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A Note on Varieties of Groupoids Arising from m-Cycle Systems*

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Abstract. Decompositions of the complete graph with n vertices K_n into edge disjoint cycles of length m whose union is K_n are commonly called m-cycle systems. Any m-cycle system gives rise to a groupoid defined on the vertex set of K_n via a well known construction. Here, it is shown that the groupoids arising from all m-cycle systems are precisely the finite members of a variety (of groupoids) for $m = 3$ and 5 only.

Keywords: m-cycle system, variety, equationally defined, groupoid

1. Closed trail systems and groupoids

A *closed trail* in a graph G is a connected subgraph all of whose vertices have even degree. The closed m-trail (closed trail of length m) with edges $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_m, x_1\}$ is denoted by $\langle x_1, x_2, \ldots, x_m\rangle$. A *traverse* of a closed m-trail is an Euler walk in the trail. The traverse with vertex sequence $x_1, x_2, \ldots, x_m, x_1$ is denoted by the m-tuple $(x_1, x_2, \ldots, x_m)$. Note, for example, that (a, b, c, a, d, e) and (a, b, c, a, e, d) are distinct traverses of the 6-trail $\langle a, b, c, a, d, e\rangle = \langle a, b, c, a, e, d\rangle$.

A *closed trail system* of order n is a pair (S, T) where S is the vertex set of a complete graph $G \cong K_n$ ($|S| = n$) and T is a set of edge disjoint closed trails (in G) whose union is G. If the closed trails of a closed trail system are all of the same length, m say, then the system is called a *closed m-trail system.* Two obvious necessary conditions for the existence of a closed m-trail system of order n are:

(1) n is odd; and
(2) $n(n-1)/2m$ is an integer.

Closed m-trail systems are also known as *neighbour designs* and it has been shown (see [3]) that the above necessary conditions for the existence of a closed m-trail system of order n are also sufficient.

A closed m-trail with all vertices of degree 2 is an *m-cycle.* A *cycle system* (S, C) is a closed trail system in which all the closed trails are cycles. If there exists an m-cycle system of order n then, as well as the above two conditions it is also necessary that $n \geq m$ (if $n > 1$). Whether or not these three necessary conditions are also sufficient for the existence of an m-cycle system is in general an unsolved problem. For a survey of cycle systems see [5].

Given a closed trail system (S, T) and a traverse $\tau(t)$ of each trail $t \in T$, we can define a binary operation (denoted by juxtaposition) on S as follows:

(1) $x^2 = x$, for all $x \in S$; and
(2) if $x \neq y, xy = z$ and $yx = w$ if and only if $(\ldots, w, x, y, z, \ldots) \in \tau(T)$, where $\tau(T) = \{\tau(t): t \in T\}$.

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Clearly, if (S, T) is a cycle system then the binary operation is independent of the choice of the traverses. However, this is not the case in general.

It is easy to verify that any groupoid $(S, \cdot)$ obtained in the manner described above satisfies the identities $x^2 = x$, $(xy)y = x$ and the quasi-identity $xy = x \to x = y$. It is not difficult to see that the converse of this is also true. That is, any finite groupoid satisfying the above identities and quasi-identity gives rise to a closed trail system. The closed trail system (S, T) is constructed from the groupoid $(S, \cdot)$ by letting $\langle \ldots, x(yx), yx, x, y, xy, y(xy), \ldots \rangle \in T$ for each distinct $x, y \in S$. The quasi-identity $xy = x \to x = y$ is necessary to ensure that T consists entirely of closed trails. For example, if the quasi-identity is not satisfied, the (open) trail consisting of the edges $\{a, b\}$ and $\{b, c\}$ could be included in the set T; it would be denoted by $\langle a, b, c, b \rangle$. These ideas were first introduced in 1970 by Kotzig [4].

This note deals with universal algebraic properties of the groupoids corresponding to m-cycle systems. For the standard definitions and results of universal algebra which are used here see the text [2]. If the groupoids corresponding to all finite m-cycle systems are precisely the finite members of a variety then we say that m-cycle systems are *equationally defined.* In this paper we will prove the following theorem:

Theorem 1.1 *The only values of m for which m-cycle systems can be equationally defined are $m = 3$ and $m = 5$.*

For $m = 6$ and for all $m \geq 8$, it has been shown [1] that there is an m-cycle system whose corresponding groupoid has a homomorphism onto a groupoid which does not correspond to an m-cycle system. In fact, the result was proven for a special class of m-cycle systems (called *2-perfect*) whose corresponding groupoids are quasigroups. The result means that for these values of m, m-cycle systems cannot be equationally defined. Hence we need only consider the cases $m = 3, 4, 5$ and 7.

2. The $m = 3, 5$ and 7 cases

The identities $x^2 = x$, $(xy)y = x$ and $xy = yx$ equationally define 3-cycle systems. The identity $xy = yx$ ensures that the closed trails are of length 3, and hence are cycles. It also implies that the quasi-identity $xy = x \to x = y$ holds;

$$xy = x \to yx = x \to (yx)x = x^2 \to y = x.$$

The identities $x^2 = x$, $(xy)y = x$ and $x(yx) = y(xy)$ equationally define 5-cycle systems. The identity $x(yx) = y(xy)$ ensures that the closed trails are of length 5, and hence are cycles. It also implies that the quasi-identity $xy = x \to x = y$ holds;

$$xy = x \to x(yx) = yx \to (x(yx))(yx) = (yx)^2 \to x = yx \to x^2 = (yx)x \to x = y.$$

We now show that 7-cycle systems cannot be equationally defined. We construct a 7-cycle system of order 49 whose corresponding groupoid $\mathbf{G}_{49}$ has a homomorphism onto a groupoid $\mathbf{G}_7$ of order 7 which does not correspond to a 7-cycle system. Let $\mathbf{G}_7$ be the groupoid corresponding to the closed 7-trail system $(\mathbb{Z}_7, T)$ where

$$T = \{\langle 0, 1, 3, 0, 4, 5, 6 \rangle, \langle 1, 4, 6, 1, 2, 3, 5 \rangle, \langle 2, 0, 5, 2, 4, 3, 6 \rangle\}.$$

For each $(a, b, c, a, d, e, f) \in \{(0, 1, 3, 0, 4, 5, 6), (1, 4, 6, 1, 2, 3, 5), (2, 0, 5, 2, 4, 3, 6)\}$ and each $(x, y) \in \mathbb{Z}_7 \times \mathbb{Z}_7$ let

$$\langle (a, x), (b, y), (c, x + y), (a, x + 1), (d, y), (e, x), (f, y) \rangle \in C.$$

Also, for each $a \in \mathbb{Z}_7$, let

$$\begin{aligned}&\langle (a, 0), (a, 1), (a, 2), (a, 3), (a, 4), (a, 5), (a, 6) \rangle,\\ &\langle (a, 0), (a, 2), (a, 4), (a, 6), (a, 1), (a, 3), (a, 5) \rangle, \quad \text{and}\\ &\langle (a, 0), (a, 3), (a, 6), (a, 2), (a, 5), (a, 1), (a, 4) \rangle \in C.\end{aligned}$$

Then $(\mathbb{Z}_7 \times \mathbb{Z}_7, C)$ is a 7-cycle system of order 49 (with corresponding groupoid $\mathbf{G}_{49}$ say). Moreover, the map $(p, q) \mapsto p$ is a homomorphism from $\mathbf{G}_{49}$ onto $\mathbf{G}_7$. Hence, 7-cycle systems cannot be equationally defined.

3. The $m = 4$ case

It is easy to see that a groupoid corresponds to a 4-cycle system if and only if it satisfies the identities $x^2 = x$, $(xy)y = x$, $x(yx) = xy$ and the quasi-identity $xy = x \rightarrow x = y$. The extra identity $x(yx) = xy$ ensures that the closed trails have length 4 and any closed trail of length 4 is necessarily a 4-cycle. However we will show that unlike the cases $m = 3$ and $m = 5$, the quasi-identity cannot be deduced from the three identities. First we prove the following theorem:

Theorem 3.1 *Any homomorphic image of a groupoid corresponding to a finite* 4*-cycle system is the groupoid of another* 4*-cycle system.*

Proof: Suppose **G** is the groupoid corresponding to a finite 4-cycle system and suppose ϕ is a homomorphism from **G** onto **H**. We show that **H** corresponds to a 4-cycle system. Since ϕ is a homomorphism we need only check that the quasi-identity $xy = x \rightarrow x = y$ holds in **H**.

Let $a, b \in \mathbf{H}$ be distinct and suppose $ab = a$. Now, let $\phi^{-1}(a) = \{a_1, a_2, \dots, a_r\}$ and let $b^* \in \phi^{-1}(b)$. Now, for all $i \in \{1, 2, \dots, r\}$

$$\phi(a_i b^*) = \phi(a_i)\phi(b^*) = ab = a$$

That is, for all $i \in \{1, 2, \dots, r\}$, $a_i b^* \in \phi^{-1}(a)$. Since **G** corresponds to a 4-cycle system, we must have $a_i b^* = a_j$ for some $j \neq i$. Hence $|\phi^{-1}(a)|$ is even; we can pair off its elements by making $\{a_i, a_j\}$ a pair if and only if $a_i b^* = a_j$ (note that $a_i b^* = a_j \leftrightarrow (a_i b^*)b^* = a_j b^* \leftrightarrow a_i = a_j b^*$). But, if $a_i, a_j \in \phi^{-1}(a)$ then

$$\phi(a_i a_j) = \phi(a_i)\phi(a_j) = a^2 = a$$

and so $a_i a_j \in \phi^{-1}(a)$. Hence, $(\phi^{-1}(a), \cdot)$ corresponds to a 4-cycle system (a subsystem of the system corresponding to **G**) and so $|\phi^{-1}(a)|$ is odd ... a contradiction. □

We now show that 4-cycle systems cannot be equationally defined by showing that a 2-element groupoid (which clearly does not correspond to a 4-cycle system) is in the variety $\mathcal{V}$ generated by the class of groupoids corresponding to finite 4-cycle systems. Let $(\mathbb{N} \times \{a\}, C_a)$ and $(\mathbb{N} \times \{b\}, C_b)$ be infinite 4-cycle systems. Let $C_c = \{\langle (x, a), (y, b), (x+1, a), (y+1, b)\rangle \mid x, y \text{ are odd}, x, y \in \mathbb{N}\}$. Then $(\mathbb{N} \times \{a, b\}, C)$ where $C = C_a \cup C_b \cup C_c$ is an infinite 4-cycle system. Let **G** be the groupoid corresponding to $(\mathbb{N} \times \{a, b\}, C)$ and consider the map $\phi: \mathbb{N} \times \{a, b\} \to \{1, 2\}$ defined by $\phi(x, a) = 1$ and $\phi(x, b) = 2$ for all $x \in \mathbb{N}$. Clearly, ϕ is a homomorphism from **G** onto a 2-element groupoid **H**.

We now show, that **G**, and hence **H**, is in $\mathcal{V}$. If $\mathbf{G} \notin \mathcal{V}$, then there exists an identity I which holds in $\mathcal{V}$ but not in **G**. Since I does not hold in **G** there is a finite collection of 4-cycles in the 4-cycle system corresponding to **G** (that is, a finite partial 4-cycle system) which defines a finite partial groupoid in which I fails. This partial 4-cycle system can be embedded in a finite 4-cycle system, (S, D) say, see [6]. Hence, I fails in the finite groupoid corresponding to (S, D). This is a contradiction and so $\mathbf{H} \in \mathcal{V}$. Hence, 4-cycle systems cannot be equationally defined.

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References

1. D.E. Bryant and C.C. Lindner, "2-perfect m-cycle systems can be equationally defined for $m = 3, 5$, and 7 only," *Algebra Universalis* (to appear).
2. S. Burris and H.P. Sankappanavar, "A course in universal algebra," *Graduate Texts in Mathematics* **78**, Springer-Verlag, New York, 1981.
3. F.K. Hwang and S. Lin, "Neighbour designs," *J. Combin. Theory Ser. A* **23** (1977), 302–313.
4. A. Kotzig, "Groupoids and partitions of complete graphs," *Combinatorial Structures and Their Applications* (Proc. Calgary Internat. Conf., Calgary, Alta), Gordon and Breach, New York (1970), 215–221.
5. C.C. Lindner and C.A. Rodger, "Decompositions into cycles II: Cycle systems," in *Contemporary design theory: a collection of surveys* (eds. J.H. Dinitz and D.R. Stinson), John Wiley and Sons, New York (1992), 325–369.
6. C.C. Lindner, C.A. Rodger, and D.R. Stinson, "Embedding cycle systems of even length," *J. Combin. Math. Combin. Comput.* **3** (1988), 65–69.
7. R.M. Wilson, "Decompositions of edge-colored complete graphs," *Journal of Combinatorial Designs* (to appear).