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Group Weighted Matchings in Bipartite Graphs

R. AHARONI, R. MESHULAM* AND B. WAJNRYB
Department of Mathematics, Technion, Haifa 32000, Israel

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Abstract. Let G be a bipartite graph with bicoloration $\{A, B\}$, $|A| = |B|$, and let $w : E(G) \to \mathbf{K}$ where $\mathbf{K}$ is a finite abelian group with k elements. For a subset $S \subset E(G)$ let $w(S) = \prod_{e \in S} w(e)$. A perfect matching $M \subset E(G)$ is a w-*matching* if $w(M) = 1$.

A characterization is given for all w's for which every perfect matching is a w-matching.

It is shown that if $G = K_{k+1,k+1}$ then either G has no w-matchings or it has at least 2 w-matchings.

If $\mathbf{K}$ is the group of order 2 and $\deg(a) \geq d$ for all $a \in A$, then either G has no w-matchings, or G has at least $(d-1)!$ w-matchings.

Keywords: bipartite matching, Abelian group

1 Introduction

Let G be a bipartite graph with bicoloration $\{A, B\}$, $|A| = |B| = n$. Let $E(G) \subset A \times B$ denote the edge set of G, $e(G) = |E(G)|$.

Let $\mathbf{K}$ be a (multiplicative) finite abelian group $|\mathbf{K}| = k$, and let $w: E(G) \to \mathbf{K}$ be a weight assignment on the edges of G. For a subset $S \subset E(G)$ let $w(S) = \prod_{e \in S} w(e)$.

A perfect matching M of G is a w-*matching* if $w(M) = 1$. We shall consider several problems concerning w-matchings.

Let $F(G) = \mathbf{K}^{E(G)}$ denote all mappings $w: E(G) \to \mathbf{K}$. and let $M(G)$ denote all $w \in F(G)$ which satisfy $w(M) = 1$ for *all* perfect matchings M of G.

Aharoni, Manber and Wajnryb [1] obtained a concise description of $M(G)$ when $\mathbf{K} = \mathbf{C}_2$ is the group of order 2. Here we give a new proof and an extension to arbitrary abelian groups.

One simple way of obtaining elements of $M(G)$ is the following: Choose $\alpha: A \to \mathbf{K}$, $\beta: B \to \mathbf{K}$ which satisfy $\prod_{a \in A} \alpha(a) \prod_{b \in B} \beta(b) = 1$, and define $w: E(G) \to \mathbf{K}$ by $w(a, b) = \alpha(a)\beta(b)$. Clearly $w \in M(G)$.

Denote by $U(G) \subset M(G)$ the set of all w's obtained this way.

Theorem 1.1 *If every edge in G is contained in a perfect matching then* $U(G) = M(G)$.

The case $\mathbf{K} = \mathbf{C}_2$ of Theorem 1.1 was proved by Aharoni, Manber and Wajnryb [1].

Next we consider w-matchings in complete bipartite graphs.

Let $K_{k+1,k+1}$ denote the complete bipartite graph on $\{A, B\}$, $|A| = |B| = k + 1$, and let $w: E(K_{k+1,k+1}) \to \mathbf{K}$.

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Theorem 1.2 *If $K_{k+1,k+1}$ has a w-matching, then it has at least two w-matchings.*

Finally we consider the number of w-matchings in bipartite graphs.

M. Hall (see exercise 7.15 in [4]) proved that if G has a perfect matching and if $\deg(a) \geq d$ for all $a \in A$, Then G has at least $d!$ perfect matchings.

Here we show

Theorem 1.3 *Let $w: E(G) \rightarrow \mathbf{C}_2$. If G has a w-matching and* $\deg(a) \geq d$ *for all $a \in A$, then G has at least $(d-1)!$ w-matchings.*

Theorem 1.1 is proved in section 2. In section 3 we apply the group algebra of $\mathbf{K}$ to w-matchings in complete bipartite graphs. In section 4 we prove a result on $\mathbf{C}_2$-weighted digraphs which implies Theorem 1.3. A special case of Theorem 1.3 is then applied to a problem of Rinnot on random matrices. We conclude in section 5 with a conjecture which extends the results of sections 3 and 4.

2 Proof of Theorem 1.1

We may clearly assume that G is an *elementary* bipartite graph, i.e. G is connected and every edge of G is contained in a perfect matching.

By a result of Hetyei (exercise 7.7 in [4]) G satisfies

$$|\Gamma(X)| > |X| \text{ whenever } \emptyset \neq X \subseteq A \text{ or } \emptyset \neq X \subseteq B \tag{2.1}$$

where $\Gamma(X)$ denotes the neighbors of X.

Note that $U(G) \subset M(G) \subset F(G)$ are abelian groups with respect to pointwise multiplication: $w_1 w_2(e) = w_1(e) w_2(e)$.

We first prove a lower bound on $|U(G)|$.

Claim 2.2. $|U(G)| \geq k^{2n-2}$.

Proof: Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$. Denote by K^m the direct product $\mathbf{K} \times \cdots \times \mathbf{K}$ (m times). Define a homomorphism $\Phi: \mathbf{K}^{n-1} \times \mathbf{K}^{n-1} \rightarrow U(G)$ as follows: Let $u = (u_1, \ldots, u_{n-1})$, $v = (v_1, \ldots, v_{n-1}) \in \mathbf{K}^{n-1}$, and set $u_n = \prod_{i=1}^{n-1} u_i^{-1} v_i^{-1}$, $v_n = 1$.

Define $\Phi(u, v) \in M(G)$ by $\Phi(u, v)(a_i, b_j) = u_i v_j$ for $(a_i, b_j) \in E(G)$.

We show that Φ is 1-1. Suppose to the contrary that $(\mathbf{1}, \mathbf{1}) \neq (u, v) \in \ker \Phi \subset \mathbf{K}^{n-1} \times \mathbf{K}^{n-1}$. Let $X = \{a_i : u_i \neq 1\}$, $Y = \{b_j : v_j \neq 1\}$. If $|X| \leq |Y|$ then since $|Y| \leq n-1$ it follows from 2.1 that $|\Gamma(Y)| > |X|$. Therefore there exists an edge $(a_i, b_j) \in E(G)$ such that $a_i \notin X$ and $b_j \in Y$. Thus $1 = \Phi(u, v)(a_i, b_j) = u_i v_j = v_j$, a contradiction. The case $|X| > |Y|$ is similiar. Therefore Φ is 1-1 and the Claim follows. □

Denote by $\widehat{H}$ the character group of a finite abelian group H. For a subgroup $\Lambda \subset \widehat{H}$ let $\Lambda^{\perp} = \{h \in H: \chi(h) = 1 \text{ for all } \chi \in \Lambda\}$. $\Lambda^{\perp}$ is a subgroup of H and $|\Lambda||\Lambda^{\perp}| = |H|$.

For each $\chi \in \widehat{\mathbf{K}}$ and a perfect matching M of G, let $c(M, \chi) \in \widehat{F(G)}$ be defined by $c(M, \chi)(w) = \chi(w(M))$. Let $P(G) \subset \widehat{F(G)}$ be the subgroup generated by all the $c(M, \chi)$'s. Clearly $P(G)^{\perp} = M(G)$. We now prove a lower bound on $|P(G)|$.

Claim 2.3. $|P(G)| \geq k^{e(G)-2(n-1)}$.

Proof: We argue by induction on $e(G)$. If $e(G) = 1$ then $n = 1$ and $P(G) \cong \widehat{\mathbf{K}}$. Suppose $e(G) > 1$. By a theorem of Hetyei on the structure of elementary bipartite graphs (exercise 7.8 in [4]) G decomposes as $G = G' \cup C$, where G' is again elementary, and C is an odd path joining $x \in V(G') \cap A$ and $y \in V(G') \cap B$ such that $V(C) \cap V(G') = \{x, y\}$.

To simplify notation assume that for some $1 \leq m \leq n$.

$V(G') = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$, $V(C) = \{a_m, \ldots, a_n\} \cup \{b_m, \ldots, b_n\}$ and $E(C) = \{(a_i, b_{i-1})\}_{i=m+1}^{n} \cup \{(a_i, b_i)\}_{i=m+1}^{n} \cup \{(a_m, b_n)\}$

We also choose a (fixed) perfect matching $\overline{M}$ of G which contains the edge (a_m, b_n).

Every perfect matching M' of G' can be extended to a perfect matching $\epsilon(M') = M$ by $M = M' \cup \{(a_i, b_i)\}_{i=m+1}^{n}$.

Define $h: P(G') \times \widehat{\mathbf{K}} \to P(G)$ as follows: Let $\varphi = \prod_{i=1}^{t} c(M_i', \chi_i) \in P(G')$ where $\chi_i \in \widehat{K}$ and the M_i''s are perfect matchings of G'. Define $h(\varphi, \chi) = \prod_{i=1}^{t} c(\epsilon(M_i'), \chi_i) c(\overline{M}, \chi)$.

We check that h is 1-1. Suppose $(\psi, \eta) = (\prod_{j=1}^{t} c(N_j', \eta_j), \eta) \in P(G') \times \widehat{\mathbf{K}}$ where $\eta_j \in \widehat{\mathbf{K}}$ and the N_j' 's are perfect matchings of G'.

If $\chi \neq \eta$ then $\chi(z) \neq \eta(z)$ for some $z \in K$. Define $w \in F(G)$ by $w(e) = z$ if $e = (a_m, b_n)$ and $w(e) = 1$ otherwise. Clearly $h(\varphi, \chi)(w) = \chi(z) \neq \eta(z) = h(\psi, \eta)(w)$. If on the other hand $\chi = \eta$, then $\varphi \neq \psi$ and so $\varphi(w') \neq \psi(w')$ for some $w' \in F(G')$. Defining $w \in F(G)$ by $w(e) = w'(e)$ for $e \in E(G')$ and $w(e) = 1$ otherwise, we obtain $h(\varphi, \chi)(w) = \varphi(w')\chi(\omega(\overline{M})) \neq \psi(w')\chi(\omega(\overline{M})) = h(\psi, \eta)(w)$.

The injectivity of h together with the induction hypothesis imply:

$$|P(G)| \geq |P(G')| \cdot |\widehat{\mathbf{K}}| \geq k^{e(G')-2(m-1)+1} = k^{e(G)-2(n-1)} \qquad \square$$

Claims 2.2 and 2.3 imply

$$k^{2n-2} \leq |U(G)| \leq |M(G)| = |P(G)^{\perp}| = (\widehat{F(G)} : P(G)) = k^{e(G)}/|P(G)| \leq k^{2n-2}$$

Therefore $U(G) = M(G)$. $\square$

3 w-matchings in complete bipartite graphs

Let $M_m(S)$ denote all $m \times m$ matrices with entries in S.

For $Q = (q_{ij}) \in M_m(\mathbf{K})$ and a permutation $\sigma \in S_m$, let $Q(\sigma) = \prod_{i=1}^{m} a_{i\sigma(i)}$. For $x \in \mathbf{K}$ let $S(Q, x) = \{\sigma \in S_m : Q(\sigma) = x\}$.

Let $t = t(\mathbf{K})$ denote the minimal t such that for any $Q \in M_t(\mathbf{K})$, either $S(Q, 1) = \emptyset$ or $|S(Q, 1)| \geq 2$.

A mapping $w: E(K_{m,m}) \to \mathbf{K}$ naturally corresponds to a matrix $Q \in M_m(\mathbf{K})$. We prove the following matrix version of Theorem 1.2.

Theorem 3.1 $t(\mathbf{K}) \leq k + 1$.

Proof: Let $Q = (q_{ij}) \in M_{k+1}(\mathbf{K})$. Denote by $\mathbf{C}[\mathbf{K}]$ the complex group algebra of $\mathbf{K}$ and let $\widehat{\mathbf{K}} = \{\chi_1, \ldots, \chi_k\}$.

Define $(\lambda_{ij}) \in M_{k+1}(\mathbf{C}^*)$ by $\lambda_{1j} = 1$ and $\lambda_{ij} = \chi_{i-1}(q_{1j}q_{ij}^{-1})$ for all $2 \le i \le k+1$, $1 \le j \le k+1$. Let $R = (r_{ij}) \in M_{k+1}(\mathbf{C}[\mathbf{K}])$ be defined by $r_{ij} = \lambda_{ij}q_{ij}$. Note that $\det R \in \mathbf{C}[\mathbf{K}]$.

Claim 3.2. $\det R = 0$.

Proof: Let $1 \le l \le k$ and consider the matrix $\chi_l(R) = (\chi_l(r_{ij})) \in M_{k+1}(\mathbf{C})$.

Clearly $\chi_l(r_{1j}) = \chi_l(r_{l+1,j})$ for all $1 \le j \le k+1$, therefore $\chi_l(R)$ is singular and $\chi_l(\det R) = \det(\chi_l(R)) = 0$. Since this holds for all $1 \le l \le k$ it follows that $\det R = 0$. □

Therefore

$$0 = \det R = \sum_{x \in \mathbf{K}} \left(\sum_{\sigma \in S(Q,x)} Sg(\sigma) \prod_{i=1}^{k+1} \lambda_{i\sigma(i)} \right) x$$

So that for each $x \in \mathbf{K}$ either $S(Q, x) = \emptyset$ or $|S(Q, x)| \ge 2$. □

A lower bound on $t(\mathbf{K})$ may be obtained as follows: Let $s = s(\mathbf{K})$ denote the maximal s for which there exists a sequence $x_1, \dots, x_s \in \mathbf{K}$ such that $\prod_{i \in I} x_i \ne 1$ for all $\emptyset \ne I \subset \{1, \dots, s\}$.

Define $Q = (q_{ij}) \in M_{s+1}(\mathbf{K})$ by $q_{ij} = 1$ if $i = j$ or $i = s+1$, and $q_{ij} = x_i$ otherwise. Clearly $S(Q, 1)$ contains only the identity permutation, so $t(\mathbf{K}) \ge s(\mathbf{K}) + 2$. Note that for the cyclic group $\mathbf{K} = \mathbf{C}_k$ this lower bound is tight by Theorem 3.1.

$s(\mathbf{K})$ was studied by a number of authors ([6], [3], [2], [5]). We shall need the following result of Olson. Let $\mathbf{Z}_p[\mathbf{K}]$ denote the group algebra of $\mathbf{K}$ with coefficients in $\mathbf{Z}_p$.

Theorem (Olson [6]) *Let* $\mathbf{K}$ *be an abelian p-group* $\mathbf{K} = \mathbf{C}_{p^{e_1}} \times \cdots \times \mathbf{C}_{p^{e_l}}$. *Then* $s = s(\mathbf{K}) = \sum_{i=1}^{l}(p^{e_i} - 1)$ *and for every* $x_1, \dots, x_{s+1} \in \mathbf{K}$, $\prod_{i=1}^{s+1}(x_i - 1) = 0$ *in* $\mathbf{Z}_p[\mathbf{K}]$. □

We now show

Theorem 3.3 *If* $\mathbf{K}$ *is an abelian p-group, then* $t(\mathbf{K}) = s(\mathbf{K}) + 2$.

Proof: Let $s = s(\mathbf{K})$ and let $Q = (q_{ij}) \in M_{s+2}(\mathbf{K})$. As in Theorem 3.1 it suffices to show that $\det Q = 0$ in $\mathbf{Z}_p[\mathbf{K}]$. Multiplying rows and columns by appropriate elements of $\mathbf{K}$ we may assume that $q_{1i} = q_{i1} = 1$ for all $1 \le i \le s+2$. Subtracting the first row from the others, we obtain:

$$\det Q = \sum_{\sigma} Sg(\sigma) \prod_{i=2}^{s+2} (q_{i\sigma(i)} - 1)$$

where σ ranges over all permutations of $2, \dots, s+2$. By Olson's Theorem all products on the right vanish and so $\det Q = 0$. □

In section 4 we shall need a version of Theorem 3.1 for directed graphs. Let $\vec{K}_{k+1}$ denote the complete directed graph on $V = \{1, \dots, k+1\}$, $E(\vec{K}_{k+1}) = \{(i, j): 1 \le i \ne j \le k+1\}$. For $w: E(\vec{K}_{k+1}) \to \mathbf{K}$ and $S \subset E(\vec{K}_{k+1})$ let $w(S) = \prod_{e \in S} w(e)$.

Corollary 3.4 *For any* $w: E(\vec{K}_{k+1}) \to \mathbf{K}$ *there exist vertex disjoint directed cycles* $C_1, \ldots, C_l$ *in* $\vec{K}_{k+1}$ *such that* $\prod_{i=1}^{l} w(C_i) = 1$.

Proof: Define $Q = (q_{ij}) \in M_{k+1}(\mathbf{K})$ by $q_{ii} = 1$ and $q_{ij} = w(i, j)$ for $1 \leq i \neq j \leq k+1$. Since the identity permutation belong to $S(Q, 1)$, it follows from Theorem 3.1 that there exists a $1 \neq \sigma \in S(Q, 1)$.

$V_0 = \{i: \sigma(i) \neq i\}$ clearly decomposes into vertex disjoint directed cycles $C_1, \ldots, C_l$ such that $\prod_{i=1}^{l} w(C_i) = \prod_{j=1}^{n} q_{j\sigma(j)} = 1$. □

4 On the number of w-matchings

Let $D = (V, E)$ be a directed graph, possibly with loops but with no multiple edges in the same direction.

The proof of Theorem 1.3 depends on the following result which combines an idea of Thomassen [8] with Corollary 3.4.

Proposition 4.1 *Let* $D = (V, E)$ *be a digraph (as above), and let* $w: E \to \mathbf{C}_2$. *If* $\deg^+(v) = 2$ *for all* $v \in V$, *then there exist vertex disjoint directed cycles* $C_1, \ldots, C_l$ *such that* $\prod_{i=1}^{l} w(C_i) = 1$.

Proof: Let D be a minimal counterexample. If C_1, C_2 are two vertex disjoint directed cycles then either $w(C_i) = 1$ for some i, or $w(C_1)w(C_2) = 1$. It follows that any two dicycles intersect. If D has a loop $C_1 = (v, v)$ then $D - v$ has a directed cycle C_2, thus D is loopless.

Suppose there is an edge $(x, y) \in E$ such that for no $v \in V$ both (v, x) and (v, y) are edges. We form a new digraph $D' = (V', E')$ on $V' = V - x$ by deleting x and all edges incident with it, and replacing each edge of the form $(v, x) \in E$ by a new edge $(v, y) \in E'$. Note that $\deg^+(v') = 2$ for all $v' \in V'$. Define $w': E' \to \mathbf{C}_2$ by $w'(e') = w(e')$ for $e' \in E$, and $w'(v, y) = w(v, x)w(x, y)$ if $(v, x) \in E$.

With each directed cycle C' in D' we associate a directed cycle C in D. If C' contains a new edge $(v, y) \in E'$ (where $(v, x) \in E$), let

$C = C' - (v, y) + (v, x) + (x, y)$. Otherwise $C = C'$. Clearly $w(C) = w'(C')$ and $V(C_1') \bigcap V(C_2') = \emptyset$ implies $V(C_1) \bigcap V(C_2) = \emptyset$. Therefore if D' satisfies the conclusions of the Theorem, so does D—in contradiction with the minimality assumption.

Therefore for every $(x, y) \in E$ there exists a vertex $z \neq x, y$ such that $(z, x), (z, y) \in E$. It follows that each $v \in V$ is dominated by a directed cycle, and in particular $\deg^-(v) \geq 2$. Since $\deg^+(v) = 2$ for all v, it follows that there exists a v such that $\deg^-(v) = 2$. Thus there is a cycle $C_1 = \{(x, y), (y, x)\}$ such that $(x, v), (y, v) \in E$.

Let C_2 be a cycle which dominates x. Clearly $y \in V(C_2)$ for otherwise C_1 and C_2 are vertex disjoint. Therefore $v \in V(C_2)$ too, and so $(v, x) \in E$. Similarly we conclude that $(v, y) \in E$.

Therefore the complete directed graph on $\{x, y, v\}$ is contained in D, in contradiction with Corollary 3.4 (for the group $\mathbf{K} = \mathbf{C}_2$). □

Returning to the number of w-matchings, let G be a bipartite graph on $\{A, B\}$, $|A| = |B| = n$ and $w: E(G) \to \mathbf{C}_2$. For $a \in A$ let $U_G(a, w)$ denote the set of all edges incident with a which participate in a w-matching of G, $|U_G(a, w)| = u_G(a, w)$.

The following result clearly implies Theorem 1.3 by induction on d.

Theorem 4.2 *If G has a w-matching then there exists an $a \in A$ such that $u_G(a, w) \geq \deg_G(a) - 1$.*

Proof: We argue by induction on $e(G)$. Let $\delta(G) = \min\{\deg_G(a): a \in A\}$. The assertion is clear if $\delta(G) \leq 2$, so we assume $\delta(G) \geq 3$.

Suppose there exists an $a \in A$ with $\deg_G(a) \geq 4$ and distinguish two cases:

a) $U_G(a, w) = \{e\}$. Choose $e' \neq e$ incident with a and let $G' = G - e'$. By induction there exists an $a' \in A$ such that $u_{G'}(a', w) \geq \deg_{G'}(a') - 1$. Since $u_G(a, w) = 1$ and $\deg_{G'}(a) \geq 3$, it follows that $a' \neq a$ and so $u_G(a', w) = u_{G'}(a', w) \geq \deg_G(a') - 1$.

b) $U_G(a, w) \supset \{e, e'\}$. Again let $G' = G - e'$ and choose by induction an $a' \in A$ such that $u_{G'}(a', w) \geq \deg_{G'}(a') - 1$. If $a' \neq a$ we are done as before. Otherwise $a' = a$ and so $U_G(a, w) = U_{G'}(a, w) \bigcup \{e'\}$. Therefore

$$u_G(a, w) = u_{G'}(a, w) + 1 \geq (\deg_{G'}(a) - 1) + 1 = \deg_G(a) - 1.$$

We thus remain with the case $\deg(a) = 3$ for all $a \in A$.

Let $M = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ be a w-matching of G. With no loss of generality we may assume that $w(a_i, b_i) = 1$ for all i. Construct a directed graph D on $\{1, \ldots, n\}$ by $(i, j) \in E(D)$ iff $i \neq j$ and $(a_i, b_j) \in E(G)$, and let $\varphi: E(D) \to \mathbf{C}_2$ be defined by $\varphi(i, j) = w(a_i, b_j)$. Since $\deg^+(v) = 2$ for all $v \in V(D)$, it follows from Proposition 4.1 that there exist vertex disjoint cycles $C_1, \ldots, C_l$ such that $\prod_{i=1}^{l} w(C_i) = 1$. Let $V_0 = \bigcup_{i=1}^{l} V(C_i)$ and define a permutation σ on V_0 by $\sigma(v_1) = v_2$ if $(v_1, v_2) \in \bigcup_{i=1}^{l} E(C_i)$. Consider the perfect matching

$$M' = \{(a_i, b_i): i \notin V_0\} \bigcup \{(a_i, b_{\sigma(i)}): i \in V_0\}.$$

Clearly $M' \neq M$ and $w(M') = \prod_{i=1}^{l} \varphi(C_i) = 1$. □

Applying Theorem 1.3 to the complete bipartite graph $K_{n,n}$ we obtain

Corollary 4.3 *Let $Q = (q_{ij}) \in M_n(\mathbf{C}_2)$. Then either $S(Q, 1) = \emptyset$ or $|S(Q, 1)| \geq (n-1)!$.*

We conclude this section with an application of Corollary 4.3.

Let $X = (X_{ij})$ be an $n \times n$ matrix of independent random variables X_{ij} such that $\Pr(X_{ij} = 1) = \Pr(X_{ij} = -1) = 1/2$. For $\sigma \in S_n$, define a random variable $X(\sigma) = \prod_{i=1}^{n} X_{i\sigma(i)}$ and let id be the identity permutation in S_n.

Denote by $f(n)$ the maximal cardinality of a family of permutations $S \subset S_n$ such that $X(\text{id})$ is independent of $\{X(\sigma): \sigma \in S\}$. Y. Rinnot [7] noted that $S = \{\sigma \in S_n: \sigma(1) \neq 1\}$ satisfies this independence condition and thus $f(n) \geq |S| = n! - (n-1)!$. Here we show that Rinnot's construction is optimal:

Theorem 4.4 *If $X(\text{id})$ is independent of $\{X(\sigma): \sigma \in S\}$, then $|S| \leq n! - (n-1)!$.*

Proof: The events $A = \{X(\sigma) = -1 \textit{ for all } \sigma \in S\}$ and $B = \{X(\text{id}) = 1\}$ are clearly independent and both have positive probability, therefore $\Pr(A \bigcap B) = \Pr(A)\Pr(B) > 0$.

Hence there exists a matrix $Q \in M_n(\pm 1)$ such that $Q(\sigma) = -1$ for all $\sigma \in S$ and $Q(\mathrm{id}) = 1$. Therefore $S(Q, 1) \bigcap S = \emptyset$ and $S(Q, 1) \neq \emptyset$, so by Corollary 4.3

$$|S| \leq n! - |S(Q, 1)| \leq n! - (n-1)!\,.$$

□

5 Concluding remarks

Our results seem to suggest the following extension of Theorem 1.3.

Conjecture 5.1 *Let G be a bipartite graph on $\{A, B\}$, $|A| = |B|$, and let $w\colon E(G) \to \mathbf{K}$. If G has a w-matching and* $\deg(a) \geq d$ *for all $a \in A$, then G has at least* $(d - s(\mathbf{K}))!$ *w-matchings.*

The proof of Theorem 4.2 may be modified to show that Conjecture 5.1 is equivalent to

Conjecture 5.2 *Let $D = (V, E)$ be a simple digraph, and let $w\colon E \to \mathbf{K}$. If* $\deg^+(v) = s(\mathbf{K}) + 1$ *for all $v \in V$, Then there exist vertex disjoint directed cycles $C_1, \ldots, C_l$ such that* $\prod_{i=1}^{l} w(C_i) = 1$.

Remarks

a) *The lower bound $t(\mathbf{K}) \geq s(\mathbf{K}) + 2$ shows that the conjectures do not hold if $s(\mathbf{K})$ is replaced by a smaller constant.*

b) *Both conjectures hold when $s(\mathbf{K})$ is replaced by another (much larger) constant $c(\mathbf{K})$.*

Added on June 1, 1993: J. Kahn and R. Meshulam proved that both conjectures hold when $s(\mathbf{K})$ is replaced by $|\mathbf{K}| - 1$. In particular the conjectures are valid for cyclic $\mathbf{K}$. Details will appear elsewhere.

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