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On Spin Models, Triply Regular Association Schemes, and Duality

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Abstract. Motivated by the construction of invariants of links in 3-space, we study spin models on graphs for which all edge weights (considered as matrices) belong to the Bose-Mesner algebra of some association scheme. We show that for series-parallel graphs the computation of the partition function can be performed by using series-parallel reductions of the graph appropriately coupled with operations in the Bose-Mesner algebra. Then we extend this approach to all plane graphs by introducing star-triangle transformations and restricting our attention to a special class of Bose-Mesner algebras which we call exactly triply regular. We also introduce the following two properties for Bose-Mesner algebras. The planar duality property (defined in the self-dual case) expresses the partition function for any plane graph in terms of the partition function for its dual graph, and the planar reversibility property asserts that the partition function for any plane graph is equal to the partition function for the oppositely oriented graph. Both properties hold for any Bose-Mesner algebra if one considers only series-parallel graphs instead of arbitrary plane graphs. We relate these notions to spin models for link invariants, and among other results we show that the Abelian group Bose-Mesner algebras have the planar duality property and that for self-dual Bose-Mesner algebras, planar duality implies planar reversibility. We also prove that for exactly triply regular Bose-Mesner algebras, to check one of the above properties it is sufficient to check it on the complete graph on four vertices. A number of applications, examples and open problems are discussed.

Keywords: association scheme, spin model, link invariant, star-triangle transformation

1 Introduction

A spin model is defined on a directed graph G by assigning to each edge e a square matrix $w(e)$ whose rows and columns are indexed by a given finite set X. Let c: $V(G) \to X$ be an arbitrary coloring of the vertices of G with elements of X. Then with each edge e from v to v' is associated the $(c(v), c(v'))$ entry of $w(e)$. The product over all edges of these numbers is called the weight of the coloring c, and the sum of weights of all colorings is called the partition function.

These concepts are of fundamental importance in statistical mechanics (see for instance [12], [14], [46]). They are also of great interest in graph theory. In particular Tutte's dichromatic polynomial [52] is essentially equivalent to the partition function of the Potts model of statistical mechanics. More recently V. F. R. Jones [33] showed how to use spin models to define invariants of links in 3-space, such as the famous polynomial invariant previously discovered by him in a completely different setting [35]. Each link can be represented by a plane graph with signed edges, and V. F. R. Jones defines on this signed graph a spin model for which the matrix associated with any edge is chosen according to sign among two given matrices W^+, W^-. Then he gives a set of equations which, when satisfied by W^+, W^-, guarantee that the partition function (after an adequate normalization) is a link invariant. The basic tool here is the fact that the natural topological equivalence of links is represented in terms of signed plane graphs as the equivalence generated by certain simple graph transformations (among which a signed version of the ubiquitous star-triangle transformation). To each type of graph transformation corresponds an equation and in particular to the star-triangle transformation corresponds the so-called star-triangle equation. Only symmetric matrices were considered in [33], but the extension to arbitrary matrices has been introduced in [38] and we shall deal here with this more general concept which involves the representation of oriented links by directed signed plane graphs (see Proposition 1). The study of spin models which give rise to link invariants, which we call topological spin models, is the main motivation for the present paper, and Section 2 is devoted to a detailed presentation of this topic.

The matrices defining topological spin models which have been found so far belong to the Bose-Mesner algebra of some association scheme (see [33], [34], [25], [31], [26], [5], [3], [42], [39], [9]). Such algebras can be characterized as commutative algebras of complex square matrices which are closed under transposition and Hadamard product and contain the identity matrix and the all-one matrix. As already shown in [31], Bose-Mesner algebras (called here BM-algebras) provide a convenient and natural framework for the construction of topological spin models. We introduce this framework in Section 3, first recalling some basic notions on association schemes and BM-algebras. In particular we present the concepts of duality map and self-duality which will play a prominent rôle in the sequel. The reason for this is described in Proposition 2, which asserts that if the matrices W^+, W^- defining a topological spin model generate a BM-algebra in a certain sense, then this BM-algebra is self-dual with duality map exchanging (up to a multiplicative factor and transposition) W^+ and W^- (see [4], [6], [31]). When this last situation occurs we say that the topological spin model fully belongs to the self-dual BM-algebra. We show how in this case all matrix equations of Proposition 1 except the star-triangle equation reduce to one system of $d+1$ quadratic equations in $d+1$ unknowns, where $d+1$ is the dimension of the BM-algebra.

In the sequel of the paper we consider only spin models defined on (directed) graphs in such a way that all matrices assigned to edges belong to a given BM-algebra. Then, as explained in Section 4, if a graph contains a loop, a pendant edge, two edges in series or two edges in parallel, one can easily compute the partition function on a reduced graph for which the assignment of matrices to edges has been modified in an appropriate way. In particular if a graph is series-parallel the partition function can be computed by iterating this process. Such a computation, which we call series-parallel evaluation, only uses the abstract properties of the BM-algebra and not its actual representation by matrices. We introduce a convenient formalism to describe series-parallel evaluation. In this formalism, the partition function is viewed as a linear form on a tensor product of copies of the BM-algebra, with one copy for each edge of the graph. This seems to be the most natural way to incorporate into a single object all spin models defined on the graph by assigning an element of the BM-algebra to each edge. Also a significant advantage of this approach is that we can state our results more abstractly, independently of the various explicit forms which derive from different choices of a basis for the BM-algebra (even if some proofs will rely on such a choice). For instance, the concept of series-parallel evaluation is described in Proposition 3 by expressing the partition function for a series-parallel graph as a composition of linear maps. As an illustration of the usefulness of series-parallel evaluation and to introduce other results appearing in subsequent sections, we show in Proposition 4 that the partition function for a series-parallel graph is not modified if one reverses simultaneously the orientations of all edges. One interesting consequence is that a link invariant associated with a topological spin model whose matrices W^+, W^- belong to a BM-algebra will never be able to detect the reversal of orientations of all components of a link if this link can be represented by a series-parallel signed graph (this is the case for the examples of noninvertible knots given in [49]).

What we would like to do now is to extend the concept of series-parallel evaluation to all plane graphs. This will be possible if we consider only certain BM-algebras which we call exactly triply regular. The evaluation process will rely on Epifanov's Theorem [22] (see also [24], [50], [23]) which asserts that every connected plane graph can be reduced to a trivial graph by a finite number of star-triangle transformations and series-parallel reductions (see Proposition 5). Then we shall be able to define such an evaluation process (which we call star-triangle evaluation) if, given any two plane graphs related by a star-triangle transformation, the partition function for one graph can be evaluated from the partition function for the other by composing it with a certain linear map (this map relates two tensor products of three copies of the BM-algebra, one being associated with the star edges and the other with the triangle edges). In Section 5 we define exactly triply regular BM-algebras in such a way that the necessary linear maps exist and this allows the star-triangle evaluation process described in Proposition 6. Actually the concept of exactly triply regular BM-algebra is closely related to the combinatorial concept of triply regular association scheme (explored for distance-regular graphs in [48]). Informally speaking, an association scheme on X (or the corresponding BM-algebra) is triply regular if for any triple (x, y, z) of elements of X the number of elements of X satisfying given scheme relations with x, y, z only depends on the mutual relations between x, y, z. We give an algebraic formulation of this property and introduce in a natural way the dual property (see Propositions 7 and 8). We can then define an exactly triply regular BM-algebra as a BM-algebra which is both triply regular and dually triply regular. As a consequence, self-dual triply regular BM-algebras are exactly triply regular (Proposition 9) and it follows that the star-triangle evaluation process applies to many known topological spin models. We conclude Section 5 with a simple form of the star-triangle equation for the case of topological spin models which fully belong to a self-dual triply regular BM-algebra.

In Section 6 we introduce and study a property of self-dual BM-algebras which we call planar duality. This property asserts that the partition function for any connected plane graph can be computed (up to a suitable factor) by replacing the graph by its dual, replacing the matrix associated with each edge by its image under the duality map, and then evaluating the corresponding partition function. We first show, using series-parallel evaluation, that this holds for any self-dual BM-algebra, provided we restrict our attention to series-parallel graphs (Proposition 10). Then, extending an idea due to N. L. Biggs [16], we prove that any Abelian group self-dual BM-algebra has the planar duality property (Proposition 11). In the case of a plane graph to the edges of which are assigned matrices W^+ or W^- defining a topological spin model which fully belongs to the given self-dual BM-algebra, the planar duality property simply expresses the fact that the link invariant associated with the topological spin model can be equivalently computed on two dual signed graphs representing the link (see [33], Proposition 2.14). We use this fact to show that if the matrices W^+, W^- generate the BM-algebra in a certain strong sense (this property is in general stronger than the generation property considered in Proposition 2 but is equivalent to it in the symmetric case), this BM-algebra satisfies the planar duality property (Proposition 12). Then, aiming at possible extensions of Proposition 4 and its consequences for link invariants, we introduce the planar reversibility property for BM-algebras, which asserts that the partition function for any plane graph is not modified if one reverses the orientations of all edges. Applying an idea of [38] we show that in particular the BM-algebras of group schemes have this property (Proposition 13). We also prove in Proposition 14 that for a self-dual BM-algebra the planar duality property implies the planar reversibility property. It is natural to consider a weaker version of each of these properties where we restrict our attention to only one plane graph, the complete graph on 4 vertices (which is the unique minor-minimal non-series-parallel graph). We call K_4 duality and K_4 reversibility these weaker versions. We obtain algebraic formulations of these properties (Propositions 15 and 16) and show in Proposition 17 that they always hold for BM-algebras of dimension at most 3. Finally in Propositions 18 and 19 we show that for a self-dual triply regular BM-algebra, planar duality or planar reversibility are actually equivalent to K_4 duality or K_4 reversibility respectively.

The above results are illustrated in Section 7 on four examples. We examine briefly the case when the BM-algebra has dimension 2, which leads to the well known topological spin models for the Jones polynomial. In Proposition 20, we give our version of a short proof by Munemasa [40] of the fact that every BM-algebra of dimension 3 generated by a topological spin model (in the sense of Proposition 2) is exactly triply regular (in the symmetric case this also follows from the results of [31]). We also show in Proposition 21 that there is no non-symmetric triply regular BM-algebra of dimension 3 on a set of at least 4 elements (a reformulation of a result by Herzog and Reid [29]), thus obtaining another proof of the result by Ikuta [30] that these algebras are not generated by topological spin models. In Proposition 22, we use star-triangle evaluation to show that the link invariant associated with a certain topological spin model constructed by K. Nomura [42] in the BM-algebra of a given Hadamard graph only depends on the order of this Hadamard graph. This is the first step in the proof of an expression for this invariant in terms of the Jones polynomial of a link and its sublinks [32]. For our last example we give another proof of the planar duality property for Abelian group self-dual BM-algebras and we present in Proposition 23 a general family of topological spin models which fully belong to such a BM-algebra. They contain the models of [3] as special cases and can be identified with the models of [39] (see [6]).

Finally in Section 8 we conclude this paper with a discussion of our results and some open problems concerning the various properties of BM-algebras which we have introduced.

2 Spin models for graphs and links

All graphs will be finite, and loops and multiple edges will be allowed. Graphs will be directed, unless otherwise specified. The vertex-set and edge-set of the graph G will be denoted by $V(G)$ and $E(G)$ respectively. The initial (respectively: terminal) end of an edge e of G will be denoted by $i(e)$ (respectively: $t(e)$). For every finite non-empty set X, $\mathcal{M}(X)$ will denote the set of square matrices with complex entries and with rows and columns indexed by X. The entry of the matrix A with row index x and column index y will be denoted by $A[x, y]$. Recall that the Hadamard product of two matrices A, B in $\mathcal{M}(X)$ is denoted by $A \circ B$ and given by $(A \circ B)[x, y] = A[x, y]B[x, y]$ for all x, y in X. The transpose of a matrix A will be denoted by A^T and the ordinary product of two matrices A, B will be denoted by AB. We shall denote by I the identity matrix and by J the all-one matrix (i.e. the identity for the Hadamard product).

Let G be a graph and w be a mapping from $E(G)$ to $\mathcal{M}(X)$ whose values will be called *edge weights*. Let us call *state of G* any mapping σ from $V(G)$ to X. We define the *weight* $w(e \mid \sigma)$ *of the edge e with respect to the state* σ as $w(e)[\sigma(i(e)), \sigma(t(e))]$. The *weight of the state* σ is then $w(\sigma) = \prod_{e \in E(G)} w(e \mid \sigma)$ (this will be set to 1 if $E(G)$ is empty). Let $Z(G, w) = \sum_{\sigma: V(G) \to X} w(\sigma)$ be the sum of weights of states. $Z(G, w)$ is the *partition function of the spin model defined on the graph G by the system of weights w*. This concept plays an important rôle in statistical mechanics (see [12], [14], [46]) and leads to interesting invariants of graphs (see [52], [28], [27]). The main motivation of the present paper is the application of spin models to the construction of invariants of links, as initiated by V.F.R. Jones in [33].

A *link* consists of a finite collection of disjoint simple closed smooth curves (the *components* of the link) in 3-space. If each component has received an orientation, the link is said to be *oriented.* (Oriented) links can be represented by *(oriented) diagrams*. A diagram of a link is a generic plane projection (there is only a finite number of multiple points, each of which is a simple crossing), together with an indication at each crossing of which part of the link goes over the other, and, for oriented links, with some arrows which specify the orientations of the components (some examples are displayed in Figure 1). The *Tait number (or writhe)* $T(L)$ of an oriented diagram L is the sum of signs of its crossings, where the sign of a crossing is defined on Figure 2.

Two links are *ambient isotopic* if there exists an isotopy of the ambient 3-space which carries one onto the other (for oriented links, this isotopy must preserve the orientations). This natural equivalence of links is described at the diagram level by Reidemeister's Theorem, which asserts that two diagrams represent ambient isotopic links if and only if one can be obtained from the other by a finite sequence of elementary local diagram transformations, the Reidemeister moves. These moves belong to three basic types described for the unoriented case in Figure 3 (for the oriented case all possible local orientations of these configurations must be considered). A move is performed by replacing a part of diagram which is one of the configurations of Figure 3 by an equivalent configuration without altering the remaining part of the diagram. More details can be found for instance in [11] or [36].

Reidemeister's Theorem allows the definition of a *link invariant* as a valuation of diagrams which is invariant under Reidemeister moves. As shown in [33], one may use spin models to define such valuations. The construction of [33] was restricted to spin models using only symmetric matrices. We present now an extension of this construction suggested by V.F.R. Jones and due to K. Kawagoe, A. Munemasa and Y. Watatani [38]. An even more general construction was introduced recently by E. Bannai and E. Bannai [2] and some of the results to follow should be relevant to this generalization as well.

With every connected unoriented diagram L we associate an undirected plane graph $G(L)$ as follows (see for instance [11]). The regions of the plane delimited by the diagram are colored with two colors, black and white, in such a way that adjacent regions receive different colors and the infinite region is colored white. Then $G(L)$ has one vertex r° for each black region r, and one edge c° for each crossing c. The vertex r° is placed inside r. If the crossing c is incident to the (possibly identical) black regions r_1, r_2, the edge c° has ends r_1°, r_2° and is embedded as a simple curve joining these ends through c. This will be done in such a way as to obtain a plane embedding of $G(L)$.

Now each edge c° of $G(L)$ will receive a *sign* $s(c^\circ) \in \{+, -\}$ which is defined by the color of the regions first swept by the upper part of the link near the crossing c when

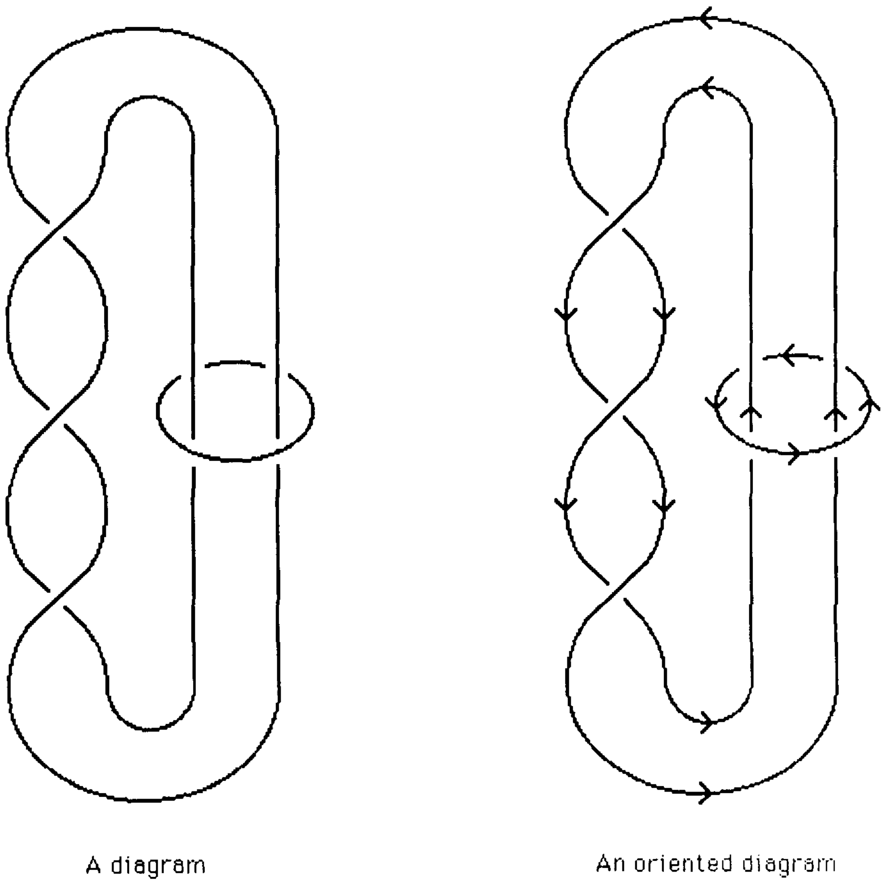

Figure 1.

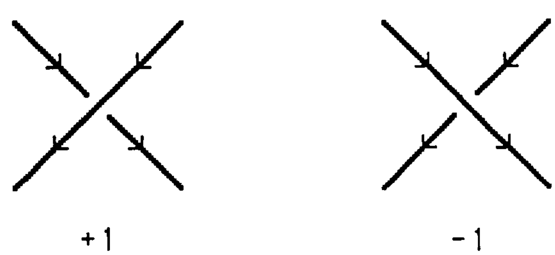

Figure 2.

slightly rotated clockwise around c (see Figure 4). Note that the sign of an edge must not be confused with the sign of the corresponding crossing.

Moreover if the link diagram is oriented, $G(L)$ will also receive an orientation. The orientation of the edge c° will be defined by that of the upper part of the link near the crossing c as shown on Figure 5.

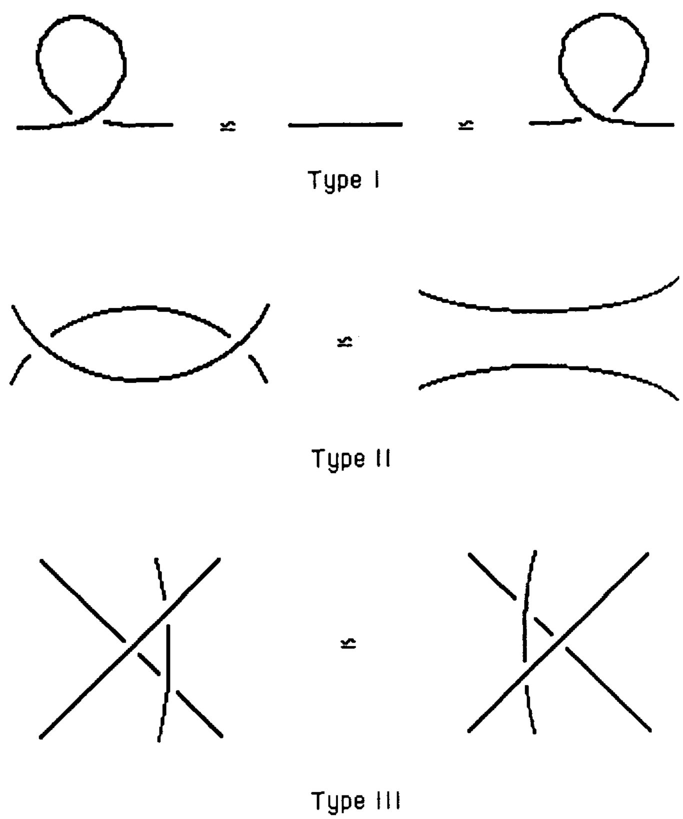

Figure 3.

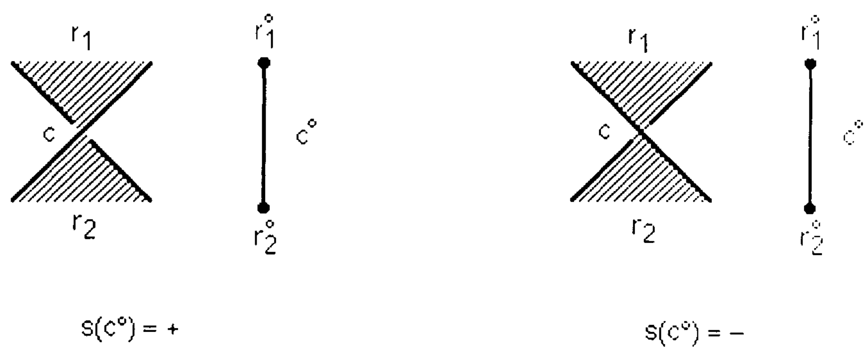

Figure 4.

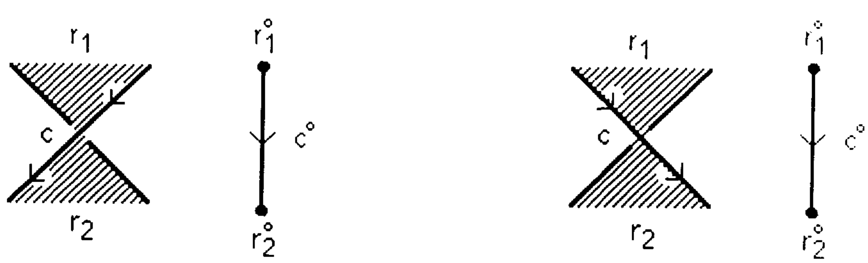

Figure 5.

We now introduce two matrices W^+ and W^- in $\mathcal{M}(X)$, where X is a set of size $n \geq 2$, and two numbers a and D, with $D^2 = n$. Let w_s be the mapping from $E(G(L))$ to $\mathcal{M}(X)$ such that for every edge $e, w_s(e) = W^+$ if $s(e) = +$ and $w_s(e) = W^-$ if $s(e) = -$. We associate with every connected oriented diagram L the complex number $Z(L) = a^{-T(L)} D^{-|V(G(L))|} Z(G(L), w_s)$.

If the oriented diagram L is not connected, we define $Z(L)$ as the product of the values of the function Z on the connected components of L.

The following result can be found in [38] (see also [33] for the symmetric case, and [2] for a generalization using four matrices).

Proposition 1 *The function Z is a link invariant whenever the following equations are satisfied.*

$$I \circ W^+ = aI, I \circ W^- = a^{-1}I. \tag{1}$$

$$JW^+ = W^+J = Da^{-1}J, JW^- = W^-J = DaJ. \tag{2}$$

$$W^+W^- = nI. \tag{3}$$

$$W^+ \circ W^{-T} = J. \tag{4}$$

(*Star-triangle equation*) *For every α, β, γ in X,*

$$\sum_{x \in X} W^+[\alpha, x]W^+[x, \beta]W^-[x, \gamma] = DW^+[\alpha, \beta]W^-[\alpha, \gamma]W^-[\beta, \gamma]. \tag{5}$$

Sketch of Proof. The invariance of $Z(L)$ under (oriented) Reidemeister moves is checked by associating with each move on L the corresponding graph transformation on $G(L)$ (two cases must be considered for each move, one for each local coloring of the regions). Equations (1), (2) (respectively: (3), (4)) guarantee invariance under Reidemeister moves of type I (respectively: II). For Reidemeister moves of type III, only one oriented version of the move described on Figure 3 needs to be considered (see for instance [51]). We shall consider the version where all arrows are oriented downwards. This leads to (5) and another similar equation where W^- is replaced by its transpose, but this second equation can be shown to derive from the first one together with (3), (4). □

We shall call a *topological spin model* a triple (X, W^+, W^-), where W^+, W^- are matrices in $\mathcal{M}(X)$ which satisfy the properties (1) to (5) for some numbers a and D with $D^2 = n = |X|$ (a is the *modulus* and D is the *loop variable* of the model).

Remarks

(i) The properties (1) *to* (5) *are not independent (see* [2], [33], [38]).

(ii) When the matrices W^+, W^- are symmetric, $Z(G(L), w_s)$ does not depend on the orientation of $G(L)$. Then we have exactly the definition of spin models given in [33], *and we shall call such models symmetric.*

(iii) It is easy to see that if (X, W^+, W^-) is a topological spin model with loop variable D, $(X, iW^+, -iW^-)$ (with $i^2 = -1$) is a topological spin model with loop variable $-D$. Hence there would be no loss of generality in considering only the case $D = \sqrt{n}$.

3 Spin models and association schemes

Every topological spin model known at the time of this writing is related (in ways to be explained below) to some association scheme. Let us recall some basic facts concerning these structures (see [8], [20], [7] for more details).

A (commutative) *d-class association scheme* on the finite non-empty set X is a partition of $X \times X$ into $d+1$ non-empty relations $R_i, i = 0, \dots, d$, where $R_0 = \{(x, x)/x \in X\}$, which satisfies the following properties:

(i) For every i in $\{0, \dots d\}$, there exists i' in $\{0, \dots, d\}$ such that $\{(y, x)/(x, y) \in R_i\} = R_{i'}$.

(ii) For every i, j, k in $\{0, \dots, d\}$ there exists an integer p_{ij}^k (called an *intersection number*) such that, for every x, y in X with (x, y) in R_k, $|\{z \in X/(x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^k$. Moreover $p_{ji}^k = p_{ij}^k$.
Define matrices $A_i, i = 0, \dots, d$, in $\mathcal{M}(X)$ by

(iii) $A_i[x, y]$ equals 1 if $(x, y) \in R_i$, and equals 0 otherwise.
The above definitions can then be reformulated as follows.

$$A_i \neq 0,\ A_i \circ A_j = \delta_{ij} A_i, \text{ where } \delta \text{ is the Kronecker symbol.} \tag{6}$$

$$A_0 = I. \tag{7}$$

$$\sum_{i=0,\dots,d} A_i = J. \tag{8}$$

$$A_i^T = A_{i'} \text{ for some } i' \text{ in } \{0, \dots, d\}. \tag{9}$$

$$A_i A_j = A_j A_i = \sum_{k=0,\dots,d} p_{ij}^k A_k. \tag{10}$$

The association scheme is said to be *symmetric* if every matrix A_i is symmetric.

Let $\mathcal{A}$ be the subspace of $\mathcal{M}(X)$ spanned by the matrices $A_i, i = 0, \dots, d$. By (6) these matrices are linearly independent and hence form a basis of $\mathcal{A}$. Then (6) and (8) imply that under Hadamard product $\mathcal{A}$ is an associative commutative algebra with unit J, and $\{A_i/i \in \{0, \dots, d\}\}$ is a basis of orthogonal idempotents of this algebra. Moreover by (9) $\mathcal{A}$ is closed under transposition. Finally it follows from (7) and (10) that under ordinary matrix product $\mathcal{A}$ is also an associative commutative algebra with unit I. The subspace $\mathcal{A}$ of $\mathcal{M}(X)$ endowed with these two algebra structures is called the *adjacency algebra*, or *Bose-Mesner algebra* (see [10]) of the association scheme and will be called here a *BM-algebra on X*. By the unicity of the basis of orthogonal idempotents for the Hadamard product, all combinatorial properties of an association scheme are encoded into its BM-algebra, and will be in the sequel identified with properties of this BM-algebra. BM-algebras on X can be characterized abstractly as those vector subspaces of $\mathcal{M}(X)$ which contain I, J, are closed under transposition, Hadamard product and ordinary matrix product, and for which the ordinary matrix product is commutative (it is easy to extend the proof given for the symmetric case in [7], Th. 2.6.1).

Classical results in linear algebra show that the BM-algebra $\mathcal{A}$ has also a (necessarily unique) basis of orthogonal idempotents for the ordinary matrix product (see [8] Section II.3). One may denote these idempotents by $E_i, i = 0, \dots, d$, in such a way that the following properties are satisfied, where as before we write $n = |X|$ (compare with (6)–(10) above).

$$E_i \neq 0,\ E_i E_j = \delta_{ij} E_i. \tag{11}$$

$$E_0 = n^{-1} J. \tag{12}$$

$$\sum_{i=0,\dots,d} E_i = I. \tag{13}$$

$$E_i^T = \overline{E}_i = E_{i^\wedge} \quad \text{for some } i^\wedge \text{ in } \{0,\dots,d\}. \tag{14}$$

$$E_i \circ E_j = n^{-1} \sum_{k=0,\dots,d} q_{ij}^k E_k \tag{15}$$

(the structure constants q_{ij}^k are called the *Krein parameters* of the scheme).

In the sequel we denote by τ the transposition map on $\mathcal{A}$ defined by $\tau(M) = M^T$ for every M in $\mathcal{A}$.

The *eigenmatrices* P and Q of the scheme relate the two bases of idempotents as follows:

$$A_j = \sum_{i=0,\dots,d} P_{ij} E_i. \tag{16}$$

$$E_j = n^{-1} \sum_{i=0,\dots,d} Q_{ij} A_i. \tag{17}$$

Hence

$$PQ = QP = nI. \tag{18}$$

The scheme is said to be *self-dual* if the two matrices P and Q are complex conjugates for an appropriate choice of the indices of the idempotents. To such a choice corresponds a linear *duality map* Ψ from $\mathcal{A}$ to itself defined by $\Psi(E_i) = A_i (i = 0,\dots,d)$, so that by (16) P is the matrix of Ψ with respect to the basis $\{E_i / i = 0,\dots,d\}$.

Taking the complex conjugate and transpose of (16) and using (14), we obtain that $A_j^T = \sum_{i=0,\dots,d} Q_{ij} E_i$. Hence, denoting the composition of maps by the symbol $\bullet$, the eigenmatrix Q of the scheme is the matrix of $\tau \bullet \Psi$ with respect to the basis $\{E_i / i = 0,\dots,d\}$. Then by (18), $\Psi \bullet \tau \bullet \Psi = \tau \bullet \Psi \bullet \Psi = n Id$ (where *Id* denotes the identity map) and hence

$$\Psi \bullet \Psi = n\tau, \tag{19}$$

$$\Psi \bullet \tau = \tau \bullet \Psi. \tag{20}$$

Applying both sides of (20) to E_i we obtain, using the notations of (9) and (14),

$$A_{i^\wedge} = \Psi(E_{i^\wedge}) = (\Psi(E_i))^T = A_{i'}$$

and hence $i' = i^\wedge$ for every i in $\{0,\dots,d\}$. So for self-dual schemes we shall use the notation $E_{i'}$ for $E_i^T = \overline{E}_i$.

It follows immediately from (6), (11) and the definition of Ψ that, for any two matrices M, N in $\mathcal{A}$,

$$\Psi(MN) = \Psi(M) \circ \Psi(N). \tag{21}$$

In the sequel we call a *duality map* on the BM-algebra $\mathcal{A}$ a linear map Ψ from $\mathcal{A}$ to itself which satisfies (19) (implying (20)) and (21) (see also [41] where a related semi-linear map is considered). Clearly if Ψ is a duality map, $\Psi \bullet \tau = \tau \bullet \Psi$ is also a duality map. By a *self-dual BM-algebra* we mean a pair $(\mathcal{A}, \Psi)$ where $\mathcal{A}$ is a BM-algebra and Ψ is a duality map on $\mathcal{A}$. Then it will always be assumed that the idempotents are indexed so that

$$\Psi(E_i) = A_i (i = 0, \ldots, d). \tag{22}$$

Then by (19),

$$\Psi(A_i) = nE_{i'} (i = 0, \ldots, d). \tag{23}$$

Applying Ψ to (10), using (21), (23), transposing and comparing with (15) we obtain (see also [8]) that

$$\text{In a self-dual scheme, } p_{ij}^k = q_{ij}^k \quad \text{for every } i, j, k \text{ in } \{0, \ldots, d\}. \tag{24}$$

All topological spin models (X, W^+, W^-) known at the time of this writing have the property that the matrix W^+ belongs to some BM-algebra $\mathcal{A}$ (and then by (4) W^- also belongs to $\mathcal{A}$). We shall then say that the topological spin model *belongs to* $\mathcal{A}$. The corresponding schemes come from strongly regular graphs ([33], [34], [31], [26]), from Hamming graphs [5], cyclic groups ([25], [3]), or Hadamard graphs [42].

In many cases the matrices J, W^+ and W^{+T} together generate the BM-algebra $\mathcal{A}$ under ordinary matrix product. It can be shown that this is equivalent to the property that I, W^+ and W^{+T} together generate the BM-algebra $\mathcal{A}$ under Hadamard product (see [4]). In that situation we shall say that the topological spin model (X, W^+, W^-) *generates* $\mathcal{A}$.

Remark It is not difficult to check that the topological spin models of [5] belonging to the BM-algebra of the Hamming scheme $H(d, q)$ do not generate this BM-algebra when q is 2, 3 or 4 and $d \geq 7 - q$.

The following result can be found in [4], [6] (see also [31] for the symmetric case).

Proposition 2 *Assume that the topological spin model (X, W^+, W^-) generates the BM-algebra $\mathcal{A}$. Then there exists a duality map Ψ on $\mathcal{A}$ satisfying*

$$\Psi(W^+) = DW^-, \Psi(W^{+T}) = DW^{-T}, \Psi(W^-) = DW^{+T}, \Psi(W^{-T}) = DW^+. \tag{25}$$

We shall say that a topological spin model (X, W^+, W^-) *fully belongs to the self-dual* BM-algebra $(\mathcal{A}, \Psi)$ if W^+ belongs to $\mathcal{A}$ and $\Psi(W^+) = DW^-$. Then (25) follows easily from (19), (20). If we look for topological spin models which fully belong to a given self-dual BM-algebra $(\mathcal{A}, \Psi)$, we have the following simple approach to the construction of solutions to equations (1), (2), (3), (4) (see also [4], [31]).

The matrix W^+ will be given by $W^+ = \sum_{i=0,\ldots,d} t_i A_i$ with $t_i \neq 0$ for $i = 0, \ldots, d$.

Then (4) reduces to $W^{-T} = \sum_{i=0,\ldots,d} t_i^{-1} A_i$. Now by (9), $W^{+T} = \sum_{i=0,\ldots,d} t_{i'} A_i$ and $W^- = \sum_{i=0,\ldots,d} t_{i'}^{-1} A_i$. Using (23) the condition $\Psi(W^+) = DW^-$ can be written $\sum_{i=0,\ldots,d} t_i n E_{i'} = DW^-$, or equivalently $W^- = D \sum_{i=0,\ldots,d} t_{i'} E_i$. By (19) we also have $\Psi(W^-) = DW^{+T}$, which similarly becomes $\sum_{i=0,\ldots,d} t_{i'}^{-1} n E_{i'} = DW^{+T}$, or equivalently $W^{+T} = D \sum_{i=0,\ldots,d} t_i^{-1} E_i$. Then by (14), $W^{-T} = D \sum_{i=0,\ldots,d} t_i E_i$ and $W^+ = D \sum_{i=0,\ldots,d} t_{i'}^{-1} E_i$. Recalling (7), (12), we see that the above formulas imply equations (1), (2) with $a = t_0$. Moreover equation (3) follows from (11) and (13). To conclude, if $W^+ = \sum_{i=0,\dots,d} t_i A_i$ and $W^- = \sum_{i=0,\dots,d} t_{i'}^{-1} A_i = D \sum_{i=0,\dots,d} t_{i'} E_i$, equations (1), (2), (3), (4) are satisfied.

Using (16), the last equality can usefully be rewritten as

$$\sum_{j=0,\dots,d} P_{ij} t_{j'}^{-1} = D t_{i'} \quad \text{for} \quad i = 0, \dots, d. \tag{26}$$

Thus any solution to (26) yields a solution to (1), (2), (3), (4) which also satisfies (25).

4 Series-parallel evaluation

Let $\mathcal{A}$ be a BM-algebra on a set X of size n. We now consider spin models for which all edge weights belong to $\mathcal{A}$.

Every graph G with non empty edge-set will be provided with an arbitrary total ordering of its edges. Let e_j be the j^{th} edge of G for $j = 1, \dots, m = |E(G)|$. Let us represent every mapping w from $E(G)$ to $\mathcal{A}$ by the vector $(w(e_1), \dots, w(e_m))$ in $\mathcal{A}^m$.

Recall from Section 2 that $Z(G, w) = \sum_{\sigma: V(G) \to X} \prod_{j \in \{1,\dots,m\}} w(e_j)[\sigma(i(e_j)), \sigma(t(e_j))]$. Then clearly the mapping $w \to Z(G, w)$ defines a m-multilinear form on $\mathcal{A}^m$ which we shall denote by Z_G. Let us denote by $\mathcal{A}_G$ the tensor product of vector spaces $\otimes_{j \in \{1,\dots,m\}} \mathcal{A}_j$, where $\mathcal{A}_j$ (which we shall call the j^{th} factor of $\mathcal{A}_G$) corresponds to the j^{th} edge of G and is identified with $\mathcal{A}$ for $j = 1, \dots, m$. We shall identify Z_G with the linear form on $\mathcal{A}_G$ which takes the value $Z(G, w)$ on $w(e_1) \otimes \cdots \otimes w(e_m)$ for every mapping w from $E(G)$ to $\mathcal{A}$.

If G has no edges, in accordance with the classical definitions of tensor algebra, $\mathcal{A}_G$ will be taken to be the one-dimensional space $\mathbf{C}$ of complex numbers, and the empty mapping from $E(G) = \emptyset$ to $\mathcal{A}$ will correspond to the number 1 in $\mathcal{A}_G$. In that case, the form Z_G is just multiplication by the scalar $Z_G(1)$. Note that

$$\text{If } G \text{ has no edges, } Z_G(1) = n^{|V(G)|}. \tag{27}$$

We now describe some rules which can be used together with (27) to compute Z_G. We observe that a change of the total ordering chosen for the edges of G corresponds to the composition of Z_G with an automorphism of the vector space $\mathcal{A}_G$ which permutes its factors. Hence, without loss of generality, for any given rule we shall always choose an ordering of the edges which yields a simple description for this rule. Also let us recall that given vector spaces S_i, S_i' and linear maps $f_i\colon S_i \to S_i' (i = 1, \dots, m)$, $f_1 \otimes \cdots \otimes f_m$ is the unique linear map from $S_1 \otimes \cdots \otimes S_m$ to $S_1' \otimes \cdots S_m'$ such that $(f_1 \otimes \cdots \otimes f_m)(s_1 \otimes \cdots \otimes s_m) = f_1(s_1) \otimes \cdots \otimes f_m(s_m)$ for every $(s_1, \dots, s_m)$ in $S_1 \times \cdots \times S_m$. In the sequel we shall use implicitly the canonical isomorphisms $\mathbf{C} \otimes S \cong S$ for complex vector spaces S.

Let $G^i (i = 1, \dots, k)$ be the connected components of G. We may identify $\mathcal{A}_G$ with $\otimes_{i \in \{1,\dots,k\}} \mathcal{A}_{G^i}$ in such a way that

$$Z_G = \otimes_{i \in \{1,\dots,k\}} Z_{G^i}. \tag{28}$$

Let $C(G, 1)$ (respectively: $D(G, 1)$) be the graph obtained from G by contracting (respectively: deleting) the edge e_1. Thus $\mathcal{A}_{C(G,1)}$ and $\mathcal{A}_{D(G,1)}$ are obtained from $\mathcal{A}_G$ by deleting the first factor. It is easy to see that for every ω in $\mathcal{A}_{C(G,1)} \simeq \mathcal{A}_{D(G,1)}$

$$Z_G(I \otimes \omega) = Z_{C(G,1)}(\omega), \tag{29}$$

$$Z_G(J \otimes \omega) = Z_{D(G,1)}(\omega). \tag{30}$$

Note that when e_1 is a loop, the left-hand sides of (29) and (30) are equal since I and J have the same diagonal elements, and the right-hand sides are equal since $C(G, 1) = D(G, 1)$.

Properties (27), (29), (30) may be used to compute Z_G when $\mathcal{A}$ is spanned by I and J, that is, when $\mathcal{A}$ is the BM-algebra of a (symmetric) 1-class association scheme. The corresponding spin models contain the *resonant models* of [14] and in particular the Potts model of statistical mechanics, which is essentially equivalent to Tutte's dichromatic polynomial (see [12], [14], [46], [52]). By (29) and (30) we get the equality $Z_G((aI + bJ) \otimes \omega) = aZ_{C(G,1)}(\omega) + bZ_{D(G,1)}(\omega)$. This is essentially the well known "deletion-contraction rule" which together with (27) leads to a recursive process to compute Z_G. Also, if we expand every element of $\mathcal{A}_G$ with respect to the natural basis $\{M_1 \otimes \cdots \otimes M_m / M_i \in \{I, J\}, i = 1, \ldots, m\}$, and use (27), (29), (30) to compute the value of Z_G on the elements of this basis, we shall obtain classical formulas for the Tutte polynomial and various extensions (such as the polynomial of [37]) or for the partition function of the Potts model.

The above considerations are relevant to the study of the Jones polynomial introduced in [35]. Indeed if the complex number α satisfies $\alpha^4 + \alpha^{-4} + 2 = n$, setting $W^+ = -(\alpha^3 + \alpha^{-1})I + \alpha^{-1}J$, $W^- = -(\alpha^{-3} + \alpha)I + \alpha J$, $D = -\alpha^2 - \alpha^{-2}$, $a = -\alpha^3$, we obtain a symmetric topological spin model whose associated link invariant is the Jones polynomial up to a change of variables ([33]; see also [31], Section 3.2).

We now present further rules for the computation of Z_G.

Let $R(G, 1)$ be the graph obtained from G by reversing the orientation of e_1. Then clearly

$$Z_G = Z_{R(G,1)} \bullet (\tau \otimes Id), \tag{31}$$

where Id denotes the identity map acting on the appropriate factors.

Note that if $\mathcal{A}$ is the BM-algebra of a symmetric association scheme, so that τ is the identity, Z_G does not depend on the orientation of G.

There exists two linear forms θ and θ^* on $\mathcal{A}$ such that, for every matrix M in $\mathcal{A}$

$$I \circ M = \theta(M)I, \quad JM = MJ = \theta^*(M)J. \tag{32}$$

Then it is easy to check that

$$\text{If } e_1 \text{ is a loop, } Z_G = Z_{D(G,1)} \bullet (\theta \otimes Id) \tag{33}$$

$$\text{If } e_1 \text{ is a pendant edge, } Z_G = Z_{C(G,1)} \bullet (\theta^* \otimes Id) \tag{34}$$

where we use implicitly the isomorphisms

$$\mathcal{A}_{C(G,1)} \cong \mathcal{A}_{D(G,1)} \cong \otimes_{i \in \{2,\ldots,m\}} \mathcal{A}_i \cong \mathbf{C} \otimes (\otimes_{i \in \{2,\ldots,m\}} \mathcal{A}_i).$$

Let μ and μ^* be the linear maps from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}$ defined by $\mu(M \otimes N) = MN$, $\mu^*(M \otimes N) = M \circ N$ for any two matrices M, N in $\mathcal{A}$. Recall that two non-loop edges e, f are said to be in series (or to form a series pair) if they have a common end which is incident to no other edges, and two edges of G are said to be parallel (or to form a parallel pair) if they have the same pair of ends. A pair of non-loop edges will be called a *strict series pair* if the terminal end of one of these edges equals the initial end of the other and is incident to no edge outside the pair. Also, we shall say that a pair of edges is a *strict parallel pair* if these two edges have the same initial end and the same terminal end.

The following statements are easy consequences of the above definitions.

$$\text{If } e_1, e_2 \text{ form a strict series pair, } Z_G = Z_{C(G,1)} \bullet (\mu \otimes Id) \tag{35}$$

$$\text{If } e_1, e_2 \text{ form a strict parallel pair, } Z_G = Z_{D(G,1)} \bullet (\mu^* \otimes Id) \tag{36}$$

where we use implicitly the isomorphisms

$$\mathcal{A}_{C(G,1)} \cong \mathcal{A}_{D(G,1)} \cong \otimes_{i\in\{2,\ldots,m\}}\mathcal{A}_i \cong \mathcal{A} \otimes (\otimes_{i\in\{3,\ldots,m\}}\mathcal{A}_i).$$

Recall that a graph G is *series-parallel* (see [21], [44]) if and only if it can be reduced to a graph with no edges by repeated application of operations of one of the following types which we shall call *extended series-parallel reductions*:

(i) deletion of a loop.
(ii) contraction of a pendant edge.
(iii) contraction of one of the edges of a series pair.
(iv) deletion of one of the edges of a parallel pair.

Proposition 3 *If G is a connected series-parallel graph, Z_G is a composition $\rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n, and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps τ, θ, θ^*, μ, μ^* on some factors of a tensor product of copies of $\mathcal{A}$.*

Proof: To each extended series-parallel reduction can be applied one of the rules (33), (34), (35), (36) (with possibly the use of rule (31) to transform a series or parallel pair into a strict one). The reduction process ends up with the trivial graph with one vertex and no edges, for which we apply rule (27). □

Remark The case of BM-algebras of strongly regular graphs was already considered with a different approach in [27].

We may consider Proposition 3 from several points of view.

Firstly, Proposition 3 provides a convenient diagrammatic description of a certain type of computation in a BM-algebra, which we shall call *series-parallel evaluation*. We shall illustrate this in the next section.

Secondly, it expresses the possibility of a "matrix-free" approach to spin models on series-parallel graphs for which all edge weights belong to a given BM-algebra (in fact it is not difficult to obtain an analogous result for arbitrary edge weights).

One aspect of this matrix-free approach is the fact that the partition function can be computed much more efficiently than by state enumeration. Actually, if we assume that all operations in the BM-algebra can be performed in constant time, we obtain a linear-time computation of the partition function. A more realistic study of the complexity of this computation should rely on adequate assumptions concerning the complex numbers involved in the BM-algebra operations.

Another aspect is the fact that Z_G only depends on the abstract BM-algebra structure (characterized by the maps τ, θ, θ^*, μ, μ^*) and not on its particular representation by matrices.

Finally, the most interesting aspect from the point of view of the present paper is that Proposition 3 provides a tool to establish properties of partition functions of spin models defined on series-parallel graphs. Let us illustrate this with the following property, which we shall call *series-parallel reversibility*.

Proposition 4 *If G is a series-parallel graph and $R(G)$ is obtained from G by reversing the orientation of every edge,* $Z_{R(G)} = Z_G$.

Proof: By (31), $Z_{R(G)} = Z_G \bullet \tau^{\otimes}$, where $\tau^{\otimes}$ denotes the action of the transposition map on all factors of $\mathcal{A}_G$. Write $Z_G = \rho_0 \bullet \rho_1 \cdots \bullet \rho_k$ as in the statement of Proposition 3. We claim that for all $i = 1, \dots k$, $\rho_i \bullet \tau^{\otimes} = \tau^{\otimes} \bullet \rho_i$, where on each side $\tau^{\otimes}$ denotes the action of τ on all factors of the relevant tensor product. This follows at once from the following easily checked identities:

$$\theta \bullet \tau = \theta, \ \theta^* \bullet \tau = \theta^*, \tag{37}$$

$$\mu \bullet (\tau \otimes \tau) = \tau \bullet \mu, \ \mu^* \bullet (\tau \otimes \tau) = \tau \bullet \mu^*. \tag{38}$$

Since the transposition map acts trivially on $\mathbf{C}$ we obtain

$$\begin{aligned} Z_{R(G)} &= \rho_0 \bullet \rho_1 \cdots \bullet \rho_k \bullet \tau^{\otimes} = \rho_0 \bullet \tau^{\otimes} \bullet \rho_1 \cdots \bullet \rho_k \\ &= \rho_0 \bullet \rho_1 \cdots \bullet \rho_k = Z_G. \end{aligned}$$

□

Proposition 4 is motivated by the following considerations. The *inverse* of an oriented link is obtained by the simultaneous reversal of orientations of all components of the link. A link is *non-invertible* if it is not ambient isotopic to its inverse, and such links are known to exist [49]. So far all known link invariants which can be described by models in the sense of statistical mechanics (see [33]) do not distinguish between inverse links. Does there exist a topological spin model whose associated link invariant distinguishes between inverse links?

If a topological spin model belongs to a BM-algebra $\mathcal{A}$ and $Z_{G(L)} = Z_{R(G(L))}$ for a given link diagram L, clearly the corresponding link invariant will not distinguish between the link represented by L and its inverse. By Proposition 4 this is the case for any BM-algebra if $G(L)$ is series-parallel (this occurs in the examples of [49]).

5 Stars, triangles, triply regular schemes and spin models for plane graphs

5.1 *Star and triangle projections in association schemes*

The following definitions are useful for the study of the star-triangle equation (5). Let $\mathcal{A}$ be a BM-algebra on X and $\mathcal{S}$ be the complex vector space with basis X. We shall provide $\mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}$ with the positive definite Hermitian form $\langle , \rangle$ such that $\{\alpha \otimes \beta \otimes \gamma / \alpha, \beta, \gamma \in X\}$ is an orthonormal basis. We define the linear maps π (*star projection*) and π^* (*triangle projection*) from $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ to $\mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}$ by

$$\pi(A \otimes B \otimes C) = \sum_{\alpha,\beta,\gamma \in X} \left(\sum_{x \in X} A[x,\alpha]B[x,\beta]C[x,\gamma] \right) \alpha \otimes \beta \otimes \gamma \tag{39}$$

$$\pi^*(A \otimes B \otimes C) = \sum_{\alpha,\beta,\gamma \in X} A[\beta,\gamma]B[\gamma,\alpha]C[\alpha,\beta] \alpha \otimes \beta \otimes \gamma \tag{40}$$

Then two elements W^+, W^- of $\mathcal{A}$ satisfy the star-triangle equation (5) if and only if

$$\pi(W^{+T} \otimes W^+ \otimes W^-) = D\pi^*(W^- \otimes W^{-T} \otimes W^+). \quad (41)$$

We now study more closely the star and triangle projections.

For i, j, k, u, v, w in $\{0, \ldots, d\}$ let $Y_{ijk} = \pi(E_i \otimes E_j \otimes E_k)$ and $\Delta_{uvw} = \pi^*(A_u \otimes A_v \otimes A_w)$. Thus $Y_{ijk} = \sum_{\alpha,\beta,\gamma \in X}(\sum_{x \in X} E_i[x,\alpha]E_j[x,\beta]E_k[x,\gamma])\ \alpha \otimes \beta \otimes \gamma$ and $\langle Y_{ijk}, Y_{rst}\rangle = \sum_{\alpha,\beta,\gamma \in X}(\sum_{x \in X} E_i[x,\alpha]E_j[x,\beta]E_k[x,\gamma])(\sum_{x \in X} \bar{E}_r[x,\alpha]\bar{E}_s[x,\beta]\,\bar{E}_t[x,\gamma])$. From (14) we get

$$\langle Y_{ijk}, Y_{rst}\rangle = \sum_{\alpha,\beta,\gamma \in X}\left(\sum_{x \in X} E_i[x,\alpha]E_j[x,\beta]E_k[x,\gamma]\right)\left(\sum_{y \in X} E_r^T[y,\alpha]E_s^T[y,\beta]E_t^T[y,\gamma]\right).$$

It follows that $\langle Y_{ijk}, Y_{rst}\rangle = Z(G, w)$, where the graph G and the edge weights $w(e)$ are depicted on Figure 6. Then series-parallel evaluation easily gives

$$\langle Y_{ijk}, Y_{rst}\rangle = \delta_{ijk,rst}\theta(E_k)q_{ij}^{k^\wedge}. \quad (42)$$

Similarly, $\Delta_{uvw} = \sum_{\alpha,\beta,\gamma \in X} A_u[\beta,\gamma]A_v[\gamma,\alpha]A_w[\alpha,\beta]\,\alpha \otimes \beta \otimes \gamma$ yields (since the matrices A_i, $i = 0, \ldots, d$, are real)

$$\langle \Delta_{uvw}, \Delta_{xyz}\rangle = \sum_{\alpha,\beta,\gamma \in X} (A_u[\beta,\gamma]A_v[\gamma,\alpha]A_w[\alpha,\beta])(A_x[\beta,\gamma]A_y[\gamma,\alpha]A_z[\alpha,\beta]).$$

Then series-parallel evaluation on the graph depicted on Figure 7 easily gives

$$\langle \Delta_{uvw}, \Delta_{xyz}\rangle = \delta_{uvw,xyz}n\theta^*(A_w)p_{uv}^{w'}. \quad (43)$$

Formulas (42), (43) for $ijk = rst$, $uvw = xyz$ can be found in [8], Chapter II, Th.3.6 (see also [17], Lemma 4.2, and [47], Lemma 3.2).

Note that $n\theta^*(A_w)$, the sum of entries of A_w, is non-zero. Thus, by (43), Δ_{uvw} is non-zero if and only if $p_{uv}^{w'} \neq 0$, and in this case we shall call (u, v, w) a *feasible triple*. From the combinatorial point of view, (u, v, w) is feasible if and only if there exists α, β, γ in X with (β, γ) in R_u, (γ, α) in R_v, (α, β) in R_w. We shall similarly call a *dually feasible triple* any triple (i, j, k) in $\{0, \ldots, d\}^3$ such that Y_{ijk} is non-zero. Since $n\theta(E_k) = \text{Trace}(E_k) = \text{Rank}(E_k)$ is non-zero, by (42) (i, j, k) is dually feasible if and only if $q_{ij}^{k^\wedge} \neq 0$. We shall denote by $F(\mathcal{A})$ the set of feasible triples and by $F^*(\mathcal{A})$ the set of dually feasible triples. It is clear from (9), (10), (14), (15) and (38) that $p_{u'v'}^{w'} = p_{uv}^{w}$ and $q_{i^\wedge j^\wedge}^{k^\wedge} = q_{ij}^{k}$. Consequently (u', v', w') is feasible if and only if (u, v, w) is feasible and similarly $(i^\wedge, j^\wedge, k^\wedge)$ is dually feasible if and only if (i, j, k) is dually feasible. Also it follows from (24) that if $\mathcal{A}$ is self-dual, $F(\mathcal{A}) = F^*(\mathcal{A})$.

Finally note that since $\{Y_{ijk}/(i, j, k) \in \{0, \ldots, d\}^3\}$ generates $\text{Im}\pi$, (42) shows that $\{Y_{ijk}/(i, j, k) \in F^*(\mathcal{A})\}$ is an orthogonal basis of this space. Similarly, (43) shows that $\{\Delta_{uvw}/(u, v, w) \in F(\mathcal{A})\}$ is an orthogonal basis of $\text{Im}\,\pi^*$.

5.2 *The star-triangle equation in self-dual schemes*

Let us now consider the star-triangle equation (41) in the case where $\mathcal{A}$ is self-dual with duality map Ψ and (X, W^+, W^-) fully belongs to the self-dual BM-algebra $(\mathcal{A}, \Psi)$. Let us

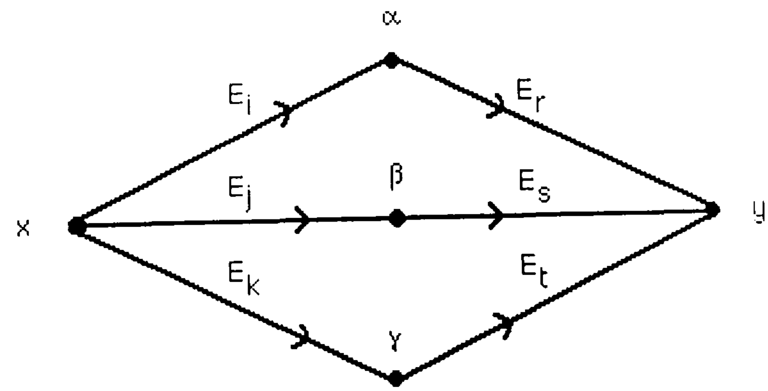

Figure 6.

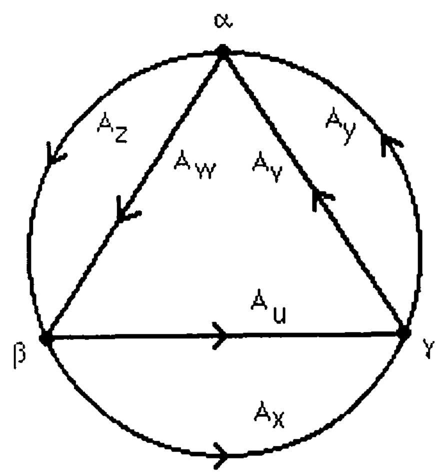

Figure 7.

recall from Section 3 the expressions

$$W^{+T} = D \sum_{i=0,\dots,d} t_i^{-1} E_i, \quad W^{+} = D \sum_{i=0,\dots,d} t_{i'}^{-1} E_i, \quad W^{-} = D \sum_{i=0,\dots,d} t_{i'} E_i$$

and

$$W^{-} = \sum_{i=0,\dots,d} t_{i'}^{-1} A_i, \quad W^{-T} = \sum_{i=0,\dots,d} t_i^{-1} A_i, \quad W^{+} = \sum_{i=0,\dots,d} t_i A_i.$$

It follows that

$$\begin{aligned} W^{+T} \otimes W^{+} \otimes W^{-} &= D^3 \sum_{(i,j,k)\in\{0,\dots,d\}^3} t_i^{-1} t_{j'}^{-1} t_{k'} \, E_i \otimes E_j \otimes E_k \\ &= D^3 \sum_{(i,j,k)\in\{0,\dots,d\}^3} t_{i'}^{-1} t_j^{-1} t_k \, E_{i'} \otimes E_{j'} \otimes E_{k'} \end{aligned}$$

and

$$W^- \otimes W^{-T} \otimes W^+ = \sum_{(u,v,w)\in\{0,\ldots,d\}^3} t_{u'}^{-1} t_v^{-1} t_w \, A_u \otimes A_v \otimes A_w.$$

Now (41) becomes

$$n \sum_{(i,j,k)\in F(\mathcal{A})} t_{i'}^{-1} t_j^{-1} t_k Y_{i'j'k'} = \sum_{(u,v,w)\in F(\mathcal{A})} t_{u'}^{-1} t_v^{-1} t_w \Delta_{uvw} \tag{44}$$

The relevance of this particularly symmetric form of the star-triangle equation will be illustrated in the sequel.

5.3 *Triply regular association schemes*

Consider an association scheme with BM-algebra $\mathcal{A}$ defined on the set X by the relations $R_i, i = 0, \ldots, d$. The scheme (and its BM-algebra $\mathcal{A}$) will be said to be *triply regular* if the following property holds.

For every (i, j, k) in $\{0, \ldots, d\}^3$ and (u, v, w) in $F(\mathcal{A})$ there exists an integer $K(ijk/uvw)$ such that, for every α, β, γ in X with (β, γ) in R_u, (γ, α) in R_v, (α, β) in R_w,

$$|\{x \in X/(x, \alpha) \in R_i, \quad (x, \beta) \in R_j, \quad (x, \gamma) \in R_k\}| = K(ijk/uvw). \tag{45}$$

This concept has been studied by Terwilliger [48] for (schemes of) distance-regular graphs.

The equality (45) can be reformulated in matrix terms as

$$\sum_{x\in X} A_i[x, \alpha] A_j[x, \beta] A_k[x, \gamma] = \sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) A_u[\beta, \gamma] A_v[\gamma, \alpha] A_w[\alpha, \beta]. \tag{46}$$

Now (46) holds for every α, β, γ in X if and only if

$$\begin{aligned}
&\pi(A_i \otimes A_j \otimes A_k)\\
&= \sum_{\alpha,\beta,\gamma\in X} \left(\sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) A_u[\beta, \gamma] A_v[\gamma, \alpha] A_w[\alpha, \beta] \right) \alpha \otimes \beta \otimes \gamma\\
&= \sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) \left(\sum_{\alpha,\beta,\gamma\in X} A_u[\beta, \gamma] A_v[\gamma, \alpha] A_w[\alpha, \beta] \, \alpha \otimes \beta \otimes \gamma \right)\\
&= \sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) \pi^*(A_u \otimes A_v \otimes A_w)\\
&= \pi^* \left(\sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) \, A_u \otimes A_v \otimes A_w \right)
\end{aligned}$$

Thus, defining the linear map κ from $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ to itself by

$$\kappa(A_i \otimes A_j \otimes A_k) = \sum_{(u,v,w)\in F(\mathcal{A})} K(ijk/uvw) A_u \otimes A_v \otimes A_w \tag{47}$$

we see that (46) holds for every α, β, γ in X and i, j, k in $\{0, \ldots, d\}$ if and only if

$$\pi = \pi^* \bullet \kappa \tag{48}$$

Conversely, assume that there exists a linear map κ from $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ to itself such that (48) holds. Let us express this map in the basis $\{A_i \otimes A_j \otimes A_k / i, j, k \in \{0, \dots, d\}\}$ of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ as $\kappa(A_i \otimes A_j \otimes A_k) = \sum_{u,v,w \in \{0,\dots,d\}} K(ijk/uvw) A_u \otimes A_v \otimes A_w$. Then applying π^* to both sides and noting that $\pi^*(A_u \otimes A_v \otimes A_w) = \Delta_{uvw}$ vanishes unless $(u, v, w) \in F(\mathcal{A})$, we easily obtain that (46) holds for every α, β, γ in X and i, j, k in $\{0, \dots, d\}$.

To conclude, an association scheme is triply regular if and only if there exists a linear map κ from $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ to itself such that (48) holds. The constants $K(ijk/uvw)$ for $(u, v, w) \in F(\mathcal{A})$ are the significant entries of the matrix of κ with respect to the basis $\{A_i \otimes A_j \otimes A_k / i, j, k \in \{0, \dots, d\}\}$ of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (the other entries can be chosen arbitrarily without altering property (48)).

Many known topological spin models belong to triply regular BM-algebras. This is easy to see for the models of [33] and [25]. For the strongly regular graphs of [31], see for instance [26] and Remark 5.5 of [18]. For the cyclic group models of [3] and the Hadamard graph models of [42] the triple regularity is used explicitly in these papers for the proof of the star-triangle equation.

5.4 Star-triangle transformations

Let G be an undirected plane graph which has a vertex v incident to exactly three edges e_1, e_2, e_3, where e_i has ends v, v_i for $i = 1, 2, 3$ and the vertices v, v_1, v_2, v_3 are distinct. Delete v, e_1, e_2, e_3 and add three new edges e'_1, e'_2, e'_3 where e'_i has ends v_j, v_k whenever $\{i, j, k\} = \{1, 2, 3\}$. The new edges will be embedded in the plane in such a way as to obtain a new plane graph G' in which e'_1, e'_2, e'_3 bound a triangular face (see Figure 8). We shall say that G' is obtained from G by a *star-to-triangle, or* $Y - \Delta$, *transformation* and that G is obtained from G' by a *triangle-to-star, or* $\Delta - Y$, *transformation*. The following result, due to Epifanov [22] (see also [24], [50], [23]) is essential in what follows.

Proposition 5 *Every connected undirected plane graph can be reduced to the trivial graph with one vertex and no edge by a finite sequence of* $Y - \Delta$ *or* $\Delta - Y$ *transformations and extended series-parallel reductions.*

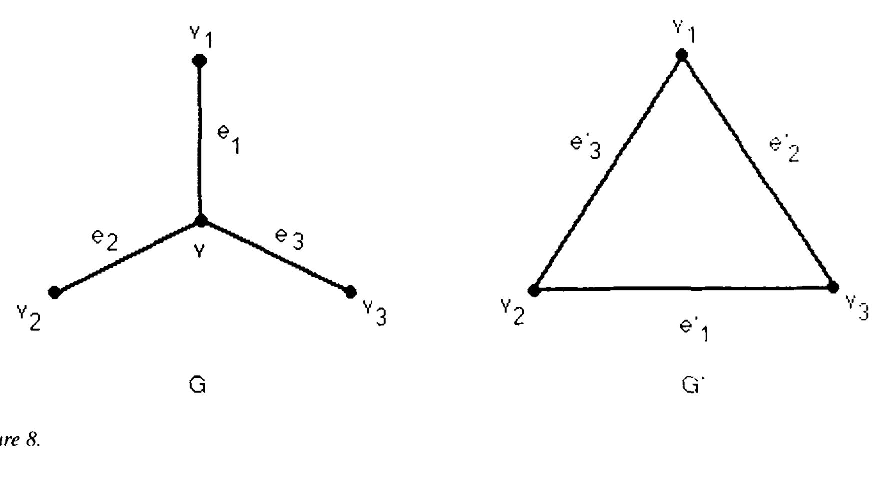

Figure 8.

We now consider a (directed) plane graph G and the associated form Z_G defined in Section 4.

Let us assume that G is obtained from G' by a $\Delta - Y$ transformation and, keeping the same notations as above, that $i(e_j) = v$, $t(e_j) = v_j (j = 1, 2, 3)$ and that e'_1, e'_2, e'_3 have initial ends v_2, v_3, v_1 respectively. Let H be the graph obtained from G (respectively: G') by deleting v, e_1, e_2, e_3 (respectively: e'_1, e'_2, e'_3). For every α, β, γ in X let $S(\alpha\beta\gamma)$ be the set of states $\sigma\colon V(H) \to X$ such that $\sigma(v_1) = \alpha, \sigma(v_2) = \beta, \sigma(v_3) = \gamma$, and for every $w\colon E(H) \to \mathcal{A}$, let $Z(H, w, \alpha\beta\gamma) = \sum_{\sigma\in S(\alpha\beta\gamma)} \prod_{e\in E(H)} w(e|\sigma)$. Let us now identify e'_i with e_i, thus identifying $E(G')$ with $E(G)$ and $\mathcal{A}_{G'}$ with $\mathcal{A}_G$. Let w be a mapping from $E(G) = E(G')$ to $\mathcal{A}$ and $w \mid H$ be its restriction to $E(H)$. Then clearly

$$Z(G, w) = \sum_{\alpha,\beta,\gamma\in X} Z(H, w \mid H, \alpha\beta\gamma)\Bigg(\sum_{x\in X} w(e_1)[x, \alpha]w(e_2)[x, \beta]w(e_3)[x, \gamma]\Bigg)$$

and

$$Z(G', w) = \sum_{\alpha,\beta,\gamma\in X} Z(H, w \mid H, \alpha\beta\gamma)(w(e_1)[\beta, \gamma]w(e_2)[\gamma, \alpha]w(e_3)[\alpha, \beta]).$$

The above equalities can be written $Z(G, w) = \langle\pi(w(e_1) \otimes w(e_2) \otimes w(e_3)), Z\rangle$ and $Z(G', w) = \langle\pi^*(w(e_1) \otimes w(e_2) \otimes w(e_3)), Z\rangle$, where the $\alpha \otimes \beta \otimes \gamma$ component of the vector Z is the complex conjugate of $Z(H, w \mid H, \alpha\beta\gamma)$.

Now if (48) holds, we have

$$Z(G, w) = \langle(\pi^* \bullet \kappa)(w(e_1) \otimes w(e_2) \otimes w(e_3)), Z\rangle.$$

It then follows that

$$Z_G = Z_{G'} \bullet (\kappa \otimes Id), \tag{49}$$

where κ acts on the first three factors of $\mathcal{A}_G = \mathcal{A}_{G'}$.

Let us now say that an association scheme or its BM-algebra $\mathcal{A}$ is *dually triply regular* if there exists a linear map κ^* from $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$ to itself such that the following property holds:

$$\pi^* = \pi \bullet \kappa^*. \tag{50}$$

In that case we shall also obtain, similarly to (49),

$$Z_{G'} = Z_G \bullet (\kappa^* \otimes Id). \tag{51}$$

We now define an *exactly triply regular scheme (or BM-algebra)* as a scheme (or BM-algebra) which is both triply regular and dually triply regular.

Proposition 6 *Let $\mathcal{A}$ be an exactly triply regular BM-algebra. If G is a connected plane graph, the linear form Z_G on $\mathcal{A}_G$ is a composition $\rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, where ρ_0 is scalar multiplication by n, and each of $\rho_1, \ldots, \rho_k$ corresponds to the action of one of the maps τ, θ, θ^*, μ, μ^*, κ, κ^* on some factors of a tensor product of copies of $\mathcal{A}$.*

Proof: The proof is exactly similar to that of Proposition 3, with the additional use of Proposition 5 and properties (49), (51). □

Thus we have also a "matrix-free" approach to spin models for plane graphs when all edge weights belong to an exactly triply regular BM-algebra. In particular, in this case we may compute a partition function $Z(G, w)$ by obtaining (using for instance the algorithms of [23] or [50]) a reduction of G to the trivial graph as described in Proposition 5 and computing at each step the action of the corresponding map ρ_i introduced in Proposition 6. We shall call this process *star-triangle evaluation*.

We now study more closely the notions of triple regularity, dual triple regularity, and exact triple regularity.

5.5 Characterizations of triple regularity

Given a BM-algebra $\mathcal{A}$, we define two linear maps κ, κ^* from $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ to itself by

$$\kappa(E_i \otimes E_j \otimes E_k) = \sum_{(u,v,w)\in F(\mathcal{A})} (\langle Y_{ijk}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle)\, A_u \otimes A_v \otimes A_w \tag{52}$$

$$\kappa^*(A_u \otimes A_v \otimes A_w) = \sum_{(i,j,k)\in F^*(\mathcal{A})} (\langle \Delta_{uvw}, Y_{ijk}\rangle / \langle Y_{ijk}, Y_{ijk}\rangle)\, E_i \otimes E_j \otimes E_k. \tag{53}$$

Proposition 7 *The following properties are equivalent:*

(i) The BM-algebra $\mathcal{A}$ is triply regular.
(ii) $\mathrm{Im}\pi \subseteq \mathrm{Im}\pi^*$.
(iii) The linear map κ defined by (52) *satisfies* (48), *that is,* $\pi = \pi^* \bullet \kappa$.

Proof: First note that by (52),

$$(\pi^* \bullet \kappa)(E_i \otimes E_j \otimes E_k) = \sum_{(u,v,w)\in F(\mathcal{A})} (\langle Y_{ijk}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle)\Delta_{uvw}.$$

Clearly, (48) implies that $\mathrm{Im}\pi \subseteq \mathrm{Im}\pi^*$. Assume now that $\mathrm{Im}\pi \subseteq \mathrm{Im}\pi^*$. Then for $(i, j, k) \in F^*(\mathcal{A})$, Y_{ijk} can be expressed in the orthogonal basis $\{\Delta_{uvw}/(u, v, w) \in F(\mathcal{A})\}$ of $\mathrm{Im}\pi^*$. Comparing this expression with the above expression for $(\pi^* \bullet \kappa)(E_i \otimes E_j \otimes E_k)$ we see that $(\pi^* \bullet \kappa)(E_i \otimes E_j \otimes E_k) = Y_{ijk} = \pi(E_i \otimes E_j \otimes E_k)$. If (i, j, k) is not dually feasible, $(\pi^* \bullet \kappa)(E_i \otimes E_j \otimes E_k) = 0 = Y_{ijk} = \pi(E_i \otimes E_j \otimes E_k)$. Hence $\pi^* \bullet \kappa = \pi$ and (iii) holds.

We can prove in exactly the same way the following dual result. □

Proposition 8 *The following properties are equivalent:*

(i) The BM-algebra $\mathcal{A}$ is dually triply regular.
(ii) $\mathrm{Im}\pi^* \subseteq \mathrm{Im}\pi$.
(iii) The linear map κ^* defined by (53) *satisfies* (50), *that is,* $\pi^* = \pi \bullet \kappa^*$.

In the sequel κ and κ^* will always denote the maps defined by (52) and (53). Combining Propositions 7 and 8 we easily obtain the following result.

Proposition 9

(i) The BM-algebra $\mathcal{A}$ is exactly triply regular if and only if $\mathrm{Im}\pi = \mathrm{Im}\pi^*$.

(ii) The triply regular BM-algebra $\mathcal{A}$ is exactly triply regular if and only if $|F(\mathcal{A})| = |F^*(\mathcal{A})|$, or equivalently
$|\{(i, j, k) \in \{0, \dots, d\}^3 / p_{ij}^k \neq 0\}| = |\{(i, j, k) \in \{0, \dots, d\}^3 / q_{ij}{}^{k^\wedge} \neq 0\}|$.
(iii) Every self-dual triply regular BM-algebra is exactly triply regular.

Remark From Remark 4.3 of [17] (see also Theorem 6.6 of [18]) it follows that non self-dual Smith graphs (see [45]) give examples of triply regular schemes which are not exactly triply regular.

Proposition 9(iii) applies to the topological spin models of [33], [25], [31], [3], [42], since each of them belongs to a self-dual triply regular BM-algebra. In these cases the matrix-free approach of Proposition 6 is valid. We shall see an example of application in Section 7.

5.6 *The star-triangle equation in self-dual triply regular BM-algebras*

Let us consider the star-triangle equation (44) for topological spin models which fully belong to a triply regular self-dual BM-algebra $(\mathcal{A}, \Psi)$. By Proposition 9, $\mathrm{Im}\pi = \mathrm{Im}\pi^*$ and we know that each of $\{\Delta_{uvw}/(u, v, w) \in F(\mathcal{A})\}$ and $\{Y_{ijk}/(i, j, k) \in F(\mathcal{A})\}$ is an orthogonal basis of this space. Hence (44) is equivalent to the equality

$$n \sum_{(i,j,k)\in F(\mathcal{A})} t_{i'}^{-1} t_j^{-1} t_k \langle Y_{i'j'k'}, \Delta_{uvw}\rangle = t_{u'}^{-1} t_v^{-1} t_w \langle \Delta_{uvw}, \Delta_{uvw}\rangle$$

for every $(u, v, w) \in F(\mathcal{A})$. We define a matrix S with rows and columns indexed by $F(\mathcal{A})$ as follows.

$$\text{For every } (u, v, w), (i, j, k) \text{ in } F(\mathcal{A}),$$
$$S_{(u,v,w)(i,j,k)} = n\langle Y_{i'j'k'}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle. \tag{54}$$

Let T be the column vector indexed by $F(\mathcal{A})$ defined by

$$\text{For every } (i, j, k) \text{ in } F(\mathcal{A}),\ T_{(i,j,k)} = t_{i'}^{-1} t_j^{-1} t_k. \tag{55}$$

Then the star-triangle equation (44) is equivalent to

$$ST = T. \tag{56}$$

A natural preliminary step in the solution of (56) would be the study of the space of fixed points of S.

6 Planar duality and reversibility

6.1 *Series-parallel duality*

Recall that given a connected undirected plane graph G, its (geometric) *dual* is a connected undirected plane graph G^* defined as follows (see for instance [43]). The graph G^* has one vertex f^* for each face f of G, and one edge e^* for each edge e of G (e and e^* are called *dual edges*). The vertex f^* is placed inside f. If the edge e belongs to the boundary of the (possibly identical) faces f_1, f_2, the dual edge e^* has ends f_1^*, f_2^* and is embedded as a simple curve joining these two ends and meeting G in only one point situated in the interior of e. This is done in such a way as to obtain a plane embedding of G^*.

Strictly speaking, G^* is not uniquely defined as a plane graph. However, by adding a point at infinity inside the infinite face, we can view every plane graph as embedded on the sphere, and then the dual G^* is uniquely defined in this setting. We shall adopt here implicitly this point of view on plane graphs. This will be consistent since the properties of plane graphs which we consider (such as partition functions) do not depend on the particular embedding chosen. Moreover it will be seen that the same point of view for link diagrams plays an essential rôle in the proof of Proposition 12.

Let G be a connected undirected series-parallel graph. As is well known, G can be embedded in the plane (this follows immediately from the constructive definition given in Section 4) and will then be said to be *plane*. It is easy to see that a plane connected undirected series-parallel graph can be reduced to the trivial graph with one vertex and no edge by a sequence of extended series-parallel reductions which satisfy the following requirements: one may delete a loop only if it has empty interior, and similarly one may delete one edge of a parallel pair only if these two edges form a closed curve with empty interior. Such extended series-parallel reductions will be called *plane*. It follows from the proofs in [24] and [50] that the extended series-parallel reductions needed in Proposition 5 can also be assumed to be plane.

To each plane extended series-parallel reduction for a connected plane series-parallel graph G corresponds another such reduction for the dual plane graph G^*, which we call the *dual* reduction, in such a way that the sequence of dual reductions also transforms G^* into the trivial graph. The dual of a loop deletion is the contraction of a pendant edge, and conversely. Similarly the dual of the deletion of one edge in a parallel pair is the contraction of one edge in a series pair, and conversely.

Given a connected directed plane graph G, we shall call its *dual* and denote by G^* the dual undirected plane graph provided with the orientation defined for each edge according to the convention described in Figure 9. Note that with this convention strict parallel pairs are dual to strict series pairs.

We now consider a self-dual BM-algebra $(\mathcal{A}, \Psi)$, and we identify $\mathcal{A}_G$ with $\mathcal{A}_{G^*}$ in such a way that dual edges correspond to the same factor in the tensor product.

Proposition 10 *Let G be a connected plane series-parallel graph. Then*

$$Z_G = n^{1-|V(G^*)|} Z_{G^*} \bullet \Psi^{\otimes} \tag{57}$$

where $\Psi^{\otimes}$ denotes the action of Ψ on each factor of the tensor product.

Proof: We apply simultaneously the proof of Proposition 3 to G and G^*, using only plane extended series-parallel reductions. Then we can write $Z_G = \rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, $Z_{G^*} = \rho_0 \bullet \rho_1^* \cdots \bullet \rho_k^*$, where ρ_0 is scalar multiplication by n, each of $\rho_1, \ldots, \rho_k, \rho_1^*, \ldots, \rho_k^*$ corresponds to the action of one of the maps $\tau, \theta, \theta^*, \mu, \mu^*$ on some factors of a tensor product of copies of $\mathcal{A}$, and for each $i = 1, \ldots, k$ the pair (ρ_i, ρ_i^*) corresponds to one of the pairs (τ, τ), (θ, θ^*), (θ^*, θ), (μ, μ^*), (μ^*, μ). We claim that for each $i = 1, \ldots, k$,

$$(i)\ \rho_i^* \bullet \Psi^{\otimes} = n^{\varepsilon(i)} \Psi^{\otimes} \bullet \rho_i,$$

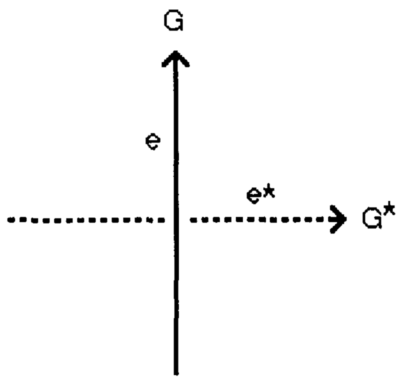

Figure 9.

where on both sides $\Psi^{\otimes}$ denotes the action of Ψ on all factors (if there are no factors $\Psi^{\otimes}$ is just the identity map on **C**) and $\varepsilon(i)$ is 1 if ρ_i^* corresponds to the action of θ^* or μ and $\varepsilon(i)$ is 0 otherwise. This will imply (57) since $\rho_0 \bullet \Psi^{\otimes} = \rho_0$ and $\sum_{i\in\{1,\ldots,k\}} \varepsilon(i) = |V(G^*)| - 1$.

Property (i) follows from (20) and from the following identities.

$$\theta^* \bullet \Psi = n\theta, \quad \theta \bullet \Psi = \theta^* \tag{58}$$

$$\Psi \bullet \mu = \mu^* \bullet (\Psi \otimes \Psi), \quad \Psi \bullet \mu^* = n^{-1}\mu \bullet (\Psi \otimes \Psi). \tag{59}$$

The first identity in (59) is just a reformulation of (21). Using (19), (20), and (38) the second one can then be derived from the first. Since $E_0 = n^{-1}J$ and $A_0 = I$, $\Psi(J) = nI$ by (22) and hence $\Psi(I) = J$ by (19). Applying Ψ to (32) and using (59) we finally obtain (58). □

6.2 *Planar duality*

A self-dual BM-algebra $(\mathcal{A}, \Psi)$ will be said to satisfy the *planar duality property* if (57) holds for every connected plane graph G.

Remark Using Euler's formula and (19) it is easy to check that a double application of (57) gives $Z_G = Z_{(G^*)^*} \bullet \tau^{\otimes}$. By (31), this is compatible with the fact that $(G^*)^*$ is the graph $R(G)$ obtained from G by reversing the orientation of all edges.

We now illustrate this definition with the example of Abelian group schemes. Let X be an Abelian group of order n written additively. Recall (see [8]) that the corresponding group scheme is defined on X by the relations $R_i = \{(x, y)/y - x = i\}$, $i \in X$, and note that $i' = -i$ for every i in X. Let χ_i, $i \in X$, be the characters of X, with indices chosen such that $\chi_i(j) = \chi_j(i)$ for all i, j in X. For every i in X let $E_i = n^{-1} \sum_{j\in X} \overline{\chi_i(j)} A_j$. Then the E_i form the basis of orthogonal idempotents of the scheme for the ordinary matrix product. Moreover $A_i = \sum_{j\in X} \chi_i(j) E_j$ for every i in X, so that the scheme is self-dual. By (14) and (23), for every i in X

$$\Psi(A_i) = nE_{i'} = n\overline{E}_i = \sum_{j\in X} \chi_i(j) A_j. \tag{60}$$

In the following we call Abelian group self-dual BM-algebra a pair $(\mathcal{A}, \Psi)$, where $\mathcal{A}$ is the BM-algebra of an Abelian group scheme and the duality map Ψ is given by (60).

Proposition 11 *Every Abelian group self-dual BM-algebra has the planar duality property.*

Proof: Let G be a connected plane graph, which we may assume non-trivial. Let us write $E(G) = \{e_1, \ldots, e_m\}$, $E(G^*) = \{e_1^*, \ldots, e_m^*\}$, where e_j and e_j^* are dual edges.

We check (57) by applying both sides to an arbitrary element of the basis $\{A_{i_1} \otimes \cdots \otimes A_{i_m} / i_1, \ldots, i_m \in X\}$ of $\mathcal{A}_G = \mathcal{A}_{G^*}$. So, by (60), we must prove that

$$Z_G(A_{i_1} \otimes \cdots \otimes A_{i_m}) = n^{1-|V(G^*)|} Z_{G^*}(nE_{i_{1'}} \otimes \cdots \otimes nE_{i_{m'}}).$$

Define the maps w from $E(G)$ to $\mathcal{A}$ and w^* from $E(G^*)$ to $\mathcal{A}$ by $w(e_j) = A_{i_j}$ and $w^*(e_j^*) = nE_{i_{j'}}$ for $j = 1, \ldots, m$. Then the above equation amounts to the equality of $Z(G, w)$ and $n^{1-|V(G^*)|} Z(G^*, w^*)$, which we now prove in a way inspired by [16] (see also [14]). Clearly,

$$\begin{aligned} Z(G, w) &= \sum_{\sigma: V(G) \to X} \prod_{j \in \{1, \ldots, m\}} A_{i_j}[\sigma(i(e_j)), \sigma(t(e_j))] \\ &= \sum_{\sigma: V(G) \to X} \prod_{j \in \{1, \ldots, m\}} \delta_{i_j, \sigma(t(e_j)) - \sigma(i(e_j))} \\ &= |\{\sigma\colon V(G) \to X / \sigma(t(e_j)) - \sigma(i(e_j)) = i_j \quad \text{for } j = 1, \ldots, m\}|. \end{aligned}$$

For every interior face r of G and index j in $1, \ldots, m$, let $\varepsilon(r, j)$ be equal to $+1$ (respectively: -1) if e_j appears exactly once in a clockwise walk around the boundary of r and is traversed according to its orientation (respectively: according to the reverse orientation), and let $\varepsilon(r, j)$ be equal to 0 otherwise. For the infinite face we exchange $+1$ and -1 in the above definition. In the sequel X is considered as a left $\mathbb{Z}$-module.

It is well known and easy to prove (see for instance [43], Chapter 7) that for any fixed vertex v_0 of G and element x_0 of X there is a bijective correspondence ω from $\{\sigma\colon V(G) \to X / \sigma(v_0) = x_0\}$ to $\{\varphi\colon E(G) \to X / \sum_{j \in \{1, \ldots, m\}} \varepsilon(r, j) \varphi(e_j) = 0$ for every region r of $G\}$ given by $\omega(\sigma)(e_j) = \sigma(t(e_j)) - \sigma(i(e_j))$ for $j = 1, \ldots, m$.

It follows that, defining the element $\langle r \rangle$ of X by $\langle r \rangle = \sum_{j \in \{1, \ldots, m\}} \varepsilon(r, j) i_j$, $Z(G, w)$ equals n if $\langle r \rangle = 0$ for every region r of G, and equals 0 otherwise. Hence,

$$\begin{aligned} Z(G, w) &= n \prod_{r \in V(G^*)} \delta_{\langle r \rangle, 0} = n \prod_{r \in V(G^*)} \left(n^{-1} \sum_{i \in X} \chi_{\langle r \rangle}(i)\right) \\ &= n^{1-|V(G^*)|} \sum_{\sigma^*: V(G^*) \to X} \prod_{r \in V(G^*)} \chi_{\langle r \rangle}(\sigma^*(r)). \end{aligned}$$

Now $\chi_{\langle r \rangle}(\sigma^*(r)) = \chi_{\sigma^*(r)}(\sum_{j \in \{1, \ldots, m\}} \varepsilon(r, j) i_j) = \prod_{j \in \{1, \ldots, m\}} \chi_{\sigma^*(r)}(\varepsilon(r, j) i_j)$ and hence $\prod_{r \in V(G^*)} \chi_{\langle r \rangle}(\sigma^*(r)) = \prod_{j \in \{1, \ldots, m\}} \prod_{r \in V(G^*)} \chi_{\sigma^*(r)}(\varepsilon(r, j) i_j) =$ $\prod_{j \in \{1, \ldots, m\}} \prod_{r \in V(G^*)} \chi_{i_j}(\varepsilon(r, j) \sigma^*(r)) = \prod_{j \in \{1, \ldots, m\}} \chi_{i_j}(\Sigma_{r \in V(G^*)} \varepsilon(r, j) \sigma^*(r)) =$ $\prod_{j \in \{1, \ldots, m\}} \chi_{i_j}(\sigma^*(t(e_j^*)) - \sigma^*(i(e_j^*)))$. Thus

$$Z(G, w) = n^{1-|V(G^*)|} \sum_{\sigma^*: V(G^*) \to X} \prod_{j \in \{1, \ldots, m\}} \chi_{i_j}(\sigma^*(t(e_j^*)) - \sigma^*(i(e_j^*))).$$

By (60), $\chi_{i_j}(\sigma^*(t(e_j^*)) - \sigma^*(i(e_j^*))) = nE_{i_{j'}}[\sigma^*(i(e_j^*)), \sigma^*(t(e_j^*))]$ and $Z(G, w) = n^{1-|V(G^*)|}\, Z(G^*, w^*)$ as required. □

Remark For self-dual BM-algebras $(\mathcal{A}, \Psi)$ and $(\mathcal{A}', \Psi')$ such that $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{A}'$ is invariant under Ψ, and Ψ' is the restriction of Ψ to $\mathcal{A}'$, the planar duality property for $(\mathcal{A}, \Psi)$ clearly implies the same property for $(\mathcal{A}', \Psi')$. It then follows from Proposition 11 that Abelian group symmetric BM-algebras ($\mathcal{A}'$ will consist of the symmetric matrices in $\mathcal{A}$), and 2-dimensional BM-algebras ($\mathcal{A}'$ will be the linear span of I and J) have the planar duality property.

In statistical mechanics strong connections have been established between the star-triangle equation, partition functions and the duality of planar graphs (see for instance [12]). We now investigate similar connections in our context.

Let us consider a topological spin model (X, W^+, W^-) which belongs to the BM-algebra $\mathcal{A}$. Thus we may write $W^+ = \sum_{i=0,\ldots,d} t_i A_i$. Assume that all coefficients t_i with $i \neq 0$ are distinct. Then it is easy to see that W^+ and I generate $\mathcal{A}$ under Hadamard product. More precisely, denoting by $\circ^i W^+$ the matrix obtained from J by a succession of i Hadamard products with W^+, $\{I\} \cup \{\circ^i W^+ / i = 1, \ldots, d\}$ is a basis of $\mathcal{A}$ (the matrix which expresses these elements in the basis $\{A_i / i = 0, \ldots, d\}$ has a non-zero Vandermonde determinant). In this case we shall say that the topological spin model (X, W^+, W^-) *strongly generates* $\mathcal{A}$. Then it also generates $\mathcal{A}$ in the sense considered in Proposition 2. A symmetric topological spin model strongly generates $\mathcal{A}$ if and only if it generates $\mathcal{A}$, but the models of [3] show that this is no longer true in general.

Proposition 12 *Let $\mathcal{A}$ be a BM-algebra strongly generated by a topological spin model (X, W^+, W^-) and let Ψ be the duality map given by Proposition 2. Then the self-dual BM-algebra $(\mathcal{A}, \Psi)$ satisfies the planar duality property.*

Proof: We know that $\mathcal{B} = \{I\} \cup \{\circ^i W^+ / i = 1, \ldots, d\}$ is a basis of $\mathcal{A}$. By (59) and (25), $\Psi(\circ^i W^+) = n^{1-i}(\Psi(W^+))^i = n^{1-i}(DW^-)^i = D^{2-i}(W^-)^i$. Also, recall that $\Psi(I) = J$. Hence $\{J\} \cup \{(W^-)^i / i = 1, \ldots, d\}$ is a basis of $\mathcal{A}$, and it easily follows from (2), (3) that $\mathcal{B}^* = \{J\} \cup \{(W^+)^i / i = 1, \ldots, d\}$ is also a basis of $\mathcal{A}$.

Let G be a connected plane graph with $E(G) = \{e_1, \ldots, e_m\}$, $E(G^*) = \{e_1^*, \ldots, e_m^*\}$, where as before e_j and e_j^* are dual edges. We want to check the planar duality identity (57) for G. Note that by (31) and (20), this identity is independent of the choice of an orientation for G. Let us first forget the orientation of G and assign to each edge a positive sign. It is then easy to construct a connected unoriented link diagram L such that G is exactly $G(L)$, with edges signed as specified in Figure 4: construct the medial graph of G (see [43], p. 47) and replace each (4-valent) vertex by a crossing in the appropriate way.

Let us choose an arbitrary orientation for L, and provide $G = G(L)$ with the corresponding orientation according to the convention of Figure 5. There are now two types of edges of G which we call vertical and horizontal, as shown in Figure 10. It is easy to see that every vertex of G has an even number of incidences with vertical edges and that similarly every face boundary of G has an even number of incidences with horizontal edges. If a bridge of G were vertical, the number of incidences between vertices and vertical edges inside each connected component with respect to that bridge would be odd, a contradiction.

Figure 10.

Consequently, every bridge of G is horizontal and a dual argument shows that every loop of G is vertical.

We shall check (57) by applying both sides to an arbitrary element of the basis $\{B_1 \otimes \cdots \otimes B_m / B_j \in \mathcal{B}$ if e_j is horizontal, $B_j \in \mathcal{B}^*$ if e_j is vertical$\}$ of $\mathcal{A}_G = \mathcal{A}_{G^*}$.

We may assume that (57) holds for all graphs with less edges than G (note that (57) is trivially true for the trivial graph).

First let us consider the case where one of the matrices B_j is I when e_j is horizontal or J when e_j is vertical. Suppose for instance that e_1 is horizontal and $B_1 = I$. By (29), $Z_G(I \otimes B_2 \otimes \cdots \otimes B_m) = Z_{C(G,1)}(B_2 \otimes \cdots \otimes B_m)$, and by (30) $Z_{G^*}(\Psi(I) \otimes \Psi(B_2) \otimes \cdots \otimes \Psi(B_m)) = Z_{G^*}(J \otimes \Psi(B_2) \otimes \cdots \otimes \Psi(B_m)) = Z_{D(G^*,1)}(\Psi(B_2) \otimes \cdots \otimes \Psi(B_m))$. $D(G^*, 1)$ is connected because e_1, being horizontal, cannot be a loop of G and hence e_1^* is not a bridge of G^*. Then $D(G^*, 1) = (C(G, 1))^*$ and (57) holds for G because it holds for the smaller graph $C(G, 1)$. We can deal with the case when e_1 is vertical and $B_1 = J$ by a similar argument.

So we may assume that for every $j \in \{1, \ldots, m\}$ there is an exponent $k_j \in \{1, \ldots, d\}$ such that $B_j = \circ^{k_j} W^+$ if e_j is horizontal and $B_j = (W^+)^{k_j}$ if e_j is vertical.

We now introduce a connected plane graph H obtained from G by replacing each edge e_j of G by k_j parallel edges with initial end $i(e_j)$ and terminal end $t(e_j)$ if e_j is horizontal and by k_j edges in series forming a directed path from $i(e_j)$ to $t(e_j)$ if e_j is vertical. It is easy to see that H^* can be similarly obtained from G^* by replacing each edge e_j^* of G^* by k_j edges in series forming a directed path from $i(e_j^*)$ to $t(e_j^*)$ if e_j is horizontal and by k_j edges with initial end $i(e_j^*)$ and terminal end $t(e_j^*)$ if e_j is vertical. Then it follows from (35) and (36) that $Z_G(B_1 \otimes B_2 \otimes \cdots \otimes B_m) = Z_H(W^+ \otimes W^+ \otimes \cdots \otimes W^+)$. Similarly, since $\Psi(\circ^i W^+) = D^{2-i}(W^-)^i$ and $\Psi((W^+)^i) = D^i \circ^i W^-$, we obtain $Z_{G^*}(\Psi(B_1) \otimes \Psi(B_2) \otimes \cdots \otimes \Psi(B_m)) = D^\lambda Z_{H^*}(W^- \otimes W^- \otimes \cdots \otimes W^-)$, where $\lambda = \sum_{e_j \text{horizontal}} (2 - k_j) + \sum_{e_j \text{vertical}} k_j$. Now it is enough to show that

$$(i)\; Z_H(W^+ \otimes W^+ \otimes \cdots \otimes W^+) = n^{1-|V(G^*)|} D^\lambda Z_{H^*}(W^- \otimes W^- \otimes \cdots \otimes W^-).$$

Consider again the oriented link diagram L such that $G = G(L)$ and replace as shown in Figure 11 each crossing corresponding to an edge e_j of G by a series of k_j crossings with positive sign if e_j is vertical and negative sign if e_j is horizontal. Let L' be the resulting oriented diagram. Then clearly $G(L') = H$ (with edges oriented according to the convention of Figure 5) and the associated link invariant $Z(L')$ (see Section 2) is given by

$$Z(L') = a^{-T(L')} D^{-|V(H)|} Z_H(W^+ \otimes W^+ \otimes \cdots \otimes W^+).$$

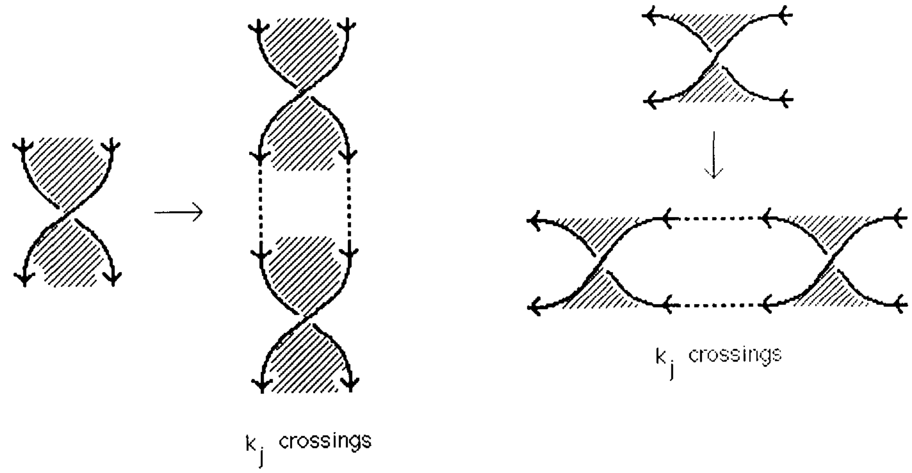

Figure 11.

We consider also the oriented link diagram L'' obtained from L' by the following process (see [33], Proposition 2.14 and Figure 2.15). We add a point at infinity inside the infinite region delimited by L', so that L' can now be viewed as a spherical link projection. Then by an isotopy of the sphere we move an edge of L' lying on the boundary of the infinite region through the point at infinity, thus obtaining L'' which is again considered as embedded in the plane. Using Figures 4, 5 and 9 it is easy to see that $G(L'') = H^*$ and that the associated link invariant $Z(L'')$ is given by

$$Z(L'') = a^{-T(L'')} D^{-|V(H^*)|} Z_{H^*}(W^- \otimes W^- \otimes \cdots \otimes W^-).$$

Since clearly L' and L'' represent ambient isotopic links and $T(L') = T(L'')$, it follows from Proposition 1 that

$$Z_H(W^+ \otimes W^+ \otimes \cdots \otimes W^+) = D^{|V(H)|-|V(H^*)|} Z_{H^*}(W^- \otimes W^- \otimes \cdots \otimes W^-).$$

Moreover $|V(H)| = |V(G)| + \sum_{e_j \text{vertical}} (k_j - 1)$ and $|V(H^*)| = |V(G^*)| + \sum_{e_j \text{horizontal}} (k_j - 1)$, so that
$|V(H)| - |V(H^*)| = |V(G)| - |V(G^*)| + \sum_{e_j \text{vertical}} (k_j - 1) - \sum_{e_j \text{horizontal}} (k_j - 1) = |V(G)| - |V(G^*)| + \lambda - |E(G)|$. Hence by Euler's formula $|V(H)| - |V(H^*)| = 2 - 2|V(G^*)| + \lambda$ and (i) follows. □

Remark Proposition 12 could be extended to self-dual BM-algebras which are "generated" by a topological spin model in a weaker sense than the one used above. All we need is a basis consisting of matrices which can be associated with some link diagrams corresponding to series-parallel graphs, in the same way as the elements of the bases $\mathcal{B}$ and $\mathcal{B}^*$ different from I and J can be associated with the diagrams of Figure 11. However it seems difficult to obtain an intrinsic characterization of self-dual BM-algebras which have such a basis.

6.3 *Planar reversibility*

In this section we consider the possibility of extending Proposition 4, and we are led to the following general question: for which BM-algebras $\mathcal{A}$ and graphs G does the equality $Z_G = Z_{R(G)}$ hold?

Let us first give a result closely analogous to Lemma 4.1 in [38] for which we use the same proof. We shall say that an association scheme or BM-algebra has the *full reversibility property* if $Z_G = Z_{R(G)}$ for every graph G. Let us call *anti-automorphism* of an association scheme (or its BM-algebra $\mathcal{A}$) on X a permutation φ of X such that $M[\varphi(x), \varphi(y)] = M[y, x]$ for every x, y in X and M in $\mathcal{A}$.

Proposition 13 *If an association scheme has an anti-automorphism it has the full reversibility property.*

Proof: Given the same edge weights for G and $R(G)$, with every state σ of G we associate the state $\varphi \bullet \sigma$ of $R(G)$. Then it is easy to see that for every edge e, $w(e \mid \sigma)$ computed in G equals $w(e \mid \varphi \bullet \sigma)$ computed in $R(G)$, so that the weight of σ in G and the weight of $\varphi \bullet \sigma$ in $R(G)$ are the same. □

For instance, in a group scheme (these schemes generalize Abelian group schemes, see [8] p. 54, example (2)) the inversion is an anti-automorphism, so the full reversibility property holds.

The link invariants associated with topological spin models which belong to a BM-algebra satisfying the full reversibility property do not distinguish between inverse links. However for this application to link invariants the full reversibility property can be replaced by the following weaker property.

We shall say that an association scheme (or its BM-algebra) satisfies the *planar reversibility property* if $Z_G = Z_{R(G)}$ for every plane graph G.

Proposition 14 *If a self-dual BM-algebra has the planar duality property, it also has the planar reversibility property.*

Proof: Let G be a connected plane graph. We can check that $Z_G = Z_{R(G)}$ by applying both sides to an arbitrary element of the basis $\{E_{i_1} \otimes \cdots \otimes E_{i_m} / i_1, \ldots, i_m \in \{0, \ldots, d\}\}$ of $\mathcal{A}_G = \mathcal{A}_{R(G)}$. Indeed it follows easily from (31) and (14) that $Z_G(E_{i_1} \otimes \cdots \otimes E_{i_m})$ and $Z_{R(G)}(E_{i_1} \otimes \cdots \otimes E_{i_m})$ are complex conjugates, while by (57) and (22) both numbers are real. □

Thus Propositions 12 and 14 together show that the link invariant associated with a topological spin model which strongly generates a BM-algebra cannot distinguish between inverse links.

6.4 *K_4 versus planar duality and reversibility*

In view of Propositions 4 and 10 and of the fact that the complete graph on 4 vertices K_4 is the smallest (actually the unique minor-minimal) non-series-parallel graph it is natural to consider the following properties. We shall say that a self-dual BM-algebra $(\mathcal{A}, \Psi)$ satisfies

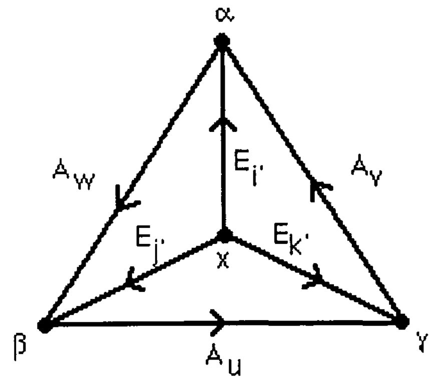

Figure 12.

the K_4 *duality property* if (57) holds when G is K_4 embedded in the plane. Similarly an association scheme (or its BM-algebra) will be said to satisfy the K_4 *reversibility property* if $Z_G = Z_{R(G)}$ when G is K_4. Note that by (31) and (20) the orientation of K_4 can be chosen arbitrarily in the statement of the above properties. Also it is clear from the proof of Proposition 14 that the K_4 duality property implies the K_4 reversibility property.

In the following result S is the matrix defined when $\mathcal{A}$ is self-dual and triply regular by (54) in Section 5.6.

Proposition 15 *The following properties are equivalent:*

(i) The self-dual BM-algebra $(\mathcal{A}, \Psi)$ has the K_4 duality property
(ii) For every (i, j, k) and (u, v, w) in $F(\mathcal{A})$ the following equality holds:

$$\langle Y_{i'j'k'}, \Delta_{uvw}\rangle = \langle \Delta_{ijk}, Y_{u'v'w'}\rangle \tag{61}$$

Moreover if $\mathcal{A}$ is triply regular these properties are equivalent to
(iii) The matrix S is an involution.

Proof: Let G be a complete graph on four vertices with edges oriented as shown on Figure 12. To study (57) we consider as before the application of both sides to the elements of an appropriate basis of $\mathcal{A}_G$. The corresponding edge weights are also displayed on Figure 12. Then clearly the associated value of Z_G is $\langle Y_{i'j'k'}, \Delta_{uvw}\rangle$. On Figure 13 is displayed G^* with edge weights obtained from the edge weights of G by application of Ψ (see (22), (23)). Reversing all arrows and using (31), we see that the associated value of Z_{G^*} is $n^3\langle Y_{uvw}, \Delta_{ijk}\rangle = n^3\overline{\langle \Delta_{ijk}, Y_{uvw}\rangle}$. By (14) this is also equal to $n^3\langle \Delta_{ijk}, Y_{u'v'w'}\rangle$ and hence (57) holds if and only if (61) holds for every i, j, k, u, v, w in $\{0, \ldots, d\}$. Moreover, if (i, j, k) or (u, v, w) does not belong to $F(\mathcal{A}) = F^*(\mathcal{A})$, both sides of (61) will be zero. Hence (i) and (ii) are equivalent.

Let us now assume that $\mathcal{A}$ is triply regular and consider property (iii). Note that since $\mathrm{Im}\pi = \mathrm{Im}\pi^*$ by Proposition 9, for every (u, v, w), (i, j, k) in $F(\mathcal{A})$,

$$\sum_{(u,v,w)\in F(\mathcal{A})} (\langle Y_{i'j'k'}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle)\Delta_{uvw} = Y_{i'j'k'}$$

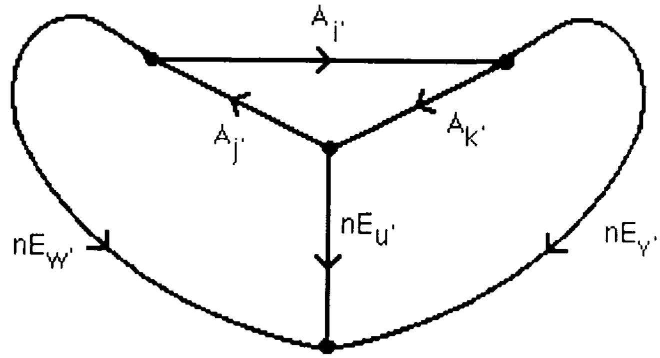

Figure 13.

and similarly

$$\sum_{(u,v,w)\in F(\mathcal{A})} (\langle \Delta_{ijk}, Y_{u'v'w'}\rangle / \langle Y_{u'v'w'}, Y_{u'v'w'}\rangle) Y_{u'v'w'} = \Delta_{ijk}.$$

Since by (54), $\langle Y_{i'j'k'}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle = n^{-1} S_{(u,v,w)(i,j,k)}$, we see that $\langle \Delta_{ijk}, Y_{u'v'w'}\rangle /$ $\langle Y_{u'v'w'}, Y_{u'v'w'}\rangle = n S^{-1}_{(u,v,w)(i,j,k)}$. Thus (iii) is equivalent to $n\langle Y_{i'j'k'}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle$ $= n^{-1}\langle \Delta_{ijk}, Y_{u'v'w'}\rangle / \langle Y_{u'v'w'}, Y_{u'v'w'}\rangle$ for every (u, v, w), (i, j, k) in $F(\mathcal{A})$.

Using Proposition 10 and Figures 6, 7, it is easy to show that

$$\langle \Delta_{uvw}, \Delta_{uvw}\rangle = n^2 \langle Y_{uvw}, Y_{uvw}\rangle = n^2 \langle Y_{u'v'w'}, Y_{u'v'w'}\rangle. \tag{62}$$

It follows that (iii) is equivalent to (ii). □

Propositions 12 and 15 show that if we look for a topological spin model which strongly generates a triply regular BM-algebra, we may restrict our attention to the self-dual ones for which S is an involution. Then we may take advantage of this property in the study of the star-triangle equation (56) (in particular, we have a simple description of the space of fixed points of S). We now state the following immediate result for the sake of completeness.

Proposition 16 *The following properties are equivalent:*

(i) The BM-algebra $\mathcal{A}$ has the K_4 reversibility property
(ii) For every (i, j, k) in $F^*(\mathcal{A})$ and (u, v, w) in $F(\mathcal{A})$ the following equality holds:

$$\langle Y_{i^\wedge j^\wedge k^\wedge}, \Delta_{uvw}\rangle = \langle Y_{ijk}, \Delta_{u'v'w'}\rangle \tag{63}$$

Ikuta has shown in [30] that for $n \geq 4$ a non-symmetric BM-algebra of dimension 3 (which is necessarily self-dual) is never generated by a topological spin model. However the following result holds.

Proposition 17 *Any self-dual BM-algebra of dimension at most 3 has the K_4 duality property. Any BM-algebra of dimension at most 3 has the K_4 reversibility property.*

Proof: We first check (57) for K_4 and $(\mathcal{A}, \Psi)$ using an appropriate basis. The edges incident to x in Figure 12 will receive a weight I, J, or E_1, while the other edges will receive a weight I, J, or A_1 (when the dimension is 2 we shall not need E_1 or A_1). Then if a weight I or J is used we may derive (57) from (29), (30) and Proposition 10. Otherwise we have only to check the equality (61) for $(i', j', k') = (u, v, w) = (1, 1, 1)$, that is $\langle Y_{111}, \Delta_{111}\rangle = \langle \Delta_{1'1'1'}, Y_{1'1'1'}\rangle$. This equality holds since a plane reflection shows that in general $\langle Y_{ijk}, \Delta_{uvw}\rangle = \langle Y_{ikj}, \Delta_{u'w'v'}\rangle$, and conjugation together with (14) shows that in general $\langle Y_{ijk}, \Delta_{uvw}\rangle = \langle \Delta_{uvw}, Y_{i'j'k'}\rangle$, so that $\langle Y_{ijk}, \Delta_{uvw}\rangle = \langle \Delta_{u'w'v'}, Y_{i'k'j'}\rangle$.

Then K_4 reversibility is trivial in the symmetric case and follows from the proof of Proposition 14 in the non-symmetric case. □

Clearly planar duality (respectively: reversibility) implies K_4 duality (respectively: reversibility). We have the following partial converses.

Proposition 18 *Let $(\mathcal{A}, \Psi)$ be a self-dual triply regular BM-algebra. The following properties are equivalent*:

(i) $(\mathcal{A}, \Psi)$ *has the* K_4 *duality property.*
(ii) $(\Psi \otimes \Psi \otimes \Psi) \bullet \kappa = n\kappa^* \bullet (\tau \otimes \tau \otimes \tau) \bullet (\Psi \otimes \Psi \otimes \Psi)$
(iii) $(\mathcal{A}, \Psi)$ *has the planar duality property.*

Proof: By (52) and (23), $((\Psi \otimes \Psi \otimes \Psi) \bullet \kappa)(E_i \otimes E_j \otimes E_k)$ equals
$\sum_{(u,v,w)\in F(\mathcal{A})} (\langle Y_{ijk}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle) n^3 E_{u'} \otimes E_{v'} \otimes E_{w'}$
$= n^3 \sum_{(u,v,w)\in F(\mathcal{A})} (\langle Y_{ijk}, \Delta_{u'v'w'}\rangle / \langle \Delta_{u'v'w'}, \Delta_{u'v'w'}\rangle)\ E_u \otimes E_v \otimes E_w$. Similarly, by (22), (9) and (53),

$$n(\kappa^* \bullet (\tau \otimes \tau \otimes \tau) \bullet (\Psi \otimes \Psi \otimes \Psi))(E_i \otimes E_j \otimes E_k) = n\kappa^*(A_{i'} \otimes A_{j'} \otimes A_{k'})$$
$$= n \sum_{(u,v,w)\in F(\mathcal{A})} (\langle \Delta_{i'j'k'}, Y_{uvw}\rangle / \langle Y_{uvw}, Y_{uvw}\rangle) E_u \otimes E_v \otimes E_w.$$

Since $(i', j', k') \in F(\mathcal{A})$ if and only if $(i, j, k) \in F(A)$, property (ii) reduces to (ii′) For every (i, j, k) and (u, v, w) in $F(\mathcal{A})$,

$$n^3 \langle Y_{ijk}, \Delta_{u'v'w'}\rangle / \langle \Delta_{u'v'w'}, \Delta_{u'v'w'}\rangle = n\langle \Delta_{i'j'k'}, Y_{uvw}\rangle / \langle Y_{uvw}, Y_{uvw}\rangle$$

By (62), $\langle \Delta_{u'v'w'}, \Delta_{u'v'w'}\rangle = n^2 \langle Y_{uvw}, Y_{uvw}\rangle$. This shows that (ii′) is equivalent to property (ii) of Proposition 15, and hence (i) and (ii) are equivalent.

It remains to show that (ii) implies the planar duality property. The proof is similar to that of Proposition 10. Let G be a connected plane graph. We apply simultaneously to G and G^* the proof of Proposition 6. This yields expressions $Z_G = \rho_0 \bullet \rho_1 \cdots \bullet \rho_k$, $Z_G^* = \rho_0 \bullet \rho_1^* \cdots \bullet \rho_k^*$, where ρ_0 is scalar multiplication by n, each of $\rho_1, \ldots, \rho_k, \rho_1^*, \ldots, \rho_k^*$ corresponds to the action of one of the maps τ, θ, θ^*, μ, μ^*, κ, κ^* on some factors of a tensor product of copies of $\mathcal{A}$, and for each $i = 1, \ldots, k$ the pair (ρ_i, ρ_i^*) corresponds to one of the pairs (τ, τ), (θ, θ^*), (θ^*, θ), (μ, μ^*), (μ^*, μ), or (see Figure 14 and use (49), (51), (31)) $(\kappa, \kappa^* \bullet (\tau \otimes \tau \otimes \tau))$, $(\kappa^*, (\tau \otimes \tau \otimes \tau) \bullet \kappa)$. It is easy to see that in order to complete the proof in the same way as for Proposition 10 we only need two relations, namely (ii) and $(\Psi \otimes \Psi \otimes \Psi) \bullet \kappa^* = n^{-1}(\tau \otimes \tau \otimes \tau) \bullet \kappa \bullet (\Psi \otimes \Psi \otimes \Psi)$, which can easily be derived from (ii) and (19), (20). □

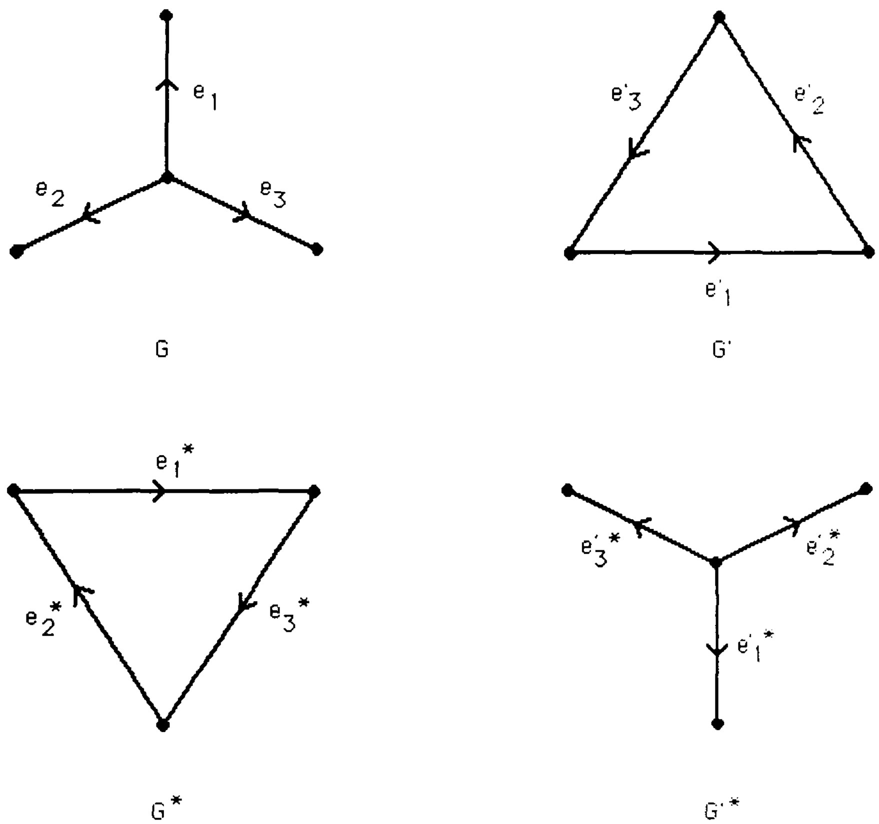

Figure 14.

Proposition 19 *Let $\mathcal{A}$ be an exactly triply regular BM-algebra. The following properties are equivalent:*

(i) *$\mathcal{A}$ has the K_4 reversibility property.*
(ii) $(\tau \otimes \tau \otimes \tau) \bullet \kappa = \kappa \bullet (\tau \otimes \tau \otimes \tau)$
(iii) $(\tau \otimes \tau \otimes \tau) \bullet \kappa^* = \kappa^* \bullet (\tau \otimes \tau \otimes \tau)$
(iv) *$\mathcal{A}$ has the planar reversibility property.*

Proof: By (9) and (52), $((\tau \otimes \tau \otimes \tau) \bullet \kappa)(E_i \otimes E_j \otimes E_k)$ equals $\sum_{(u,v,w)\in F(\mathcal{A})}(\langle Y_{ijk}, \Delta_{uvw}\rangle/\langle \Delta_{uvw}, \Delta_{uvw}\rangle)\, A_{u'} \otimes A_{v'} \otimes A_{w'} =$ $\sum_{(u,v,w)\in F(\mathcal{A})}(\langle Y_{ijk}, \Delta_{u'v'w'}\rangle/\langle \Delta_{u'v'w'}, \Delta_{u'v'w'}\rangle)\, A_u \otimes A_v \otimes A_w$. Similarly, by (14) and (52), $(\kappa \bullet (\tau \otimes \tau \otimes \tau))(E_i \otimes E_j \otimes E_k) = \kappa(E_{i^\wedge} \otimes E_{j^\wedge} \otimes E_{k^\wedge}) = \sum_{(u,v,w)\in F(\mathcal{A})}(\langle Y_{\hat{i}\hat{j}\hat{k}}, \Delta_{uvw}\rangle/\langle \Delta_{uvw}, \Delta_{uvw}\rangle)\, A_u \otimes A_v \otimes A_w$. Since $\langle \Delta_{u'v'w'}, \Delta_{u'v'w'}\rangle = \langle \Delta_{uvw}, \Delta_{uvw}\rangle$, property (ii) reduces to (ii′) For every (i, j, k) in $F^*(\mathcal{A})$ and (u, v, w) in $F(\mathcal{A})$, $\langle Y_{ijk}, \Delta_{u'v'w'}\rangle = \langle Y_{i^\wedge j^\wedge k^\wedge}, \Delta_{uvw}\rangle$. Hence by Proposition 16, (i) and (ii) are equivalent. One can show in exactly the same way that (i) and (iii) are equivalent. Finally it is easy to show that (ii) and (iii) together imply the planar reversibility property: the identities (ii) and (iii) are exactly those needed to extend the proof of Proposition 4 to all plane graphs by using Proposition 6. □

7 Examples and consequences

7.1 *Dimension 2*

We have $A_0 = I, A_1 = J - I, E_0 = n^{-1}J, E_1 = I - n^{-1}J$. The map Ψ defined by (22) is a duality map, and it is easy to check triple regularity. Using (29), (30), elementary computations give for the matrix S defined in (54) the following value:

$$S = n^{-1}\begin{bmatrix} 1 & n-1 & n-1 & n-1 & n^2-3n+2 \\ 1 & n-1 & -1 & -1 & 2-n \\ 1 & -1 & n-1 & -1 & 2-n \\ 1 & -1 & -1 & n-1 & 2-n \\ 1 & -1 & -1 & -1 & 2 \end{bmatrix}$$

where the triples indexing rows and columns are $(0,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)$ in this order. Checking that S is an involution gives by Propositions 15 and 18 another proof of the planar duality property.

The vector T defined in (55) is $T = \left(t_0^{-1}, t_0^{-1}, t_0^{-1}, t_0 t_1^{-2}, t_1^{-1}\right)$. Then it is easy to check that the star-triangle equation (56) holds if and only if $n = 2 - t_1 t_0^{-1} - t_0 t_1^{-1}$ and that this is a consequence of (26). Solving this last equation we obtain the topological spin model already described in Section 4, i.e. $t_0 = -\alpha^3, t_1 = \alpha^{-1}$, with $D = -\alpha^2 - \alpha^{-2}$ (and consequently $n = \alpha^4 + \alpha^{-4} + 2$).

As far as we know, the possibility to compute the Jones polynomial of a link or the Tutte polynomial of a planar graph by star-triangle evaluation has not been considered before, except in the case $n = 2$ (the Ising model) where it yields a polynomial-time algorithm (see [19]).

7.2 *Dimension 3*

The following result was known in the symmetric case [31]. A nice proof which works also in the non-symmetric case was found by Munemasa [40] using the framework of the algebras introduced by Terwilliger in [47]. We now reformulate this idea within the framework of Section 5.

Proposition 20 *Every BM-algebra of dimension* 3 *generated by a topological spin model is exactly triply regular.*

Proof: The following identities are easy to check using (39), (40):

$$\pi(I \otimes B \otimes C) = \pi^*(J \otimes C^T \otimes B),$$
$$\pi(J \otimes B \otimes C) = \pi^*(B^T C \otimes J \otimes J),$$
$$\pi^*(I \otimes B \otimes C) = \pi(B_\circ C^T \otimes I \otimes I).$$

Observe that I, J together with any one of the matrices W^+, W^{+T}, W^-, W^{-T} form a basis of $\mathcal{A}$. Thus using the star-triangle relation (41) together with the above identities (and those obtained by permuting their factors) we can show that an appropriate basis $\mathcal{B}$ of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

satisfies $\pi(\mathcal{B}) \subseteq \mathrm{Im}\pi^*$ and similarly another appropriate basis $\mathcal{B}'$ satisfies $\pi^*(\mathcal{B}') \subseteq \mathrm{Im}\pi$. The result now follows from Proposition 9(i). □

It is shown in [31] (see also [26]) that conversely every symmetric triply regular self-dual BM-algebra of dimension 3 is generated by a topological spin model, provided the scheme is primitive when $n \geq 5$. On the other hand it was shown by Ikuta [30] that a non-symmetric BM-algebra of dimension 3 with $n \geq 4$ is never generated by a topological spin model. Using Proposition 20 this is also a consequence from the following result due to Herzog and Reid (see [29]).

Proposition 21 *The only non-symmetric triply regular BM-algebra of dimension* 3 *is the BM-algebra of the cyclic group of order* 3.

Proof: The following properties of non-symmetric BM-algebras of dimension 3 are well known but we justify them briefly for the sake of completeness.

Since $A_2 = A_1^T$ and $J = A_0 + A_1 + A_2, \theta^*(A_1) = \theta^*(A_2) = \frac{1}{2}\theta^*(J - I) = \frac{1}{2}(n-1)$. Also since $A_1 A_2 = A_1 A_1^T$ is symmetric, it is of the form $kI + \lambda(J - I)$. Clearly $k = \theta^*(A_1) = \frac{1}{2}(n-1)$. Now applying θ^* to the equation $A_1 A_2 = \frac{1}{2}(n-1)I + \lambda(J - I)$ we obtain $(\frac{1}{2}(n-1))^2 = \frac{1}{2}(n-1) + \lambda(n-1)$ and hence $\lambda = \frac{1}{4}(n-3)$. Thus $n = 4\lambda + 3$, $A_1 A_2 = (2\lambda + 1)I + \lambda(A_1 + A_2)$ and it easily follows that $(A_1)^2 = \lambda A_1 + (\lambda + 1)A_2$ and $(A_2)^2 = (\lambda + 1)A_1 + \lambda A_2$.

We now assume that the BM-algebra is triply regular and that $\lambda \geq 1$. The following argument is essentially the same as the one given for Theorem 2.1 in [29].

Let T be the tournament on the vertex-set X with arc-set $E(T)$ defined by the adjacency matrix A_1. Let us fix $(x, y) \in E(T)$ and define $C = \{z \in X/(y, z) \in E(T), (z, x) \in E(T)\}$, $D = \{z \in X/(y, z) \in E(T), (x, z) \in E(T)\}$. Since $|C| = \lambda + 1$ and $|D| = \lambda$, C and D are not empty. The triple regularity property as defined in Section 5.3 implies that C and D induce regular tournaments, and hence $|C|$ and $|D|$ are both odd, a contradiction. □

7.3 Nomura's "Hadamard graph" topological spin models

A *Hadamard graph* is a distance-regular graph of diameter 4 on a set X of $n = 16m$ vertices (m a positive integer) with intersection array $\{4m, 4m-1, 2m, 1; 1, 2m, 4m-1, 4m\}$ (see [15], [8], [7] for definitions). This means that if we define the matrix $A_i (i = 0, \ldots, 4)$ in $\mathcal{M}(X)$ by setting the entry $A_i[x, y]$ to 1 if the vertices x, y are at distance i and to 0 otherwise, the matrices $A_i (i = 0, \ldots, 4)$ form the basis of Hadamard idempotents of a BM-algebra $\mathcal{A}$ whose parameters can be deduced from the following equations:

$$(A_1)^2 = 4mA_0 + 2mA_2, \quad A_1 A_2 = (4m-1)A_1 + (4m-1)A_3,$$
$$A_1 A_3 = 2mA_2 + 4mA_4, \quad A_1 A_4 = A_3. \tag{64}$$

It is then easy to see that the BM-algebra $\mathcal{A}$ has two self-dual structures for which the eigenmatrix P is equal to

$$P_1 = \begin{bmatrix} 1 & 4m & 8m-2 & 4m & 1 \\ 1 & 2\sqrt{m} & 0 & -2\sqrt{m} & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2\sqrt{m} & 0 & 2\sqrt{m} & -1 \\ 1 & -4m & 8m-2 & -4m & 1 \end{bmatrix}$$

or to the matrix P_2 obtained from P_1 by exchanging the second and fourth row.

K. Nomura has recently constructed for every Hadamard graph some topological spin models which belong to its BM-algebra $\mathcal{A}$. More precisely these topological spin models are given by (see [42])

$$W^+ = t_0A_0 + t_1A_1 + st_0A_2 - t_1A_3 + t_0A_4,$$

where

$$s^2 + 2(2m-1)s + 1 = 0,$$

$$t_0^2 = \frac{2\sqrt{m}}{(4m-1)s+1}, t_1^4 = 1 \quad \text{and } D = 4\sqrt{m}.$$

It is easy to check that the models with $t_1^2 = 1$ (respectively: $t_1^2 = -1$) fully belong to the self-dual BM-algebra $(\mathcal{A}, \Psi_1)$ (respectively: $(\mathcal{A}, \Psi_2)$), where the duality map Ψ_i corresponds to P_i for $i = 1, 2$. In other words, (26) holds for P_1 (respectively: P_2) when $t_1^2 = 1$ (respectively: $t_1^2 = -1$), and hence, as seen in Section 3, this establishes equations (1), (2), (3), (4). However the verification of the star-triangle equation (5) is more difficult. The proof given in [42] relies on the fact that $\mathcal{A}$ is triply regular, which is established by computing explicitly the parameters $K(ijk/uvw)$ introduced in Section 5.3 above.

It is interesting to know what link invariants are associated with the Nomura models. We found recently an explicit formula for these invariants in terms of the Jones polynomial of a link and its sublinks [32]. The first step in the proof of this formula is the following result.

Proposition 22 *For any two Hadamard graphs on the same number of vertices, the associated Nomura topological spin models (with corresponding values of* t_0, t_1, s*) yield the same link invariant.*

Proof: Let $\mathcal{A}^1, \mathcal{A}^2$ be the BM-algebras of two Hadamard graphs with $n = 16m$ vertices. For $i = 1, 2$ the representative in $\mathcal{A}^i$ of any object associated with a BM-algebra $\mathcal{A}$ in the previous sections will receive a superscript i. In particular $A_j^i (j = 0, \dots, 4)$ will be the Hadamard idempotents of $\mathcal{A}^i$ (with notation chosen in such a way that (64) holds with A_j^i replacing A_j) and for a graph G, Z_G^i will be the multilinear form on the BM-algebra $\mathcal{A}^i$ defined in Section 4. We shall also choose the indices of the idempotents so that corresponding eigenmatrices of $\mathcal{A}^1, \mathcal{A}^2$ are equal, i.e. $P^1 = P^2$ and $Q^1 = Q^2$. Let φ be the linear map from $\mathcal{A}^1$ to $\mathcal{A}^2$ defined by $\varphi(A_j^1) = A_j^2$ for $j = 0, \dots, 4$. We shall show that, for any connected plane graph G, $Z_G^1 = Z_G^2 \bullet \phi^{\otimes}$, where $\varphi^{\otimes}$ denotes the action of φ on each factor of the relevant tensor product. The result will then follow immediately from the definitions.

Since $\mathcal{A}^i$ is triply regular and self-dual, it is exactly triply regular by Proposition 9. Hence we may apply Proposition 6 and write $Z_G^i = \rho_0^i \bullet \rho_1^i \cdots \bullet \rho_k^i (i = 1, 2)$, where ρ_0^i is scalar multiplication by n, and each of $\rho_1^i, \ldots, \rho_k^i$ corresponds to the action of one of the maps $\tau^i, \theta^i, \theta^{*i}, \mu^i, \mu^{*i}, \kappa^i, \kappa^{*i}$ on some factors of a tensor product of copies of $\mathcal{A}^i$, in such a way that each pair (ρ_j^1, ρ_j^2) corresponds to one of the pairs (τ^1, τ^2), (θ^1, θ^2), $(\theta^{*1}, \theta^{*2})$, (μ^1, μ^2), (μ^{*1}, μ^{*2}), (κ^1, κ^2), $(\kappa^{*1}, \kappa^{*2})$. Clearly it will be enough to show that $\varphi^{\otimes} \bullet \rho_j^1 = \rho_j^2 \bullet \varphi^{\otimes}$ for $j = 1, \ldots, k$. This is immediate from the definition of φ if ρ_j^1 is one of the maps $\tau^1, \theta^1, \theta^{*1}, \mu^1, \mu^{*1}$. We claim that we also have

$$(\varphi \otimes \varphi \otimes \varphi) \bullet \kappa^1 = \kappa^2 \bullet (\varphi \otimes \varphi \otimes \varphi) \tag{65}$$

$$(\varphi \otimes \varphi \otimes \varphi) \bullet \kappa^{*1} = \kappa^{*2} \bullet (\varphi \otimes \varphi \otimes \varphi). \tag{66}$$

By Propositions 7 and 8 we may assume that the maps κ^i and κ^{*i} are given by (52) and (53). Since $F(\mathcal{A}^1) = F(\mathcal{A}^2)$, loosely speaking the equalities (65), (66) mean that the coefficients $\langle Y_{ijk}, \Delta_{uvw}\rangle / \langle \Delta_{uvw}, \Delta_{uvw}\rangle$ and $\langle \Delta_{uvw}, Y_{ijk}\rangle / \langle Y_{ijk}, Y_{ijk}\rangle$ (with(i, j, k) and (u, v, w) feasible) which appear in (52), (53) have the same values for $\mathcal{A}^1$ and $\mathcal{A}^2$. By (42), (43) this already holds for the denominators of these coefficients. Thus it will be enough to prove that, for any (i, j, k) and (u, v, w) in $F(\mathcal{A}^1) = F(\mathcal{A}^2)$,
$\langle \pi(E_i^1 \otimes E_j^1 \otimes E_k^1), \pi^*(A_u^1 \otimes A_v^1 \otimes A_w^1)\rangle = \langle \pi(E_i^2 \otimes E_j^2 \otimes E_k^2), \pi^*(A_u^2 \otimes A_v^2 \otimes A_w^2)\rangle$ (the other equality which is needed for (66) will be obtained from this one by complex conjugation).

Now by (17) we may write

$$E_i^1 \otimes E_j^1 \otimes E_k^1 = n^{-3} \sum_{r,s,t \in \{0,\ldots,4\}} Q_{ri} Q_{sj} Q_{tk}\, A_r^1 \otimes A_s^1 \otimes A_t^1$$

and similarly

$$E_i^2 \otimes E_j^2 \otimes E_k^2 = n^{-3} \sum_{r,s,t \in 0,\ldots,4} Q_{ri} Q_{sj} Q_{tk}\, A_r^2 \otimes A_s^2 \otimes A_t^2,$$

where $Q = Q^1 = Q^2$.

Hence it will be sufficient to show that, for any r, s, t, u, v, w in $0, \ldots, 4$ with (u, v, w) in $F(\mathcal{A}^1) = F(\mathcal{A}^2)$,

$$\langle \pi(A_r^1 \otimes A_s^1 \otimes A_t^1), \pi^*(A_u^1 \otimes A_v^1 \otimes A_w^1)\rangle = \langle \pi(A_r^2 \otimes A_s^2 \otimes A_t^2), \pi^*(A_u^2 \otimes A_v^2 \otimes A_w^2)\rangle.$$

Using (46) and (43) this can be reduced to

$$\begin{aligned} &K^1(rst/uvw)\langle \pi^*(A_u^1 \otimes A_v^1 \otimes A_w^1), \pi^*(A_u^1 \otimes A_v^1 \otimes A_w^1)\rangle \\ &\quad = K^2(rst/uvw)\langle \pi^*(A_u^2 \otimes A_v^2 \otimes A_w^2), \pi^*(A_u^2 \otimes A_v^2 \otimes A_w^2)\rangle. \end{aligned}$$

The equality of the parameters $K^1(rst/uvw)$ and $K^2(rst/uvw)$ for every r, s, t, u, v, w in $\{0, \ldots, 4\}$ with (u, v, w) feasible in $\mathcal{A}^1$ and $\mathcal{A}^2$ is established in [42] (where these parameters are given as functions of m). Since by (43) $\langle \Delta_{uvw}, \Delta_{uvw}\rangle$ has the same values for $\mathcal{A}^1$ and $\mathcal{A}^2$, the proof is complete. □

7.4 Topological spin models in Abelian group schemes

Let us recall some notations from Section 6.2. X is a finite Abelian group written additively, and for $i \in X$ the matrix A_i is defined by $A_i[x, y] = \delta_{i,y-x}$ for every x, y in X. We also have $i' = -i$ and $E_i = n^{-1} \sum_{j\in X} \overline{\chi_i(j)} A_j$, $A_i = \sum_{j\in X} \chi_i(j) E_j$ where the χ_i, $i \in X$, are the characters of X, with indices chosen such that $\chi_i(j) = \chi_j(i)$ for all i, j in X. Defining Ψ by (22) we obtain a self-dual BM-algebra $(\mathcal{A}, \Psi)$.

Then for any u, v, w in X,

$$\Delta_{uvw} = \pi^*(A_u \otimes A_v \otimes A_w) = \sum_{\alpha,\beta,\gamma\in X} A_u[\beta, \gamma] A_v[\gamma, \alpha] A_w[\alpha, \beta]\, \alpha \otimes \beta \otimes \gamma$$
$$= \delta_{0,u+v+w} \sum_{\alpha\in X} \alpha \otimes (\alpha + w) \otimes (\alpha - v).$$

So (u, v, w) is feasible if and only if $u + v + w = 0$ and in this case $\langle \Delta_{uvw}, \Delta_{uvw} \rangle = n$. Similarly, $\pi(A_u \otimes A_v \otimes A_w) = \sum_{\alpha,\beta,\gamma\in X} (\sum_{x\in X} A_u[x, \alpha] A_v[x, \beta] A_w[x, \gamma]) \alpha \otimes \beta \otimes \gamma = \sum_{\alpha\in X} \alpha \otimes (\alpha + v - u) \otimes (\alpha + w - u) = \pi^*(A_{w-v} \otimes A_{u-w} \otimes A_{v-u})$. By Proposition 9(i), this shows that $\mathcal{A}$ is exactly triply regular.

Also $Y_{ijk} = \pi(E_i \otimes E_j \otimes E_k) = n^{-3} \sum_{u,v,w\in X} \overline{\chi_i(u)}\, \overline{\chi_j(v)}\, \overline{\chi_k(w)}\, \pi(A_u \otimes A_v \otimes A_w) = n^{-3} \sum_{u,v,w\in X} \sum_{\alpha\in X} \overline{\chi_i(u)}\, \overline{\chi_j(v)}\, \overline{\chi_k(w)}\, \alpha \otimes (\alpha + v - u) \otimes (\alpha + w - u) = n^{-3} \sum_{\alpha\in X} \sum_{u,x,y\in X} \overline{\chi_i(u)}\, \overline{\chi_j(u + x)}\, \overline{\chi_k(u + y)}\, \alpha \otimes (\alpha + x) \otimes (\alpha + y)$.

Now $\sum_{u\in X} \overline{\chi_i(u)}\, \overline{\chi_j(u + x)}\, \overline{\chi_k(u + y)} = \sum_{u\in X} \overline{\chi_{i+j+k}(u)}\, \overline{\chi_j(x)}\, \overline{\chi_k(y)} = n\delta_{0,i+j+k} \overline{\chi_j(x)}\, \overline{\chi_k(y)}$ and thus $Y_{ijk} = n^{-2} \delta_{0,i+j+k} \sum_{\alpha,x,y\in X} \overline{\chi_j(x)}\, \overline{\chi_k(y)}\, \alpha \otimes (\alpha + x) \otimes (\alpha + y)$.

So we verify that $F(\mathcal{A}) = F^*(\mathcal{A}) = \{(i, j, k) \in X^3 / i + j + k = 0\}$. Moreover if (i, j, k) is feasible, $Y_{i'j'k'} = n^{-2} \sum_{\alpha,x,y\in X} \chi_j(x) \chi_k(y)\, \alpha \otimes (\alpha + x) \otimes (\alpha + y)$. Hence for feasible (i, j, k) and (u, v, w), $\langle Y_{i'j'k'}, \Delta_{uvw} \rangle = n^{-1} \chi_j(w) \chi_k(-v)$. It is now easy to check the equality (61) of Proposition 15 and to obtain that the self-dual BM-algebra $(\mathcal{A}, \Psi)$ has the K_4 duality property. It then follows from Proposition 18 that $(\mathcal{A}, \Psi)$ has the planar duality property and thus we have obtained another proof of Proposition 11. Similarly the K_4 reversibility property is immediate from Proposition 16 and yields planar reversibility by Proposition 19.

The matrix S defined by (54) in Section 5.6 has rows and columns indexed by $F(\mathcal{A})$ and has entries $S_{(u,v,w)(i,j,k)} = n^{-1} \chi_j(w) \chi_k(-v)$. We use this to establish the following result inspired by [3].

Proposition 23 *Let t_i, $i \in X$, be non-zero complex numbers such that*

$$t_i t_j = \chi_i(j) t_0 t_{i+j} \text{ for every } i, j \text{ in } X \tag{67}$$
$$\sum_{i\in X} t_i^{-1} = D t_0, \text{ where } D^2 = n. \tag{68}$$

Then setting $W^+ = \sum_{i\in X} t_i A_i$ and $W^- = \sum_{i\in X} t_{i'}^{-1} A_i$ yields a topological spin model (X, W^+, W^-) with loop variable D which fully belongs to $(\mathcal{A}, \Psi)$.

Proof: Let us first check (26), which reads here:

$$\sum_{j\in X} \chi_i(j) t_{j'}^{-1} = D t_{i'} \text{ for every } i \text{ in } X.$$

By (67) we have

$$\sum_{j\in X} \chi_i(j) t_{i'}^{-1} t_{j'}^{-1} = \sum_{j\in X} \chi_i(j)\chi_{i'}(j')^{-1} t_0^{-1}(t_{i'+j'})^{-1} = t_0^{-1}\sum_{j\in X}(t_{i'+j'})^{-1}$$

and hence (26) follows from (68). We now establish (56), that is $ST = T$, where T is the column vector indexed by $F(\mathcal{A})$ with entries $T_{(i,j,k)} = t_{i'}^{-1}t_j^{-1}t_k$. Thus we must show that for every (u, v, w) in $F(\mathcal{A})$, $n^{-1}\sum_{(i,j,k)\in F(\mathcal{A})} \chi_j(w)\chi_k(-v)t_{i'}^{-1}t_j^{-1}t_k = t_{u'}^{-1}t_v^{-1}t_w$, or equivalently $n^{-1}\sum_{j,k\in X}\chi_j(w)\chi_k(-v)(t_{j+k})^{-1}t_j^{-1}t_k = (t_{v+w})^{-1}t_v^{-1}t_w$. Using (67) this reduces to

$$n^{-1}\sum_{j,k\in X}\chi_j(w)\chi_k(-v)t_j^{-1}t_k^{-1}\chi_j(k)t_0t_j^{-1}t_k = t_v^{-1}t_w^{-1}\chi_v(w)t_0t_v^{-1}t_w$$

or equivalently

$$n^{-1}\sum_{j,k\in X}\chi_j(w)\chi_k(-v)\chi_j(k)t_j^{-2} = t_v^{-2}\chi_v(w).$$

Since $\Sigma_{k\in X}\chi_k(-v)\chi_j(k) = \Sigma_{k\in X}\chi_k(j - v) = n\delta_{j,v}$ this becomes $\sum_{j\in X}\chi_j(w)\delta_{j,v}\, t_j^{-2} = t_v^{-2}\chi_v(w)$ and (56) is proved. □

Note that if one finds a solution to (67) such that $\Sigma_{i\in X}t_i^{-1}$ is non-zero, we can normalize it so that (68) holds as well.

When X is a cyclic group, explicit solutions to (67), (68) can be found in [3]. In [6], the relationship of equations (67), (68) with the modular invariance property considered in [1], [3] and [5] is clarified. Moreover these equations are explicity solved for general Abelian groups and the corresponding spin models are identified with those recently constructed by Kac and Wakimoto from any even rational lattice [39].

8 Conclusion

The concept of partition function of a spin model defines an interaction between graphs and BM-algebras. We can investigate this interaction from two main points of view.

From the first point of view we shall be mainly interested in the invariants of graphs and links that can be evaluated as partition functions of spin models with all edge weights in a given BM-algebra $\mathcal{A}$. For graphs it would be natural to assign the same matrix $\Sigma t_i A_i$ to each edge and to consider the resulting value of the partition function as a polynomial in the variables t_i (see [28]) which we may call the $\mathcal{A}$-polynomial. When $t_i = 1$ and $t_j = 0$ for $j \neq i$, the $\mathcal{A}$-polynomial of a graph G gives the number of homomorphisms from G to the graph with adjacency matrix A_i. Such invariants are studied in [27] and it would be interesting to investigate the $\mathcal{A}$-polynomials in the same spirit.

For series-parallel graphs Proposition 3 gives a matrix-free approach to the computation of the partition function (which can be applied to the $\mathcal{A}$-polynomial, or to link invariants).

For plane graphs Proposition 6 also gives a matrix-free approach using star-triangle evaluation if we restrict our attention to exactly triply regular BM-algebras. Clearly star-triangle evaluation whenever possible will be more efficient (except for very small graphs or link diagrams) than the brute force computation based on state enumeration. It would be interesting to study rigorously the computational complexity of star-triangle evaluation. Also, Proposition 22 is only a first example of a contribution of the matrix-free approach to a better understanding of the link invariants associated with topological spin models, and we plan to develop further this line of research in the near future.

From the second point of view we shall be mostly interested in properties of BM-algebras which are relevant to the computation of partition functions. We have studied a number of such properties here: self-duality, exact triple regularity, generation by a topological spin model, planar duality, planar reversibility and related notions. We have established a number of logical implications between these properties. For instance Proposition 12 establishes a connection between (strong) generation by a topological spin model and planar duality, Proposition 14 shows that planar duality implies planar reversibility, and Propositions 18, 19 assert that for exactly triply regular BM-algebras, planar duality or reversibility are equivalent to their specializations to K_4. We have also given a few examples to illustrate our results. However we are very far from a clear picture of the relations existing between the above properties and much more examples would be needed. In particular it would be interesting to have examples of the following types of BM-algebras, or to prove that there are none:

(a) exactly triply regular, but not self-dual
(b) self-dual, satisfying the K_4 duality property but not the planar duality property
(b′) self-dual, not satisfying the K_4 duality property
(b″) self-dual, triply regular, not satisfying the K_4 duality property
(c) satisfying the K_4 reversibility property but not the planar reversibility property
(c′) not satisfying the K_4 reversibility property
(c″) triply regular, not satisfying the K_4 reversibility property

Natural candidates for (b) and (c) would be non-symmetric 3-dimensional BM-algebras with no anti-automorphism (it is easy to see that the full reversibility property does not hold for these BM-algebras, since, with the notations of Proposition 21, $Z_T(A_1 \otimes \cdots \otimes A_1) \geq 1$ while $Z_T(A_2 \otimes \cdots \otimes A_2) = 0$). An example for $n = 15$ is described in [13].

By Propositions 18 and 19, for exactly triply regular BM-algebras, planar duality or planar reversibility can be checked on a single graph, namely K_4. Is there also such a finite decision procedure for general BM-algebras? That is, does there exist a computable function f such that for every BM-algebra on X, the planar duality or planar reversibility property is equivalent to its specialization to planar graphs with at most $f(|X|)$ edges?

In the context of self-dual triply regular BM-algebras we have obtained some simple forms for the equations defining topological spin models, and we have described some interesting solutions in the case of Abelian group schemes. We hope to be able to use this approach for certain other self-dual triply regular BM-algebras. But it would be also interesting to obtain general results concerning the existence of solutions for these equations.

Finally, we believe that Proposition 12 could be significantly generalized. This would give a larger class of topological spin models for which the associated link invariants do not distinguish between inverse links.

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