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Classifying a family of edge-transitive metacirculant graphs

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Abstract A characterization is given of the class of edge-transitive Cayley graphs of Frobenius groups $\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ with p, q odd prime, of valency coprime to p. This characterization is then used to study an isomorphism problem regarding Cayley graphs, and to construct new families of half-arc-transitive graphs.

Keywords Edge-transitive · Metacirculant · Half-arc-transitive · Cayley graph

1 Introduction

A graph $\Gamma = (V, E)$ is called X*-edge-transitive* if $X \leq \mathsf{Aut}\,\Gamma$ is transitive on the edge set E. A graph $\Gamma = (V, E)$ is a Cayley graph if there exists a group G and a subset $S \subset G$ with $S = S^{-1} = \{s^{-1} \mid s \in S\}$ such that the vertex set V can be identified with G and x is adjacent to y if and only if $yx^{-1} \in S$. A *circulant* is a Cayley graph of a cyclic group. Edge-transitive circulants have been characterized by Kovács [6] and Li

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[12] independently. It would be a natural next step toward a characterization of edge-transitive metacirculants (which admit a vertex transitive metacyclic group), see [7] for the study of arc-regular dihedrants. In this paper, we characterize a special class of metacirculants.

A group X with $\mathrm{PSL}(d,q) \le X \le \mathrm{P\Gamma L}(d,q)$ naturally acts on the set Ω of 1-subspaces of the vector space $\mathbb{F}_q^d$. Then a point $\omega \in \Omega$ is a 1-dimensional subspace, and a point stabilizer of X has the form $[q^{d-1}].\frac{1}{(d,q-1)}\mathrm{GL}(d-1,q).o$, where $o = X/\mathrm{PSL}(d,q)$, called a *parabolic subgroup* and denoted by P_1.

Theorem 1.1 *Let $G = \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group, with p, q odd primes. Let Γ be an X-edge-transitive Cayley graph of G of valency coprime to p, where $G < X \le \mathsf{Aut}\,\Gamma$. Then one of the following statements holds*:

- (i) *X is almost simple and quasiprimitive on V, and either $\Gamma = \mathbf{K}_{p^d q}$ or $d = 1$*;
- (ii) *$\Gamma = \mathbf{C}_q[\overline{\mathbf{K}}_{p^d}] - p^d\mathbf{C}_q$, and $X = \mathrm{PGL}(q, r^q).q$, and $X_\alpha = P_1 \cap \mathrm{PGL}(q, r^q)$, or $\Gamma = \mathbf{K}_{p^d} \times \mathbf{C}_q$, and X is as in Lemma 2.7*(iii);
- (iii) *G is normal in X*;
- (iv) *X has a normal subgroup which is cyclic and regular on the vertex set, and the valency of Γ is divisible by q*;
- (v) *$\Gamma = \Sigma[\overline{\mathbf{K}}_q]$, and X has a minimal normal subgroup which is not simple, where Σ is an edge-transitive circulant of order p^d and of valency divisible by q.*

Remarks on Theorem 1.1

(1) Edge-transitive graphs of order pq are characterized in [17], and thus the graphs in (i) are known.

(2) A graph is called a *normal Cayley graph* if $\mathsf{Aut}\,\Gamma$ contains a normal regular subgroup. In particular, if the normal regular subgroup is cyclic, then Γ is called *normal circulant*. For normal Cayley graph Γ, $\mathsf{Aut}\,\Gamma$ is determined by $\mathsf{Aut}(G)$, see Lemma 4.1. Thus in part (iii)–(iv), the group X is well-characterized.

A graph $\Gamma = (V, E)$ is called *half-arc-transitive* if $\mathsf{Aut}\,\Gamma$ is transitive on V and E but intransitive on the arcs (recall that an *arc* is an ordered pair of adjacent vertices). In the literature of algebraic graph theory, studying half-arc-transitive graphs is a hot topic, see [2, 14, 15, 18] for references. The characterization given in Theorem 1.1 enables us to construct half-arc-transitive graphs.

Theorem 1.2 *Let $G = \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group, where p, q are odd primes. Then for each integer k such that $k > 1$, $2k \mid (p-1)$ and $(k, q) = 1$, there are exactly $\frac{q-1}{2}$ non-isomorphic connected edge-transitive Cayley graphs of G of valency $2k$, which are all half-arc-transitive.*

Next we apply Theorem 1.1 to study an isomorphism problem of Cayley graphs, that is, Problem 6.3 of the survey [10]. This was one of the main motivations for this work. The isomorphism problem for graphs is fundamental and difficult. One way to study the isomorphism problem for Cayley graphs is to determine whether the isomorphism between two Cayley graphs is determined by an automorphism of the defining group.

Table 1 $\mathsf{Aut}\,\Gamma$ with an insoluble subgroup

X	G	X_α	$\mathsf{val}(\Gamma)$	$\mathsf{Aut}\,\Gamma$
$\mathrm{PSL}(2,p)$	$p{:}\frac{p-1}{2}$	D_{p+1}	$\frac{p+1}{2}$	$\mathrm{PGL}(2,p)$
			$\frac{p+1}{4}$	X
$\mathrm{PGL}(2,p)$	$p{:}\frac{p-1}{2}$	$\mathrm{D}_{2(p+1)}$	$\frac{p+1}{2}$	X
$\mathrm{PSL}(2,11)$	$11{:}5$	A_4	$4, 6$	$\mathrm{PGL}(2,11)$
$\mathrm{PGL}(2,11)$	$11{:}5$	S_4	$4, 6, 8$	X
$\mathrm{PSL}(2,23)$	$23{:}11$	S_4	$4, 6, 8, 12$	X
$\mathrm{PSL}(2,29)$	$29{:}7$	A_5	12	X
$\mathrm{PSL}(2,59)$	$59{:}29$	A_5	$6, 10, 12, 20, 24, 30$	X

A Cayley graph $\mathsf{Cay}(G,S)$ is called a *CI-graph* of G if, for any Cayley graph $\mathsf{Cay}(G,T)$, whenever $\mathsf{Cay}(G,S)\cong\mathsf{Cay}(G,T)$ we have $S^\sigma=T$ for some $\sigma\in\mathsf{Aut}\,(G)$. (CI stands for a Cayley isomorphism.)

Let G be a group and q the smallest prime divisor of $|G|$. It was shown in [8] that each connected Cayley graph of G of valency less than q is a CI-graph, and then in [9] examples were constructed to show that connected Cayley graphs of valency q are not necessarily CI-graphs. Problem 6.3 in [10] proposed to characterize such Cayley graphs. Theorem 1.1 enables us to construct another type of example, given in the following theorem.

Theorem 1.3 *Let $G=\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group, where p,q are odd primes. Let Γ be a connected undirected Cayley graph of G of valency at most $2q$. Then one of the following occurs*:

(i) *G is a Hall normal subgroup of $\mathsf{Aut}\,\Gamma$, and Γ is a CI-graph of G*;
(ii) *Γ is arc-transitive of valency $2q$, and $\mathsf{Aut}\,\Gamma\cong(\mathbb{Z}_{p^d}{:}\mathbb{Z}_{2q})\times\mathbb{Z}_q$; furthermore, if $q\geq 5$, Γ is not a CI-graph of G.*
(iii) *$G=\mathbb{Z}_p{:}\mathbb{Z}_q$, and lies in Table 1.*

Comparing with Theorem 1.2, the next result is somehow a bit surprising.

Corollary 1.4 *All connected edge-transitive Cayley graphs of valency $2q$ of the Frobenius group $\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ with p,q odd primes are isomorphic and arc-transitive circulants.*

2 Transitive permutation groups of degree p^dq

Let X be a transitive permutation group on Ω. If Ω has a non-trivial X-invariant partition $\mathcal{B}$ say, then X is called *imprimitive*, and X induces a transitive permutation group on $\mathcal{B}$, denoted by $X^{\mathcal{B}}$. On the other hand, if Ω has no non-trivial X-invariant partition then X is said to be *primitive*. An X-invariant partition $\mathcal{B}$ is called *minimal* if for a block $B\in\mathcal{B}$, the induced action X_B^B is primitive.

Lemma 2.1 *Let $X \le \mathrm{Sym}(\Omega)$ be transitive. Let $\mathcal{B}$ be a minimal X-invariant partition of Ω, and let K be the kernel of X acting on $\mathcal{B}$. Assume that $\mathsf{soc}(X_B^B)$ is simple for $B \in \mathcal{B}$, and $K \ne 1$. Then the following statements hold*:

(i) *K acts on B faithfully if and only if $\mathsf{soc}(K)$ is simple.*
(ii) *$\mathsf{soc}(K)$ is a characteristically simple group.*
(iii) *In the case where $\mathsf{soc}(K)$ is not simple, for each $C \in \mathcal{B}$, the action of $K_{(B)}$ on C is trivial or transitive, and there exists at least one C on which $K_{(B)}$ is transitive.*

Proof Since $\mathcal{B}$ is a minimal X-invariant partition of Ω, the induced permutation group X_B^B is primitive. Let $N = \mathsf{soc}(K)$. Then $N \ne 1$, and it follows that $N^B \ne 1$, that is, N^B is a non-trivial normal subgroup of X_B^B. Since $\mathsf{soc}(X_B^B)$ is simple, so is N^B.

Suppose that $N = \mathsf{soc}(K)$ is simple. Since N is normal in X, the actions of N on all blocks in $\mathcal{B}$ are equivalent. Thus, if N acts trivially on B, then N is trivial on every block in $\mathcal{B}$, which is not possible. So N is non-trivial on B, and it follows that K is faithful on B. Suppose on the other hand that N is not simple. Since N^B is simple, we have $K_{(B)} \ge N_{(B)} \ne 1$, as in part (i).

Let S be a simple direct factor of N. Then $S \lhd N \lhd X$, and S acts non-trivially on some block $C \in \mathcal{B}$. Thus, $S \cong S^C \lhd N^C \lhd X_C^C$. Since $\mathsf{soc}(X_C^C) \cong \mathsf{soc}(X_B^B)$ is simple, it follows that S is isomorphic to $\mathsf{soc}(X_C^C)$. Hence $N = \mathsf{soc}(K)$ is a direct product of isomorphic simple groups, that is, N is characteristically simple, as in part (ii).

Suppose that for a block $C \in \mathcal{B}$, the action $N_{(B)}$ on C is non-trivial. Then $N_{(B)}^C \lhd N^C \lhd X_C^C$. Since $\mathsf{soc}(X_C^C)$ is simple, it follows that $N_{(B)}^C \ge \mathsf{soc}(X_C^C)$ and $N_{(B)}^C$ is transitive, and so is $K_{(B)}^C$.

Since $N_{(B)} \ne 1$, there exists at least one block $B' \in \mathcal{B}$ on which $N_{(B)}$ acts non-trivially, as in part (iii). □

If $N \lhd X$, a normal subgroup of X, then either N is transitive on Ω, or the set of N-orbits on Ω is an X-invariant partition of Ω, called a *normal partition*. It follows that any non-identity normal subgroup of a primitive permutation group is transitive. For any permutation group X, if each non-trivial normal subgroup of X is transitive then X is called *quasiprimitive*. Hence primitive groups are quasiprimitive; however, the converse statement is not true.

Next we assume that X contains a metacyclic regular subgroup

$$G = \langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_{p^d}{:}\mathbb{Z}_q,$$

where p, q are odd primes. Then $|\Omega| = p^d q$. Assume further that G is a Frobenius group, equivalently in this case, the center $\mathbf{Z}(G) = 1$.

The primitive case is known by a recent publication [13]. The following lemma gives a list of triples (X, G, X_ω) where $\omega \in \Omega$, which is read out of Tables 16.1–16.3 of [13].

Lemma 2.2 *If X is primitive on Ω, then either $G \le X \le \mathrm{AGL}(1, p)$ or (X, G, X_ω) is one of the triples in Table 2.*

Table 2 Primitive group X with subgroup G

X	G	X_ω	condition
$\mathrm{PGL}(q,r)$	$\frac{r^q-1}{(r-1)q}{:}q$	P_1	r prime, $q \mid (r-1)$
$\mathrm{PGL}(2,11)$	$11{:}5$	S_4	
$\mathrm{PGL}(2,23)$	$23{:}11$	S_4	
$\mathrm{PSL}(2,29)$	$29{:}7$	A_5	
$\mathrm{PSL}(2,59)$	$59{:}29$	A_5	
$\mathrm{PSL}(5,2)$	$31{:}5$	P_2	
M_{11}	$11{:}5$	$\mathrm{M}_9.2$	
M_{23}	$23{:}11$	$\mathrm{M}_{21}.2$	
	$23{:}11$	$2^4{:}\mathrm{A}_7$	
A_p	$p{:}\frac{p-1}{2}$	S_{p-2}	$q=\frac{p-1}{2}$

Table 3 Primitive group X of order p^d

P	P_ω	p^d	Condition
$\mathrm{A}_{p^d}, \mathrm{S}_{p^d}$	$\mathrm{A}_{p^d-1}, \mathrm{S}_{p^d-1}$		
$\mathrm{PGL}(n,r)$	P_1	$\frac{r^n-1}{r-1}$	n, r prime
$\mathrm{PGL}(n,r^n), \mathrm{PGL}(n,r^n).n$	P_1	$\frac{r^{n^2}-1}{r^n-1}$	n, r prime
$\mathrm{PSL}(2,11)$	A_5	11	
M_{11}	M_{10}	11	
M_{23}	M_{22}	23	

Next, we assume that X is imprimitive on Ω. The following result is a special case in [5] and [11], which will be frequently used.

Lemma 2.3 *Let P be a quasiprimitive permutation group on Ω that contains a regular subgroup $\mathbb{Z}_{p^d}$ with p prime. Then P is primitive on Ω, and further, either $d=1$ and $P \le \mathrm{AGL}(1,p)$, or P is almost simple and 2-transitive, listed in Table 3.*

Let $\mathcal{B}$ be a non-trivial X-invariant partition of Ω, and let K be the kernel of X acting on $\mathcal{B}$.

Lemma 2.4 *Using the above notation, either*

(i) $|\mathcal{B}| = p^d$ *and* $\langle a\rangle$ *is regular on* $\mathcal{B}$, *or*
(ii) $1 \ne K\cap G \le \langle a\rangle$, *and* $G^{\mathcal{B}}$ *or* $\langle a\rangle^{\mathcal{B}}$ *is regular.*

Proof Since $K \lhd X$, we have $K\cap G \lhd G$. Since G is transitive on Ω, the factor group $G/(K\cap G)$ is a transitive Frobenius group on $\mathcal{B}$. If $K\cap G\ne 1$, then since $K\cap G$ is a proper normal subgroup of G and G is a Frobenius group, we have $K\cap G\le\langle a\rangle$. Since $G^{\mathcal{B}} \cong G/(K\cap G)$ is transitive, it follows that $G_B^{\mathcal{B}} = 1$ or $\mathbb{Z}_q$, and hence $\langle a\rangle^{\mathcal{B}}$ or $G^{\mathcal{B}}$ is regular, respectively.

Suppose that $K\cap G = 1$. Then G is faithful on $\mathcal{B}$, and $G\cong G^{\mathcal{B}}$. Thus, the point stabilizer $G_B^{\mathcal{B}}$ of G acting on $\mathcal{B}$ is core free. Since the only core free subgroups of G

Table 4 Quasiprimitive and not primitive group X

X	G	X_ω	Comments
$\mathrm{PGL}(q,r)$	$\frac{r^q-1}{r-1}{:}q$	$[q^{q-1}]{:}\frac{1}{q}\mathrm{GL}(q-1,r)$	
$\mathrm{PGL}(q,r^q)$	$\frac{r^{q^2}-1}{r^q-1}{:}q$	$[q^{q-1}]{:}\frac{1}{q}\mathrm{GL}(q-1,r^q)$	r prime, $q \mid (r-1)$
$\mathrm{PGL}(q,r^q).q$	$\frac{r^{q^2}-1}{r^q-1}{:}q$	$[q^{q-1}]{:}\frac{1}{q}\mathrm{GL}(q-1,r^q).q$	
$\mathrm{PSL}(2,11)$	$11{:}5$	A_4	

are isomorphic to $\mathbb{Z}_q$, we conclude that $G_B \cong \mathbb{Z}_q$, and thus $\langle a\rangle \cong \mathbb{Z}_{p^d}$ acts regularly on $\mathcal{B}$. □

The following lemma determines the quasiprimitive case.

Lemma 2.5 *If X is quasiprimitive and imprimitive on Ω, then (X, G, X_ω) is one of the triples in Table 4.*

Proof Let $\mathcal{B}$ be a non-trivial X-invariant partition of Ω. Since X is quasiprimitive on Ω, $X \cong X^{\mathcal{B}}$ is faithful, and by Lemma 2.4, we have $|\mathcal{B}| = p^d$ and $X^{\mathcal{B}}$ has a cyclic regular subgroup. By Lemma 2.3, $X \cong X^{\mathcal{B}}$ lies in Table 3, and $G \le \mathbf{N}_X(\langle a\rangle)$. Furthermore, for $B \in \mathcal{B}$, we have $X_B = G_B X_\omega$ with $G_B = \mathbb{Z}_q$. It then follows that (X, G, X_ω) lies in Table 4. □

We next consider minimal block systems. Assume that $\mathcal{B}$ is a minimal X-invariant partition of Ω. Take a block $B \in \mathcal{B}$. Then the induced permutation group X_B^B is primitive.

Lemma 2.6 *Assume that $K \ne 1$ and $K \cap G = 1$. Then $|B| = q$, and either $\mathsf{soc}(K)$ is non-simple and acts on B unfaithfully, or $K = \mathbb{Z}_q$ and $\langle a\rangle \times K \cong \mathbb{Z}_{p^d q}$ is regular on Ω.*

Proof By Lemma 2.4, $|B| = q$, and $\langle a\rangle$ is regular on $\mathcal{B}$. Hence $\mathbb{Z}_q \cong \langle b\rangle \le G_B$, and $\mathsf{soc}(X_B^B)$ is simple. If $\mathsf{soc}(K)$ is not simple, then by Lemma 2.1, $\mathsf{soc}(K)$ is unfaithful on B.

Assume that $\mathsf{soc}(K)$ is simple. Then by Lemma 2.1, $K \cong K^B$. By Lemma 2.3, either $K \cong K^B \lhd X_B^B \le \mathrm{AGL}(1,q)$, or K is almost simple. It follows that $KG = K \times G$. Since G is regular on Ω, we conclude that $K \cong \mathbb{Z}_q$, and K is semiregular on Ω. So $\langle a\rangle \times K$ is regular on Ω. □

Lemma 2.7 *Assume that $K \cap G \ne 1$. Then $\mathsf{soc}(K)$ is characteristically simple, and one of the following holds:*

(i) $\mathsf{soc}(K)$ *is not simple and acts on B unfaithfully*;
(ii) $X = \mathrm{PGL}(q,r^q).q$, $G = \mathbb{Z}_{\frac{r^{q^2}-1}{r^q-1}}{:}\mathbb{Z}_q$, *and* $X_\omega = P_1 \cap \mathrm{PGL}(q,r^q)$, *where r is prime*;

Table 5

X	G	X_ω	*Condition*
$(\mathrm{A}_{p^d} \times \mathbb{Z}_q).o$	$\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$	$\mathrm{A}_{p^d-1}.o$	$q \mid (p-1)$, $o = 1$ *or* 2
$\mathrm{PGL}(q, r) \times \mathbb{Z}_q$	$\mathbb{Z}_{\frac{r^q-1}{r-1}}{:}\mathbb{Z}_q$		
$\mathrm{PGL}(q, r^q) \times \mathbb{Z}_q$	$\mathbb{Z}_{\frac{r^{q^2}-1}{r^q-1}}{:}\mathbb{Z}_q$	P_1 (*parabolic*)	*r prime*
$\mathrm{PGL}(q, r^q).q \times \mathbb{Z}_q$	$\mathbb{Z}_{\frac{r^{q^2}-1}{r^q-1}}{:}\mathbb{Z}_q$		
$\mathrm{PSL}(2, 11) \times \mathbb{Z}_5$	11:5	A_5	
$\mathrm{M}_{11} \times \mathbb{Z}_5$	11:5	A_5	
$\mathrm{M}_{23} \times \mathbb{Z}_{11}$	23:11	M_{22}	

(iii) (X, G, X_ω) *is a triple of Table* 5;
(iv) $G = \mathbb{Z}_p{:}\mathbb{Z}_q$, *and* $X = \mathbb{Z}_p{:}\mathbb{Z}_{qk} \le \mathrm{AGL}(1, p)$;
(v) $K \cong \mathbb{Z}_p$.

Proof The primitive permutation group X_B^B contains a regular subgroup G_B^B, which is cyclic or Frobenius. By Lemmas 2.2 and 2.3, the socle of X_B^B is simple. Since $K \ne 1$, by Lemma 2.1, $\mathsf{soc}(K)$ is characteristically simple; furthermore, if $\mathsf{soc}(K)$ is not simple, then $\mathsf{soc}(K)$ acts on B unfaithfully, as in part (i).

Assume next that $\mathsf{soc}(K)$ is simple. By Lemma 2.1, we have $K_{(B)} = 1$, and $K \cong K^B \lhd X_B^B$. Then by Lemmas 2.2 and 2.3, either $K = \mathbb{Z}_p{:}\mathbb{Z}_l$ with $l \mid (p-1)$, or $K \cong K^B$ is almost simple and lies in Table 2 or 3.

If $K = \mathbb{Z}_p$, then part (v) is satisfied. Assume that $K = \mathbb{Z}_p{:}\mathbb{Z}_l \le \mathrm{AGL}(1, p)$ with $l \ne 1$. Then $K \cap G = \mathbb{Z}_p$, $K = (K \cap G){:}K_\omega$, with $K_\omega \cong \mathbb{Z}_l$, and $K \cap G \lhd X$ since $K \cap G$ is a characteristic subgroup of K. Suppose that $d > 1$. Then $\langle a\rangle \le \mathbf{C}_X(K \cap G) \lhd X$, and so $(K\langle a\rangle)/(K \cap G) \cong \langle \overline{a}\rangle \times K_\omega$, where $\langle \overline{a}\rangle = \langle a\rangle/K \cap G$. It follows that K_ω centralizes a, and hence $K_\omega \mathsf{char}\, K \lhd X$, which is not possible. So $d = 1$, and $G = \mathbb{Z}_p{:}\mathbb{Z}_q$ and $X = \mathbb{Z}_p{:}\mathbb{Z}_{qk} \le \mathrm{AGL}(1, p)$, as in part (iv).

Thus, we may assume that K is almost simple.

Case 1. Suppose $\mathbf{C}_X(K) = 1$.

Then $X \le \mathsf{Aut}(K)$, and X is almost simple of which the socle equals $\mathsf{soc}(K)$. Since $K \cong K^B \lhd X_B^B$, the primitive group X_B^B is almost simple and lies in Table 2 or Table 3, and so is K^B ($\cong K$). Since $K \cap G$ is normal in G, it follows that K lies in Table 3. As K is the kernel of a minimal invariant partition of Ω, K is intransitive on Ω, and thus $K \cap G$ is a proper subgroup of G. Now $|B| = |K : K_\alpha|$, $|\Omega| = |X : X_\alpha|$, and $|\mathcal{B}|$ divides $|\Omega|$. Noticing that $|K : K_\alpha| = |B|$ divides $|X : X_\alpha| = |\Omega|$, we have $|\mathcal{B}| = \frac{|X:X_\alpha|}{|K:K_\alpha|} = \frac{|X|}{|K|} \cdot \frac{|K_\alpha|}{|X_\alpha|}$. Furthermore, as $|\Omega| = p^d q$ and $(pq, |X_\alpha|) = 1$, p or q divides $\frac{|X|}{|K|} = |X/K|$. Since p, q are odd primes, by Table 2 and Table 3, we conclude that $\mathsf{soc}(K) = \mathrm{PSL}(n, r)$ or $\mathrm{PSL}(n, r^n)$, where n, r are primes. For the former, $X \le K.\mathsf{Out}(K) = \mathrm{PSL}(n, r).(n, r-1)$, or $\mathrm{PSL}(n, r).(n, r-1).2$. Thus $X = \mathrm{PSL}(n, r)$, $\mathrm{PSL}(n, r).2$, $\mathrm{PSL}(n, r).n$, or $\mathrm{PSL}(n, r).n.2$ with $p^d = \frac{r^n-1}{r-1}$. Since $\mathbb{Z}_{p^d}{:}\mathbb{Z}_q \cong G \le X$, we have $n = q$. Similarly, for the latter case for which $soc(K) = \mathrm{PSL}(n, r^n)$, we have $n=q$. On the other hand, X_B^B lies in Table 2 or Table 3, and $X_B^B \le X$. Thus $X = \mathrm{PGL}(q, r^q).q$ satisfying part (ii).

Case 2. Let $C := \mathbf{C}_X(K) \ne 1$.

Then $X = (KC).H = (K \times C).H$, where $H \le \mathsf{Out}(K)$. If $C \cap G \ne 1$, then $(K \cap G) \times (C \cap G)$ is a normal subgroup of G, which is not possible. Thus, $C \cap G = 1$, and by Lemma 2.4, an orbit of C on V is of length q. Let $\mathcal{C}$ be the set of C-orbits on V, and let L be the kernel of X acting on $\mathcal{C}$. By Lemmas 2.4, 2.2 and 2.3, either L is almost simple which contains a regular cyclic group $\mathbb{Z}_q$, and has the form in Table 3, or $L \le \mathrm{AGL}(1,q)$. Since $G = \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$, the intersection $L \cap G = 1$. Moreover, as $L \lhd LG$, we have $LG = L{:}G$. In both cases, there is no automorphism of L of order q or p. Thus $LG = L \times G$, that is, G centralizes L, and since G is regular on V, we conclude that $L = C = \mathbb{Z}_q$. Thus, $X = (K \times \mathbb{Z}_q).H$, and $X^{\mathcal{C}} \cong X/L \cong K.H$. Now $|\mathcal{C}| = p^d$, and $X^{\mathcal{C}}$ is a permutation group on $\mathcal{C}$ and contains a cyclic regular subgroup. Since K is almost simple, so is $X^{\mathcal{C}}$. By [11, Corollary 1.4], $X^{\mathcal{C}}$ acts on $\mathcal{C}$ primitively, and thus $X^{\mathcal{C}}$ satisfies Lemma 2.3. Now X is an extension of $\mathbb{Z}_q$ by $X^{\mathcal{C}}$. Noticing that $\mathbb{Z}_{p^d} \le G \le X^{\mathcal{C}}$, it is easily shown that X satisfies part (iii). □

3 Proof of Theorem 1.1

We prove Theorem 1.1 in this section. We first study some examples appearing in the theorem.

For graphs $\Delta = (U, E)$ and $\Sigma = (W, F)$ with vertex sets U and W, respectively, we define *lexicographic product* $\Delta[\Sigma]$ and *direct product* $\Delta \times \Sigma$, both with vertex set $V = U \times W = \{(u, w) \mid u \in U, w \in W\}$. The adjacency is defined as follows: given two vertices $v_1 = (u_1, w_1)$ and $v_2 = (u_2, w_2)$ in V,

(a) for $\Delta[\Sigma]$, two vertices (u_1, w_1) and (u_2, w_2) are adjacent if and only if either u_1, u_2 are adjacent in Δ or $u_1 = u_2$ and w_1, w_2 are adjacent in Σ;
(b) for $\Gamma \times \Sigma$, two vertices (u_1, w_1) and (u_2, w_2) are adjacent if and only if $\{u_1, u_2\} \in E$ and $\{w_1, w_2\} \in F$.

Example 3.1 Let $X = \mathrm{PGL}(q, r^q).q$ where q is an odd prime, let $X_\omega = P_1 \cap \mathrm{PGL}(q, r^q) = [r^{q(q-1)}]{:}\mathrm{GL}(q-1, r^q)$, and let $\Omega = [X : X_\omega]$. Then $|\Omega| = \frac{r^{q^2}-1}{r^q-1}$, and X has a subgroup $G = \mathbb{Z}_{\frac{r^{q^2}-1}{r^q-1}}{:}\mathbb{Z}_q$ which is regular on Ω.

Assume that $\frac{r^{q^2}-1}{r^q-1} = p^d$ for some prime p. Let $K = \mathrm{PGL}(q, r^q) \lhd X$. Since $X_\omega < K$, the normal subgroup K is intransitive and has exactly q orbits on Ω. Let $\mathcal{B}$ be the set of K-orbits on Ω, and let $\mathcal{B} = \{B_1, B_2, \dots, B_q\}$. Then $|\mathcal{B}| = q$, and $|B| = \frac{r^{q^2}-1}{r^q-1}$. Let Γ be a connected orbital graph. Then the quotient graph Γ_K is a circulant of order q.

Since $q \ge 3$, the linear group K has exactly two inequivalent 2-transitive permutation representations of degree $\frac{r^{q^2}-1}{r^q-1}$. Suppose that K^{B_i} and K^{B_j} are not equivalent for some blocks B_i and B_j. Then, relabeling if necessary, assume that $K^{B_1}, \dots, K^{B_t}$ are equivalent to K^{B_i}, and $K^{B_{t+1}}, \dots, K^{B_q}$ are equivalent to K^{B_j}. Thus, the quotient Γ_K is bipartite, which is not possible. So the actions of K on B_i are all equivalent.

Let B_1 and B_j be such that the induced subgraph $[B_1 \cup B_j]$ has at least one edge. Let $\alpha \in B_1$ and $\beta \in B_j$ be such that $K_\alpha = K_\beta$. Suppose that α is adjacent to β. Since Γ is X-edge-transitive and $K_\alpha = X_\alpha$, it follows that K_α fixes $\Gamma(\alpha)$ pointwise. Since Γ is connected, it follows that K_α fixes all vertices of Γ, and thus K is semiregular on Ω, which is a contradiction. Thus α is not adjacent to β. Since K is 2-transitive on B_j, the stabilizer $K_\alpha = K_\beta$ is transitive on $B_j \setminus \{\beta\}$. It follows that $[B_1, B_j] = \mathbf{K}_{p^d,p^d} - p^d\mathbf{K}_2$. On the other hand, since $K_\alpha = X_\alpha$ and Γ is X-edge-transitive, we conclude that $\Gamma_K = \mathbf{C}_q$, and $\Gamma = \mathbf{C}_q[\overline{\mathbf{K}}_{p^d}] - p^d\mathbf{C}_q$.

Example 3.2 Let $\Gamma = \mathbf{K}_{p^d} \times \mathbf{C}_q$, where q divides $p-1$. Then $\mathsf{Aut}\,\Gamma = \mathrm{S}_{p^d} \times \mathrm{D}_{2q}$. Let $P \le \mathrm{S}_{p^d}$ be a 2-transitive group which contains a regular subgroup $R \cong \mathbb{Z}_{p^d}$ such that $\mathbf{N}_P(R)$ contains a Frobenius group $R{:}\langle z\rangle$ where $\langle z\rangle \cong \mathbb{Z}_q$. Let $X = P \times C$, where $C = \langle c\rangle = \mathbb{Z}_q$. Then Γ is X-edge-transitive, and the subgroup $R{:}\langle zc\rangle < X$ is a Frobenius group and regular on $V\Gamma$.

To prove Theorem 1.1, we refine a result for edge-transitive circulants.

Lemma 3.3 *Let $\Gamma = (V, E)$ be a connected X-edge-transitive circulant of order p^d with p prime such that X contains a regular cyclic subgroup G. Assume that the valency of Γ is coprime to p. Then either $\Gamma = \mathbf{K}_{p^d}$ and X is almost simple and 2-transitive, or G is normal in X.*

Proof If X is primitive on V, then either X is almost simple and 2-transitive, so Γ is a complete graph, or $X \le \mathrm{AGL}(1, p)$. Then the lemma holds.

Thus, we next assume that X is imprimitive. Suppose that X has two minimal normal subgroups M, N. Let $\mathcal{B}$ and $\mathcal{C}$ be the sets of M-orbits and N-orbits, respectively, on V. Let K, L be the kernels of X acting on $\mathcal{B}, \mathcal{C}$, respectively. By [11, Lemma 3.1], we have $K \cap G \ne 1$, and $L \cap G \ne 1$. Hence $(K \cap G) \times (L \cap G) \le G \cong \mathbb{Z}_{p^d}$, which is not possible. Thus, X has a unique minimal normal subgroup, say M.

Let $\mathcal{B}$ be a minimal X-invariant partition of V, and let $B \in \mathcal{B}$. Let K be the kernel of X acting on $\mathcal{B}$, and let $\overline{X} = X/K$. Then X_B^B is primitive, and by Lemma 2.3, $\mathsf{soc}(X_B^B)$ is simple.

Suppose that $\mathsf{soc}(K)$ is not simple. By Lemma 2.1, $\mathsf{soc}(K)$ is characteristically simple and acts non-trivial on B, and there exists $B' \in \mathcal{B}$ on which $K_{(B)}$ is transitive. It follows that the induced subgraph $[B, B'] = \mathbf{K}_{p^c,p^c}$, where $p^c = |B|$. Thus, $\Gamma = \Gamma_{\mathcal{B}}[\overline{\mathbf{K}}_{p^c}]$, which is not possible since the valency of Γ should be coprime to p.

Hence $\mathsf{soc}(K)$ is simple. Then $K \cong K^B \lhd X_B^B$. Hence either K is almost simple or $K = \mathbb{Z}_p{:}\mathbb{Z}_l \le \mathrm{AGL}(1, p)$. Let $C = \mathbf{C}_X(K)$. Then $X = (CK).H$, where $H \le \mathsf{Out}(K)$. If $C = 1$, then either X is almost simple, or $X = \mathbb{Z}_p{:}\mathbb{Z}_l \le \mathrm{AGL}(1, p)$, and H has order coprime to p. So X is primitive, which is a contradiction. Hence $C \ne 1$.

Suppose that K is not abelian. Then $CK = C \times K$, and hence X has a minimal normal subgroup that is contained in C, which contradicts the previous conclusion that X has only one minimal normal subgroup. Thus, $K \cong \mathbb{Z}_p$.

By the inductive assumption, $\Gamma_{\mathcal{B}}$ and $\overline{X}$ satisfy the lemma. Suppose that $\overline{X}$ is almost simple. By Lemma 2.3, $\overline{X}$ lies in Table 3. It follows that $K \times \mathsf{soc}(\overline{X}) \lhd K.\overline{X}$, and G has two minimal normal subgroups. This is not possible since X has no subgroup which is isomorphic to $\mathbb{Z}_p^2$. Thus, G/K is normal in $\overline{X} = X/K$, and so G is normal in X, as claimed. □

The following conclusion is a consequence of Lemma 3.3.

Lemma 3.4 *Let $\Gamma = (V, E)$ be a connected circulant of order p^d and valency at most $p-1$, with p odd prime. Assume that $\mathbb{Z}_{p^d} \cong G \le \mathsf{Aut}\,\Gamma$ and G is regular on V. Then either $\Gamma = \mathbf{K}_p$, or G is normal in $\mathsf{Aut}\,\Gamma$.*

Proof Let $R = \langle a\rangle \cong \mathbb{Z}_{p^d}$ be such that $\Gamma = \mathsf{Cay}(R, S)$, and let $X = \mathsf{Aut}\,\Gamma$. Then $\langle S\rangle = R$, and so S contains an element of order p^d. Without loss of generality, assume $a \in S$. Let Σ be the graph with vertex set V and edge set $\{1, a\}^X$. Then Σ is a connected X-edge-transitive Cayley graph of R, and $X \le \mathsf{Aut}\,\Sigma$.

If $\Sigma = \mathbf{K}_{p^d}$, then since the valency of Σ is at most $p-1$, we conclude that $d = 1$ and $\Gamma = \Sigma = \mathbf{K}_p$. Assume that Σ is not a complete graph. By Lemma 3.3, G is normal in X. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let G, Γ and X be as in the theorem. If X is primitive on V, then by Lemma 2.2, either $G \le X \le \mathrm{AGL}(1, p)$ or (X, G, X_α), where α is a vertex, lies in Table 2 and part (i) of Theorem 1.1 is satisfied.

Thus, we assume that X is imprimitive on V. Let $\mathcal{B}$ be a minimal X-invariant partition of Ω, and let $B \in \mathcal{B}$. Let K be the kernel of X acting on $\mathcal{B}$. If $\mathsf{soc}(K)$ is not simple, then by Lemma 2.1, $K_{(B)} \ne 1$. It follows that $K_{(B)}$ is transitive on C for some block $C \in \mathcal{B}$ which is adjacent to B. Thus, the induced subgraph $[B, C] \cong \mathbf{K}_{l,l}$, where $l = |B|$. Since the valency of Γ is not divisible by p, we have $(l, p) = 1$. Hence $l = q$, and so $\Gamma = \Sigma[\overline{\mathbf{K}}_q]$, as in part (v).

If $K = 1$, then $X \cong X^{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ is an edge-transitive circulant of order p^d. By Lemma 3.3, either $\Gamma_{\mathcal{B}} = \mathbf{K}_{p^d}$ and $X^{\mathcal{B}}$ is almost simple 2-transitive, or $G \cong G^{\mathcal{B}} \lhd X^{\mathcal{B}} \cong X$. For the former, X itself is almost simple and quasiprimitive, as in part (i).

Thus, we next assume that $K \ne 1$ and $\mathsf{soc}(K)$ is simple. Then $K \cong K^B$ is faithful.

Assume now that $K \cap G = 1$. By Lemma 2.6, we have $K = \mathbb{Z}_q$. Hence, G centralizes K, and so $\langle a\rangle \times K \cong \mathbb{Z}_{p^d q}$ is regular on V. Further, $\langle a\rangle$ is normal in X/K, and $\langle a\rangle \times K$ is normal in X. Hence q divides the valency of Γ as in part (iv).

Assume that $K \cap G \ne 1$. Then one of parts (ii)–(iv) of Lemma 2.7 is satisfied.

First consider the triple (X, G, X_α) in part (ii) of Lemma 2.7. Then $X = \mathrm{PGL}(q, r^q).q$, $p^d = \frac{r^{q^2}-1}{r^q-1}$, and $K = \mathrm{PGL}(q, r^q)$. For two adjacent orbits B, B' of K, the actions of K on B and B' are equivalent and 2-transitive. It follows that the induced subgraph $[B, B'] \cong \mathbf{K}_{l,l} - l\mathbf{K}_2$ with $l = |B| = p^d$, and hence $\Gamma \cong \mathbf{C}_q[\overline{\mathbf{K}}_{p^d}] - p^d\mathbf{C}_q$.

Now consider part (iii) of Lemma 2.7. It is shown that $\Gamma = \mathbf{K}_{p^d} \times \mathbf{C}_q$, as in Example 3.2.

For part (iv) of Lemma 2.7, we have $X \le \mathrm{AGL}(1, p)$, and G is normal in X.

Finally, for part (v) of Lemma 2.7, we have $\mathbb{Z}_p = K < \langle a\rangle$. If V has another minimal X-invariant partition $\mathcal{C}$ with L being the kernel of X on $\mathcal{C}$. Then $L \not\cong \mathbb{Z}_p$ since $\mathbb{Z}_p^2 \not\leq X$, and hence p^d divides $|\mathcal{C}|$. Then the previous argument with $\mathcal{C}$ in place of $\mathcal{B}$ shows that the theorem holds. Thus to complete the proof of Theorem 1.1, we may further assume that $\mathcal{B}$ is the unique minimal X-invariant partition of V. It follows that K is the unique minimal normal subgroup of X. Thus, X is an extension of K by $\overline{X} := X/K \cong X^{\mathcal{B}} \leq \mathsf{Aut}\,\Gamma_K$. We assume inductively that Γ_K satisfies the statement of the theorem.

Let $C = \mathbf{C}_X(K)$. Then $a \in C \lhd X$, and $X/C \leq \mathsf{Aut}(K) \cong \mathbb{Z}_{p-1}$. If $\overline{X}$ has a non-abelian minimal normal subgroup N, then $K.N$ is a non-split extension of $K = \mathbb{Z}_p$ by N. Hence $\mathbb{Z}_p$ is a factor group of the Schur multiplier of N, which is not possible, refer to [4]. It follows that $\langle\overline{a}\rangle$ is normal in $\overline{X}$, and thus $\langle a\rangle$ is normal in X. Then either G is normal in X, or X has a cyclic regular subgroup. □

4 Normal Cayley graphs

Here we study some properties of Cayley graphs, and give a proof of Theorem 1.2.

Let Γ be a Cayley graph of a group G. Then the right multiplications of elements of G induce automorphisms of Γ, that is,

$$\hat{g}:\ x \mapsto xg, \quad \text{for all } g, x \in G.$$

Further, $G \cong \hat{G} = \{\hat{g} \mid g \in G\}$, and $\hat{G} \leq \mathsf{Aut}\,\Gamma$.

For an element $g \in G$, the left multiplication:

$$\check{g}:\ x \mapsto g^{-1}x, \quad x \in G$$

is not necessarily an automorphism of Γ. However, inside $\mathrm{Sym}(G)$, there is a relation between $\hat{G}$ and $\check{G}$: $\hat{G}$ centralizes $\check{G}$, namely, $\hat{G} \circ \check{G} = \langle \hat{G}, \check{G}\rangle < \mathrm{Sym}(G)$.

We observe that, for an element $g \in G$,

$$\check{g}\hat{g}:\ x \mapsto g^{-1}xg,$$

which is an inner automorphism of G induced by g, denoted by $\tilde{g}$. Let $\tilde{G} = \{\tilde{g} \mid g \in G\}$. Then $\tilde{G} = \mathsf{Inn}(G)$.

For a subgroup H of a group X, denote by $\mathbf{N}_X(H)$ and $\mathbf{C}_X(H)$ the normalizer and the centralizer of H in X, respectively. It is easily shown that $\mathbf{C}_{\mathrm{Sym}(G)}(\hat{G}) = \check{G}$, and $\hat{G}\mathbf{C}_{\mathrm{Sym}(G)}(\hat{G}) = \hat{G}\check{G} = \hat{G}{:}\tilde{G} = \hat{G}{:}\mathsf{Inn}(G)$. Moreover, for Cayley graphs, we have the following statements, refer to [3].

Lemma 4.1 *For a Cayley graph* $\Gamma = \mathsf{Cay}(G, S)$, *we have the following property*:

$$\mathbf{N}_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G}{:}\mathsf{Aut}(G, S), \quad \textit{and} \quad \hat{G}\mathbf{C}_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G}{:}\mathsf{Inn}(G, S).$$

To prove Theorem 1.2, we need to find $\mathsf{Aut}(G)$ for the Frobenius group $G = \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$.

Lemma 4.2 *Let $G = \langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group with p, q primes. Then $\mathsf{Aut}(G) = \langle \rho, \tau\rangle \cong \mathbb{Z}_{p^d}{:}\mathbb{Z}_{p^{d-1}(p-1)}$, where*

$$\begin{aligned}&\rho : a \to a^s,\ b \to b, \quad \text{where } (s, p) = 1,\\ &\tau : a \to a,\ b \to ab.\end{aligned}$$

In particular, an element of G of order q is not conjugate to its inverse under $\mathsf{Aut}(G)$, and elements of $\mathsf{Aut}(G)$ of order q are inner automorphisms of G.

Proof By definition, we have $b^{-1}ab = a^r$, where $r^q \equiv 1 \pmod{p^d}$. Let $\sigma \in \mathsf{Aut}(G)$. Since $\langle a\rangle$ is a characteristic subgroup of G, σ maps a to a^x, where $(x, p^d) = 1$, and σ maps b to $a^y b^m$ for some integers y, m, with $m \neq 0$. We claim $m = 1$, that is, σ maps b to $a^y b$.

Now $(ab)^\sigma = a^\sigma b^\sigma = a^x a^y b^m = a^y a^x b^m$. Since $ab = ba^r$, we also have $(ab)^\sigma = (ba^r)^\sigma = a^y b^m a^{rx}$. Thus, $a^y a^x b^m = a^y b^m a^{rx}$, and hence $a^x b^m = b^m a^{rx}$, and $b^{-m} a^x b^m = a^{rx}$. Because $b^{-1}ab = a^r$, we obtain $b^{-m} a^x b^m = a^{xr^m}$. Thus $rx \equiv xr^m \pmod{p^d}$, which implies that $r^{m-1} \equiv 1 \pmod{p^d}$. Hence $m - 1 \equiv 0 \pmod{p^d}$ and σ maps b to $a^y b$.

We claim that σ is uniquely determined by the parameters x and y. Let

$$\begin{aligned}&\sigma_1 :\ a \to a^{x_1},\ b \to a^{y_1} b,\\ &\sigma_2 :\ a \to a^{x_2},\ b \to a^{y_2} b.\end{aligned}$$

If $\sigma_1 = \sigma_2$, then $a^{x_1} = a^{\sigma_1} = a^{\sigma_2} = a^{x_2}$ and $a^{y_1} b = b^{\sigma_1} = b^{\sigma_2} = a^{y_2} b$. So $x_1 - x_2 \equiv 0 \pmod{p^d}$ and $y_1 - y_2 \equiv 0 \pmod{p^d}$. Thus,

$$\mathsf{Aut}(G) = \{\sigma \mid \sigma : a \to a^x, b \to a^y b \text{ such that } (x, p^d) = 1\},$$

and in particular, $|\mathsf{Aut}(G)| = p^d . p^{d-1}(p-1)$.

Let $\rho, \tau \in \mathsf{Aut}(G)$ be such that

$$\begin{aligned}&\rho :\ a \to a^s,\ b^\rho \to b, \quad \text{where } (s, p) = 1,\\ &\tau :\ a \to a,\ b \to ab.\end{aligned}$$

Then $o(\rho) = p^{d-1}(p-1)$, and $o(\tau) = p^d$. Calculation shows that $\rho^{-1}\tau\rho = \tau^\delta$. It follows that $\langle\tau\rangle$ is a normal subgroup of $\mathsf{Aut}(G)$, and hence $\mathsf{Aut}(G) = \langle\tau\rangle{:}\langle\rho\rangle \cong \mathbb{Z}_{p^d}{:}\mathbb{Z}_{p^{d-1}(p-1)}$. □

The following lemma will be used to determine isomorphism classes of Cayley graphs.

Lemma 4.3 ([8]) *Let G be a finite group, and let $\Gamma = \mathsf{Cay}(G, S)$. Assume that G is of odd order, and assume further that $\hat{G}$ is a Hall subgroup of $\mathsf{Aut}\,\Gamma$. Then for subset $S' \subset G$ such that $\Gamma \cong \mathsf{Cay}(G, S')$, there exists $\sigma \in \mathsf{Aut}(G)$ such that $S^\sigma = S'$.*

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Let $G=\langle a\rangle{:}\langle b\rangle\cong\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group. Suppose that k is a divisor of $p-1$ which is bigger than 2 and coprime to pq. Let $\Gamma=\mathsf{Cay}(G,S)$ be connected, undirected, edge-transitive, and of valency $2k$. Since $(2k,pq)=1$ and $2k\,\big|\,(p-1)$, by [16, 17], this is not possible. Thus by Theorem 1.1 $\hat{G}$ is normal in $\mathsf{Aut}\,\Gamma$. Thus, by Lemma 4.1, we conclude that

$$\mathsf{Aut}\,\Gamma=\hat{G}{:}\mathsf{Aut}(G,S).$$

Since Γ is connected, we have $\langle S\rangle=G$, and as Γ is undirected, $S=\{s_1,s_1^{-1},\dots,s_k,s_k^{-1}\}$. Further, as Γ is edge-transitive and $\mathsf{Aut}\,\Gamma=\hat{G}{:}\mathsf{Aut}(G,S)$, we conclude that the subsets $\{s_i,s_i^{-1}\}$ with $1\le i\le k$ are all conjugate in $\mathsf{Aut}(G,S)$. Therefore, since $G=\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ is Frobenius, all elements of S are of order q. As $|S|=2k$ is coprime to pq, it follows from Lemma 4.2 that $\mathsf{Aut}(G,S)=\langle\tau\rangle\cong\mathbb{Z}_k$, and so $\mathsf{Aut}\,\Gamma=\hat{G}{:}\langle\tau\rangle\cong\hat{G}{:}\mathbb{Z}_k$. In particular, an element of order q is not conjugate in $\mathsf{Aut}\,\Gamma$ to its inverse, and therefore, Γ is not arc-transitive.

Subgroups of X of order q are Sylow q-subgroups, and hence they are conjugate. Thus, we may assume that s_1 is conjugate to b^i, where $1\le i\le\frac{q-1}{2}$. Then S is conjugate to

$$S_i=\{b^i,b^{-i}\}^{\langle\tau\rangle}.$$

Noticing that $\hat{G}$ is a Hall subgroup of $\mathsf{Aut}\,\Gamma$, by Lemma 4.3, Γ is a CI-graph. It is easily shown that the subsets S_i with $1\le i\le\frac{q-1}{2}$ are pairwise non-conjugate under $\mathsf{Aut}(G)$. So $\mathsf{Cay}(G,S_i)$ with $1\le i\le\frac{q-1}{2}$ are pairwise non-isomorphic. Therefore, there are exactly $\frac{q-1}{2}$ non-isomorphic edge-transitive Cayley graphs of G of valency $2k$. □

5 Proof of Theorem 1.3

We first state a simple property about edge-transitive graphs. Recall that a permutation group P on Ω is called *bi-transitive* if P has exactly two orbits on Ω and the two orbits have equal size.

Lemma 5.1 *A graph Γ is X-edge-transitive if and only if one of the following two cases happens, where α is a vertex*:

1. *X_α is transitive on $\Gamma(\alpha)$*;
2. *X_α is bi-transitive on $\Gamma(\alpha)$ with two orbits Δ_1 and Δ_2, and there exists $\sigma\in\mathsf{Aut}\,\Gamma$ such that $(\alpha,\beta)^\sigma=(\gamma,\alpha)$ where $\beta\in\Delta_1$ and $\gamma\in\Delta_2$.*

As before, let $G=\mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group, where p,q are odd primes. Let Γ be a connected undirected Cayley graphs of G of valency at most $2q$. Denote by α the vertex of Γ corresponding to the identity of G. Let $X\leqslant\mathsf{Aut}\,\Gamma$ contain $\hat{G}$. Then we have $X=\hat{G}X_\alpha$.

Lemma 5.2 *Assume that q divides $|X_\alpha|$. Then Γ is edge-transitive and of valency $2q$.*

Proof Since q divides $|X_\alpha|$, it follows that $\mathbb{Z}_q \leqslant X_\alpha^{\Gamma(\alpha)}$. Thus X_α has an orbit Δ of size at least q. For $\beta \in \Delta$, we have $\beta^{X_\alpha} = \Delta$. Define a graph $\Gamma_0 := (V, E_0)$, where $E_0 = \{\alpha, \beta\}^X$. Let Σ be a connected component of Γ_0 which contains α. Then the vertex set $V\Sigma$ is a subgroup of G, and so $|V\Sigma|$ divides $p^d q$. Let $Y = \mathsf{Aut}\,\Sigma$. Then q divides $|Y_\alpha|$. It follows that the valency of Σ is at least q, and hence $|V\Sigma| > q$ and p divides $|V\Sigma|$. Moreover, Σ is Y-edge-transitive. By Lemma 5.1, Y_α is transitive or bi-transitive on $\Sigma(\alpha)$.

Assume that $|\Sigma(\alpha)| < 2q$. If Y_α is bi-transitive on $\Sigma(\alpha)$, then the two orbits of Y_α on $\Sigma(\alpha)$ have equal size which is at least q, which is not possible. Thus Y_α is transitive on $\Sigma(\alpha)$. Suppose that Y_α is imprimitive on $\Gamma(\alpha)$. Then $\Gamma(\alpha)$ has a Y_α-invariant partition $\mathcal{B}$ which has l blocks and each block has size m. Thus, $q \mid l$, or $q \mid m$, which is not possible since $lm = |\Gamma(\alpha)| < 2q$. So Y_α is primitive on $\Gamma(\alpha)$. Let N be the maximal intransitive normal subgroup of Y. Let $\mathcal{B}$ be the set of N-orbits on $V\Sigma$, and let K be the kernel of Y on $\mathcal{B}$. Since Y_α is primitive on $\Sigma(\alpha)$ and $|V\Sigma|$ is odd, it follows that Σ is a cover of Σ_N, and $K_\alpha = 1$. Thus, $K = N$, and $\overline{Y} := Y/K \leqslant \mathsf{Aut}\,\Sigma_N$. Now $\overline{Y}$ is quasiprimitive on $\mathcal{B}$. By Lemmas 2.2, 2.3 and 2.5, $\overline{Y}$ is almost simple, and lies in Table 2 or 3. Since Σ_N has valency less than $2q$ and q divides $|\overline{Y}_{\overline{\alpha}}|$, we conclude that this is not possible.

Therefore, Σ is of valency $2q$, and so $\Gamma = \Sigma$ is edge-transitive. $\square$

Moreover, we have a characterization of the valency $2q$ case.

Lemma 5.3 *Let Γ be a connected edge-transitive of valency $2q$. Then the following statements hold*:

(1) $\mathsf{Aut}\,\Gamma = \langle \hat{a}\check{b}\rangle{:}\langle \hat{b}\check{b}\tau\rangle \cong \mathbb{Z}_{p^d q}{:}\mathbb{Z}_{2q}$, *where τ is an involution, and*

$$\tau : \begin{cases} \hat{a}\check{b} \to (\hat{a}\check{b})^{-1}, \\ \hat{b}\check{b} \to \hat{b}\check{b}. \end{cases}$$

(2) $\mathsf{Aut}\,\Gamma$ *has exactly $\frac{q-1}{2}$ non-conjugate subgroups which are isomorphic to H and regular on V*:

$$\langle \hat{a}\rangle{:}\langle \hat{b}\check{b}^j\rangle, \quad 2 \le j \le \frac{q+1}{2}.$$

(3) *If $q \ge 5$, then Γ is not a CI-graph.*

Proof If $\Gamma = \Sigma[\overline{\mathbf{K}}_q]$, as in part (v) of Theorem 1.1, then Σ is of valency divisible by q. Hence Γ has valency divisible by q^2, which is a contradiction since $q > 2$. It is easily shown that none of the graphs in parts (i)–(ii) of Theorem 1.1 has valency $2q$. Thus, part (iii) or (iv) of Theorem 1.1 is satisfied.

Suppose that $\hat{G}$ is normal in $X = \mathsf{Aut}\,\Gamma$. Write $\Gamma = \mathsf{Cay}(G, S)$, where $|S| = 2q$. By Lemma 4.1, $X = \hat{G}{:}\mathsf{Aut}(G, S)$. Since $(2q, p) = 1$, by Lemma 4.2, $\mathsf{Aut}(G, S)$ is isomorphic to a subgroup $\mathbb{Z}_{p-1}$. Further, $\mathsf{Aut}(G, S)$ is faithful on S, and S contains at least one element of order q since $\langle S\rangle = G$. Thus $\mathsf{Aut}(G, S) \le \mathbb{Z}_{2q}$. If $\mathsf{Aut}(G, S) = \mathbb{Z}_{2q}$, then each element $z \in S$ is conjugate to z^{-1} under $\mathsf{Aut}(G, S)$, which is not possible because an element of G of order q is not conjugate to its inverse. Therefore,

$\mathsf{Aut}(G,S)=\mathbb{Z}_q$, and hence it follows from Lemma 4.2 that $\mathsf{Aut}(G,S)$ is conjugate to $\langle \hat{b}\check{b}\rangle$. So $X=\hat{G}\times\langle\check{b}\rangle$, and Γ is not arc-transitive. However, $\langle\hat{a}\rangle\times\langle\check{b}\rangle\cong\mathbb{Z}_{p^dq}$ is regular on V, and so Γ is a circulant. It is well-known that edge-transitive circulant is arc-transitive, which is a contradiction. Thus, $\hat{G}$ is not normal in $\mathsf{Aut}\,\Gamma$.

By Theorem 1.1, $X=\mathsf{Aut}\,\Gamma$ contains a regular normal cyclic subgroup. It follows from Lemma 4.1 that $X_\alpha\le\mathbb{Z}_{p-1}$. Since Γ is arc-transitive, X_α is transitive on $\Gamma(\alpha)$. Hence $X_\alpha=\mathbb{Z}_{2q}$, and $\langle\hat{a}\rangle\lhd X$. Let H be a Hall $\{2,q\}$-subgroup of X that contains $\hat{b}$. Then $X=\langle\hat{a}\rangle{:}H$, and $H\cong\mathbb{Z}_q^2{:}\mathbb{Z}_2$. Let $C=\mathbf{C}_X(\hat{a})$. Then $C=\langle\hat{a}\rangle\times C_{p'}$ $\lhd X$, and $X/C\le\mathsf{Aut}(\langle\hat{a}\rangle)$. Hence $C_{p'}$ has order q or $2q$, and contains a characteristic subgroup $\langle\sigma\rangle\cong\mathbb{Z}_q$. It follows that $\hat{G}$ centralizes $C_{p'}$. Since $\hat{G}$ is regular, we conclude that $C_{p'}=\langle\sigma\rangle=\langle\check{b}\rangle$. Thus, $\hat{G}\times\langle\check{b}\rangle$ is a subgroup of $\mathsf{Aut}\,\Gamma$ of index 2, and $X/C_{p'}\cong\mathbb{Z}_{p^d}{:}\mathbb{Z}_{2q}$. It then follows that X has an involution τ which maps $\hat{a}$ to $\hat{a}^{-1}$ and centralizes the inner automorphism $\hat{b}\check{b}$. It is easily shown that τ does not centralize $\check{b}$, and hence $\check{b}^\tau=\check{b}^{-1}$, as in part (1).

Notice that τ sends $\hat{b}\check{b}^i$ to $\hat{b}\check{b}.\check{b}^{-i+1}$, which equals $\hat{b}\check{b}^{q+2-i}$, where $2\le i\le\frac{q+1}{2}$. In particular, $\tau:\ \langle\hat{a}\rangle{:}\langle\hat{b}\check{b}^2\rangle\to\hat{G}$. It follows that $\mathsf{Aut}\,\Gamma$ has exactly $\frac{q-1}{2}$ non-conjugate subgroups which are isomorphic to H and regular on V, which are $\langle\hat{a}\rangle{:}\langle\hat{b}\check{b}^j\rangle$, where $2\le j\le\frac{q+1}{2}$, as in part (2).

Finally, for $q\ge 5$, $\mathsf{Aut}\,\Gamma$ has at least two non-conjugate subgroups which are isomorphic to G and regular on V. Hence, by Babai's criterion given in [1], Γ is not a CI-graph, as in part (3). □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let k be the valency of Γ, where $k\ge 3$. Suppose that X is primitive on V. Then by Lemma 2.2, either X is 2-transitive on V and Γ is a complete graph, or $|V|=pq$, and Γ is characterized in [16, 17]. So part (iii) holds. Thus, in the following we assume that X is imprimitive. If q divides $|X_\alpha|$, by Lemma 5.2, Γ is edge-transitive of valency $2q$. Then by Lemma 5.3, Theorem 1.3 holds.

Hence we now assume that q does not divide $|X_\alpha|$. Then G is a Hall $\{p,q\}$-subgroup of X. Hence by Lemma 4.3, Γ is a CI-graph of G.

We next determine the automorphism group $X\le\mathsf{Aut}\,\Gamma$. Let $\mathcal{B}$ be a minimal X-invariant partition of V, and Let K be the kernel of X acting on $\mathcal{B}$. Then X_B^B is a primitive group, where $B\in\mathcal{B}$.

If $K=1$, then $X\cong X^{\mathcal{B}}$ is faithful, and by Lemma 2.4, $\Gamma_{\mathcal{B}}$ is a circulant of order p^d. By Lemma 3.4, either $\Gamma_{\mathcal{B}}=\mathbf{K}_p$, or $G\cong G^{\mathcal{B}}\lhd X^{\mathcal{B}}\cong X$. The latter case is as in part (i) of the theorem. For the former, it follows that X is quasiprimitive, given in Lemma 2.5.

Thus, we assume that $K\ne 1$ in the following.

Suppose that $\mathsf{soc}(K)$ is not simple. By Lemma 2.1, $K_{(B)}\ne 1$ and $K_{(B)}$ is transitive on some $B'\in\mathcal{B}$. Since p or q divides $|B'|$, p or q divides $|K_{(B)}|$, which is not possible as $K_{(B)}\lhd K_\alpha$ and $(|K_\alpha|,pq)=1$. Thus, $\mathsf{soc}(K)$ is simple. Suppose that q divides $|K|$. Since $K\cap G\lhd G$ and G is a Frobenius group, q does not divide $|K\cap G|$, and thus q^2 divides $|KG|(=\frac{|K||G|}{|K\cap G|})$, which is a contradiction. Thus, q does not divide $|K|$. By Lemma 2.4, $|B| = p^c$, and X_B^B is a primitive group and contains a cyclic regular subgroup of order p^c.

Suppose that X_B^B is almost simple. Then so is K^B, and thus $K \cong K^B$ lies in Table 3. Further, q is odd. It is easily shown that $K_\alpha = K_\alpha^B$ has no permutation representation of degree less than $p-1$, neither does X_α, which is not possible.

Thus, X_B^B is affine, and $\mathbb{Z}_p \cong K \lhd X_B$. Now X is an extension of $K = \mathbb{Z}_p$ by $\overline{X} := X/K$. By induction, we assume that Γ_K satisfies Theorem 1.3. Hence $\overline{G} \lhd \overline{X}$, and so G is normal in X. □

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4 As before, let $G = \langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_{p^d}{:}\mathbb{Z}_q$ be a Frobenius group, where p, q are odd primes. Let $z = b^a$, and $T_j = \{z^j, z^{-j}\}^{\langle b\rangle}$ where $1 \le j \le \frac{q-1}{2}$. Let $\Sigma_j = \mathsf{Cay}(G, T_j)$. Then the Σ_j are X-edge-transitive, where $X = \hat{G}{:}\langle\tilde{b}\rangle \le \mathsf{Aut}\,\Sigma_j$ and $T_1, T_2, \ldots, T_{\frac{q-1}{2}}$ are pairwise non-conjugate under $\mathsf{Aut}(G)$.

Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected edge-transitive graph of valency $2q$. Then S contains an element x of order q. Since all subgroups of G of order q are conjugate, there exists $\varsigma \in \mathsf{Aut}(G)$ such that $x^\varsigma = z^i$ for some i with $1 \le i \le \frac{q-1}{2}$. By Lemma 5.3, $\mathsf{Aut}\,\Gamma = \langle \hat{a}\check{b}\rangle : \langle \hat{b}\check{b}\tau\rangle > \hat{G} : \langle\tilde{b}\rangle$. Thus $\tilde{b}$ fixes S^ς, and so $(z^i)^{\langle\tilde{b}\rangle} \subseteq S^\varsigma$. Since Γ is undirected, $(z^{-i})^{\langle\tilde{b}\rangle} \subseteq S^\varsigma$. As $|S| = 2q$, $S^\varsigma = \{z^i, z^{-i}\}^{\langle\tilde{b}\rangle} = T_i$, and so $\Gamma^\varsigma = \Sigma_i$. That is to say, under $\mathsf{Aut}(G)$, there are exactly $\frac{q-1}{2}$ classes of edge-transitive Cayley graphs of G of valency $2q$, of which $\Sigma_1, \Sigma_2, \ldots, \Sigma_{\frac{q-1}{2}}$ are representatives.

The automorphism group $\mathsf{Aut}\,\Gamma = \langle\hat{a}\check{b}\rangle{:}\langle\hat{b}\check{b}\tau\rangle \cong \mathbb{Z}_{p^d q}{:}\mathbb{Z}_{2q}$ has exactly $\frac{q-1}{2}$ non-conjugate subgroups which are isomorphic to G and regular on V. Let H be one of these subgroups such that $\hat{G} \ne H \cong \hat{G}$. Then H is conjugate in the symmetric group $\mathrm{Sym}(G)$ to $\hat{G}$, namely,

$$H = \hat{G}^\rho, \quad \text{for some permutation } \rho \in \mathrm{Sym}(G).$$

Let $\Sigma = \Gamma^\rho$, which is the graph with vertex set G^ρ $(= G)$ and edge set consisting of $\{u^\rho, v^\rho\}$ for edges $\{u, v\}$ of Γ. Then $\rho^{-1}\mathsf{Aut}\,\Gamma\rho = \mathsf{Aut}\,\Sigma$, and hence $\rho^{-1}\hat{G}\rho$ is a subgroup of $\mathsf{Aut}\,\Sigma$ and regular on $V\Sigma = G$. Thus, Σ is a Cayley graph of G, namely, $\Sigma = \mathsf{Cay}(G, S')$ for some subset S'. Since it is isomorphic to Γ, Σ is edge-transitive and of valency $2q$, and $S' = S'^{-1}$. It follows that the q pairs $\{s, s^{-1}\}$ with $s \in S'$ are conjugate under a subgroup of $\mathsf{Aut}(G)$.

Suppose that $S' = S^\xi$ for some automorphism $\xi \in \mathsf{Aut}(G)$. Then $\Gamma^\xi = \Sigma = \Gamma^\rho$, and hence $\Gamma^{\rho\xi^{-1}} = \Gamma$, that is, $\rho\xi^{-1} \in \mathsf{Aut}\,\Gamma$ is an automorphism of Γ. Since $\hat{G} \lhd \mathsf{Aut}\,\Gamma$, the element $\rho\xi^{-1}$ normalizes $\hat{G}$. However, $\xi \in \mathsf{Aut}(G)$ normalizes $\hat{G}$, and hence ρ normalizes $\hat{G}$, which is a contradiction to the fact that $\hat{G}^\rho = H \ne \hat{G}$. Thus, S' and S are not conjugate in $\mathsf{Aut}(G)$.

Similarly, it is easily shown that the $\frac{q-1}{2}$ non-conjugate regular subgroups of $\mathsf{Aut}\,\Gamma$ correspond to $\frac{q-1}{2}$ Cayley graphs $\Gamma_j = \mathsf{Cay}(G, S_j)$ which are isomorphic such that $S_1 (= S), S_2, \ldots, S_{\frac{q-1}{2}}$ are pairwise non-conjugate in $\mathsf{Aut}(G)$. Hence $\Gamma_j \cong \Sigma_i$ for some $1 \leq i \leq \frac{q-1}{2}$. It follows that all edge-transitive graphs of G of valency $2q$ are isomorphic. □

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