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Equivariant Pieri Rule for the homology of the affine Grassmannian

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Abstract An explicit rule is given for the product of the degree two class with an arbitrary Schubert class in the torus-equivariant homology of the affine Grassmannian. In addition a Pieri rule (the Schubert expansion of the product of a special Schubert class with an arbitrary one) is established for the equivariant homology of the affine Grassmannians of SL_n and a similar formula is conjectured for Sp_{2n} and SO_{2n+1}. For SL_n the formula is explicit and positive. By a theorem of Peterson these compute certain products of Schubert classes in the torus-equivariant quantum cohomology of flag varieties. The SL_n Pieri rule is used in our recent definition of k-double Schur functions and affine double Schur functions.

Keywords Schubert calculus · Affine Grassmannian · Pieri rule · Quantum cohomology

1 Introduction

Let G be a semisimple algebraic group over $\mathbb{C}$ with a Borel subgroup B and maximal torus T. Let $\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ be the affine Grassmannian of G. The T-equivariant homology $H_T(\mathrm{Gr}_G)$ and cohomology $H^T(\mathrm{Gr}_G)$ are dual Hopf algebras over $S = H^T(\mathrm{pt})$ with Pontryagin and cup products, respectively. Let W_{af}^0 be the minimal length cosets in W_{af}/W where W_{af} and W are the affine and finite Weyl groups. Let $\{\xi_w \mid w \in W_{\mathrm{af}}^0\}$ be the Schubert basis of $H_T(\mathrm{Gr}_G)$. Define the equivariant

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Schubert homology structure constants $d_{uv}^w \in S$ by

$$\xi_u \xi_v = \sum_{w \in W_{\mathrm{af}}^0} d_{uv}^w \xi_w \tag{1}$$

where $u, v \in W_{\mathrm{af}}^0$. One interest in the polynomials d_{uv}^w is the fact that they are precisely the Schubert structure constants for the T-equivariant quantum cohomology rings $QH^T(G/B)$ [9, 13]. Due to a result of Mihalcea [12], they have the positivity property

$$d_{uv}^w \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]. \tag{2}$$

Our first main result (Theorem 6) is an "equivariant homology Chevalley formula", which describes $d_{r_0,v}^w$ for an arbitrary affine Grassmannian. Our second main result (Theorem 20) is an "equivariant homology Pieri formula" for $G = SL_n$, which is a manifestly positive formula for $d_{\sigma_m,v}^w$ where the homology classes $\{\xi_{\sigma_m} \mid 1 \leq m \leq n-1\}$ are the special classes that generate $H_T(\mathrm{Gr}_{SL_n})$. In a separate work [10] we use this Pieri formula to define new symmetric functions, called k-double Schur functions and affine double Schur functions, which represent the equivariant Schubert homology and cohomology classes for Gr_{SL_n}.

2 The equivariant homology of Gr_G

We recall Peterson's construction [13] of the equivariant Schubert basis $\{j_w \mid w \in W_{\mathrm{af}}^0\}$ of $H_T(\mathrm{Gr}_G)$ using the level-zero variant of the Kostant and Kumar (graded) nilHecke ring [6]. We also describe the equivariant localizations of Schubert cohomology classes for the affine flag ind-scheme in terms of the nilHecke ring; these are an important ingredient in our equivariant Chevalley and Pieri rules.

2.1 Peterson's level-zero affine nilHecke ring

Let I and $I_{\mathrm{af}} = I \cup \{0\}$ be the finite and affine Dynkin node sets and $(a_{ij} \mid i, j \in I_{\mathrm{af}})$ the affine Cartan matrix.

Let $P_{\mathrm{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i \in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$ be the affine weight lattice, with δ the null root and Λ_i the affine fundamental weight. The dual lattice $P_{\mathrm{af}}^* = \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{af}}, \mathbb{Z})$ has dual basis $\{d\} \cup \{\alpha_i^\vee \mid i \in I_{\mathrm{af}}\}$ where d is the degree generator and $\alpha_i^\vee$ is a simple coroot. The simple roots $\{\alpha_i \mid i \in I_{\mathrm{af}}\} \subset P_{\mathrm{af}}$ are defined by $\alpha_j = \delta_{j0}\delta + \sum_{i \in I_{\mathrm{af}}} a_{ij} \Lambda_i$ for $j \in I_{\mathrm{af}}$ where $(a_{ij} \mid i, j \in I_{\mathrm{af}})$ is the affine Cartan matrix. Then $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ for all $i, j \in I_{\mathrm{af}}$. Let $(a_i \mid i \in I_{\mathrm{af}})$ (resp. $(a_i^\vee \mid i \in I_{\mathrm{af}})$) be the tuple of relatively prime positive integers giving a relation among the columns (resp. rows) of the affine Cartan matrix. Then $\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i$. Let $c = \sum_{i \in I_{\mathrm{af}}} a_i^\vee \alpha_i^\vee \in P_{\mathrm{af}}^*$ be the canonical central element. The level of a weight $\lambda \in P_{\mathrm{af}}$ is defined by $\langle c, \lambda \rangle$.

There is a canonical projection $P_{\mathrm{af}} \to P$ where P is the finite weight lattice, with kernel $\mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$. There is a section $P \to P_{\mathrm{af}}$ of this projection whose image lies in the sublattice of $\bigoplus_{i \in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$ consisting of level-zero weights. We regard $P \subset P_{\mathrm{af}}$ via this section.

Let W and W_{af} denote the finite and affine Weyl groups. Denote by $\{r_i \mid i \in I_{\mathrm{af}}\}$ the simple generators of W_{af}. W_{af} acts on P_{af} by $r_i \cdot \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for $i \in I_{\mathrm{af}}$ and $\lambda \in P_{\mathrm{af}}$. W_{af} acts on P_{af}^* by $r_i \cdot \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee$ for $i \in I_{\mathrm{af}}$ and $\mu \in P_{\mathrm{af}}^*$. There is an isomorphism $W_{\mathrm{af}} \cong W \ltimes Q^\vee$ where $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset P_{\mathrm{af}}^*$ is the finite coroot lattice. The embedding $Q^\vee \to W_{\mathrm{af}}$ is denoted $\mu \mapsto t_\mu$. The set of real affine roots is $W_{\mathrm{af}} \cdot \{\alpha_i \mid i \in I_{\mathrm{af}}\}$. For a real affine root $\alpha = w \cdot \alpha_i$, the associated coroot is well-defined by $\alpha^\vee = w \cdot \alpha_i^\vee$.

Let $S = \mathrm{Sym}(P)$ be the symmetric algebra, and $Q = \mathrm{Frac}(S)$ the fraction field. $W_{\mathrm{af}} \cong W \ltimes Q^\vee$ acts on P (and therefore on S and on Q) by the level-zero action:

$$wt_\mu \cdot \lambda = w \cdot \lambda \quad \text{for } w \in W \text{ and } \mu \in Q^\vee. \tag{3}$$

Since $t_{-\theta^\vee} = r_\theta r_0$ we have

$$r_0 \cdot \lambda = r_\theta \cdot \lambda \quad \text{for } \lambda \in P. \tag{4}$$

Finally, we have $\delta = \alpha_0 + \theta$ where $\theta \in P$ is the highest root. So under the projection $P_{\mathrm{af}} \to P$, $\alpha_0 \mapsto -\theta$.

Let $Q_{W_{\mathrm{af}}} = \bigoplus_{w \in W_{\mathrm{af}}} Qw$ be the skew group ring, the Q-vector space $Q \otimes_{\mathbb{Q}} \mathbb{Q}[W_{\mathrm{af}}]$ with Q-basis W_{af} and product $(p \otimes v)(q \otimes w) = p(v \cdot q) \otimes vw$ for $p, q \in Q$ and $v, w \in W_{\mathrm{af}}$. $Q_{W_{\mathrm{af}}}$ acts on Q: $q \in Q$ acts by left multiplication and W_{af} acts as above.

For $i \in I_{\mathrm{af}}$ define the element $A_i \in Q_{W_{\mathrm{af}}}$ by

$$A_i = \alpha_i^{-1}(1 - r_i). \tag{5}$$

A_i acts on S since

$$A_i \cdot \lambda = \langle \alpha_i^\vee, \lambda \rangle \quad \text{for } \lambda \in P \tag{6}$$

$$A_i \cdot (ss') = (A_i \cdot s)s' + (r_i \cdot s)(A_i \cdot s') \quad \text{for } s, s' \in S. \tag{7}$$

The A_i satisfy $A_i^2 = 0$ and

$$\underbrace{A_i A_j A_i \cdots}_{m_{ij} \text{ times}} = \underbrace{A_j A_i A_j \cdots}_{m_{ij} \text{ times}}$$

where

$$\underbrace{r_i r_j r_i \cdots}_{m_{ij} \text{ times}} = \underbrace{r_j r_i r_j \cdots}_{m_{ij} \text{ times}}.$$

For $w \in W_{\mathrm{af}}$ we define A_w by

$$A_w = A_{i_1} A_{i_2} \cdots A_{i_\ell} \quad \text{where} \tag{8}$$

$$w = r_{i_1} r_{i_2} \cdots r_{i_\ell} \quad \text{is reduced.} \tag{9}$$

The level-zero graded affine nilHecke ring $\mathbb{A}$ (Peterson's [13] variant of the nilHecke ring of Kostant and Kumar [6] for an affine root system) is the subring of $Q_{W_{\mathrm{af}}}$ generated by S and $\{A_i \mid i \in I_{\mathrm{af}}\}$. In $\mathbb{A}$ we have the commutation relation

$$A_i\lambda = (A_i \cdot \lambda)1 + (r_i \cdot \lambda)A_i \quad \text{for } \lambda \in P. \tag{10}$$

In particular

$$\mathbb{A} = \bigoplus_{w \in W_{\mathrm{af}}} SA_w. \tag{11}$$

2.2 Localizations of equivariant cohomology classes

Using the relation

$$r_i = 1 - \alpha_i A_i \tag{12}$$

$w \in W_{\mathrm{af}}$ may be regarded as an element of $\mathbb{A}$. For $v, w \in W_{\mathrm{af}}$ define the elements $\xi^v(w) \in S$ by

$$w = \sum_{v \in W} (-1)^{\ell(v)} \xi^v(w) A_v. \tag{13}$$

Using a reduced decomposition (9) for w and substituting (12) for its simple reflections, one obtains the formula [1] [2]

$$\xi^v(w) = \sum_{b \in [0,1]^\ell} \left(\prod_{j=1}^{\ell} \alpha_{i_j}^{b_j} r_{i_j} \right) \cdot 1 \tag{14}$$

where the sum runs over b such that $\prod_{b_j=1} r_{i_j} = v$ is reduced and the product over j is an ordered left-to-right product of operators. Each b encodes a way to obtain a reduced word for v as an embedded subword of the given reduced word of w: if $b_j = 1$ then the reflection r_{i_j} is included in the reduced word for v. Given a fixed b and an index j such that $b_j = 1$, the root associated to the reflection r_{i_j} is by definition $r_{i_1} r_{i_2} \cdots r_{i_{j-1}} \cdot \alpha_{i_j}$. The summand for b is the product of the roots associated to reflections in the given embedded subword.

It is immediate that

$$\xi^v(w) = 0 \quad \text{unless } v \le w \tag{15}$$

$$\xi^{\mathrm{id}}(w) = 1 \quad \text{for all } w. \tag{16}$$

The element $\xi^v(w) \in S$ has the following geometric interpretation. Let $X_{\mathrm{af}} = G_{\mathrm{af}}/B_{\mathrm{af}}$ be the Kac–Moody flag ind-variety of affine type [7]. For every $v \in W_{\mathrm{af}}$ there is a T-equivariant cohomology class $[X_v] \in H^T(X_{\mathrm{af}})$ and for each $w \in W_{\mathrm{af}}$ there is an associated T-fixed point (denoted w) in X_{af} and a localization map $i_w^* : H^T(X_{\mathrm{af}}) \to H^T(w) \simeq H^T(\mathrm{pt})$ [7]. Then $\xi^v(w) = i_w^*([X_v])$. Moreover, the map $H^T(X_{\mathrm{af}}) \to H^T(W_{\mathrm{af}}) \cong \mathrm{Fun}(W_{\mathrm{af}}, S)$ given by restriction of a class to the T-fixed subset $W_{\mathrm{af}} \subset X_{\mathrm{af}}$, is an injective S-algebra homomorphism where $\mathrm{Fun}(W_{\mathrm{af}}, S)$ is the S-algebra of functions $W_{\mathrm{af}} \to S$ with pointwise product. The function $\xi^v \in \mathrm{Fun}(W_{\mathrm{af}}, S)$ is the image of $[X_v]$. The image Φ of $H^T(X_{\mathrm{af}})$ in $\mathrm{Fun}(W_{\mathrm{af}}, S)$ satisfies the GKM condition [3] [6]: For $f \in \Phi$ we have[1]

$$f(w) - f(r_\beta w) \in \beta S \quad \text{for all } w \in W_{\mathrm{af}} \text{ and affine real roots } \beta. \tag{17}$$

Lemma 1 *Suppose $u, v \in W_{\mathrm{af}}$ with $\ell(uv) = \ell(u) + \ell(v)$. Then*

$$\xi^{uv}(uv) = \xi^u(u)\big(u \cdot \xi^v(v)\big). \tag{18}$$

Lemma 2 *Suppose $v, w \in W_{\mathrm{af}}$. Then*

$$\xi^v(w) = (-1)^{\ell(v)} w \cdot \big(\xi^{v^{-1}}\big(w^{-1}\big)\big). \tag{19}$$

2.3 Peterson subalgebra and Schubert homology basis

Let $K \subset G$ denote the maximal compact subgroup of G. The homotopy equivalence between Gr_G and the based loop space ΩK endows the equivariant homology $H_T(\mathrm{Gr}_G)$ and cohomology $H^T(\mathrm{Gr}_G)$ with the structure of dual Hopf algebras. The Pontryagin multiplication in the homology $H_T(\mathrm{Gr}_G)$ is induced by the group structure of ΩK. We let $\{\xi_w\}$ denote the equivariant Schubert basis of $H_T(\mathrm{Gr}_G)$, dual (via the cap product) to the basis $\{\xi^w\}$ of $H^T(\mathrm{Gr}_G)$.

The Peterson subalgebra of $\mathbb{A}$ is the centralizer subalgebra $\mathbb{P} = Z_{\mathbb{A}}(S)$ of S in $\mathbb{A}$.

Theorem 3 [13] *There is an isomorphism $H_T(\mathrm{Gr}_G) \to \mathbb{P}$ of commutative Hopf algebras over S. For $w \in W_{\mathrm{af}}^0$ let j_w denote the image of ξ_w in $\mathbb{P}$. Then j_w is the unique element of $\mathbb{P}$ with the property that $j_w^w = 1$ and $j_w^x = 0$ for any $x \in W_{\mathrm{af}}^0 \setminus \{w\}$ where $j_w^x \in S$ are defined by*

$$j_w = \sum_{x \in W_{\mathrm{af}}} j_w^x A_x. \tag{20}$$

Moreover, if $j_w^x \neq 0$ then $\ell(x) \geq \ell(w)$ and j_w^x is a polynomial of degree $\ell(x) - \ell(w)$.

The Schubert structure constants for $H_T(\mathrm{Gr}_G)$ are obtained as coefficients of the elements j_w.

Proposition 4 ([13]) *Let $u, v, w \in W_{\mathrm{af}}^0$. Then*

$$d_{uv}^w = \begin{cases} j_u^{wv^{-1}} & \text{if } \ell(w) = \ell(v) + \ell(wv^{-1}) \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

[1] Using equivariance for the maximal torus $T_{\mathrm{af}} \subset G_{\mathrm{af}}$, the GKM condition characterizes the image of localization to torus fixed points. However, after forgetting equivariance down to the smaller torus T, elements of Φ are characterized by additional conditions, which were determined in [4].

Due to the fact [9, 13] that the collections of Schubert structure constants for $H_T(\mathrm{Gr}_G)$ and $QH^T(G/B)$ are the same and Mihalcea's positivity theorem for equivariant quantum Schubert structure constants, we have the positivity property

Proposition 5 *$j_w^x \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]$ for all $w \in W_{\mathrm{af}}^0$ and $x \in W_{\mathrm{af}}$.*

Given $u \in W_{\mathrm{af}}^0$ let $t^u = t_\lambda$ where $\lambda \in Q^\vee$ is such that $t_\lambda W = uW$.

Since the translation elements act trivially on S and $W_{\mathrm{af}} \subset \mathbb{A}$ via (12), we have $t_\lambda \in \mathbb{P}$ for all $\lambda \in Q^\vee$, so that $t_\lambda \in \bigoplus_{v \in W_{\mathrm{af}}^0} S j_v$. For any $w \in W_{\mathrm{af}}^0$, we have

$$t^w = \sum_{v \in W_{\mathrm{af}}^0} (-1)^{\ell(v)} \xi^v\big(t^w\big) j_v = \sum_{v \in W_{\mathrm{af}}^0} (-1)^{\ell(v)} \xi^v(w) j_v$$

by the definitions and Lemma 1.

Define the $W_{\mathrm{af}}^0 \times W_{\mathrm{af}}^0$-matrices

$$A_{wv} = (-1)^{\ell(v)} \xi^v(w) \tag{22}$$

$$B = A^{-1}. \tag{23}$$

The matrix A is lower triangular by (15) and has nonzero diagonal terms, and is hence invertible over $Q = \mathrm{Frac}(S)$. We have

$$j_v = \sum_{\substack{w \in W_{\mathrm{af}}^0 \\ w \leq v}} B_{wv} t^w.$$

Taking the coefficient of A_x for $x \in W_{\mathrm{af}}$, we have

$$j_v^x = (-1)^{\ell(x)} \sum_{\substack{w \in W_{\mathrm{af}}^0 \\ w \leq v}} B_{wv} \xi^x\big(t^w\big). \tag{24}$$

Note that if $\Omega \subset W_{\mathrm{af}}^0$ is any order ideal (downwardly closed subset) then the restriction $A|_{\Omega \times \Omega}$ is invertible. In the sequel we choose certain such order ideals and find a formula for the inverse of this submatrix. Since the values of ξ^x are given by (14) we obtain an explicit formula for j_v^x for $v \in \Omega$ and all $x \in W_{\mathrm{af}}$.

3 Equivariant homology Chevalley rule

Theorem 6 *For every $x \in W_{\mathrm{af}} \setminus \{\mathrm{id}\}$, $\xi^{x^{-1}}(r_\theta) \in \theta S$ and*

$$j_{r_0} = \sum_{x \in W \setminus \{\mathrm{id}\}} \big(\theta^{-1} \xi^{x^{-1}}(r_\theta) A_x + \xi^{x^{-1}}(r_\theta) A_{r_0 x}\big). \tag{25}$$

Proof For $x \neq \mathrm{id}$, the GKM condition (17) and (15) implies that $\xi^{x^{-1}}(r_\theta) \in \theta S$. $\Omega = \{\mathrm{id}, r_0\} \subset W_{\mathrm{af}}^0$ is an order ideal. The matrix $A|_{\Omega \times \Omega}$ and its inverse are given by

$$\begin{pmatrix} 1 & 0 \\ 1 & \theta \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ -\theta^{-1} & \theta^{-1} \end{pmatrix}.$$

Since $\mathrm{id} = t^{\mathrm{id}}$ and $t_{\theta^\vee} = t^{r_0}$ (as $t_{\theta^\vee} = r_0 r_\theta$), we have

$$(-1)^{\ell(y)} j_{r_0}^y = -\theta^{-1}\xi^y(\mathrm{id}) + \theta^{-1}\xi^y(t_{\theta^\vee}).$$

By the length condition in Theorem 3 we have

$$(-1)^{\ell(y)} j_{r_0}^y = \theta^{-1}\xi^y(t_{\theta^\vee}) \quad \text{for } y \neq \mathrm{id}.$$

By (15) $j_{r_0}^y = 0$ unless $y \leq t_{\theta^\vee} = r_0 r_\theta$. So assume this.

Suppose $r_0 y < y$. Write $y = r_0 x$. Then

$$(-1)^{\ell(y)}\xi^y(t_{\theta^\vee}) = (-1)^{\ell(y)}(\alpha_0)\big(r_0 \cdot \xi^x(r_\theta)\big) = (-1)^{\ell(x)}\theta\big(r_\theta \cdot \xi^x(r_\theta)\big) = \theta\,\xi^{x^{-1}}(r_\theta).$$

If $r_0 y > y$ then we write $y = x \leq r_\theta$ and

$$(-1)^{\ell(x)}\xi^x(t_{\theta^\vee}) = (-1)^{\ell(x)} r_0 \cdot \xi^x(r_\theta) = (-1)^{\ell(x)} r_\theta \cdot \xi^x(r_\theta) = \xi^{x^{-1}}(r_\theta)$$

as required. □

The formula (14) shows that $\xi^{x^{-1}}(r_\theta) \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]$. The same holds for $\theta^{-1}\xi^{x^{-1}}(r_\theta)$. Indeed,

Lemma 7 $\alpha^{-1}\xi^x(r_\alpha) \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]$ *for any positive root* α.

Proof The reflection r_α has a reduced word $\mathbf{i} = i_1 i_2 \cdots i_{r-1} i_r i_{r-1} \cdots i_1$ which is symmetric. Consider the different embeddings $\mathbf{j}$ of reduced words of x into $\mathbf{i}$, as in (14). If $\mathbf{j}$ uses the letter i_r, then the corresponding term in (14) has θ as a factor. Otherwise, $\mathbf{j}$ uses i_s but not $i_{s+1}, \ldots, i_r$, for some s. But then there is another embedding of $\mathbf{j}'$ of the same reduced word of x into $\mathbf{i}$, which uses the other copy of the letter i_s in $\mathbf{i}$. The two terms in (14) which correspond to $\mathbf{j}$ and $\mathbf{j}'$ contribute $A(\beta - r_\alpha \cdot \beta) = A(\langle \alpha^\vee, \beta\rangle \alpha)$ where $A \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]$, and β is an inversion of r_α. It follows that $\langle \alpha^\vee, \beta\rangle > 0$. The lemma follows. □

Remark 8 The polynomials $\xi^{x^{-1}}(r_\theta)$ appearing in (25) may be computed entirely in the finite Weyl group and finite weight lattice.

Remark 9 In [8, Proposition 2.17], we gave an expression for the non-equivariant part of j_{r_0}, consisting of the terms $j_{r_0}^x A_x$ where $\ell(x) = 1 = \ell(r_0)$. This follows easily from Theorem 6 and the fact [6] that $\xi^{r_i}(w) = \omega_i - w \cdot \omega_i$, where ω_i is the ith fundamental weight.

3.1 Application to quantum cohomology

The equivariant homology Chevalley rule (Theorem 6) may be used to obtain a new formula for some Gromov–Witten invariants for $QH^T(G/P)$ where $P \subsetneq G$ is a parabolic subgroup.[2]

For this subsection we adopt the notation of [9], some of which we recall presently. Our goal is Proposition 10, which is the equivariant generalization of [9, Prop. 11.2].

Consider the Levi factor of P. It has Dynkin node subset $I_P \subset I$, Weyl group $W_P \subset W$, coroot lattice $Q_P^\vee \subset Q^\vee$, root system $R_P \subset R$ and positive roots R_P^+. Denote the natural projection $Q_{\mathrm{af}} \to Q$ by $\beta \mapsto \overline{\beta}$. Define

$$
\begin{aligned}
(W_P)_{\mathrm{af}} &= W_P \ltimes Q_P^\vee \\
\left(R_P^+\right)_{\mathrm{af}} &= \left\{\beta \in R_{\mathrm{af}}^+ \mid \overline{\beta} \in R_P\right\} \\
\left(W^P\right)_{\mathrm{af}} &= \left\{x \in W_{\mathrm{af}} \mid x \cdot \beta > 0 \text{ for all } \beta \in \left(R_P^+\right)_{\mathrm{af}}\right\}.
\end{aligned}
$$

Every element $w \in W_{\mathrm{af}}$ has a unique expression $w = w_1 w_2$ with $w_1 \in (W^P)_{\mathrm{af}}$ and $w_2 \in (W_P)_{\mathrm{af}}$; denote by $\pi_P : W_{\mathrm{af}} \mapsto (W^P)_{\mathrm{af}}$ the map that sends $w \mapsto w_1$.

Recall that the ring $H_T(\mathrm{Gr}_G)$ has an S-basis $\{\xi_x \mid x \in W_{\mathrm{af}}^-\}$. It has an ideal

$$
J_P = \bigoplus_{x \in W_{\mathrm{af}}^- \setminus (W^P)_{\mathrm{af}}} S\xi_x.
$$

The set $\mathcal{T} = \{\xi_{\pi_P(t_\lambda)} \mid \lambda \in \tilde{Q}\}$ is multiplicatively closed, where $\tilde{Q} = \{\lambda \in Q^\vee \mid \langle \lambda, \alpha_i \rangle \leq 0 \text{ for all } i \in I\}$ is the set of antidominant elements of $Q^\vee$. Finally let $\eta_P : Q^\vee \to Q^\vee / Q_P^\vee$ be the natural projection. Then by [9, Thm. 10.16] there is an isomorphism

$$
\Psi_P : \left(H_T(\mathrm{Gr}_G)/J_P\right)\left[\xi_{\pi_P(t_\lambda)}^{-1} \mid \lambda \in \tilde{Q}\right] \cong QH^T(G/P)_{(q)}
$$

where (q) denotes localization at the quantum parameters. For $x \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ with $x = wt_\lambda$ for $w \in W$ and $\lambda \in Q^\vee$, we have $w \in W^P$ and $\lambda \in \tilde{Q}$. Then $\Psi_P(\xi_x) = q_{\eta_P(\lambda)} \sigma_P^w$ where σ_P^w is the quantum Schubert class in $QH^T(G/P)$ associated with $w \in W^P$.

Proposition 10 *Let $w \in W^P$. Then*

$$
\begin{aligned}
\sigma_P^{\pi_P(r_\theta)} \sigma_P^w = &\sum_{\substack{\mathrm{id} \neq u \leq r_\theta \\ \ell(uw) = \ell(w) - \ell(u)}} \theta^{-1} \xi^{u^{-1}}(r_\theta) q_{\eta_P(\theta^\vee)} \sigma_P^{uw} \\
&+ \sum_{\substack{\mathrm{id} \neq u \leq r_\theta \\ \ell(uw) = \ell(w) - \ell(u) \\ (uw)^{-1}\theta \in R^+ \setminus R_P^+}} \xi^{u^{-1}}(r_\theta) q_{\eta_P(\theta^\vee - (uw)^{-1}\theta^\vee)} \sigma_P^{\pi_P(r_\theta uw)}.
\end{aligned}
$$

[2] This notation for P will be used only in this subsection and should not cause confusion for the reader with its previous use as the weight lattice of G.

Proof Choose $\lambda \in Q^\vee$ such that $\langle \lambda, \alpha_i \rangle = 0$ for $i \in I_P$ and $\langle \lambda, \alpha_i \rangle \ll 0$ for $i \in I \setminus I_P$. Then $\langle \lambda, \alpha \rangle = 0$ for $\alpha \in R_P$ and $\langle \lambda, \alpha \rangle \ll 0$ for $\alpha \in R^+ \setminus R_P^+$.

We have $x = wt_\lambda \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$ by [9, Lemmata 3.3, 10.1]. Define the set

$$\mathcal{A}_x = \left\{ u \in W_{\mathrm{af}} \mid \ell(ux) = \ell(u) + \ell(x) \text{ and } ux \in W_{\mathrm{af}}^- \right\}. \tag{26}$$

Using the characterization of the Schubert basis in Theorem 3, for $z \in W_{\mathrm{af}}^-$ the coefficient of j_z in $j_{r_0} j_x$ is given by the coefficient of A_z in $j_{r_0} A_x$. We obtain

$$\xi_{r_0}\xi_x = \sum_{\substack{1 \neq u \leq r_\theta \\ u \in \mathcal{A}_x}} \left(\theta^{-1} \xi^{u^{-1}}(r_\theta) \xi_{ux} + \chi(r_0 \in \mathcal{A}_{ux}) \xi^{u^{-1}}(r_\theta) \xi_{r_0 ux} \right) \tag{27}$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. We shall apply the map Ψ_P to the above expression. First it is desirable to factor out the dependence of the right hand side on λ.

Suppose $u \in W$ (which holds for $u \leq r_\theta \in W$). We claim that $u \in \mathcal{A}_x$ if and only if $\ell(uw) = \ell(w) - \ell(u)$. Suppose $u \in \mathcal{A}_x$. Since $ux \in W_{\mathrm{af}}^-$ we have $\ell(ux) = \ell(uwt_\lambda) = \ell(t_\lambda) - \ell(uw)$ and $\ell(u) + \ell(x) = \ell(u) + \ell(t_\lambda) - \ell(w)$. Since $\ell(ux) = \ell(u) + \ell(x)$ it follows that $\ell(uw) = \ell(w) - \ell(u)$. Conversely suppose $\ell(uw) = \ell(w) - \ell(u)$. Since $w \in W^P$ it follows that $uw \in W^P$. In particular $uwt_\lambda \in W_{\mathrm{af}}^-$. Therefore $\ell(ux) = \ell(u) + \ell(x)$ and $u \in \mathcal{A}_x$.

Let us fix the assumption that $u \in W$ and $\ell(uw) = \ell(w) - \ell(u)$. Then $u \in \mathcal{A}_x$ and $ux \in (W^P)_{\mathrm{af}}$ since $uw \in W^P$. One may show that:

(1) $r_0 ux > ux$ if and only if $(uw)^{-1} \cdot \theta \in R^+$ and $(ux)^{-1} \cdot \alpha_0 \in \mathbb{Z}_{>0}\delta - (uw)^{-1} \cdot \theta$.
(2) $r_0 ux \notin (W^P)_{\mathrm{af}}$ if and only if $(uw)^{-1} \cdot \theta \in R_P^+$.
(3) $r_0 ux \notin W_{\mathrm{af}}^-$ if and only if $ux\alpha_i = \alpha_0$ for some $i \in I$.

It follows that under the assumption on u, $(uw)^{-1}\theta \in R^+ \setminus R_P^+$ if and only if $r_0 ux > ux$, $r_0 ux \in W_{\mathrm{af}}^-$, and $r_0 ux \in (W^P)_{\mathrm{af}}$.

We now apply the map Ψ_P. By [9, Remark 10.1] $r_0 \in W_{\mathrm{af}}^- \cap (W^P)_{\mathrm{af}}$. Since $r_0 = r_\theta t_{-\theta^\vee}$ we have $\Psi_P(\xi_{r_0}) = q_{\eta_P(-\theta^\vee)} \sigma_P^{\pi_P(r_\theta)}$.

By [9, Prop. 10.5, 10.8] $\pi_P(w) = w$, $\pi_P(t_\lambda) = t_\lambda$ and $\pi_P(x) = x$. Therefore $\Psi_P(\xi_x) = q_{\eta_P(\lambda)} \sigma_P^w$.

Let $1 \neq u \leq r_\theta$ and $u \in \mathcal{A}_x$. It follows that $uw \in W^P$ and $ux = uwt_\lambda \in (W^P)_{\mathrm{af}}$. Then $\Psi_P(\xi_{ux}) = q_{\eta_P(\lambda)} \sigma_P^{uw}$.

Finally let $1 \neq u \leq r_\theta$ be such that $u \in \mathcal{A}_x$, $r_0 \in \mathcal{A}_{ux}$, and $r_0 ux \in (W^P)_{\mathrm{af}}$. We have $r_0 ux = r_\theta t_{-\theta^\vee} uwt_\lambda = r_\theta uwt_{\lambda - (uw)^{-1}\theta^\vee}$. Therefore $\Psi_P(r_0 ux) = q_{\eta_P(\lambda - (uw)^{-1}\theta^\vee)} \sigma_P^{\pi_P(r_\theta uw)}$. Applying Ψ_P to (27) yields the required equation. □

4 Alternating equivariant Pieri rule in classical types

We first establish some notation for $G = SL_n$, Sp_{2n}, and SO_{2n+1}. Our root system conventions follow [5].

4.1 Special classes

We give explicit generating classes for $H_T(\mathrm{Gr}_G)$.

4.1.1 $H_T(\mathrm{Gr}_{SL_n})$

Define the elements

$$\hat{\sigma}_p = r_{p-1} \cdots r_1 \tag{28}$$

$$\sigma_p = r_{p-1} \cdots r_1 r_0 = \hat{\sigma}_p r_0. \tag{29}$$

So $\ell(\hat{\sigma}_p) = p - 1$ and $\ell(\sigma_p) = p$. These elements have associated translations

$$t_p := t^{\sigma_{p+1}} = t_{r_p \cdots r_2 r_1 \cdot \theta^\vee} \quad \text{for } 0 \le p \le n-2. \tag{30}$$

4.1.2 $H_T(\mathrm{Gr}_{Sp_{2n}})$

For $1 \le p \le 2n-1$ we define the elements $\hat{\sigma}_p \in W$ by

$$\hat{\sigma}_p = r_{p-1} \cdots r_2 r_1 \quad \text{for } 1 \le p \le n$$

$$\hat{\sigma}_p = r_{2n-p-1} \cdots r_{n-2} r_{n-1} \cdots r_2 r_1 \quad \text{for } n+1 \le p \le 2n-1.$$

For $1 \le p \le 2n-1$ define $\sigma_p \in W_{\mathrm{af}}^0$ and $t_{p-1} \in W_{\mathrm{af}}$ by

$$\sigma_p = \hat{\sigma}_p r_0 \tag{31}$$

$$t_{p-1} = t^{\sigma_p} = t_{\hat{\sigma}_p \cdot \theta^\vee}. \tag{32}$$

4.1.3 $H_T(\mathrm{Gr}_{SO_{2n+1}})$

For $1 \le p \le 2n-1$ we define the elements $\hat{\sigma}_p \in W_{\mathrm{af}}^0$ by

$$\hat{\sigma}_p = \begin{cases} \mathrm{id} & \text{if } p = 1 \\ r_p r_{p-1} \cdots r_3 r_2 & \text{if } 2 \le p \le n \\ r_{2n-p} r_{2n-p+1} \cdots r_{n-1} r_n r_{n-1} \cdots r_3 r_2 & \text{if } n+1 \le p \le 2n-2 \\ r_0 r_2 r_3 \cdots r_{n-1} r_n r_{n-1} \cdots r_3 r_2 & \text{if } p = 2n-1. \end{cases}$$

For $1 \le p \le 2n-1$ define $\sigma_p \in W_{\mathrm{af}}^0$ by

$$\sigma_p = \hat{\sigma}_p r_0. \tag{33}$$

For $1 \le p \le 2n-2$ define $t_{p-1} \in W_{\mathrm{af}}$ by

$$t_{p-1} = t^{\sigma_p} = t_{\hat{\sigma}_p \cdot \theta^\vee}. \tag{34}$$

For $1 \le p \le 2n-1$ let σ'_p be σ_p but with every r_0 replaced by r_1. Then define

$$t_{2n-2} = t_{2\omega_1^\vee} = \sigma_{2n-1} \sigma'_{2n-1}.$$

Then we conjecture that

$$B_{\sigma_{2n-1},\sigma_q} = \pm \frac{1}{\xi^{\sigma_{2n-1}}(\sigma_q'\sigma_{2n-1})} \quad \text{for } 1 \le q \le 2n-1 \tag{35}$$

where B is defined in (23). The sign is $-$ for $q \le 2n-2$ and $+$ for $q = 2n-1$.

4.1.4 *Special classes generate*

Let $k' = n-1$ for $G = SL_n$ and $k' = 2n-1$ for $G = Sp_{2n}$ or $G = SO_{2n+1}$. Let $\hat{\mathbb{P}} := S[[j_{\sigma_m} \mid 1 \le m \le k']]$ be the completion of $\mathbb{P} \cong H_T(\mathrm{Gr}_G)$ generated over S by series in the special classes. It inherits the Hopf structure from $\mathbb{P}$. The Hopf structure on $\mathbb{P}$ is determined by the coproduct on the special classes.

Proposition 11 *For* $G = SL_n, Sp_{2n}, SO_{2n+1}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{P} \subset \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{P}}$.

Proof It is known that the special classes generate the homology $H_*(\mathrm{Gr}_G)$ non-equivariantly for $G = SL_n, Sp_{2n}, SO_{2n+1}$ see [11, 14]. Furthermore, the equivariant homology Schubert structure constant d_{uv}^w is a polynomial in the simple roots of degree $\ell(w) - \ell(u) - \ell(v)$, and when $\ell(w) = \ell(u) + \ell(v)$, it is equal to the non-equivariant homology Schubert structure constant. It follows easily from this that each equivariant Schubert class can be expressed as a formal power series in the equivariant special classes. □

Remark 12 For $G = SL_n$ and $G = Sp_{2n}$ the special classes generate $H_*(\mathrm{Gr}_G)$ over $\mathbb{Z}$.

4.2 The alternating equivariant affine Pieri rule

Let $k = n-1$ for $G = SL_n$, $k = 2n-1$ for $G = Sp_{2n}$, and $k = 2n-2$ for $G = SO_{2n+1}$. Our goal is to compute $j_{\sigma_m}^x$ for $1 \le m \le k$; note that for $G = SO_{2n+1}$, the element σ_{2n-1} has been treated in (35). For this purpose consider the Bruhat order ideal $\Omega = \{\mathrm{id} = \sigma_0, \sigma_1, \dots, \sigma_k\}$ in W_{af}^0. Since $j_0 = \mathrm{id}$, to compute $j_{\sigma_p}^x$ for $p \ge 1$ we may assume $x \ne \mathrm{id}$ by length considerations. It suffices to invert the matrix A given in (22) over $\Omega \setminus \{\mathrm{id}\} \times \Omega \setminus \{\mathrm{id}\}$.

Define the matrices $M_{pm} = (-1)^m \xi^{\sigma_m}(\sigma_p)$ for $1 \le p, m \le k$, $N_{mq} = \xi^{\hat{\sigma}_m r_\theta}(\hat{\sigma}_q r_\theta)$ for $1 \le m, q \le k$, and the diagonal matrix $D_{pq} = \delta_{pq}\, \xi^{t_{p-1}}(t_{p-1})$ for $1 \le p, q \le k$.

Conjecture 13

$$MN = D. \tag{36}$$

Conjecture 14 For $1 \le m \le k$ and $x \ne \mathrm{id}$ we have

$$j_{\sigma_m}^x = (-1)^{\ell(x)} \sum_{q=0}^{m-1} \frac{\xi^{\hat{\sigma}_m r_\theta}(\hat{\sigma}_{q+1} r_\theta)}{\xi^{t_q}(t_q)} \xi^x(t_q). \tag{37}$$

In particular $j_{\sigma_m}^x = 0$ unless $\ell(x) \ge m$ and $x \le t_q$ for some $0 \le q \le m-1$.

Conjecture 14 follows immediately from Conjecture 13: we have $M^{-1} = ND^{-1}$, and (37) follows from (24).

Theorem 15 *Conjecture 14 holds for $G = SL_n$.*

The proof appears in Appendix A. Examples of (36) appear in Appendix B.

5 Effective Pieri rule for $H_T(\mathrm{Gr}_{SL_n})$

The goal of this section is to prove a formula for $j^x_{\sigma_m}$ that is manifestly positive. In this section we work with $G = SL_n$, $W = S_n$, and $W_{\mathrm{af}} = \tilde{S}_n$. We first establish some notation. For $a \le b$ write

$$u_a^b = r_a r_{a+1} \cdots r_b \tag{38}$$

$$d_a^b = r_b r_{b-1} \cdots r_a \tag{39}$$

$$\alpha_a^b = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b \tag{40}$$

for upward and downward sequences of reflections and for sums of consecutive roots. In particular we have $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \alpha_1^{n-1}$.

5.1 V's and Λ's

The support $\mathrm{Supp}(b)$ of a word b is the set of letters appearing in the word. For a permutation w, $\mathrm{Supp}(w)$ is the support of any reduced word of w. A V is a reduced word (for some permutation) that decreases to a minimum and increases thereafter. Special cases of V's include the empty word, any increasing word and any decreasing word. A Λ is a reduced word that increases to a maximum and decreases thereafter. A (reverse) N is a reduced word consisting of a V followed by a Λ, such that the support of the V is contained in the support of the Λ. For example, the words 32012, 23521, and 32012453 are a V, Λ, and N, respectively.

By abuse of language, we say a permutation is a V if it admits a reduced word that is a V. We use similar terminology for Λ's and N's.

A permutation is connected if its support is connected (that is, is a subinterval of the integers). The following basic facts are left as an exercise.

Lemma 16 *A permutation that is a V, admits a unique reduced word that is a V. Similarly for a connected Λ or a connected N.*

Lemma 17 *A connected permutation is a V if and only if it is a Λ, if and only if it is an N.*

5.2 t_q-factorizations

For $0 \leq q \leq n-2$, we call

$$q(q-1)\cdots 101\cdots(n-1)(n-2)\cdots(q+1) \tag{41}$$

the standard reduced word for t_q. Since this word is an N it follows that any $x \leq t_q$ is an N. We call the subwords $q(q-1)\cdots 1$, $12\cdots(n-2)$ and $(n-2)\cdots(q+1)$ the left, middle, and right branches.

Lemma 18 *If $x \in \tilde{S}_n$ admits a reduced word in which $i+1$ precedes i for some $i \in \mathbb{Z}/n\mathbb{Z}$ then $x \not\leq t_i$.*

Proof Suppose $x \leq t_i$. Since the standard reduced word of t_i has all occurrences of i preceding all occurrences of $i+1$, it follows that x has a reduced word with that property. But this property is invariant under the braid relation and the commuting relation, which connect all reduced words of x. □

Let $c(x)$ denote the number of connected components of $\mathrm{Supp}(x)$. If J and J' are subsets of integers then we write $J < J'-1$ if $\max(J) < \min(J')-1$. The following result follows easily from the definitions.

Lemma 19 *Suppose $x \leq t_q$. Then x has a unique factorization $x = v_1\cdots v_r y_1 \times y_2\cdots y_s$, called the q-factorization, where each v_i, y_i has connected support such that*

(1) $\mathrm{Supp}(v_i) < \mathrm{Supp}(v_{i+1}) - 1$ *and* $\mathrm{Supp}(y_i) < \mathrm{Supp}(y_{i+1}) - 1$
(2) $\mathrm{Supp}(v_1\cdots v_r) \subset [0,q]$
(3) $\mathrm{Supp}(y_1\cdots y_s) \subset [q+1, n-1]$
(4) *Each v_i is a V*
(5) *Each y_i is a Λ.*

We say that v_r and y_1 *touch* if $q \in \mathrm{Supp}(v_r)$ and $q+1 \in \mathrm{Supp}(y_1)$. We denote

$$\epsilon(x,q) = \begin{cases} 1 & \text{if } v_r \text{ and } y_1 \text{ touch} \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Note that $\epsilon(x,q)$ depends only on $\mathrm{Supp}(x)$ and q.

Each k in the q-factorization of $x \leq t_q$, is (S1) in the left branch of some v_i, or (S2) in the right branch of some v_i, or (S3) at the bottom of a v_i, or (S1$'$) in the left branch of some y_i, or (S2$'$) in the right branch of some y_i, or (S3$'$) at the top of a y_i. We call these sets $S1$, $S2$, $S3$, $S1'$, $S2'$, and $S3'$. Note that k can belong to both $S1$ and $S2$, or both $S1'$ and $S2'$.

For each x and each q such that $x \leq t_q$, we define the polynomials

$$M(x,q) = \left(\alpha_0^q\right)^{\epsilon(x,q)} \prod_{k\in S2} \alpha_0^{k-1} \prod_{k \in S1'} \alpha_0^k$$

$$L(x,q)=\prod_{k\in S1}\alpha_k^q$$
$$R(x,q)=\prod_{k\in S2'}\left(-\alpha_{q+1}^k\right).$$

We also define $R(x,q,m)=\prod_{k\in S2'\cap[m,n-1]}(-\alpha_{q+1}^k)$.

5.3 The equivariant Pieri rule

Let

$$\left\{q\in[0,m-1]\mid x\le t_q\right\}=\{q_1<q_2<\cdots<q_p\} \tag{43}$$

and

$$\beta_i=\alpha_{1+q_i}^{q_{i+1}} \tag{44}$$

be the root associated with the reflection r_{β_i} that exchanges the numbers $1+q_i$ and $1+q_{i+1}$. For a root β and $f\in S$ define

$$\partial_\beta f=\beta^{-1}(f-r_\beta f).$$

Theorem 20 *We have*

$$j_{\sigma_m}^x=(-1)^{\ell(x)-m+p-1}M(x,q_1)\partial_{\beta_{p-1}}\cdots\partial_{\beta_2}\partial_{\beta_1}Y(x,m)$$

where $Y(x,m)=(\alpha_0^{q_1})^{c(x)-1}R(x,q_1,m)$.

The proof of Theorem 20 is given in Sect. 6.

5.4 Positive formula

Define $\tilde{S2}'=S2'\cap[m,n-1]$, and let $K=\tilde{S2}'\cup\{n-1,\dots,n-1\}=\{k_1\ge k_2\ge\cdots\ge k_d\}$ be the multiset where the element $(n-1)$ is added to $\tilde{S2}'$ $(c(x)-1)$ times.

Theorem 21

$$j_{\sigma_m}^x=\left(\alpha_{q_1+1}^{n-1}\right)^{\epsilon(x,q)}\prod_{k\in S2}\alpha_k^{n-1}\prod_{k\in S1'}\alpha_{k+1}^{n-1}\sum_{\substack{R\subset[1,|K|]\\|R|=p-1}}\prod_{i\in[1,|K|]\setminus R}\alpha_{q_{s(i,R)}+1}^{k_i} \tag{45}$$

where $s(i,R)=\#\{r\in R\mid i<r\}+1$.

The proof of Theorem 21 is given in Sect. 6.

Example 22 Let $n=8$, $m=4$, and $x=r_0r_4r_5r_7r_4r_2r_1$. The components of $\mathrm{Supp}(x)$ are $[0,2]$, $[4,5]$, and $[7]$ so that $c(x)=3$. We have $p=3$ with $(q_1,q_2,q_3)=(0,2,3)$, $v_1=r_0$, $y_1=r_2r_1$, $y_2=r_4r_5r_4$, $y_3=r_7$, $\epsilon(x,q_1)=1$, $S1=S2=\emptyset$, $S3=\{0\}$, $S1'=\{4\}$, $S2'=\{1,4\}$, $S3'=\{2,5,7\}$, $S2'\cap[m,n-1]=\{4\}$. Thus $K=\{7,7,4\}$. Then writing $\alpha_a^b=x_a-x_{b+1}$, and noting that $\alpha_0^{n-1}=0$, Theorem 20 yields

$$\begin{aligned}
j^x_{\sigma_m} &= \big(\alpha^7_1\big)^1 \alpha^7_5 \partial_{\alpha_3}\partial_{\alpha_1+\alpha_2}\big(\alpha^7_1\big)^2\alpha^4_1 \\
&= (x_1-x_8)(x_5-x_8)\partial_{x_3-x_4}\partial_{x_1-x_3}(x_1-x_8)^2(x_1-x_5) \\
&= (x_1-x_8)(x_5-x_8)\partial_{x_3-x_4}\big((x_1-x_8)(x_1-x_5)+(x_3-x_8)(x_1-x_5) \\
&\quad + (x_3-x_8)^2\big) \\
&= (x_1-x_8)(x_5-x_8)\big((x_1-x_5)+(x_3-x_8)+(x_4-x_8)\big) \\
&= \big(\alpha^7_1\big)\big(\alpha^7_5\big)\big(\alpha^4_1+\alpha^7_3+\alpha^7_4\big)
\end{aligned}$$

agreeing with Theorem 21.

6 Proof of Theorems 20 and 21

6.1 Simplifying (37)

Let $0 \le q \le m-1$. By (14) and Lemma 2 we have

$$\begin{aligned}
\xi^{\hat{\sigma}_m r_\theta}(\hat{\sigma}_{q+1} r_\theta) &= u^{m-1}_{q+1} \cdot \xi^{\hat{\sigma}_m r_\theta}(\hat{\sigma}_m r_\theta) \\
&= (-1)^m u^{m-1}_{q+1}\hat{\sigma}_m r_\theta \cdot \xi^{r_\theta \hat{\sigma}_m^{-1}}\big(r_\theta \hat{\sigma}_m^{-1}\big) \\
&= (-1)^m \hat{\sigma}_{q+1} r_\theta \cdot \xi^{r_\theta \hat{\sigma}_m^{-1}}\big(r_\theta \hat{\sigma}_m^{-1}\big).
\end{aligned}$$

We also have

$$\begin{aligned}
\xi^{t_q}(t_q) &= \xi^{\sigma_{q+1}}(\sigma_{q+1})\big(\sigma_{q+1}\cdot \xi^{r_\theta \hat{\sigma}_m^{-1}}\big(r_\theta \hat{\sigma}_m^{-1}\big)\big)\big(\sigma_{q+1} r_\theta \hat{\sigma}_m^{-1} \cdot \xi^{d^{m-1}_{q+1}}\big(d^{m-1}_{q+1}\big)\big) \\
&= \xi^{\sigma_{q+1}}(\sigma_{q+1})\big(\hat{\sigma}_{q+1} r_\theta \cdot \xi^{r_\theta \hat{\sigma}_m^{-1}}\big(r_\theta \hat{\sigma}_m^{-1}\big)\big)\big(u^{m-1}_{q+1}\cdot \xi^{d^{m-1}_{q+1}}\big(d^{m-1}_{q+1}\big)\big) \\
&= (-1)^{m-q-1}\xi^{\sigma_{q+1}}(\sigma_{q+1})\big(\hat{\sigma}_{q+1} r_\theta \cdot \xi^{r_\theta \hat{\sigma}_m^{-1}}\big(r_\theta \hat{\sigma}_m^{-1}\big)\big)\xi^{u^{m-1}_{q+1}}\big(u^{m-1}_{q+1}\big).
\end{aligned}$$

Define

$$D(q,m) = \xi^{\sigma_{q+1}}(\sigma_{q+1})\xi^{u^{m-1}_{q+1}}\big(u^{m-1}_{q+1}\big). \tag{46}$$

so that by Theorem 15,

$$j^x_{\sigma_m} = (-1)^{\ell(x)} \sum_{q=0}^{m-1} \frac{(-1)^{q+1}}{D(q,m)} \xi^x(t_q). \tag{47}$$

Explicitly we have

$$\xi^{\sigma_{q+1}}(\sigma_{q+1}) = \alpha_q \alpha^q_{q-1}\cdots\alpha^q_1\alpha^q_0 \tag{48}$$

$$\xi^{u^{m-1}_{q+1}}\big(u^{m-1}_{q+1}\big) = \alpha_{q+1}\alpha^{q+2}_{q+1}\cdots\alpha^{m-1}_{q+1}. \tag{49}$$

6.2 Evaluation at t_q

Proposition 23 *If $x \leq t_q$, then*

$$\xi^x(t_q) = \left(\alpha_0^q\right)^{c(x)} M(x,q)\ L(x,q)\ R(x,q). \tag{50}$$

Proof We compute $\xi^x(t_q)$ using (14) by computing all embeddings of reduced words of x into the standard reduced word (41) of t_q. We refer to the q-factorization of x. Each $k \in S1$ must embed into the left branch of the N, and has associated root α_k^q. Each $k \in S2$ embeds into the middle branch of the N and has associated root α_0^{k-1}. Each $k \in S1'$ embeds into the middle branch of the N and has associated root α_0^k. Each $k \in S2'$ embeds into the right branch of the N and has associated root $-\alpha_{q+1}^k$. Each $k \in S3$ is either 0 and has associated root α_0^q, or can be embedded into the left or middle branch of the N, and the sum of the two associated roots for these positions is $\alpha_k^q + \alpha_0^{k-1} = \alpha_0^q$. Each $k \in S3'$ is either $n-1$, which has associated root $-\alpha_{q+1}^{n-1} = \alpha_0^q$, or can be embedded into the middle or right branch of the N, and the sum of associated roots is $\alpha_0^k - \alpha_{q+1}^k = \alpha_0^q$. Since all the various choices for embeddings of elements of $S3$ and $S3'$ can be varied independently, the value of $\xi^x(t_q)$ is the product of the above contributions. Each minimum of a v_i and maximum of a y_j contributes α_0^q. If there is a component of x which contains both q and $q+1$ (that is, if v_r and y_1 touch) then it is unique and contributes two copies of α_0^q. All this yields (50). □

6.3 Rotations

We now relate $\xi^x(t_q)$ with $\xi^x(t_{q'})$. Let $r_{p,q}$ denote the transposition that exchanges the integers p and q.

Proposition 24 *Let $x \leq t_q$ and consider the q-factorization of x. Let a be such that this reduced word of x contains the decreasing subword $(q+a)(q+a-1)\cdots(q+1)$ but not $(q+a+1)(q+a)\cdots(q+1)$. If $q+1 \notin \mathrm{Supp}(x)$, then set $a=1$. Then*

$$\xi^x(t_{q+1}) = \xi^x(t_{q+2}) = \cdots = \xi^x(t_{q+a-1}) = 0 \tag{51}$$

and

$$\xi^x(t_{q+a}) = M(x,q)\, r_{1+q,1+q+a}\left(\alpha_0^q\right)^{c(x)} L(x,q)\ R(x,q) \tag{52}$$

Let $y^{\uparrow}$ denote y with every r_i changed to r_{i+1}. The following lemma follows easily by induction.

Lemma 25 *Let y be increasing with support in $[b, a-1]$. Then*

$$y d_b^a = d_b^a y^{\uparrow}$$

Proof of Proposition 24 We assume that $q+1 \in \mathrm{Supp}(x)$, for otherwise the claim is easy.

By Lemma 18 we have $x \not\leq t_{q+i}$ for $1 \leq i \leq a-1$. Equation (51) follows from (15). We now prove (52). The first goal is to compute the $q+a$-factorization of x. Since $x \leq t_q$ we may consider the q-factorization of x. The decreasing word $(q+a-1)\cdots(q+2)(q+1)$ must embed into the right hand branch, that is, $[q+1, q+a-1] \subset S2'$. The hypotheses imply that $q+a \notin S2'$. There are two cases: either $q+a \in S1'$ or $q+a \in S3'$ (so that $q+a+1 \notin \mathrm{Supp}(x)$). We treat the former case, as the latter is similar: the two cases correspond to the touching and nontouching cases for the $q+a$-factorization of x, whose existence we now demonstrate.

Suppose $q+a \in S1'$. Then there is a y_1' with $\mathrm{Supp}(y_1') \subset [q+a+1, n-1]$ and a y with an increasing reduced word such that $\mathrm{Supp}(y) \subset [q+1, q+a-1]$ and $y_1 = y r_{q+a} y_1' d_{q+1}^{q+a-1} = y d_{q+1}^{q+a} y_1'$. Suppose v_r and y_1 touch. Then $v_r' := v_r y d_{q+1}^{q+a}$ is an N and therefore a V. Moreover $x \leq t_{q+a}$ since x has a $q+a$-factorization given by the q-factorization of x but with v_r and y_1 replaced by v_r' and y_1', respectively. To verify that v_r' is a V, by the touching assumption, $q \in \mathrm{Supp}(v_r)$ and we have $v_r' = v_r y d_{q+1}^{q+a} = v_r d_{q+1}^{q+a} y^{\uparrow} = d_{q+2}^{q+a} v_r r_{q+1} y^{\uparrow}$ which expresses v_r' in a V.

Suppose v_r and y_1 do not touch, that is, $q \notin \mathrm{Supp}(v_r)$. We have the V given by $v_{r+1}' = y d_{q+1}^{q+a} = d_{q+1}^{q+a} y^{\uparrow}$. Then $x \leq t_{q+a}$, as x has the $q+a$ factorization given by the q-factorization of x except that there is a new V, namely, v_{r+1}' and the first y is y_1' instead of y_1.

In every case we calculate that

$$
\begin{aligned}
M(x, q+a) &= M(x, q) \\
L(x, q+a) &= \left(\prod_{k=q+2}^{q+a} \alpha_k^{q+a}\right) d_{q+1}^{q+a} L(x, q) \\
R(x, q+a) &= d_{q+1}^{q+a} \left(\prod_{k=q+1}^{q+a-1} \left(-\alpha_{q+1}^k\right)^{-1}\right) R(x, q) \\
&= \left(\prod_{k=q+1}^{q+a-1} \left(\alpha_k^{q+a}\right)^{-1}\right) d_{q+1}^{q+a} R(x, q).
\end{aligned}
$$

The calculation for L and R follows from the fact that $[q+2, q+a] \subset S1_{q+a}$, but $[q+1, q+a-1] \subset S2_q'$. The calculation for M follows from the fact that $\mathrm{Supp}(y) \subset S2_q$ and $\mathrm{Supp}(y^{\uparrow}) \subset S2_{q+a}$, together with the following boundary cases:

If $q+a+1 \in \mathrm{Supp}(x)$ then $q+a \in S1_{q+a} \cap S1_q'$. Thus $q+a$ contributes a factor of α_0^{q+a} to $M(x, q)$. This factor appears in $M(x, q+a)$ as the factor $(\alpha_0^{q+a})^{\epsilon(x, q+a)}$, since $\epsilon(x, q+a) = 1$.

If $q \in \mathrm{Supp}(x)$ one has $\epsilon(x, q) = 1$ and $q+1 \in S2_{q+a}$ contributes a factor of α_0^q to $M(x, q+a)$. This factor appears in $M(x, q)$ as the factor $(\alpha_0^q)^{\epsilon(x, q)} = \alpha_0^q$.

Using that $d_{q+1}^{q+a} \alpha_0^q = \alpha_0^{q+a}$, $d_{q+1}^{q+a}(-\alpha_{q+1}^{q+a}) = \alpha_{q+a}$, and $r_{1+q, 1+q+a} \alpha_{q+1}^{q+a} = -\alpha_{q+1}^{q+a}$, the above relations between $M(x, q)$, $L(x, q)$, $R(x, q)$ and their counterparts for $q+a$, together with Proposition 23, yield

$$\xi^x(t_{q+a}) = \left(\alpha_{q+1}^{q+a}\right)^{-1} M(x,q)\, d_{q+1}^{q+a} \left(-\alpha_{q+1}^{q+a}\right)\left(\alpha_0^q\right)^{c(x)} L(x,q)\; R(x,q).$$

To obtain (52), since $r_{1+q,1+q+a} = d_{q+1}^{q+a} u_{q+2}^{q+a}$, it suffices to show that

$$\left(-\alpha_{q+1}^{q+a}\right)\left(\alpha_0^q\right)^{c(x)} L(x,q) R(x,q) \text{ is invariant under } u_{q+2}^{q+a}.$$

However, it is clear that α_0^q and $L(x,q)$ are invariant, and the only part of $R(x,q)$ that must be checked is the product $\prod_{k\in S2'\cap[q+1,q+a]}(-\alpha_{q+1,k})$. However, we have $S2'\cap[q+1,q+a] = [q+1,q+a-1]$, and indeed the product $\prod_{k=q+1}^{q+a}(-\alpha_{q+1}^k)$ is invariant under u_{q+2}^{q+a}, as required. □

Recall the definition of q_j from (43). In light of the proof of Proposition 24, we write

$$M(x) = M(x,q_j) \quad \text{for any } 1\le j\le p. \tag{53}$$

Recall the definition of β_i from (44). For $i\le j$ we also define

$$\beta_i^j = \beta_i + \beta_{i+1} + \cdots + \beta_j = \alpha_{q_i+1}^{q_{j+1}}.$$

Let

$$Y_i(x,m) = \left(\alpha_0^{q_i}\right)^{c(x)-1} R(x,q_i,m) \quad \text{for } 1\le i\le p \tag{54}$$

so that $Y_i(x,m) = r_{\beta_{i-1}} Y_{i-1}(x,m)$.

Recall the definitions of $D(q,m)$ and $Y_i(x,m)$ from (46).

Lemma 26

$$(-1)^{m-1-q_j-p+j}\frac{\xi^x(t_{q_j})}{D(q_j,m)} = \frac{M(x)Y_j(x,m)}{(\beta_1^{j-1}\beta_2^{j-1}\cdots\beta_{j-1}^{j-1})(\beta_j^j\beta_j^{j+1}\cdots\beta_j^{p-1})}.$$

Proof The proof proceeds by induction on j. Let D_j be the denominator of the right hand side. Suppose first that $j=1$. Consider the embedding of x into t_{q_1}. By the definition of q_1, it follows that $L(x,q_1)\alpha_0^{q_1} = \xi^{\sigma_{q_1+1}}(\sigma_{q_1+1})$. By the definition of the q_j, we also have $S2'\cap[q_1+1,m-1] = [q_1+1,m-1]\setminus\{q_2,q_3,\dots,q_p\}$. These considerations and Proposition 23 imply that

$$\begin{aligned}
\xi^x(t_{q_1}) &= \left(\alpha_0^{q_1}\right)^{c(x)} M(x)L(x,q_1)R(x,q_1)\\
&= (-1)^{m-1-q_1}\left(\alpha_0^{q_1}\right)^{c(x)} M(x)D(q_1,m)R(x,q_1,m)\prod_{j=2}^{p}\left(-\alpha_{q_1+1}^{q_j}\right)^{-1}\\
&= (-1)^{m-1-q_1-p+1} D(q_1,m)M(x)Y_1(x,m)D_1^{-1}.
\end{aligned}$$

This proves the result for $j=1$. Suppose the result holds for $1 \le j \le p-1$. We show it holds for $j+1$. By induction we have

$$\big(\alpha_0^{q_j}\big)^{c(x)} L(x,q_j) R(x,q_j) = \frac{D(q_j,m) Y_j(x,m)}{D_j}.$$

Proposition 24 yields

$$\begin{aligned}
\frac{\xi^x(t_{q_{j+1}})}{D(q_{j+1},m)} &= \frac{M(x) r_{\beta_j} (\alpha_0^{q_j})^{c(x)} L(x,q_j) R(x,q_j)}{D(q_{j+1},m)} \\
&= \frac{M(x)}{D(q_{j+1},m)} r_{\beta_j} \frac{D(q_j,m) Y_j(x,m)}{D_j} \\
&= \frac{M(x) Y_{j+1}(x,m)}{D(q_{j+1},m)} r_{\beta_j} \frac{D(q_j,m)}{D_j}.
\end{aligned}$$

It remains to show

$$(-1)^{q_{j+1}-q_j-1} \frac{D(q_{j+1},m)}{D_{j+1}} = r_{\beta_j} \frac{D(q_j,m)}{D_j}.$$

We have $D(q_j,m) = \prod_{k=0}^{q_j} \alpha_k^{q_j} \prod_{k=q_j+1}^{m-1} \alpha_{q_j+1}^k$. For $k \in [0,q_j]$ we have $r_{\beta_j} \alpha_k^{q_j} = \alpha_k^{q_{j+1}}$. For $k \in [q_j+1, q_{j+1}-1]$ we have $r_{\beta_j} \alpha_{q_j+1}^k = -\alpha_{k+1}^{q_{j+1}}$, $r_{\beta_j} \alpha_{q_j+1}^{q_{j+1}} = -\alpha_{q_j+1}^{q_{j+1}}$, and for $k \in [q_{j+1}+1, m-1]$ we have $r_{\beta_j} \alpha_{q_j+1}^k = \alpha_{q_{j+1}+1}^k$. Therefore

$$\begin{aligned}
r_{\beta_j} D(q_j,m) &= (-1)^{q_{j+1}-q_j} \prod_{k=0}^{q_j} \alpha_k^{q_{j+1}} \prod_{k=q_j}^{q_{j+1}-1} \alpha_{k+1}^{q_{j+1}} \prod_{k=q_{j+1}+1}^{m-1} \alpha_{q_{j+1}+1}^k \\
&= (-1)^{q_{j+1}-q_j} D(q_{j+1},m).
\end{aligned}$$

We also have $r_{\beta_j} \beta_{j-1}^i = \beta_j^i$ for $1 \le i \le j-1$ and $r_{\beta_j} \beta_j^i = \beta_{j+1}^i$ for $j+1 \le i \le p-1$. Therefore

$$r_{\beta_j} D_j = \left(\prod_{i=1}^{j-1} \beta_i^j \right) (-\beta_j) \left(\prod_{i=j+1}^{p-1} \beta_{j+1}^i \right) = -D_{j+1}.$$

□

The following result is immediate from the definitions.

Lemma 27 *$r_{\beta_j} Y_i(x,m) = Y_i(x,m)$ for $j \ge i+2$.*

6.4 Proof of Theorem 20

Note that if $r_{\beta_{j+1}} Y = Y$ and $i \le j$ then

$$\frac{1}{\beta_{j+1}} (1 - r_{\beta_{j+1}}) \frac{Y}{\beta_i^j} = \frac{Y}{\beta_i^j \beta_i^{j+1}}.$$

So using Lemma 27 we have

$$\begin{aligned}
&\partial_{\beta_{p-1}}\cdots\partial_{\beta_2}\partial_{\beta_1}Y(x,m)\\
&\quad=\frac{1}{\beta_{p-1}}(1-r_{\beta_{p-1}})\cdots\frac{1}{\beta_1}(1-r_{\beta_1})Y_1(x,m)\\
&\quad=\frac{1}{\beta_{p-1}}(1-r_{\beta_{p-1}})\cdots\frac{1}{\beta_2}(1-r_{\beta_2})\left(\frac{Y_1(x,m)}{\beta_1}-\frac{Y_2(x,m)}{\beta_1}\right)\\
&\quad=\frac{1}{\beta_{p-1}}(1-r_{\beta_{p-1}})\cdots\frac{1}{\beta_3}(1-r_{\beta_3})\left(\frac{Y_1(x,m)}{\beta_1\beta_1^2}-\frac{Y_2(x,m)}{\beta_1\beta_2}+\frac{Y_3(x,m)}{\beta_1^2\beta_2}\right)\\
&\quad=\cdots\\
&\quad=\frac{Y_1(x,m)}{\beta_1\beta_1^2\cdots\beta_1^{p-1}}-\frac{Y_2(x,m)}{\beta_1\beta_2\beta_2^3\cdots\beta_2^{p-1}}+\cdots\\
&\qquad+(-1)^j\frac{Y_{j+1}(x,m)}{\beta_1^j\cdots\beta_{j-1}^j\beta_j\beta_{j+1}\beta_{j+1}^{j+2}\cdots\beta_{j+1}^{p-1}}\\
&\qquad+\cdots+(-1)^{p-1}\frac{Y_p(x,m)}{\beta_1^{p-1}\cdots\beta_{p-2}^{p-1}\beta_{p-1}}.
\end{aligned}$$

Thus

$$\begin{aligned}
M(x)\partial_{\beta_{p-1}}\cdots\partial_{\beta_2}\partial_{\beta_1}Y(x,m)&=\sum_{j=1}^{p}(-1)^{j-1}\frac{M(x)Y_j(x,m)}{D_j}\\
&=(-1)^{m-p}\sum_{j=1}^{p}(-1)^{q_j}\frac{\xi^x(t_{q_j})}{D(q_j,m)}\\
&=(-1)^{m-p}\sum_{i=0}^{m-2}(-1)^i\frac{\xi^x(t_i)}{D(i,m)}\\
&=(-1)^{m-p+1}(-1)^{\ell(x)}j^x_{\sigma_m}
\end{aligned}$$

by (47), as required. □

6.5 Proof of Theorem 21

We first count the gratuitous negative signs in $M(x)=M(x,q_1)$ and $Y(x,m)$. Letting $q=q_1$, using the q_1-factorization of x, and recalling that $\tilde{S2}'=S2'\cap[m,n-1]$, this number is

$$\begin{aligned}
&\epsilon(x,q)+|S2|+|S1'|+c(x)-1+|\tilde{S2}'|\\
&\quad=|S2|+|S1'|+|S3|+|S3'|-1+|\tilde{S2}'|
\end{aligned}$$

$$
\begin{aligned}
&= \ell(x) - 1 - |S1| - |S2' \setminus \tilde{S2}'| \\
&= \ell(x) - 1 - q_1 - |[q_1+1, m-1] \setminus \{q_2, q_3, \dots, q_p\}| \\
&= \ell(x) - 1 - q_1 - \bigl(m - 1 - q_1 - (p-1)\bigr) \\
&= \ell(x) - m + p - 1.
\end{aligned}
$$

Therefore all signs cancel and we have

$$
j^x_{\sigma_m} = \bigl(\alpha^{n-1}_{q+1}\bigr)^{\epsilon(x,q)} \prod_{k\in S2} \alpha^{n-1}_k \prod_{k\in S1'} \alpha^{n-1}_{k+1}\, \partial_{\beta_{p-1}} \cdots \partial_{\beta_1} \bigl(\alpha^{n-1}_{q+1}\bigr)^{c(x)-1} \prod_{k\in \tilde{S2}'} \alpha^k_{q+1}. \quad (55)
$$

Let x_i be the standard basis of the finite weight lattice $\mathbb{Z}^n$ with $\alpha_i = x_i - x_{i+1}$. Then r_{β_j} acts by exchanging x_{q_j+1} and $x_{q_{j+1}+1}$. Let us write

$$
Z = \bigl(\alpha^{n-1}_{q+1}\bigr)^{c(x)-1} \prod_{k\in \tilde{S2}'} \alpha^k_{q+1} = \alpha^{k_1}_{q+1}\alpha^{k_2}_{q+1}\cdots\alpha^{k_d}_{q+1} = \prod_{i=1}^{d} (x_{q_1+1} - x_{k_i+1}).
$$

where $n-1 \ge k_1 \ge k_2 \ge \cdots \ge k_d \ge m$. Note that $q_j + 1 \le q_p + 1 \le m$. Since

$$
\partial_i \cdot (fg) = (\partial_i \cdot f)g + (r_i \cdot f)(\partial_i \cdot g),
$$

and since $\partial_i 1 = 0$, we have

$$
\begin{aligned}
\partial_{\beta_1} Z &= \bigl(\partial_{\beta_1} \cdot (x_{q_1+1} - x_{k_1+1})\bigr)(x_{q_1+1} - x_{k_2+1}) \cdots (x_{q_1+1} - x_{k_d+1}) \\
&\quad + (x_{q_2+1} - x_{k_1+1})\bigl(\partial_{\beta_1} \cdot (x_{q_1+1} - x_{k_2+1})\bigr)(x_{q_1+1} - x_{k_3+1}) \cdots (x_{q_1+1} - x_{k_d+1}) \\
&\quad + \cdots \\
&\quad + (x_{q_2+1} - x_{k_1+1}) \cdots (x_{q_2+1} - x_{k_{d-1}})\partial_{\beta_1}(x_{q_1+1} - x_{k_d+1}) \\
&= \sum_{i=1}^{d} (x_{q_2+1} - x_{k_1+1}) \cdots (x_{q_2+1} - x_{k_{i-1}+1}) \\
&\quad \times (x_{q_1+1} - x_{k_{i+1}+1}) \cdots (x_{q_1+1} - x_{k_d+1}).
\end{aligned}
$$

So ∂_{β_1} can act on any factor (giving the answer 1 and thus effectively removing the factor), and to the left each variable x_{q_1+1} is reflected to x_{q_2+1}. Next we apply ∂_{β_2}. It kills any factor $x_{q_1+1} - x_{k_i+1}$. Therefore we may assume it acts on a factor of the form $x_{q_2+1} - x_{k_i+1}$ which is to the left of the factor removed by ∂_{β_1}. Continuing in this manner we see that $\partial_{\beta_{p-1}} \cdots \partial_{\beta_1} Z$ is the sum of products of positive roots, where a given summand corresponds to the selection of $p-1$ of the factors, which are removed, and between the rth and $r+1$th removed factor from the right, an original factor $x_{q_1+1} - x_{k_i+1}$ is changed to $x_{q_{r+1}+1} - x_{k_i+1}$.

It follows that Theorem 20 yields Theorem 21. □

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Appendix A: Proof of Theorem 15

In this section we assume that $G = SL_n$ and prove (36).

The matrices M and N are easily seen to be lower triangular. We first check the diagonal:

$$\begin{aligned} M_{pp}N_{pp} &= (-1)^p \xi^{\sigma_p}(\sigma_p)\xi^{\hat{\sigma}_p r_\theta}(\hat{\sigma}_p r_\theta) \\ &= \xi^{\sigma_p}(\sigma_p)\big(\hat{\sigma}_p r_\theta \cdot \xi^{r_\theta \hat{\sigma}_p^{-1}}\big(r_\theta \hat{\sigma}_p^{-1}\big)\big) \\ &= \xi^{\sigma_p}(\sigma_p)\big(\sigma_p \cdot \xi^{r_\theta \hat{\sigma}_p^{-1}}\big(r_\theta \hat{\sigma}_p^{-1}\big)\big) \\ &= \xi^{t_{p-1}}(t_{p-1}), \end{aligned}$$

by (2), (4), and Lemma 1.

It remains to check below the diagonal. Let $p > q$ and $p \ge k \ge q$. We have

$$\begin{aligned} M_{pk} &= (-1)^k \xi^{\sigma_k}(\sigma_p) \\ &= (-1)^k d_k^{p-1} \cdot \xi^{\sigma_k}(\sigma_k) \\ &= (-1)^k d_k^{p-1} \cdot \big(\xi^{d_q^{k-1}}\big(d_q^{k-1}\big) d_q^{k-1} \cdot \xi^{\sigma_q}(\sigma_q)\big) \\ &= (-1)^k \big(d_k^{p-1} \cdot \xi^{d_q^{k-1}}\big(d_q^{k-1}\big)\big)\big(d_q^{p-1} \cdot \xi^{\sigma_q}(\sigma_q)\big). \end{aligned}$$

Note that the second factor is independent of k. We also have

$$\begin{aligned} N_{kq} &= \xi^{\hat{\sigma}_k r_\theta}(\hat{\sigma}_q r_\theta) \\ &= u_q^{k-1} \cdot \big(\xi^{\hat{\sigma}_k r_\theta}(\hat{\sigma}_k r_\theta)\big) \\ &= u_q^{k-1} \cdot \big(\xi^{u_k^{p-1}}\big(u_k^{p-1}\big)\big(u_k^{p-1} \cdot \xi^{\hat{\sigma}_p r_\theta}(\hat{\sigma}_p r_\theta)\big)\big) \\ &= \big(u_q^{k-1} \cdot \xi^{u_k^{p-1}}\big(u_k^{p-1}\big)\big)\big(u_q^{p-1} \cdot \xi^{\hat{\sigma}_p r_\theta}(\hat{\sigma}_p r_\theta)\big) \end{aligned}$$

with the second factor independent of k. Therefore, to prove that

$$\sum_{q \le k \le p} M_{pk} N_{kq} = 0$$

it is equivalent to show that

$$0 = \sum_{q \le k \le p} (-1)^k \big(d_k^{p-1} \cdot \xi^{d_q^{k-1}}\big(d_q^{k-1}\big)\big)\big(u_q^{k-1} \cdot \xi^{u_k^{p-1}}\big(u_k^{p-1}\big)\big). \tag{56}$$

The above identity can be rewritten as

$$0 = \sum_{q \le k \le p} (-1)^k \prod_{i=q}^{k-1} \alpha_i^{p-1} \prod_{m=k}^{p-1} \alpha_q^m. \tag{57}$$

To prove this last identity, let q' be such that $q < q' \leq p$. It is easy to show by descending induction on q' that

$$\sum_{q' \leq k \leq p} (-1)^k \prod_{i=q}^{k-1} \alpha_i^{p-1} \prod_{m=k}^{p-1} \alpha_q^m = (-1)^{q'} \prod_{i=q+1}^{q'-1} \alpha_i^{p-1} \prod_{m=q'-1}^{p-1} \alpha_q^m. \tag{58}$$

Then for $q' = q + 1$ the sum is the negative of the $k = q$ summand of (57) as required. □

Appendix B: Examples of (36)

Example 28 $G = SL_3$ has affine Cartan matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The column dependencies give the coefficients of the null root $\delta = \alpha_0 + \theta = \alpha_0 + \alpha_1 + \alpha_2$ which is set to zero due to the finite torus equivariance.

p	$\hat{\sigma}_p$	σ_p	t_{p-1}	$\hat{\sigma}_p r_\theta$
1	id	r_0	$r_0 r_1 r_2 r_1$	$r_1 r_2 r_1$
2	r_1	$r_1 r_0$	$r_1 r_0 r_1 r_2$	$r_2 r_1$

We compute the matrices

$$M = \begin{pmatrix} \alpha_1 + \alpha_2 & 0 \\ \alpha_2 & -\alpha_1\alpha_2 \end{pmatrix} \qquad N = \begin{pmatrix} \alpha_1\alpha_2(\alpha_1 + \alpha_2) & 0 \\ \alpha_2(\alpha_1 + \alpha_2) & \alpha_2(\alpha_1 + \alpha_2) \end{pmatrix}$$

$$D = \begin{pmatrix} \alpha_1\alpha_2(\alpha_1 + \alpha_2)^2 & 0 \\ 0 & -\alpha_1\alpha_2^2(\alpha_1 + \alpha_2) \end{pmatrix}$$

$$ND^{-1} = \begin{pmatrix} (\alpha_1 + \alpha_2)^{-1} & 0 \\ (\alpha_1(\alpha_1 + \alpha_2))^{-1} & -(\alpha_1\alpha_2)^{-1} \end{pmatrix}.$$

For $x = r_1 r_2$ we compute the column vector with values $(-1)^{\ell(x)} \xi^x(t_j)$ for $j = 1, 2$. Acting on this column vector by ND^{-1}, we obtain the coefficients of A_x in j_1 and j_2.

$$(-1)^{\ell(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \end{pmatrix} = \begin{pmatrix} \alpha_2(\alpha_1 + \alpha_2) \\ \alpha_2^2 \end{pmatrix} \qquad \begin{pmatrix} j_{\sigma_1}^x \\ j_{\sigma_2}^x \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix}.$$

Doing the same thing for $x = r_1 r_0 r_2$ we have

$$(-1)^{\ell(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha_1\alpha_2^2 \end{pmatrix} \qquad \begin{pmatrix} j_{\sigma_1}^x \\ j_{\sigma_2}^x \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix}.$$

Example 29 Sp_{2n} for $n=2$ has affine Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

We have $\delta = \alpha_0 + \theta = \alpha_0 + 2\alpha_1 + \alpha_2$.

p	$\hat{\sigma}_p$	σ_p	t_{p-1}	$\hat{\sigma}_p r_\theta$
1	id	r_0	$r_0 r_1 r_2 r_1$	$r_1 r_2 r_1$
2	r_1	$r_1 r_0$	$r_1 r_0 r_1 r_2$	$r_2 r_1$
3	$r_2 r_1$	$r_2 r_1 r_0$	$r_2 r_1 r_0 r_1$	r_1

We have

$$M = \begin{pmatrix} 2\alpha_1+\alpha_2 & 0 & 0 \\ \alpha_2 & -\alpha_1\alpha_2 & 0 \\ -\alpha_2 & \alpha_2(\alpha_1+\alpha_2) & -\alpha_2^2(\alpha_1+\alpha_2) \end{pmatrix}$$

$$N = \begin{pmatrix} \alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2) & 0 & 0 \\ (\alpha_1+\alpha_2)(2\alpha_1+\alpha_2) & \alpha_2(\alpha_1+\alpha_2) & 0 \\ 2\alpha_1+\alpha_2 & \alpha_1+\alpha_2 & \alpha_1 \end{pmatrix}$$

$$D = \begin{pmatrix} \alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)^2 & 0 & 0 \\ 0 & -\alpha_1\alpha_2^2(\alpha_1+\alpha_2) & 0 \\ 0 & 0 & -\alpha_1\alpha_2^2(\alpha_1+\alpha_2) \end{pmatrix}$$

$$ND^{-1} = \begin{pmatrix} (2\alpha_1+\alpha_2)^{-1} & 0 & 0 \\ (\alpha_1(2\alpha_1+\alpha_2))^{-1} & -(\alpha_1\alpha_2)^{-1} & 0 \\ (\alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2))^{-1} & -(\alpha_1\alpha_2^2)^{-1} & -(\alpha_2^2(\alpha_1+\alpha_2))^{-1} \end{pmatrix}.$$

Now let $x = r_0 r_1 r_2$. We have

$$(-1)^{\ell(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \\ \xi^x(t_3) \end{pmatrix} = \begin{pmatrix} (\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)^2 \\ \alpha_2^2(\alpha_1+\alpha_2) \\ 0 \end{pmatrix}.$$

The matrix ND^{-1} acting on the above column vector, gives the vector

$$\begin{pmatrix} j^x_{\sigma_1} \\ j^x_{\sigma_2} \\ j^x_{\sigma_3} \end{pmatrix} = \begin{pmatrix} (\alpha_1+\alpha_2)(2\alpha_1+\alpha_2) \\ 2(\alpha_1+\alpha_2) \\ 1 \end{pmatrix}.$$

Now let $x = r_1 r_2 r_1$. We have

$$(-1)^{\ell(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \\ \xi^x(t_3) \end{pmatrix} = \begin{pmatrix} \alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2) \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} j^x_{\sigma_1} \\ j^x_{\sigma_2} \\ j^x_{\sigma_3} \end{pmatrix} = \begin{pmatrix} \alpha_1(\alpha_1+\alpha_2) \\ (\alpha_1+\alpha_2) \\ 1 \end{pmatrix}.$$

Example 30 SO_{2n+1} for $n=3$ has affine Cartan matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

We have $\delta = \alpha_0 + \theta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3$.

p	$\hat{\sigma}_p$	σ_p	t_{p-1}	$\hat{\sigma}_p r_\theta$
1	id	r_0	$r_0r_2r_3r_2r_1r_2r_3r_2$	$r_2r_3r_2r_1r_2r_3r_2$
2	r_2	r_2r_0	$r_2r_0r_2r_3r_2r_1r_2r_3$	$r_3r_2r_1r_2r_3r_2$
3	r_3r_2	$r_3r_2r_0$	$r_3r_2r_0r_2r_3r_2r_1r_2$	$r_2r_1r_2r_3r_2$
4	$r_2r_3r_2$	$r_2r_3r_2r_0$	$r_2r_3r_2r_0r_2r_3r_2r_1$	$r_1r_2r_3r_2$
5	$r_0r_2r_3r_2$	$r_0r_2r_3r_2r_0$	$r_0r_2r_3r_2r_0r_1r_2r_3r_2r_1$	$r_2r_3r_2$

To save space let us write $\alpha_{ijk} := i\alpha_1 + j\alpha_2 + k\alpha_3$. We have

$$M = \begin{pmatrix} \alpha_{122} & & & \\ \alpha_{112} & -\alpha_{010}\alpha_{112} & & \\ \alpha_{110} & -\alpha_{110}\alpha_{012} & \alpha_{110}\alpha_{012}\alpha_{001} & \\ \alpha_{100} & -2\alpha_{100}\alpha_{011} & \alpha_{100}\alpha_{011}\alpha_{012} & -\alpha_{100}\alpha_{010}\alpha_{011}\alpha_{012} \end{pmatrix}$$

$$N = \begin{pmatrix} \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} & & & \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{011}\alpha_{012} & \alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{012}\alpha_{001} & & \\ 2\alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{011} & \alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{012} & \alpha_{100}\alpha_{110}\alpha_{111}\alpha_{122}\alpha_{010} & \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122} & \alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122} & \alpha_{100}\alpha_{110}\alpha_{111}\alpha_{122} & \alpha_{100}\alpha_{110}\alpha_{111}\alpha_{112} \end{pmatrix}$$

D has diagonal entries

$$\alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}^2\alpha_{010}\alpha_{011}\alpha_{012}$$

$$-\alpha_{100}\alpha_{111}\alpha_{112}^2\alpha_{122}\alpha_{010}\alpha_{012}\alpha_{001}$$

$$\alpha_{100}\alpha_{110}^2\alpha_{111}\alpha_{122}\alpha_{010}\alpha_{012}\alpha_{001}$$

$$-\alpha_{100}^2\alpha_{110}\alpha_{111}\alpha_{112}\alpha_{010}\alpha_{011}\alpha_{012}$$

One may verify that $MN = D$.

References

1. Andersen, H.H., Jantzen, J.C., Soergel, W.: Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. Astérisque **220**, 321 (1994)
2. Billey, S.: Kostant polynomials and the cohomology ring for G/B. Duke Math. J. **96**, 205–224 (1999)
3. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. **131**(1), 25–83 (1998)
4. Goresky, M., Kottwitz, R., MacPherson, R.: Homology of affine Springer fibers in the unramified case. Duke Math. J. **121**, 509–561 (2004)
5. Kac, V.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
6. Kostant, B., Kumar, S.: The nil Hecke ring and cohomology of G/P for a Kac–Moody group G. Adv. Math. **62**(3), 187–237 (1986)
7. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204. Birkhäuser, Boston (2002), pp. xvi+606
8. Lam, T., Shimozono, M.: Dual graded graphs for Kac-Moody algebras. Algebra Number Theory **1**(4), 451–488 (2007)
9. Lam, T., Shimozono, M.: Quantum cohomology of G/P and homology of affine Grassmannian. Acta Math. **204**, 49–90 (2010)
10. Lam, T., Shimozono, M.: k-Double Schur functions and equivariant (co)homology of the affine Grassmannian. preprint, arXiv:1105.2170
11. Lam, T., Schilling, A., Shimozono, M.: Schubert polynomials for the affine Grassmannian of the symplectic group. Math. Z. **264**(4), 765–811 (2010)
12. Mihalcea, L.: Positivity in equivariant quantum Schubert calculus. Am. J. Math. **128**(3), 787–803 (2006)
13. Peterson, D.: Lecture Notes at MIT (1997)
14. Pon, S.: Affine Stanley symmetric functions for classical groups. Ph.D. thesis, University of California, Davis (2010)