example 2.1 Consider $K = (1, 1, 1, 3, 5, 5)$. Then

$$L = (1, 2, 3, 6, 9, 10), \quad M = (1, 0, 3, 2, 5, 4), \quad \sigma(M) = (0, 1, 2, 3, 4, 5),$$

$$\epsilon_2(K) = -1, \quad K_0 = \frac{1}{2}[(2, 6, 10) - (0, 2, 4)] = (1, 2, 3),$$

$$K_1 = \frac{1}{2}[(1, 3, 9) - (1, 3, 5)] = (0, 0, 2).$$

Thus the 2-quotient of K is $((1, 2, 3), (2))$.

A key observation is that this process can be reversed, that is, given an arbitrary ordered pair of partitions (I, J), there is a unique partition K of weight $2|I| + 2|J|$ whose 2-quotient is equal to (I, J). Thus one can write $S_IS_J = \epsilon_2(K)\phi^2(S_K)$. As a consequence the multiplicity c_{IJ}^H is equal to the scalar product

$$c_{IJ}^H = \epsilon_2(K)\left(\phi^2(S_K), S_H\right). \tag{2}$$

Example 2.2 The particular case when $I = J$ is important for us. One can check that the partition K whose 2-quotient is (I, I) is nothing but $K = 2I \vee 2I := (2i_1, 2i_1, \ldots, 2i_n, 2i_n)$. Moreover the 2-sign of $2I \vee 2I$ is $+1$ for all I. Thus

$$c_{II}^H = \left(\phi^2(S_{2I\vee 2I}), S_H\right). \tag{3}$$

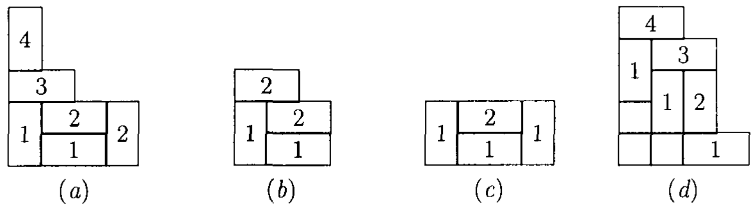


Figure 1.

Example 2.3 More generally, let $K = 2I = (2i_1, 2i_2, \ldots)$ be a partition whose parts are all even. Then, setting $I_o = (i_1, i_3, i_5, \ldots)$, $I_e = (i_2, i_4, i_6, \ldots)$, we have, $\phi^2(S_K) = S_{I_o} S_{I_e}$.

We shall return to the 2-quotient in Section 6, where a bijection is described which rests entirely upon this operation. This will give an alternative description of it, in terms of domino tableaux.

3. Domino tableaux

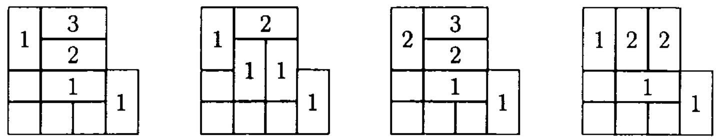
A *domino tableau of shape I* is a tiling of this shape by means of 2×1 or 1×2 rectangles called *dominoes*. Each domino is numbered by a nonnegative integer, and it is required that these integers be weakly increasing along the rows (from left to right), and strictly increasing along the columns (from bottom to top). For example Fig. 1a shows a domino tableau of shape 11244, but Fig. 1b does not represent a domino tableau (the second column is not strictly increasing), neither Fig. 1c (the second row is not weakly increasing). We shall also use domino tableaux of skew shape I/J, as illustrated in Fig. 1d.

As in the case of ordinary tableaux, a commutative monomial in $Z[A]$ is associated with each domino tableau T. It is defined by $a^T := a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$, where k_i is the number of dominoes of T numbered i. The sequence $k_n k_{n-1} \cdots k_1$ is called the *evaluation* of T. We denote by $Tab_2(I/J; K)$ the set of all domino tableaux of shape I/J and evaluation K. Thus the tableau of Fig. 1d belongs to $Tab_2(23334/12; 1113)$. In Sections 6 and 7, we shall also associate with a domino tableau two noncommutative monomials in the super plactic algebra and the plactic algebra.

The *column reading* of a domino tableau T is the word obtained by reading the successive columns of T from top to bottom and left to right. Horizontal dominoes, which belong to two successive columns i and $i+1$, are read only once, when reading column i. Thus the column reading of the domino tableau of Fig. 1d is 413121.

A *Yamanouchi word* is a word $w = w_1 w_2 \cdots w_n$ such that each right factor $w_i w_{i+1} \cdots w_n$ contains at least as many letters j than $j+1$, and this for every j. For example, $w = 31423211$ is a Yamanouchi word, while $w' = 431223211$ is not because its right factor 223211 contains more 2 than 1. A *Yamanouchi domino tableau* is a domino tableau whose column reading is a Yamanouchi word. The set of all Yamanouchi domino tableaux of shape I/J and evaluation K is denoted by $Yam_2(I/J; K)$.

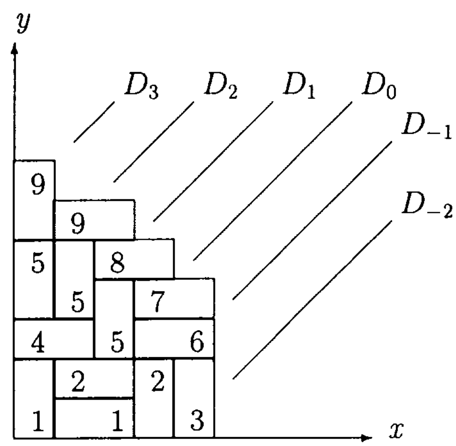
Example 3.1 There are four Yamanouchi tableaux of shape 3344/13



Column reading	13211	12111	23211	12121

We define the *spin of a domino tableau* as half the number of its vertical dominoes. Thus the four domino tableaux of Example 3.1 have respective spins 1, 2, 1, 2. The spin is in general a half-integer. It is a classical result that the parity of the number of vertical dominoes of a domino tableau T depends only on its shape I/J [12, 32]. Set $\epsilon_2(I/J) = (-1)^{2\mathrm{Spin}(T)}$. This is the 2-*sign* of I/J, and it coincides with the definition of Section 2 in the case when $J = \emptyset$. Also we define $\epsilon_2(I/J) = 0$ if there is no domino tableau of shape I/J. For instance, one has $\epsilon_2(3344/13) = +1$.

Let T be a domino tableau of shape I. Each domino of T is intersected by a unique straight line D_k of equation $y = x + 2k$, k integer. It is very convenient, as will be demonstrated in Sections 6 and 8, to materialize this intersection by writing the number attached to each domino in the square box cut by the line D_k. This is illustrated by the following picture.



The *diagonals* of a domino tableau are the sequences of numbers read along the lines D_k. Thus the preceding tableau has the following diagonals

$$9;\, 5, 9;\, 4, 5, 8;\, 1, 2, 5, 7;\, 1, 2, 6;\, 3.$$

We shall distinguish between 2 types of dominoes, according to the square box cut by the diagonal D_k. A domino is said to have the *colour* 0 if this box is the bottom or right

Colour 0 *Colour* 0 *Colour* 1 *Colour* 1

box, and to have the colour 1 otherwise (we prefer the name "colour" to the name "orientation" used by Stanton and White, which may create confusion with the different distinction between horizontal and vertical dominoes).

4. An analogue of the Littlewood-Richardson rule for domino tableaux

Recall that according to the classical Littlewood-Richardson rule, the multiplicity

$$c_{IJ}^{K} = (S_I S_J, S_K) = (S_{K/I}, S_J) = (S_{K/J}, S_I)$$

is equal to $\sharp Yam(K/I;\, J)$, that is, the number of Yamanouchi (ordinary) tableaux of shape K/I and evaluation J. Now we claim

Theorem 4.1 *Let I, J, K be three partitions and set*

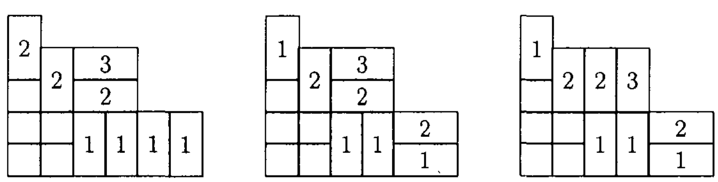
$$d_{IJ}^{K} = \sharp Yam_2(K/I;\, J),$$

the number of Yamanouchi domino tableaux of shape K/I and evaluation J. Then

$$\epsilon_2(K/I) d_{IJ}^{K} = \left(S_I \psi^2(S_J), S_K\right) = \left(\phi^2(S_{K/I}), S_J\right) = (D_{\psi^2(S_J)} S_K, S_I).$$

The proof of Theorem 4.1 is postponed to Section 7.

Example 4.2 There are three Yamanouchi domino tableaux of shape 14466/122 and evaluation 134



which correspond to the scalar product $(S_{122}\psi^2(S_{134}),\ S_{14466}) = 3$.

Corollary 4.3 *Yamanouchi domino tableaux provide combinatorial descriptions of the following expansions*:

$$S_I \psi^2(S_J) = \sum_K \epsilon_2(K/I) d_{IJ}^{K} S_K, \tag{4}$$

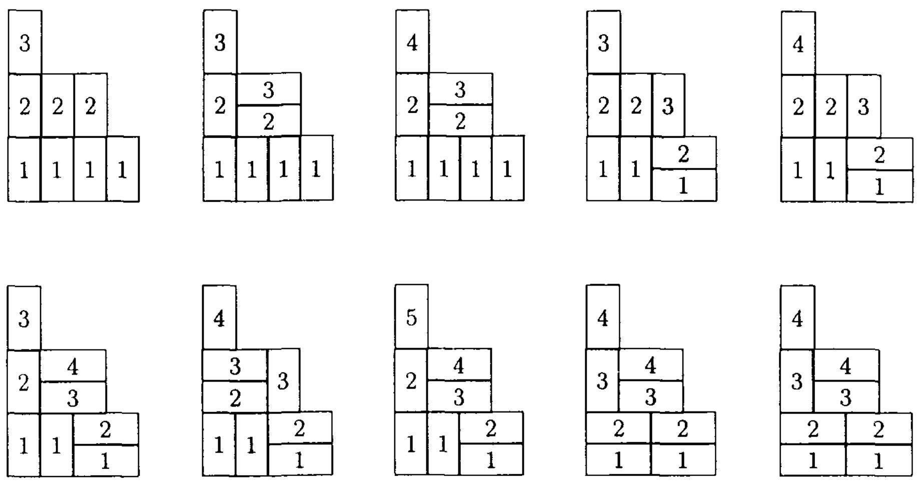
$$\phi^2(S_{K/I}) = \sum_J \epsilon_2(K/I) d_{IJ}^{K} S_J, \tag{5}$$

$$D_{\psi^2(S_J)} S_K = \sum_I \epsilon_2(K/I) d^K_{IJ} S_I. \tag{6}$$

Note that putting $I = \emptyset$ in (4) and (5) one obtains the expansions on the basis of Schur functions of the plethysm $\psi^2(S_J)$ and of the adjoint $\phi^2(S_K)$. Taking into account (2), we also deduce

Corollary 4.4 (Multiplication of Schur functions) *Let I, J be two partitions and let H be the partition of weight $2|I| + 2|J|$ whose 2-quotient is equal to (I, J). Then for any partition K, the multiplicity $c^K_{IJ} = (S_I S_J, S_K)$ is equal to $d^H_{\emptyset K} = \sharp Yam_2(H; K)$, the number of Yamanouchi domino tableaux of shape H and evaluation K.*

Example 4.5 We choose $I = 122$ and $J = 12$. Then $H = 113344$. There are ten Yamanouchi domino tableaux of shape H



from which we deduce that

$$S_{122}S_{12} = \phi^2(S_{113344}) = S_{134} + S_{224} + S_{1124} + S_{233} + S_{1133} + 2S_{1223} + S_{11123} + S_{2222} + S_{11222}.$$

In a recent paper a new expression was found for P Schur functions [24]. Recall that P_I is not zero only if all parts of I are different, that is, if

$$I = \rho_n + J = (0, 1, 2, \ldots, n-1) + (j_1, j_2, \ldots, j_n)$$

with $0 \le j_1 \le j_2 \le \cdots \le j_n$. Now there holds

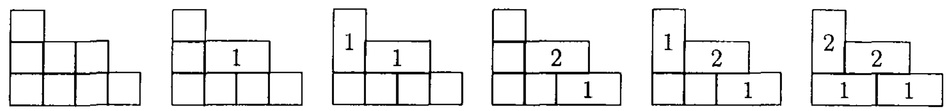
$$P_{\rho_n + J} = \sum_H \Lambda_{\rho_n + 2H} D_{\psi^2(S_H)} S_J, \tag{7}$$

where $\rho_n + 2H = (0, 1, 2, \ldots, n-1) + (2h_1, 2h_2, \ldots, 2h_n)$. We deduce from (6) and (7) the following

Corollary 4.6 (Quadratic expansion of P Schur functions) *For every partition J of length less or equal to n, one has*

$$P_{\rho_n+J} = \sum_{H,K} \epsilon_2(J/K) d^J_{K\,H} S_K \Lambda_{\rho_n+2H}.$$

Example 4.7 We shall explicitly expand the symmetric function P_{257}. We can write $(0, 2, 5, 7) = (0, 1, 2, 3) + (0, 1, 3, 4)$ so that $J = 134$, and we have to enumerate all Yamanouchi domino tableaux of shape $134/K$. These are



whence

$$P_{257} = S_{134}\Lambda_{123} + S_{114}\Lambda_{125} - S_4\Lambda_{127} + S_{112}\Lambda_{145} - S_2\Lambda_{147} - \Lambda_{167}.$$

5. A q-analogue of the multiplicity c^K_{IJ}—Splitting of the square of a Schur function

Corollary 4.4 gives a new combinatorial rule for describing the multiplicities (or Clebsch-Gordan numbers) $c^K_{IJ} = (S_I S_J, S_K) = Mult_{V_K}(V_I \otimes V_J)$ in a tensor product of irreducible representations of the general linear group. Several different combinatorial expressions of these multiplicities are already known, including the Littlewood-Richardson rule in terms of Young tableaux [22], the James-Peel rule in terms of pictures [13, 35], the Gel'fand-Zelevinsky rule in terms of Gel'fand-Tsetlin schemes [10, 11], the Lascoux-Schützenberger rule in terms of Schubert polynomials [28], Kirillov's rule in terms of configurations [14] and the Berenstein-Zelevinsky rule in terms of Berenstein-Zelevinsky triangles [2]. We mention that an explicit bijection between the Littlewood-Richardson rule and the Berenstein-Zelevinsky rule has been constructed in [3].

As explained in Section 1, our rule in terms of domino tableaux has the advantage of discriminating very naturally between the symmetric and antisymmetric parts of the square of a Schur function. Moreover Corollary 4.3 and 4.6 show that the same rule applies also to the expansion of other symmetric functions, such as the plethysms $\psi^2(S_I)$ and the P Schur functions.

We believe, however, that its main interest lies in the fact that it enables a q-analogue of c^K_{IJ} to be defined, in a purely combinatorial way, which seems to possess an algebraic meaning.

Definition 5.1 Let I, J, K be three partitions, and q an indeterminate. Denote by H the partition whose 2-quotient is equal to (I, J). We define

$$c^K_{IJ}(q) = \sum q^{\mathrm{Spin}(T)},$$

the sum being over all Yamanouchi domino tableaux T of shape H and evaluation K.

Example 5.2 We consider the case of the multiplicity $c^{33356}_{1234\,1234} = 5$. Using our rule, this multiplicity is computed by enumerating the five Yamanouchi domino tableaux of shape

22446688 and evaluation 33356 (cf Example 2.2). These five tableaux have respective spins 4, 5, 5, 6, 6, and thus $c_{1234\,1234}^{33356} = q^4 + 2q^5 + 2q^6$.

We note that our polynomials $c_{IJ}^K(q)$ do not coincide with the Clebsch-Gordan polynomials defined by Kirillov and Reshetikhin by means of the q-Kostant partition function and the Kostant-Steinberg formula [18]. Indeed Clebsch-Gordan polynomials can have negative coefficients which is not the case for ours, as is clear from their combinatorial definition.

The splitting of the square of a Schur function announced in Section 1 may be stated now. Recall that by Corollary 4.4, $c_{II}^J(1) = (S_I S_I, S_J)$.

Theorem 5.3 *Let I, J be two partitions. There holds*

$$c_{II}^J(-1) = (-1)^{|I|}\left(\psi^2(S_I), S_J\right) = (-1)^{|I|}\left((S^2(S_I), S_J) - (\Lambda^2(S_I), S_J)\right),$$

where $|I| = i_1 + i_2 + \cdots$ is the weight of the partition I.

The proof of Theorem 5.3 is relegated to Section 8.

It is important to note that the two combinatorial descriptions of $\psi^2(S_I)$ provided by Corollary 4.3 and Theorem 5.3 are different. This is illustrated by the next example.

Example 5.4 There are five Yamanouchi domino tableaux of shape 22446688 and evaluation 33356, among which three have even spin and two have odd spin (see Example 5.2). Therefore, $c_{1234\,1234}^{33356}(-1) = 1 = (\psi^2(S_{1234}), S_{33356})$, by Theorem 5.3. On the other hand, there is only one Yamanouchi domino tableau of shape 33356 and evaluation 1234, which gives also $(\psi^2(S_{1234}), S_{33356}) = d_{\emptyset\,1234}^{33356} = 1$ by Corollary 4.3.

An immediate consequence of Theorem 5.3 and Example 2.2 is the following

Corollary 5.5 (Combinatorial description of $S^2(S_I)$ and $\Lambda^2(S_I)$) *Let I be a partition, and put*

$$S^2(S_I) = \sum_J \alpha_{IJ} S_J, \qquad \Lambda^2(S_I) = \sum_J \beta_{IJ} S_J.$$

Then α_{IJ} (resp. β_{IJ}) is the number of Yamanouchi domino tableaux of shape $2I \vee 2I := (2i_1, 2i_1, 2i_2, 2i_2, \ldots)$ and evaluation J, whose number of horizontal dominoes is $\equiv 0 \bmod 4$ (resp. $\equiv 2 \bmod 4$).

The reader is referred to Section 1 for an illustration of this Corollary.

We mention that some efficient algorithms for the computation of the plethysms $\psi^k(S_I)$, $\Lambda^k(S_I)$, $S^k(S_I)$ on the basis of Schur functions are described in [4] and [5], and have been implemented in the system *SYMMETRICA*. However, these algorithms are not based upon a combinatorial description of the coefficients such as those given in this article.

The next sections are devoted to proving Theorem 4.1 and Theorem 5.3. This is achieved by lifting the symmetric functions $\phi^2(S_I)$ to the noncommutative super plactic algebra defined in Section 6, and then mapping them to the plactic algebra as described in Section 7. Lastly, Section 8 is dedicated to the additional properties involving the spin.

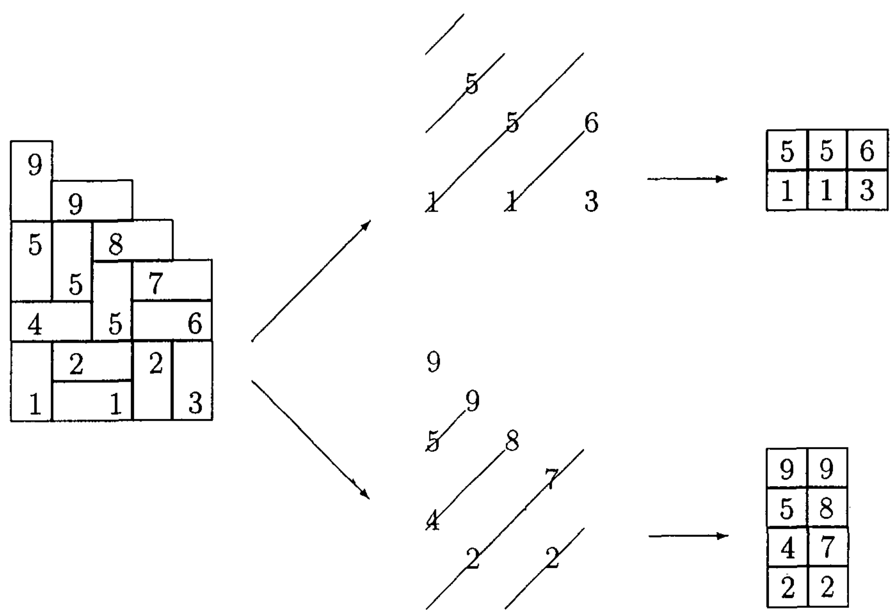
6. The super plactic monoid and algebra

In this section we describe, in the particular case of domino tableaux, a bijection discovered by Stanton and White between k-ribbon tableaux and k-uples of ordinary tableaux [33]. Our description differs from the one of these authors in two points. First we need to consider tableaux of arbitrary evaluation, while Stanton and White developed their algorithm for standard evaluation only. Secondly we follow a different combinatorial method explained to us by M.P. Schützenberger which is more simple and more illuminating. This method appears also in [6]. The algorithm is the following.

Algorithm 6.1

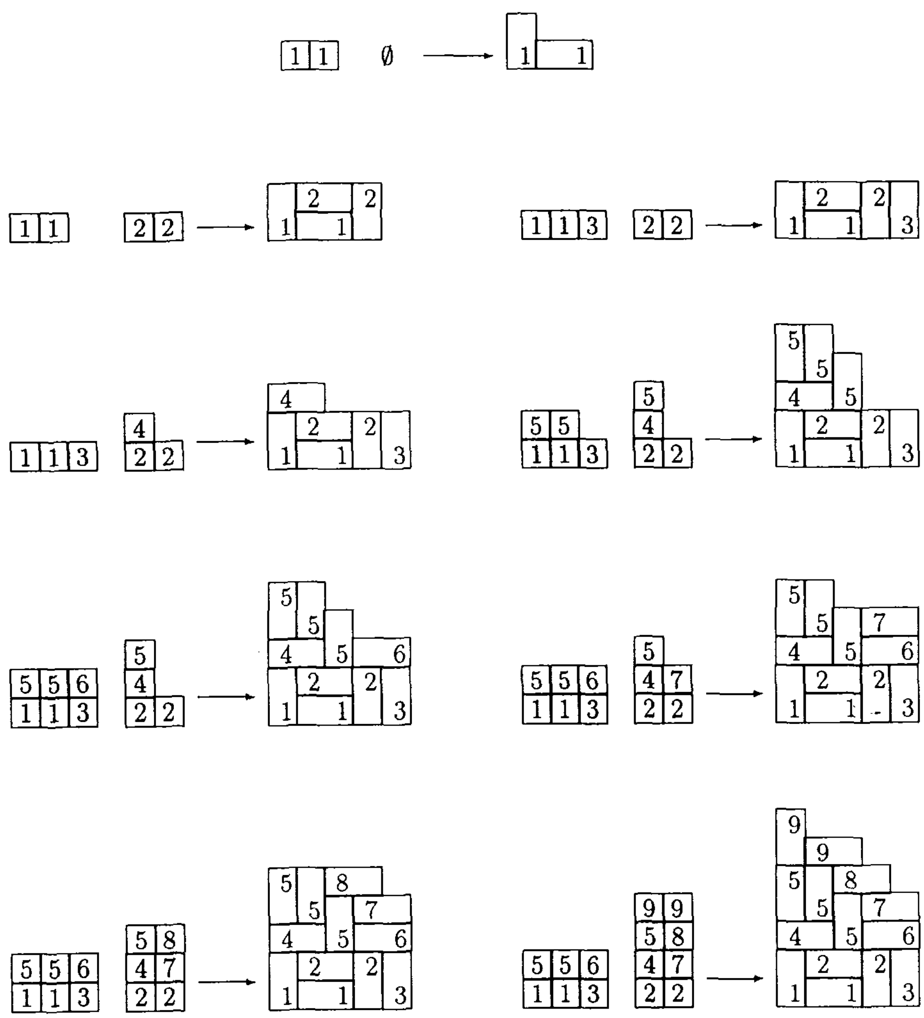
- **Input** T, a domino tableau of shape I.
- Delete on each diagonal of T all dominoes of colour 1. Let the remaining numbers slide down along their diagonals. The new sequence of diagonals so obtained is the sequence of diagonals of an ordinary tableau t_0.
- Proceeding in the same way but deleting now all dominoes of colour 0, construct a second ordinary tableau t_1.
- **Output**
  - — A tableau t_0 of shape I_0
  - — A tableau t_1 of shape I_1

Example 6.2 Let us apply this algorithm to the domino tableau shown in Section 3.



Theorem 6.3 *Algorithm* 6.1 *realizes a bijection* $T \to (t_0, t_1)$ *between domino tableaux of shape* I *and ordered pairs of ordinary tableaux of shape* (I_0, I_1), *the* 2*-quotient of* I.

Proof: It is enough to describe the reverse algorithm. So suppose that the pair (t_0, t_1) is given. The tableau T can be reconstructed step by step. The key observation is that the correspondence between an ordered pair of partitions (I_0, I_1) and the partition I whose 2-quotient is (I_0, I_1), is compatible with the adjonction of one box. Namely, suppose that a box is glued to I_0 on diagonal D, which gives a new partition I_0'. The partition I' whose 2-quotient is (I_0', I_1) is obtained from I by glueing a domino of colour 0 on the diagonal corresponding to D. And the same, of course, for colour 1. Clearly this fact may be used to reconstruct T by induction, for a tableau is nothing but a chain of partitions. This will be illustrated by working out the previous example.



□

Theorem 6.3 admits the following formulation in terms of symmetric functions.

Corollary 6.4 *Let I be a partition. One has*

$$\sum_T a^T = \epsilon_2(I)\phi^2(S_I(A)),$$

where the sum runs over all domino tableaux T of shape I.

Proof: By Section 2, $\epsilon_2(I)\phi^2(S_I(A)) = S_{I_0}(A)S_{I_1}(A)$, where (I_0, I_1) denotes the 2-quotient of I. Now it is a classical fact that $S_J(A) = \sum_t a^t$, t running over all ordinary tableaux of shape J. Thus, putting $J = I_0, I_1$, one finds

$$S_{I_0}(A)S_{I_1}(A) = \sum_{\substack{\text{shape}(t_0)=I_0\\ \text{shape}(t_1)=I_1}} a^{t_0}a^{t_1} = \sum_{\text{shape}(T)=I} a^T,$$

the last equality following from Theorem 6.3. □

Thus the sum in Corollary 6.4 is a symmetric function in the variables $a_1, a_2, \ldots$ whose expansion on the basis of monomial functions is given by

Corollary 6.5 *Define $K_{IJ}^{(2)}$ as the number of domino tableaux of shape I and evaluation J.Then,*

$$\epsilon_2(I)\phi^2(S_I) = \sum_J K_{IJ}^{(2)}\psi_J,$$

the sum being over all partitions J.

Using the fact that ϕ^2 is the adjoint morphism of ψ^2, we have the following equivalent formulation.

Corollary 6.6 *For every partition J one has*

$$\psi^2(S^J) = \sum_I \epsilon_2(I)K_{IJ}^{(2)}S_I.$$

Proof: As is well known, $(\psi_J, S^I) = \delta_{IJ}$. □

Corollary 6.5 and 6.6 show that the numbers $K_{IJ}^{(2)} = \sharp Tab_2(I, J)$ are to be seen as the domino analogues of the Kostka numbers $K_{IJ} = \sharp Tab(I, J)$. Indeed $K_{IJ} = (S^J, S_I)$, while $K_{IJ}^{(2)} = |(\psi^2(S^J), S_I)| = |(S^J, \phi^2(S_I))|$. Moreover, using the results of [25], one sees that, denoting by $J \vee J$ the partition $(j_1, j_1, j_2, j_2, \ldots)$, there holds $K_{IJ}^{(2)} = |K_{I,J\vee J}(-1)|$, where $K_{H,L}(q)$ is the Kostka-Foulkes polynomial (cf. [29]). Finally, in the particular case of a standard evaluation (i.e. $J = (1, 1, \ldots, 1)$), it was shown by Morris and Sultana [30] that $K_{I1\ldots1}^{(2)}$ is given by a "modular hook formula", obtained by specializing to $q = -1$ the q-hook formula for standard Kostka-Foulkes polynomials.

We shall end this Section by an algebraic formulation of Theorem 6.3. To this end we must first recall the main lines of the proof of Littlewood-Richardson rule given by Lascoux and

Schützenberger in [27]. It consists in interpreting Young tableaux as elements of a monoid, the so-called *plactic monoid*, which is defined as the quotient of the free monoid by Knuth's relations [17]. The combinatorial expressions of Schur functions S_I as sums of all tableaux of shape I, are then lifted to the algebra of the plactic monoid, generating a subalgebra of this noncommutative algebra, isomorphic to the algebra of symmetric functions. In this setting, the multiplicity c_{IJ}^{K} is nothing but the number of ways of writing a tableau of shape K as a product of two tableaux of shape I and J.

Consider now the direct product $Pl(A^0) \times Pl(A^1)$ of two plactic monoids on the alphabets $A^0 = \{a_1^0 < a_2^0 < \cdots\}$, $A^1 = \{a_1^1 < a_2^1 < \cdots\}$. This monoid may be described as the quotient of the free monoid on $A^0 \cup A^1$ by the following relations

$$\begin{array}{lll} a_j^0 a_i^0 a_k^0 \equiv a_j^0 a_k^0 a_i^0 & a_j^1 a_i^1 a_k^1 \equiv a_j^1 a_k^1 a_i^1, & i < j < k \\ a_i^0 a_k^0 a_j^0 \equiv a_k^0 a_i^0 a_j^0 & a_i^1 a_k^1 a_j^1 \equiv a_k^1 a_i^1 a_j^1, & i < j < k \\ a_j^0 a_j^0 a_i^0 \equiv a_j^0 a_i^0 a_j^0 & a_j^1 a_j^1 a_i^1 \equiv a_j^1 a_i^1 a_j^1, & i < j \\ a_j^0 a_i^0 a_i^0 \equiv a_i^0 a_j^0 a_j^0 & a_j^1 a_i^1 a_i^1 \equiv a_i^1 a_j^1 a_j^1, & i < j \end{array}$$

$$a_i^0 a_j^1 \equiv a_j^1 a_i^0 \quad \forall\, i, j$$

It will be called the *super plactic monoid* and denoted by $SPl(A)$. Theorem 6.3 shows that the elements of this monoid can be viewed as domino tableaux. For instance the domino tableau of Example 6.2 represents the following element of $SPl(A)$

$$a_5^0 a_1^0 a_5^0 a_1^0 a_6^0 a_3^0 a_9^1 a_5^1 a_4^1 a_2^1 a_9^1 a_8^1 a_7^1 a_2^1 \equiv a_5^0 a_5^0 a_6^0 a_1^0 a_1^0 a_3^0 a_9^1 a_9^1 a_5^1 a_8^1 a_4^1 a_7^1 a_2^1 a_2^1.$$

The Z-algebra $Z[SPl(A)]$ of the super plactic monoid contains now some remarkable elements, namely, for each partition I, the sum Σ_I of all domino tableaux of shape I. Corollary 6.4 shows that the projection on the commutative algebra $T \to a^T$ sends Σ_I onto the symmetric function $\epsilon_2(I)\phi^2(S_I)$. Thus, we see that the elements Σ_I generate a commutative subalgebra of the super plactic algebra, isomorphic to the tensor product $Sym(A) \otimes Sym(A)$ of two copies of the algebra of symmetric functions of A.

More generally the *jeu de taquin* for dominoes, as described by Stanton and White [33], enables skew domino tableaux to be interpreted as elements of the super plactic monoid. It can be shown similarly that the sum of all domino tableaux of shape I/J is a lifting in $Z[SPl(A)]$ of the symmetric function $\epsilon_2(I/J)\phi^2(S_{I/J})$.

7. An analogue of the Robinson-Littlewood bijection for domino tableaux

The proofs of the Littlewood-Richardson rule in [31], [22] and [29] result from the existence of a bijection, due to Robinson and Littlewood,

$$Tab(I/J; K) \to \bigsqcup_{L} Yam(I/J; L) \times Tab(L; K).$$

Their method consists in starting with a tableau t of shape I/J and evaluation K, and successively modifying it until its column reading becomes a Yamanouchi word. Simultaneously, a tableau is built up, which serves to record the sequence of moves made (see [29], p. 69). We shall explain now an analogue of this bijection for domino tableaux.

We need first some definitions. A *pseudo-tableau* is a generalized Young tableau, in the sense that we allow its shape to be any sequence of nonnegative integers, and that we only require that the rows and columns be weakly increasing from left to right and bottom to top. Thus, the following picture shows a pseudo-tableau of shape $(1, 2, 0, 4, 2)$ and evaluation $(1, 1, 3, 2, 1, 1)$.

$$\begin{array}{|c|c|c|c|}
\cline{1-1} 6 \\
\cline{1-2} 4 & 5 \\
\cline{1-2} \multicolumn{4}{c}{} \\
\hline 2 & 3 & 4 & 4 \\
\hline 1 & 3 \\
\cline{1-2}
\end{array}$$

The *canonical pseudo tableau* associated with a sequence $K = (k_1, \ldots, k_n)$ is the pseudo tableau having k_n letters 1 in its first row, k_{n-1} letters 2 in its second row, etc. When K is a partition this is just the Yamanouchi tableau of shape and evaluation K.

Given a word on the alphabet $\{1, 2, \ldots, n\}$, that is, a sequence $w = i_1 i_2 \cdots i_r$ of letters i between 1 and n, we associate to each letter $i > 1$ in w, an *index* equal to one plus the difference between the number of letters i and the number of letters $i - 1$ situated to the right of this letter in w. For example the indexes of $w = 321341233$ are

letter	3	2	1	3	4	1	2	3	3
index	2	0	/	2	−1	/	1	2	1

A word is a Yamanouchi word iff all its indexes are ≤ 0.

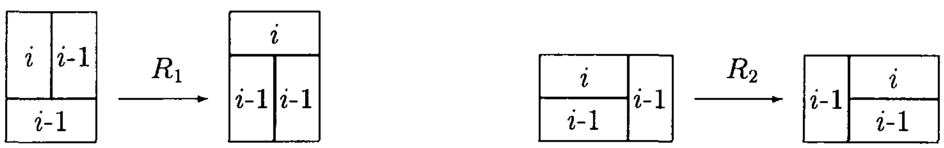
Algorithm 7.1

- **Input** T, a domino tableau of shape I/J and evaluation K.
- The algorithm recursively modifies the data (Y, t, w) composed of a domino tableau Y of shape I/J, *a pseudo-tableau* t of evaluation K, *and a word* w.
- `Initialize`
  - $Y := T$
  - t `:= the canonical pseudo-tableau associated with` K
  - w `:= the column reading of` T
- `While` w `is not a Yamanouchi word do the following`
  - `Begin`
    * `Select in` w `the letter` i `satisfying`
      1. i `has positive index and no letter` $<i$ `has;`
      2. `among letters` i`, it is the rightmost with greatest index.`
    * `Change in` Y `the corresponding domino` i `to` $i - 1$

```
* Take the rightmost letter in the i-th row of t and put
  it at the rightmost place in the (i − 1)-th row
```

— If Y is no longer a domino tableau, then the last change $i \to i-1$ has resulted into one of the following situations:

We must then apply an *elementary transformation* of one of the two following types:



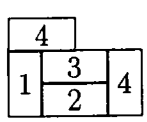

```
* While Y is not a domino tableau apply an elementary
  transformation R1 or R2
* Put in w the column reading of Y
```

— End

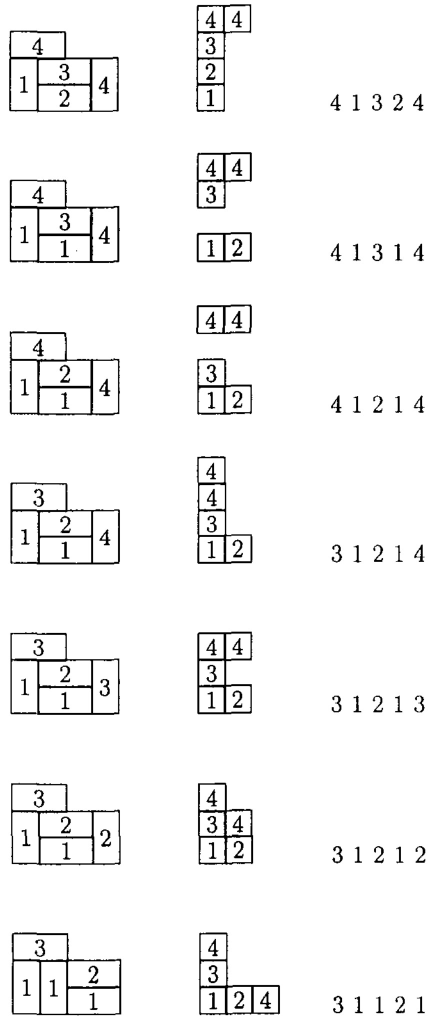
- **Output**
  — Y, a Yamanouchi domino tableau of shape I/J and evaluation L,
  — t, an ordinary tableau of shape L and evaluation K.

Example 7.2

1. The execution of Algorithm 7.1 for

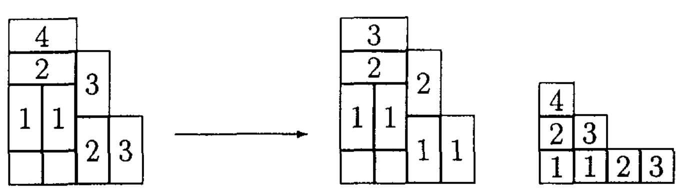
$T =$ 

goes through the following steps.

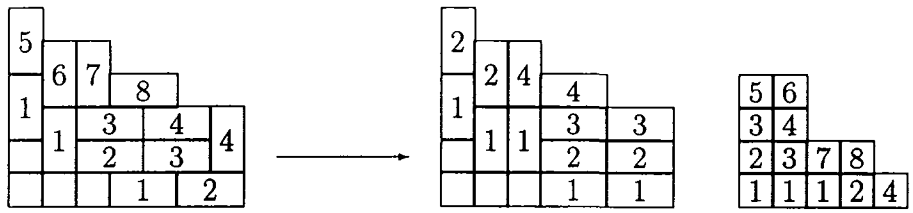


Note that the last step requires a transformation of type R_2.

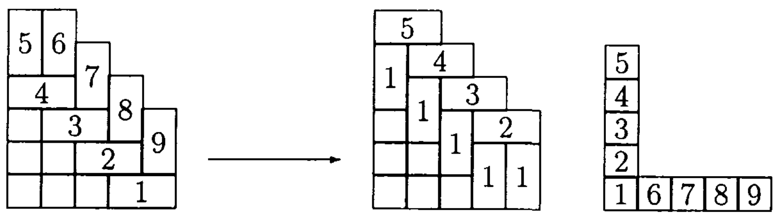
2. A simple example without transformation R_1 or R_2



3. An example requiring several transformations R_2



4. An example requiring several transformations R_1



Note that a domino tableau whose dominoes are all horizontal (or all vertical) may be replaced by an ordinary tableau in the obvious way. In this particular case, no transformation R_1, R_2 occurs during the execution of Algorithm 7.1, and we recover the usual Robinson-Littlewood algorithm.

We are now in a position to state the main theorem of this Section.

Theorem 7.3 *Algorithm* 7.1 *realizes a bijection*

$$Tab_2(I/J;\,K) \to \bigsqcup_L Yam_2(I/J;\,L) \times Tab(L;\,K)$$

$$T \to (Y, t)$$

Proof: The proof proceeds along the same lines as the detailed proof given by Macdonald [29] pp. 69–73 for the Robinson-Littlewood correspondence, the only difference lying in the transformations R_1 and R_2 of Algorithm 7.1 which are specific to the domino case. These transformations are required to obtain after each step a domino tableau Y, while the corresponding condition in the original Robinson-Littlewood algorithm is automatically satisfied (cf. [29] (9.6)). Hence after the last step, by construction, Y is a Yamanouchi domino tableau. Moreover, it follows from [29] that after this last step, t is an ordinary tableau of evaluation K and shape equal to the evaluation of the Yamanouchi word w. It remains only to prove that the mapping $T \to (Y, t)$ is a bijection, which can be done by showing that each step of the algorithm is reversible. Here again, we can imitate the argument in [29] and read off from the pseudo-tableau t which letter i must be changed into $i + 1$ to return to the preceding step, and this completes the proof. □

We shall now develop some important consequences of Theorem 7.3. We introduce a notation for the tableau t obtained from the domino tableau T by application of Algorithm 7.1.

Definition 7.4 The Z-linear map $\pi: Z[SPl(A)] \to Z[Pl(A)]$ which sends a domino tableau T on the ordinary tableau t produced by Algorithm 7.1, is called the natural projection of the super plactic algebra on the plactic algebra.

The situation is summarized in the following commutative diagram

$$\begin{tikzcd}
Z[SPl(A)] \arrow[dr, "\pi"] \arrow[dd] & \\
& Z[Pl(A)] \arrow[dl] \\
Z[A] &
\end{tikzcd}$$

where the two arrows pointing on $Z[A]$ denote the commutative evaluation of noncommutative polynomials. Theorem 4.1 amounts to the following property of π, which is readily implied by Theorem 7.3.

Corollary 7.5 *The image under π of the sum of all domino tableaux of a given shape I/J is a sum of plactic Schur functions.*

Proof: The plactic Schur function S_L is the sum in the plactic algebra of all tableaux of shape L. Theorem 7.3 shows that

$$\sum_{\text{shape}(T)=I/J} T \xrightarrow{\pi} \sum_L d^I_{JL} S_L$$

□

We can now give the

Proof of Theorem 4.1: Evaluating the last formula in the commutative algebra $Z[A]$ and using the results of Section 6, one obtains

$$\epsilon_2(I/J)\phi^2(S_{I/J}) = \sum_L d^I_{JL} S_L,$$

where S_L denotes now the usual Schur polynomial. Thus $\epsilon_2(I/J)d^I_{JL} = (\phi^2(S_{I/J}), S_L)$.

□

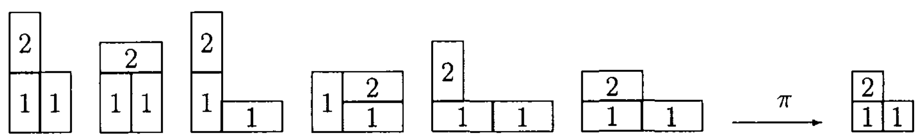
Let us now give an illustration of Theorem 6.3 in terms of plethysms. It follows from Corollary 6.6 that the set of all domino tableaux of evaluation J is a combinatorial description of the expansion on the basis of Schur functions of the plethysm

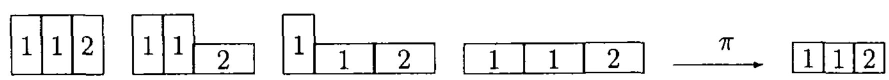
$$\psi^2(S^J) = \psi^2\left(\sum_I K_{IJ} S_I\right) = \sum_I K_{IJ}\psi^2(S_I). \tag{8}$$

Moreover, by Corollary 4.3 the dominant term $\psi^2(S_J)$ of this sum corresponds to the subset of Yamanouchi domino tableaux of evaluation J, i.e. to the subset of domino tableaux which are sent by the natural projection π onto the Yamanouchi ordinary tableau of shape J. More generally we have

Corollary 7.6 *The projection π provides a splitting of the set of domino tableaux of evaluation J, into subsets corresponding to each summand of Eq. 8. In other words, given a tableau t of shape I, the preimage $\pi^{-1}(t)$ is a combinatorial description of the expansion of $\psi^2(S_I)$ on the basis of Schur functions.*

Example 7.7 The equality $\psi^2(S^{12}) = \psi^2(S_{12}) + \psi^2(S_3)$ corresponds to the following splitting of the ten domino tableaux of evaluation 12:





Hence we have

$$\psi^2(S_{12}) = -S_{1122} + S_{222} + S_{1113} - S_{33} - S_{114} + S_{24},$$
$$\psi^2(S_3) = -S_{33} + S_{24} - S_{15} + S_6.$$

Proof: Let t and t' be two tableaux of the same shape I. Theorem 7.3 shows that the preimages $\pi^{-1}(t)$ and $\pi^{-1}(t')$ are isomorphic, that is, there exists a bijection $g: \pi^{-1}(t) \to \pi^{-1}(t')$ leaving the shape of domino tableaux invariant. Explicitly, g is defined as follows. If T is sent on (Y, t) by Algorithm 7.1, then $g(T)$ is the unique domino tableau sent on (Y, t') by the same algorithm. Now if t is the Yamanouchi tableau of shape I, the statement is true by Theorem 4.1. Therefore it is also true for t'. $\square$

Algorithm 7.1 may also be used to define an action of the symmetric group on the set of domino tableaux, which permutes the evaluation and leaves invariant the shape and the spin. Recall that Lascoux and Schützenberger defined an action of $\mathfrak{S}_n$ on the set of tableaux over the alphabet $\{1, \ldots, n\}$, which leaves invariant the shape and the charge [27], [19]. For t a tableau and μ a permutation, let t^μ denote the image of t under μ for this action.

Corollary 7.8 *The symmetric group $\mathfrak{S}_n$ acts on the set of domino tableaux filled with numbers in $\{1, \ldots, n\}$ by:*

$$T \xrightarrow{\text{Alg. 7.1}} (Y, t) \xrightarrow{\mu} (Y, t^\mu) \xrightarrow{(\text{Alg. 7.1})^{-1}} T^\mu$$

The action $T \to T^\mu$ leaves invariant the shape and spin of T.

Proof: By definition of Algorithm 7.1, the shape and spin of T are equal to the shape and spin of Y, and therefore to the shape and spin of T^{μ}. $\square$

8. Distribution of spin—Labyrinths

In this Section we analyse the distribution of the statistic spin on the sets $Tab_2(I; J)$ and $Yam_2(I; J)$. For this purpose, we divide these sets into subsets on which the distribution is very regular.

Definition 8.1 Define an equivalence relation on domino tableaux of shape I by

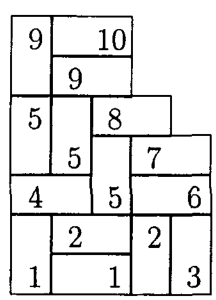
$$T \sim T' \Leftrightarrow T \text{ and } T' \text{ have the same diagonals.}$$

We have the following result concerning the equivalence classes in $Tab_2(I; J)$ and $Yam_2(I; J)$, which we call *diagonal classes*.

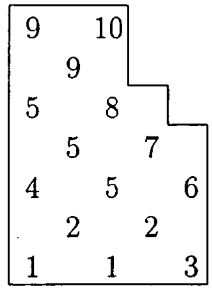
Theorem 8.2 *Let C be a diagonal class in $Tab_2(I; J)$ or $Yam_2(I; J)$. Then, the cardinality of C is a power of 2, and the* spin polynomial *of C, namely $\sum_{T \in C} q^{\mathrm{Spin}(T)}$, is equal to $q^a(1+q)^b$ for some nonnegative integer b and half-integer a.*

Proof: The main idea which is due to A. Lascoux, consists in associating with every domino tableau T a picture from which it is immediate to recover all the domino tableaux belonging to the diagonal class C of T. The starting point is the following. If one considers the domino tableaux T' which belong to the diagonal class of T, one sees that they all have in common several domino border lines of length 2. If one removes from T all the border lines which are not present in all tableaux T' of C, one obtains a skeleton of domino tableaux that we call the *labyrinth* of T. For the convenience of the reader, we shall first illustrate this principle and give the outline of the proof by working out a specific example.

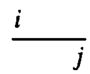
Consider the domino tableau

$T =$ 

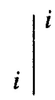
The labyrinth of T is constructed in the following way. First, draw the external shape of T and write down the numbers along the diagonals.



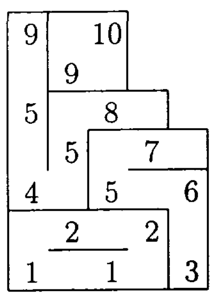
Then, read the *anti-diagonals*, that is, the diagonals going down from North-West to South-East, and between any two adjacent numbers, draw a *barrier* of length 2, according to the following two cases:

 if $i \leq j$  if $i > j$

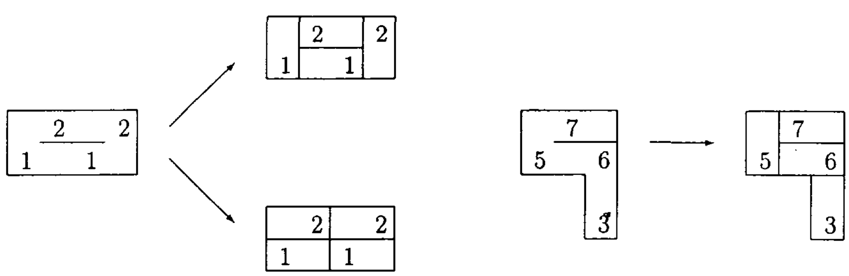
Furthermore, if two adjacent numbers on an *ascending* diagonal are equal, then place a vertical barrier between them in the following way:



The picture so obtained is the labyrinth of T. In this case, it is

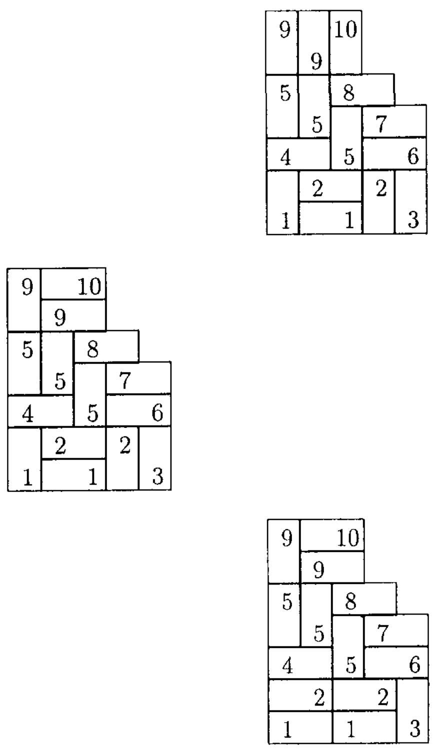


The labyrinth separates the shape of T into different *connected components*—four components in this example. There are two types of connected components. Some of them can be tiled by dominoes in two ways, and the others in only one way. For instance



two tilings *one tiling*

Moreover, for any component which can be tiled in two ways, the spins of the two tilings differ by 1. The conclusion is that, if we denote by b the number of components which can be tiled in two ways, the number of domino tableaux which compose the diagonal class of T in $Tab_2(I; J)$ is 2^b, and the spin polynomial of the class is $q^a(1+q)^b$, where a is the minimal value of the spin in that class. Thus, in our example, the diagonal class of T in $Tab_2(3345555; 1211131122)$ is



Spin

$\frac{9}{2}$

$\frac{7}{2}$

$\frac{5}{2}$

and the spin polynomial is equal to $q^{5/2}(1+q)^2$.

Let us summarize this discussion and repeat the arguments that we have used so far in the form of a series of lemmas to be proved in general. Let T be a domino tableau, and define its labyrinth as the configuration of barriers placed in the shape of T according to the rules explained above.

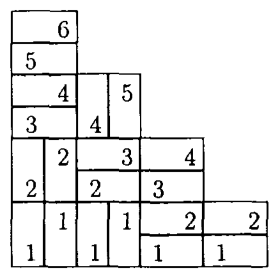
Lemma 8.3 *The diagonal class C of T in $Tab_2(I; J)$ is in one-to-one correspondence with the set of domino tilings of the shape of T compatible with the labyrinth, that is, such that no domino of the tiling crosses a barrier of the labyrinth.*

Lemma 8.4 *The labyrinth separates the shape of T into connected components which take the form of domino chains. Some of those chains are closed and admit two different domino tilings, while the others are open and admit only one tiling.*

Lemma 8.5 *The difference between the spins of the two tilings of a closed chain is equal to* 1.

The proofs of these three lemmas are presented in full in the last section, and this completes the proof of Theorem 8.2 for the diagonal class of T in $Tab_2(I;\, J)$.

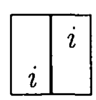
Finally, if we restrict ourselves to Yamanouchi domino tableaux, we see that among the connected components of the labyrinth which can be tiled in two ways, some are such that the two tilings are compatible with the Yamanouchi condition, and the others such that only one tiling is compatible with this condition. Anyway, we obtain again a set of tableaux whose cardinality is a power of 2, and whose spin polynomial is of the form $q^a(1+q)^b$. For instance, the diagonal classes of the Yamanouchi domino tableau

$Y =$ 

in $Tab_2(22446688;\, 123356)$ and $Yam_2(22446688;\, 123356)$, have respectively cardinalities 64 and 8, and the corresponding spin polynomials are $q^3(1+q)^6$ and $q^5(1+q)^3$. □

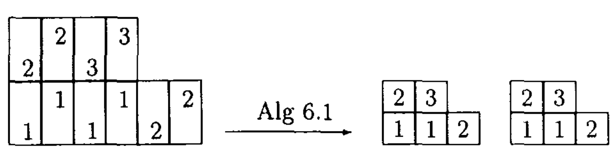
In order to prove Theorem 5.3 we make now the following simple remark, the proof of which is also explained in Section 10.

Lemma 8.6 *Let I be a partition whose parts are all even and have even multiplicities. The domino tableaux of shape I and evaluation J whose diagonal class in $Tab_2(I;\, J)$ has cardinality* 1, *are the domino tableaux entirely composed of* 2×2 *blocks of type*



In other words, these domino tableaux are exactly those sent by Algorithm 6.1 *on a pair (t, t) of two equal ordinary tableaux.*

Example 8.7 The class in $Tab_2(4466, 244)$ of

$T =$ 

is reduced to T.

Let us now deduce Theorem 5.3 from these results.

Proof of Theorem 5.3: Recall from Example 2.2 that the partition whose 2-quotient is (I, I) is $2I \vee 2I = (2i_1, 2i_1, 2i_2, 2i_2, \ldots)$, that is, a partition whose parts are all even and have even multiplicities. Let us introduce the symmetric function

$$H_{I \vee I}(q) = \sum_J c^J_{I,I}(q) S_J.$$

We know from Corollary 4.4 that $H_{I\vee I}(1) = (S_I)^2$. Now we want to prove that $H_{I\vee I}(-1) = (-1)^{|I|}\psi^2(S_I)$. This equality can be checked by comparing the expansions of both sides on the basis of monomials. First it is clear that

$$\psi^2(S_I) = S_I\left(a_1^2, a_2^2, \ldots\right) = \sum_t a^t a^t,$$

where the sum runs over all ordinary tableaux t of shape I.

On the other hand, Theorem 7.3 shows that

$$H_{I\vee I}(q) = \sum_T q^{\mathrm{Spin}(T)} a^T,$$

the sum being over all domino tableaux T of shape $2I \vee 2I$. But, Theorem 8.2 proves that when q is put equal to -1 in this last equality, one is left with

$$H_{I\vee I}(-1) = \sum_T (-1)^{\mathrm{Spin}(T)} a^T,$$

where T ranges now over all domino tableaux described in Lemma 8.6. Thus we have

$$H_{I\vee I}(-1) = \sum_t (-1)^{|I|} a^t a^t = (-1)^{|I|}\psi^2(S_I),$$

and the proof is complete. □

We end this Section by noting that generally, Theorem 8.2 provides a splitting of the polynomials $c^K_{IJ}(q)$ in elementary blocks of the type $q^a(1+q)^b$. For example, the five domino tableaux of Example 5.2 are parted into two classes of cardinality 4 and 1, yielding

$$c^{33356}_{1234,1234}(q) = q^4(1+q)^2 + q^6.$$

9. H-functions

We have demonstrated that the combinatorics of domino tableaux is strongly connected to symmetric functions and the representation theory of the general linear group. In particular we have shown that the decomposition of the square of the Schur function S_I into $S^2(S_I)$ and $\Lambda^2(S_I)$ leads to the definition of a symmetric function $H_{I\vee I}(q)$ (see proof of Theorem 5.3). More generally we set the following

Definition 9.1 Let I be a partition. Define

$$H_I(A, q) = \sum_T q^{\mathrm{Spin}(T)} a^T,$$

where the sum runs over all domino tableaux T of shape $2I$.

It is not immediate from this combinatorial definition that $H_I(A, q)$ is a symmetric function of A. This follows from the existence of the action of $\mathfrak{S}_n$ on the set of domino tableaux defined in Corollary 7.8, which permutes the evaluation and leaves invariant the shape and the spin. Moreover Theorem 4.1 and Theorem 5.3 imply

Corollary 9.2 *Let I be a partition and set $I_o = (i_1, i_3, i_5, \ldots)$, $I_e = (i_2, i_4, i_6, \ldots)$, so that (I_o, I_e) is the 2-quotient of the partition $2I$ (see Example 2.3). One has*

- $H_I(A, 0) = S_I(A)$,
- $H_I(A, 1) = S_{I_o}(A) S_{I_e}(A)$,
- $H_I(A, q) = \sum_J c^J_{I_o I_e}(q) S_J(A)$,
- $H_{I \vee I}(A, -1) = (-1)^{|I|} \psi^2(S_I(A))$,
- *The family $H_I(A, q)$ is a linear basis of the algebra of symmetric functions with coefficients in $Z[q]$. The transition matrix from the basis of Schur functions to the basis of H-functions is unitriangular, and all its entries are polynomials in q with positive integer coefficients.*

Several additional properties of H-functions will be investigated in [16], namely a "half Pieri formula", and some generating series which generalize the following classical identities of Schur and Littlewood:

$$\sum_I S_I(A) = \prod_i \frac{1}{1 - a_i} \prod_{i<j} \frac{1}{1 - a_i a_j},$$

$$\sum_I S_{I \vee I}(A) = \prod_{i<j} \frac{1}{1 - a_i a_j},$$

$$\sum_I S_{2I}(A) = \prod_{i \le j} \frac{1}{1 - a_i a_j}.$$

10. Proofs of Lemmas 8.3–8.6

Proof of Lemma 8.3: By definition of the diagonal class of T, the domino tableaux T' in this class have the same shape as T, and the same numbers at the same places. Thus these tableaux differ only by their domino configuration. In other words they can be regarded as particular domino tilings of the shape of T. It remains only to characterize those tilings. It happens that the increasing conditions on the labels of T' can be translated into some simple constraints. Namely, it is easy to check that the dominoes of the tiling corresponding to T' are not allowed to cross the barriers of length 2 placed between adjacent labels according to the rules explained in Section 8. Conversely, these constraints are enough to impose the increasing conditions on rows and columns, and therefore any such tiling corresponds to an admissible domino tableau. □

Proof of Lemma 8.4 and 8.6: Let us first assume that T has shape $H = 2I \vee 2I$, that is, a shape whose parts and multiplicities are all even. Actually, this is the only case that we need

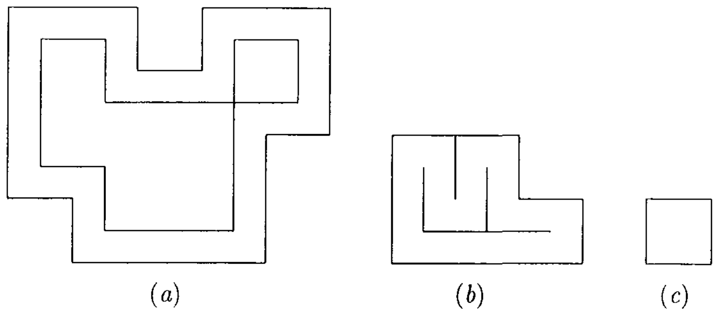
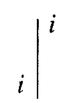


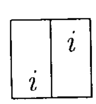
Figure 2.

for proving Theorem 5.3. Moreover, the general case results easily from this one. Let us also suppose for a moment that the labels on the ascending diagonals are strictly increasing, so that no barrier of the third type arises in the labyrinth. These assumptions imply (i) that the middle points of the barriers are all the points of the set $L = \{(x, y)/x + y \text{ is odd}\}$ lying inside the shape of T; (ii) the frontier of this shape is also formed by barriers of length 2 with middle points in the same set L. It is straightforward to see that in this case the frontiers of a connected component are made of vertical and horizontal segments of even length, the general form of a component being that of a closed chain of width 1, as shown by Fig. 2a. Note that the interior domain of the chain may well degenerate to a union of segments, as in Fig. 2b, or to a single point, as in Fig. 2c. It is clear that such a closed chain admits exactly two domino tilings.

We next relax the second assumption on T and we consider a domino tableau of shape $H = 2I \vee 2I$ with possible pairs of equal adjacent labels in ascending diagonals. Consider a barrier of the third kind placed between two such labels:



The inequalities obeyed by the neighbouring labels imply that the barrier configuration around this pair is necessarily:



In other words, each such pair of labels gives rise to a pair of connected components reduced to one domino which therefore admit only one tiling, and this already proves Lemma 8.6.

To finish the proof of Lemma 8.4 it remains to relax the assumption on the shape of T. Let then T be a domino tableau of arbitrary shape J. We can easily recover the previous situation by glueing at the periphery of T a rim of dominoes labeled by sufficiently big numbers, so as to obtain a tableau T' of shape $H = 2I \vee 2I$. This process is illustrated in Fig. 3. The connected components of the labyrinth of T' are of the two kinds described so far. When returning to T we get a new kind of connected components, by restricting those components of T' intersecting the domino rim H/J. These components take the form of an open chain and admit only one tiling. □

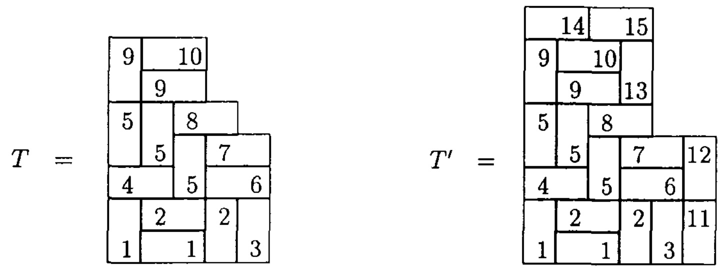


Figure 3.

Proof of Lemma 8.5: Recall from the proof of Lemma 8.4 that the outer and inner frontiers of the closed chains arising in the labyrinth are formed by vertical and horizontal segments of even length made of vertical and horizontal barriers of length 2. Consider now the two domino tilings of the chain. The main observation is the following. The vertical barriers of the outer frontier coincide with the outer borders of the vertical dominoes of one tiling, while the vertical barriers of the inner frontier (if any) coincide with the inner borders of the vertical dominoes of the other tiling. Therefore the spin difference of the chain, which is by definition half the difference between the number of vertical dominoes of the two tilings, can be expressed as

$$\Delta S = (L_o - L_i)/4, \tag{9}$$

where L_o (resp. L_i) denotes the sum of lengths of the vertical segments of the outer (resp. inner) frontier of the chain. Strictly speaking, formula (9) applies only to "generic" closed chains such as the one shown by Fig. 2a. For "singular" chains like the one in Fig. 2b, one must take into account that some segments of the frontiers are travelled up and down when one goes round the inner or outer frontiers. These segments must therefore be counted twice in the sums L_i and L_o, in order that (9) be correct. Now, with these conventions on L_o and L_i, it is an elementary geometric fact that for any closed chain, the difference $L_o - L_i$ is equal to 4. Therefore, $\Delta S = 1$ and the proof is complete. □

Acknowledgments

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