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Exponents of 2-multiarrangements and multiplicity lattices

Takuro Abe · Yasuhide Numata

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Abstract We introduce a concept of multiplicity lattices of 2-multiarrangements, determine the combinatorics and geometry of that lattice, and give a criterion and method to construct a basis for derivation modules effectively.

Keywords Hyperplane arrangements · Multiarrangements · Exponents of derivation modules · Multiplicity lattices

1 Introduction

Let $\mathbb{K}$ be a field and V a two-dimensional vector space over $\mathbb{K}$. Fix a basis $\{x, y\}$ for V^* and define $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[x, y]$. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in V. In this article, we assume that any $H \in \mathcal{A}$ contains the origin. In other words, all hyperplane arrangements are *central*. For each $H \in \mathcal{A}$, let us fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. For a hyperplane arrangement $\mathcal{A}$, a map $\mu : \mathcal{A} \to \mathbb{N} = \mathbb{Z}_{\geq 0}$ is called a *multiplicity* and a pair $(\mathcal{A}, \mu)$ a *multiarrangement*. When we want to make it clear that all multiarrangements are considered in $V \simeq \mathbb{K}^2$, we use the term 2-*multiarrangement*. (Ordinarily,

T. Abe
Department of Mechanical Engineering and Science, Kyoto University, Yoshida Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan
e-mail: abe.takuro.4c@kyoto-u.ac.jp

Y. Numata
Department of Mathematical Informatics, The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo, 113-8656, Japan

Y. Numata (✉)
JST CREST, Sanbancho, Chiyoda-ku, Tokyo, 102-0075, Japan
e-mail: numata@stat.t.u-tokyo.ac.jp

a 2-multiarrangement is defined as a pair $(\mathcal{A}, m)$ of a central hyperplane arrangement $\mathcal{A}$ and multiplicity function $m : \mathcal{A} \to \mathbb{Z}_{>0}$. From a 2-multiarrangement $(\mathcal{A}, \mu)$ in our definition, we can obtain a 2-multiarrangement $(\mathcal{A}', m)$ in the original definition by assigning $\mathcal{A}' = \mu^{-1}(\mathbb{Z}_{>0})$ and $m = \mu|_{\mathcal{A}'}$. We identify ours with the original one in this manner.) To each multiarrangement $(\mathcal{A}, \mu)$, we can associate the S-module $D(\mathcal{A}, \mu)$, called the *derivation module* in the following manner:

$$D(\mathcal{A}, \mu) := \left\{\delta \in \mathrm{Der}_{\mathbb{K}}(S) \;\middle|\; \delta(\alpha_H) \in S \cdot \alpha_H^{\mu(H)} \ (\forall H \in \mathcal{A})\right\},$$

where $\mathrm{Der}_{\mathbb{K}}(S) := S \cdot \partial_x \oplus S \cdot \partial_y$ is the module of derivations. It is known that $D(\mathcal{A}, \mu)$ is a free graded S-module because we only consider 2-multiarrangements (see [7, 8, 16]). If we choose a homogeneous basis $\{\theta, \theta'\}$ for $D(\mathcal{A}, \mu)$, then the *exponents* of $(\mathcal{A}, \mu)$, denoted by $\exp(\mathcal{A}, \mu)$, is a multiset defined by

$$\exp(\mathcal{A}) := \bigl(\deg(\theta), \deg(\theta')\bigr),$$

where the degree is a polynomial degree.

Multiarrangements were originally introduced by Ziegler in [16] and there are a lot of studies related to a multiarrangement and its derivation module. Especially, Yoshinaga characterized the freeness of hyperplane arrangements by using the freeness of multiarrangements [14, 15]. In particular, according to the results in [15], we can obtain the necessary and sufficient condition for a hyperplane arrangement in three-dimensional vector space to be free in terms of the combinatorics of hyperplane arrangements, and the explicit description of exponents of 2-multiarrangements. This is closely related to the Terao conjecture, which asserts that the freeness of hyperplane arrangements depends only on the combinatorics. However, instead of the simple description of the exponents of hyperplane arrangements, it is shown by Wakefield and Yuzvinsky in [12] that the general description of the exponents of 2-multiarrangements is very difficult. In fact, there are only few results related to them [1, 3, 10]. Recently, some theory to study the freeness of multiarrangements has been developed by the first author, Terao and Wakefield in [5, 6], and some results on the free multiplicities are appearing [2]. In these papers, the importance of the exponents of 2-multiarrangements is emphasized too. Hence it is very important to establish some general theory for the exponents of 2-multiarrangements.

The aim of this article is to give some answers to this problem. Our idea is to introduce the concept of the *multiplicity lattice* of a fixed hyperplane arrangement. The aim of the study of this lattice is similar to, but the method is contrary to the study in [12], for Wakefield and Yuzvinsky fixed one multiplicity and consider all hyperplane arrangements with it, but we fix one hyperplane arrangement and consider all multiplicities on it. Let us fix a central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ and the lattice $\Lambda = \mathbb{N}^{|\mathcal{A}|}$. We identify $\mu \in \Lambda$ with the map $\mathcal{A} \to \mathbb{N}$ such that $\mu(H_i) = \mu_i$ for $H_i \in \mathcal{A}$. Define a map $\Delta : \Lambda \to \mathbb{Z}_{\geq 0}$ by

$$\Delta(\mu) := \deg(\theta'_\mu) - \deg(\theta_\mu),$$

where $\{\theta_\mu, \theta'_\mu\}$ is a basis for $D(\mathcal{A}, \mu)$ such that $\deg(\theta_\mu) \leq \deg(\theta'_\mu)$. If we put $\Lambda' := \Lambda \setminus \Delta^{-1}(\{0\})$, then θ_μ is unique up to a scalar for each $\mu \in \Lambda'$, though θ'_μ is not.

Hence θ_μ for $\mu \in \Lambda'$ is expected to have some good properties. Our main results are the investigations of these properties through considering the shape, topology and combinatorics of Λ'. For details, see Sect. 3, or Theorems 3.1, 3.2 and 3.4. These results, combined with Saito's criterion (Theorem 4.1), allow us to construct a basis for 2-multiarrangements effectively, see Theorem 3.9 for details.

Now the organization of this article is as follows. In Sect. 2, we introduce some notation and examples related to our new definitions. In Sect. 3, we state the main results. In Sect. 4, we recall elementary results about hyperplane arrangement theory and prove the main results. In Sect. 5, we show some applications of main results, especially we determine some exponents of multiarrangements of the Coxeter type.

2 Definition and notation

In this section, we introduce some basic terms and notation. Let $\mathbb{K}$ be a field, V a two-dimensional vector space over $\mathbb{K}$, and S a symmetric algebra of V^*. By choosing a basis $\{x, y\}$ for V^*, S can be identified with a polynomial ring $\mathbb{K}[x, y]$. The algebra S can be graded by polynomial degree as $S = \bigoplus_{i\in\mathbb{N}} S_i$, where S_i is a vector space whose basis is $\{x^j y^{i-j} \,|\, j = 0, \dots, i\}$.

Let us fix a central hyperplane arrangement $\mathcal{A}$ in V, i.e., a finite collection $\{H_1, \dots, H_n\}$ of linear hyperplanes in V. For $H \in \mathcal{A}$, fix $\alpha_H \in S_1$ such that $\ker(\alpha_H) = H$. The following new definition plays the key role in this article.

Definition 2.1 We define *the multiplicity lattice* Λ of $\mathcal{A}$ by

$$\Lambda := \mathbb{N}^{|\mathcal{A}|} = \mathbb{N}^n.$$

Let us identify $\mu = (\mu_1, \dots, \mu_n) \in \Lambda$ with the multiplicity $\mu : \mathcal{A} \to \mathbb{N}$ defined by $\mu(H_i) := \mu_{H_i} = \mu_i$. Hence a pair $(\mathcal{A}, \mu)$ can be considered as a multiarrangement. The set Λ has the partial order $\subset$ defined by

$$\mu \subset \nu \quad \Longleftrightarrow \quad \mu_H \le \nu_H \quad \text{for all } H.$$

For $\mu, \nu \in \Lambda$, the binary operations $\wedge$ and $\vee$ are defined by

$$\mu \wedge \nu := \inf\{\mu, \nu\},$$
$$\mu \vee \nu := \sup\{\mu, \nu\},$$

i.e., $(\mu \wedge \nu)_H = \min\{\mu_H, \nu_H\}$ and $(\mu \vee \nu)_H = \max\{\mu_H, \nu_H\}$. For $\mu \in \Lambda$, we define the *size* $|\mu|$ of μ by $|\mu| := \sum_{H\in\mathcal{A}} \mu_H$. The element 0, which is defined by $0_H = 0$ for all $H \in \mathcal{A}$, is the minimum element. The *covering relation* $\mu \dot\subset \nu$ is defined by $\mu \subset \nu$ and $|\mu| + 1 = |\nu|$. The graph whose set of edges is $\{(\mu, \nu) \in \Lambda^2 | \mu \dot\subset \nu\}$ and whose set of vertices is Λ is called the Hasse graph of Λ. We identify Λ and its subset with (the set of vertices of) the Hasse graph and its induced subgraph, respectively. For $\mu, \nu \in \Lambda$, we define the *distance* $d(\mu, \nu)$ by $d(\mu, \nu) := \sum_{H\in\mathcal{A}} |\mu_H - \nu_H|$. For $C, C' \subset \Lambda$, we define $d(C, C')$ by $d(C, C') := \min\{d(\mu, \mu') | \mu \in C, \mu' \in C'\}$. For $\mu \in \Lambda$ and $r \in \mathbb{N}$, we define the *ball* $B(\mu, r)$ with the radius r and center μ by $B(\mu, r) := \{\nu \in \Lambda | d(\mu, \nu) < r\}$.

Definition 2.2 We define a map $\Delta : \Lambda \to \mathbb{N}$ by

$$\Delta(\mu) := |d_1 - d_2|,$$

where (d_1, d_2) are the exponents of the free multiarrangement $(\mathcal{A}, \mu)$.

Definition 2.3 Let Λ' denote the support $\Delta^{-1}(\mathbb{Z}_{>0})$. For $H \in \mathcal{A}$, let us define Λ_H to be the set

$$\left\{\mu \in \Lambda \,\middle|\, \mu_H > \frac{1}{2}|\mu|\right\}.$$

We define $\Lambda_\emptyset$ and $\Lambda'_\emptyset$ by

$$\Lambda_\emptyset := \Lambda \setminus \left(\bigcup_{H \in \mathcal{A}} \Lambda_H\right) = \left\{\mu \in \Lambda \,\middle|\, \mu_H \leq \frac{1}{2}|\mu|(\forall H \in \mathcal{A})\right\},$$
$$\Lambda'_\emptyset := \Lambda_\emptyset \cap \Lambda'.$$

Roughly speaking, $\Lambda_\emptyset$ consists of balanced elements while Λ_H consists of elements such that H monopolizes at least half of their multiplicities.

Example 2.4 Let $\mathcal{A}$ consist of three lines. In this case,

$$\begin{aligned}
\Lambda &= \big\{(\mu_1, \mu_2, \mu_3) \,\big|\, \mu_i \in \mathbb{N}\big\},\\
\Lambda_1 &= \big\{(\mu_1, \mu_2, \mu_3) \in \Lambda \,\big|\, \mu_1 > \mu_2 + \mu_3\big\},\\
\Lambda_2 &= \big\{(\mu_1, \mu_2, \mu_3) \in \Lambda \,\big|\, \mu_2 > \mu_1 + \mu_3\big\},\\
\Lambda_3 &= \big\{(\mu_1, \mu_2, \mu_3) \in \Lambda \,\big|\, \mu_3 > \mu_1 + \mu_2\big\},
\end{aligned}$$

and

$$\Lambda_\emptyset = \left\{ (\mu_1, \mu_2, \mu_3) \in \Lambda \,\middle|\, \begin{array}{c} \mu_1 \leq \mu_2 + \mu_3; \\ \mu_2 \leq \mu_1 + \mu_3; \\ \mu_3 \leq \mu_1 + \mu_2 \end{array} \right\}.$$

By the result in Wakamiko [10], the exponents in this case can be described explicitly, and we have

$$\Delta(\mu) = \begin{cases} 1 & \text{if } \mu \in \Lambda_\emptyset \text{ and } |\mu| \text{ is odd,} \\ 0 & \text{if } \mu \in \Lambda_\emptyset \text{ and } |\mu| \text{ is even,} \\ 2\mu_i - |\mu| & \text{if } 2\mu_i > |\mu|. \end{cases}$$

Hence we have $\Lambda'_\emptyset = \{\mu \in \Lambda_\emptyset | |\mu| \text{ is odd}\}$.

For each $\mu \in \Lambda'$, there exist θ_μ and θ'_μ such that $\deg(\theta_\mu) < \deg(\theta'_\mu)$ and $\{\theta_\mu, \theta'_\mu\}$ is a homogeneous basis for $D(\mathcal{A}, \mu)$. Since $\Delta(\mu) \neq 0$, θ_μ is unique up to a nonzero scalar for each $\mu \in \Lambda'$. Hence we can define a map $\theta : \Lambda' \to D(\mathcal{A}, 0) = \mathrm{Der}_{\mathbb{K}}(S)$

by $\theta(\mu) := \theta_\mu$ (up to a scalar, or regard the image of θ as a one-dimensional vector space of $D(\mathcal{A}, 0)$).

Definition 2.5 Let us define $\mathrm{cc}(\Lambda')$, $\mathrm{cc}_0(\Lambda')$ and $\mathrm{cc}_\infty(\Lambda')$ by

$$\begin{aligned}\mathrm{cc}(\Lambda') &:= \{\text{connected components of } \Lambda'\},\\ \mathrm{cc}_0(\Lambda') &:= \bigl\{C \in \mathrm{cc}(\Lambda') \bigm| |C| < \infty\bigr\},\\ \mathrm{cc}_\infty(\Lambda') &:= \bigl\{C \in \mathrm{cc}(\Lambda') \bigm| |C| = \infty\bigr\},\end{aligned}$$

where μ and ν are said to be *connected* if there exists a path from μ to ν in the induced subgraph Λ' of the Hasse graph. For $C \in \mathrm{cc}(\Lambda')$, $\mu \in C$ and $H \in \mathcal{A}$, define $C_{\mu,H}$ to be the set of $\nu \in C$ satisfying the following two conditions:

(1) $\nu_{H'} = \mu_{H'}$ for each $H' \in \mathcal{A} \setminus \{H\}$.
(2) If $\nu \subset \kappa \subset \mu$ or $\mu \subset \kappa \subset \nu$, then $\kappa \in C$.

Definition 2.6 For $C \in \mathrm{cc}_0(\Lambda')$, we define $P(C)$ by

$$P(C) := \bigl\{\mu \in C \bigm| \Delta(\mu) = \max\bigl\{\Delta(\nu) \bigm| \nu \in C\bigr\}\bigr\}$$

and $P(\Lambda')$ by

$$P(\Lambda') := \bigcup_{C \in \mathrm{cc}_0(\Lambda')} P(C).$$

Example 2.7 Let us consider the same $\mathcal{A}$ as Example 2.4, i.e., an arrangement consisting of three lines. In this case,

$$\begin{aligned}\mathrm{cc}_0(\Lambda') &= \bigl\{\{\mu\} \bigm| \mu \in \Lambda'_\emptyset\bigr\},\\ \mathrm{cc}_\infty(\Lambda') &= \{\Lambda_1, \Lambda_2, \Lambda_3\},\\ \mathrm{cc}(\Lambda') &= \mathrm{cc}_0(\Lambda') \cup \{\Lambda_1, \Lambda_2, \Lambda_3\}.\end{aligned}$$

Definition 2.8 For a saturated chain ρ in Λ, i.e., a sequence $\rho = (\rho^{(0)}, \dots, \rho^{(k)})$ of elements in Λ satisfying $\rho^{(i)} \mathrel{\dot\subset} \rho^{(i+1)}$, we define $\alpha^{(\rho)}$ by

$$\alpha^{(\rho)} = \prod_{i:\, \Delta(\rho^{(i)}) > \Delta(\rho^{(i+1)})} \alpha^{(i)},$$

where $\alpha^{(i)} = \alpha_H$ such that $\rho_H^{(i)} + 1 = \rho_H^{(i+1)}$.

3 Main Results

In this section we state the main results. First let us give three theorems which show the structure of Λ'.

Theorem 3.1 *We have the following*:

(1) *For each* $C \in \mathrm{cc}_0(\Lambda')$, *it holds that* $C \subset \Lambda'_{\emptyset}$. *Moreover,* $\bigcup_{C\in \mathrm{cc}_0(\Lambda')} C = \Lambda'_{\emptyset}$.
(2) $\mathrm{cc}_\infty(\Lambda') = \{\Lambda_H | H \in \mathcal{A}\}$.
(3) *Any maximal connected component of* $\Lambda \setminus \Lambda' = \Delta^{-1}(\{0\})$ *consists of one point.*

Theorem 3.2 *Let* $C \in \mathrm{cc}_0(\Lambda')$ *and* $\mu \in P(C)$. *Then*

$$C = B\big(\mu, \Delta(\mu)\big),$$

and, for $\nu \in C$,

$$\Delta(\nu) = \Delta(\mu) - d(\mu, \nu).$$

In particular, for $C \in \mathrm{cc}_0(\Lambda')$, $P(C)$ *consists of one point.*

Corollary 3.3 *For* $C \in \mathrm{cc}_0(\Lambda')$, $\mu \in P(C)$ *and* $\nu \in \Lambda$ *satisfying* $d(\mu, \nu) < \Delta(\mu) + 2$,

$$\Delta(\nu) = \big|\Delta(\mu) - d(\mu, \nu)\big|.$$

The following result implies the independency of "low-degree" bases.

Theorem 3.4 *Let* $C, C' \in \mathrm{cc}_0(\Lambda')$ *such that* $d(C, C') = 2$. *If* $\mu \in C$ *and* $\mu' \in C'$, *then* $\{\theta_\mu, \theta_{\mu'}\}$ *is* S*-linearly independent. Moreover, if* $C \in \mathrm{cc}(\Lambda')$ *and* $\mu, \mu' \in C$, *then* $\{\theta_\mu, \theta_{\mu'}\}$ *is* S*-linearly dependent.*

The theorems above imply the following three corollaries, which enable us to construct the basis for $D(\mathcal{A}, \mu)$ effectively.

Corollary 3.5 *Let* $N \subset \Lambda_\emptyset$ *be such that* $\Lambda_\emptyset \setminus N$ *does not have any connected component whose size is larger than* 1, *and let* $\vartheta : N \to D(\mathcal{A}, 0)$ *be such that* $\vartheta_\mu \in D(\mathcal{A}, \mu)$ *and* $\deg \vartheta_\mu < \frac{|\mu|}{2}$. *Then the following are equivalent*:

- $\{\vartheta_\mu, \vartheta_\nu\}$ *is* S*-linearly independent if*

$$\min\left\{ d(\mu', \nu') \;\middle|\; \begin{array}{l} \mu \text{ and } \mu' \text{ are in the same connected component in } N \\ \nu \text{ and } \nu' \text{ are in the same connected component in } N \end{array} \right\} = 2.$$

- $N = \Lambda'_\emptyset$.

Corollary 3.6 *Let* $N \subset \Lambda_\emptyset$ *and* $\vartheta : N \to D(\mathcal{A}, 0)$ *be such that* $\vartheta_\mu \in D(\mathcal{A}, \mu)$, $\deg \vartheta_\mu < \frac{|\mu|}{2}$ *and* $\Delta'(\mu) = |\mu| - 2\deg \vartheta_\mu > 0$. *Assume that* $B(\mu, \Delta'(\mu))$ *and* $B(\nu, \Delta'(\nu))$ *are disjoint for* $\mu \neq \nu \in N$, *and that* $\Lambda_\emptyset \setminus \bigcup_{\mu \in N} B(\mu, \Delta'(\mu))$ *has no connected components whose size is larger than* 1. *Then the following are equivalent*:

- $\{\vartheta_\mu, \vartheta_\nu\}$ *are* S*-linearly independent if* $\Delta'(\mu) + \Delta'(\nu) = d(\mu, \nu)$.
- $N = P(\Lambda')$ *and* $\vartheta_\mu = \theta_\mu$ *for each* $\mu \in P(\Lambda')$.

Corollary 3.7 *Let $N = \{\mu \in \Lambda_\emptyset \,\big|\, |\mu| \text{ is odd}\}$ and $\vartheta : N \to D(\mathcal{A}, 0)$ be such that $\vartheta_\mu \in D(\mathcal{A}, \mu)$ and $\deg \theta_\mu < \frac{|\mu|}{2}$. Define the equivalence relation $\sim$ generated by*

$$\mu \sim \nu \qquad \Longleftrightarrow \qquad \{\vartheta_\mu, \vartheta_\nu\} \text{ is } S\text{-linearly dependent and } d(\mu, \nu) = 2.$$

Then the following are equivalent for $\mu, \nu \in N$:

- $\mu \sim \nu$.
- $\mu, \nu \in C \in \mathrm{cc}_0(\Lambda')$.

Remark 3.8 In Corollaries 3.5, 3.6 and 3.7, we do not require the condition $\deg(\vartheta_\mu) = \deg(\theta_\mu)$.

Finally we state the theorems which describe the behavior of the basis near, or between the centers of connected balls.

Theorem 3.9 *Assume that $\mu, \nu \in \Lambda'$ belong to distinct connected components and satisfy $\Delta(\mu) + \Delta(\nu) = d(\mu, \nu)$. Let $\kappa \in \Lambda$ be such that $\mu \wedge \nu \subset \kappa \subset \mu \vee \nu$, and*

$$\alpha_{\mu,\kappa} = \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \mu_H, 0\}},$$

$$\alpha_{\nu,\kappa} = \prod_{H \in \mathcal{A}} \alpha_H^{\max\{\kappa_H - \nu_H, 0\}}.$$

Then $\{\alpha_{\mu,\kappa}\theta_\mu, \alpha_{\nu,\kappa}\theta_\nu\}$ is a homogeneous basis for $D(\mathcal{A}, \kappa)$.

Corollary 3.10 *For each $\mu \in \Lambda_\emptyset$, we can construct a homogeneous basis for $D(\mathcal{A}, \mu)$ from the restricted map $\theta|_{P(\Lambda')}$.*

4 Proofs of main results

In this section, we prove the main results. To prove them, first we recall a result about hyperplane arrangements and derivation modules. The following is the two-dimensional version of the famous Saito's criterion, which is very useful to find the basis for $D(\mathcal{A}, m)$. See Theorem 8 in [16] and Theorem 4.19 in [7] for the proof.

Theorem 4.1 (Saito's criterion) *Let $(\mathcal{A}, \mu)$ be a 2-multiarrangement and $\theta_1, \theta_2 \in D(\mathcal{A}, \mu)$. Then $\{\theta_1, \theta_2\}$ forms a basis for $D(\mathcal{A}, \mu)$ if and only if $\{\theta_1, \theta_2\}$ is independent and* $\deg(\theta_1) + \deg(\theta_2) = |\mu|$.

4.1 Proofs of Theorems 3.1 and 3.2

Lemma 4.2 *If $\mu, \nu \in \Lambda$ and $\mu \mathrel{\dot\subset} \nu$, then $|\Delta(\mu) - \Delta(\nu)| = 1$.*

Proof It follows from the fact that $D(\mathcal{A}, \mu) \supset D(\mathcal{A}, \nu)$ and Saito's criterion. □

Lemma 4.3 *Assume that $\mu, \nu \in \Lambda'$ and $\mu \dot{\subset} \nu$ with $\nu_H = \mu_H + 1$ for some $H \in \mathcal{A}$. Then*

$$\theta_\nu = \begin{cases} \alpha_H \theta_\mu & \text{if } \Delta(\mu) > \Delta(\nu), \\ \theta_\mu & \text{if } \Delta(\mu) < \Delta(\nu). \end{cases}$$

Proof Fix a homogeneous basis $\{\theta_\mu, \theta'\}$ for $D(\mathcal{A}, \mu)$, where $\deg(\theta_\mu) < \deg(\theta')$. If $\Delta(\mu) > \Delta(\nu)$, then Saito's criterion implies $\theta_\mu \notin D(\mathcal{A}, \nu)$. Since $\alpha_H \theta_\mu \in D(\mathcal{A}, \nu)$, Lemma 4.2 implies $\alpha_H \theta_\mu$ is a part of a homogeneous basis for $D(\mathcal{A}, \nu)$. Hence we may assume that $\{\alpha_H \theta_\mu, \theta''\}$ is a basis for $D(\mathcal{A}, \nu)$. If $\Delta(\mu) < \Delta(\nu)$, then $\theta_\mu \in D(\mathcal{A}, \nu)$, which completes the proof. □

Corollary 4.4 *Let $\mu, \nu \in C \in \mathrm{cc}(\Lambda')$ with $\mu \subset \nu$, and ρ be a saturated chain from μ to ν. Then $\theta_\nu = \alpha^{(\rho)} \theta_\mu$.*

Proof Apply Lemma 4.3 repeatedly. □

Lemma 4.5 *Let $C \in \mathrm{cc}(\Lambda')$, $\mu \in C$, and $H \in \mathcal{A}$. If $|C_{\mu,H}| < \infty$, then $\Delta|_{C_{\mu,H}}$ is unimodal, or equivalently, there exists a unique element $\kappa \in C_{\mu,H}$ such that*

$$\Delta(\nu') \leq \Delta(\nu) \quad \textit{for } \nu' \subset \nu \subset \kappa \textit{ or } \kappa \subset \nu \subset \nu'.$$

If $|C_{\mu,H}| = \infty$, then $\Delta|_{C_{\mu,H}}$ is monotonic, or equivalently,

$$\Delta(\nu') \leq \Delta(\nu) \quad \textit{for } \nu' \subset \nu.$$

Proof Let $\nu, \nu', \nu'' \in C_{\mu,H}$ satisfy $\nu \dot{\subset} \nu' \dot{\subset} \nu''$. Assume that $\Delta(\nu) > \Delta(\nu') < \Delta(\nu'')$. By Lemma 4.3, we may choose a basis $\{\alpha_H \theta_\nu, \theta'\}$ for $D(\mathcal{A}, \nu')$ such that $\{\alpha_H \theta_\nu, \alpha_H \theta'\}$ is a basis for $D(\mathcal{A}, \nu'')$. Hence $\alpha_H \theta_\nu(\alpha_H) \in S \cdot \alpha_H^{\nu''_H} = S \cdot \alpha_H^{\nu_H + 2}$ and $\theta_\nu(\alpha_H) \in S \cdot \alpha_H^{\nu_H + 1}$. Then $\theta_\nu \in D(\mathcal{A}, \nu')$, which is a contradiction. Since it follows from Lemma 4.2 that $\min\{\Delta(\mu') | \mu' \in C_{\mu,H}\} = 1$, we have the lemma. □

Definition 4.6 For $H \in \mathcal{A}$, $C \in \mathrm{cc}_0(\Lambda')$ and $\mu \in C$, we may choose, by Lemma 4.5, the unique element $\kappa \in C_{\mu,H}$ such that $\Delta(\kappa) \geq \Delta(\mu')$ for any $\mu' \in C_{\mu,H}$. We call this κ the *peak element with respect to $C_{\mu,H}$*.

Corollary 4.7 *Let $C \in \mathrm{cc}_0(\Lambda')$, $\mu \in C$ and $H \in \mathcal{A}$. Let $\kappa \in C$ be the peak element with respect to $C_{\mu,H}$. Then, for $\mu' \in C_{\mu,H}$,*

$$\theta_{\mu'} = \begin{cases} \theta_\kappa & (\mu' \subset \kappa), \\ \alpha_H^{|\mu'| - |\kappa|} \theta_\kappa & (\kappa \subset \mu'). \end{cases}$$

Proof Apply Lemmas 4.3 and 4.5. □

Lemma 4.8 *Let $C \in \mathrm{cc}(\Lambda')$, and $\kappa, \mu, \mu', \nu \in \Lambda$. Assume that $\kappa \dot{\subset} \mu \dot{\subset} \nu$, $\kappa \dot{\subset} \mu' \dot{\subset} \nu$, and $\mu \neq \mu'$.*

(1) *Assume that* $\kappa, \mu, \mu' \in C$. *Then*

$$\begin{aligned}
\Delta(\kappa) > \Delta(\mu) \text{ and } \Delta(\kappa) > \Delta(\mu') &\implies \Delta(\mu) > \Delta(\nu) \text{ and } \Delta(\mu') > \Delta(\nu);\\
\Delta(\kappa) < \Delta(\mu) \text{ and } \Delta(\kappa) < \Delta(\mu') &\implies \Delta(\mu) < \Delta(\nu) \text{ and } \Delta(\mu') < \Delta(\nu);\\
\Delta(\kappa) < \Delta(\mu) \text{ and } \Delta(\kappa) > \Delta(\mu') &\implies \Delta(\mu) > \Delta(\nu) \text{ and } \Delta(\mu') < \Delta(\nu).
\end{aligned}$$

(2) *Assume that* $\mu, \mu', \nu \in C$. *Then*

$$\begin{aligned}
\Delta(\mu) > \Delta(\nu) \text{ and } \Delta(\mu') > \Delta(\nu) &\implies \Delta(\kappa) > \Delta(\mu) \text{ and } \Delta(\kappa) > \Delta(\mu');\\
\Delta(\mu) < \Delta(\nu) \text{ and } \Delta(\mu') < \Delta(\nu) &\implies \Delta(\kappa) < \Delta(\mu) \text{ and } \Delta(\kappa) < \Delta(\mu');\\
\Delta(\mu) < \Delta(\nu) \text{ and } \Delta(\mu') > \Delta(\nu) &\implies \Delta(\kappa) > \Delta(\mu) \text{ and } \Delta(\kappa) < \Delta(\mu').
\end{aligned}$$

(3) *Assume that* $\kappa, \mu, \nu \in C$. *Then*

$$\begin{aligned}
\Delta(\kappa) > \Delta(\mu) > \Delta(\nu) &\implies \Delta(\kappa) > \Delta(\mu') > \Delta(\nu);\\
\Delta(\kappa) < \Delta(\mu) < \Delta(\nu) &\implies \Delta(\kappa) < \Delta(\mu') < \Delta(\nu);\\
\Delta(\kappa) < \Delta(\mu) > \Delta(\nu) &\implies \Delta(\kappa) > \Delta(\mu') < \Delta(\nu);\\
\Delta(\kappa) > \Delta(\mu) < \Delta(\nu) &\implies \Delta(\kappa) < \Delta(\mu') > \Delta(\nu).
\end{aligned}$$

Proof (1) Assume that $\kappa_H + 1 = \mu_H$ and $\kappa_{H'} + 1 = \mu'_{H'}$ for some $H \neq H' \in \mathcal{A}$. Since $\nu = \mu \vee \mu'$, $\mu_{H'} + 1 = \nu_{H'}$ and $\mu'_H + 1 = \nu_H$. First we consider the case when $\Delta(\kappa) > \Delta(\mu)$ and $\Delta(\kappa) > \Delta(\mu')$. Then $\Delta(\mu) = \Delta(\mu')$. It follows from Lemma 4.3 that $\theta_\mu = \alpha_H \theta_\kappa$, $\theta_{\mu'} = \alpha_{H'}\theta_\kappa$. If $\Delta(\mu) = \Delta(\mu') < \Delta(\nu)$, then $\Delta(\nu) > 0$, i.e., $\nu \in \Lambda'$. Then Lemma 4.3 implies that

$$\alpha_{H'}\theta_\kappa = \theta_{\mu'} = \theta_\nu = \theta_\mu = \alpha_H \theta_\kappa,$$

which is a contradiction.

Next we consider the case when $\Delta(\kappa) < \Delta(\mu)$ and $\Delta(\kappa) < \Delta(\mu')$. Then $\Delta(\mu) = \Delta(\mu')$ and $\Delta(\kappa) \leq \Delta(\nu)$. Hence $\nu \in C$. It follows from Lemma 4.3 that $\theta_\mu = \theta_\kappa$, $\theta_{\mu'} = \theta_\kappa$. If $\Delta(\mu) = \Delta(\mu') > \Delta(\mu)$, then Lemma 4.3 implies that

$$\alpha_H \theta_\kappa = \alpha_H \theta_{\mu'} = \theta_\nu = \alpha_{H'}\theta_\mu = \alpha_{H'}\theta_\kappa,$$

which is a contradiction.

Finally we consider the case when $\Delta(\kappa) < \Delta(\mu)$ and $\Delta(\kappa) > \Delta(\mu')$. Then $\Delta(\mu) - 1 = \Delta(\mu') + 1 = \Delta(\kappa)$. Hence $\Delta(\nu) = \Delta(\mu) - 1 = \Delta(\mu') + 1 = \Delta(\kappa)$.

The same argument is valid for (2) and (3), which completes the proof. ☐

Remark 4.9 In cases (1), (2) and (3) in Lemma 4.8, $\Delta(\nu) = 0$, $\Delta(\kappa) = 0$ and $\Delta(\mu') = 0$ may happen, respectively.

Lemma 4.10 *Let* $\mu, \mu' \in \Lambda$ *be such that* $|\mu| = |\mu'|$, $\mu \neq \mu'$ *and* $d(\mu, \mu') = 2$. *Then the following are equivalent*:

(1) *At least three of* $\{\mu \wedge \mu', \mu, \mu', \mu \vee \mu'\}$ *are in the same connected component* $C \in \mathrm{cc}(\Lambda')$.
(2) *At least three of* $\Delta(\mu \wedge \mu')$, $\Delta(\mu)$, $\Delta(\mu')$ *and* $\Delta(\mu \vee \mu')$ *are positive.*
(3) $\Delta(\mu \vee \mu') - \Delta(\mu') = \Delta(\mu) - \Delta(\mu \wedge \mu')$.
(4) $\Delta(\mu \vee \mu') - \Delta(\mu) = \Delta(\mu') - \Delta(\mu \wedge \mu')$.

Proof By the assumption, $|\mu \vee \mu'| - 1 = |\mu| = |\mu'| = |\mu \wedge \mu'| + 1$. It follows from Lemma 4.2 that $|\Delta(\mu \vee \mu') - \Delta(\mu')| = |\Delta(\mu) - \Delta(\mu \wedge \mu')| = |\Delta(\mu \vee \mu') - \Delta(\mu)| = |\Delta(\mu') - \Delta(\mu \wedge \mu')| = 1$. It is clear that Conditions (3) and (4) are equivalent. It is also clear that Conditions (1) and (2) are equivalent. It follows from Lemma 4.8 that Condition (1) implies Condition (3). Now we show that Condition (3) implies Condition (2). If two of $\Delta(\mu \vee \mu')$, $\Delta(\mu')$, $\Delta(\mu)$ and $\Delta(\mu \wedge \mu')$ are zero, then Lemma 4.2 shows that we have one of the following two:

- $\Delta(\mu \vee \mu') = \Delta(\mu \wedge \mu') = 1$ and $\Delta(\mu') = \Delta(\mu) = 0$; or
- $\Delta(\mu \vee \mu') = \Delta(\mu \wedge \mu') = 0$ and $\Delta(\mu') = \Delta(\mu) = 1$.

Both contradict Condition (3). □

Lemma 4.11 *For* $\mu \in C \in \mathrm{cc}(\Lambda')$, *define* X_μ *by* $X_\mu := \bigcup_{H \in \mathcal{A}} C_{\mu,H}$. *If* μ *satisfies* $\Delta(\mu) = \max\{\Delta(\nu) | \nu \in X_\mu\}$, *then*

$$\Delta(\kappa) = \Delta(\mu) - d(\kappa, \mu)$$

for $\kappa \in \Lambda$ *with* $d(\kappa, \mu) \leq \Delta(\mu)$. *In particular,* C *is the ball* $B(\mu, \Delta(\mu))$.

Proof If $\Delta(\mu) = 1$ then there is nothing to prove. Assume that $\Delta(\mu) > 1$. Since μ satisfies $\Delta(\mu) = \max\{\Delta(\nu) | \nu \in X_\mu\}$, it follows from Lemma 4.5 that $\Delta(\kappa) = \Delta(\mu) - d(\kappa, \mu)$ for $\kappa \in X_\mu$. In particular, we have the lemma for $d(\kappa, \mu) = 1$.

Now we prove the lemma by the induction on $d(\kappa, \mu)$. Let $d(\kappa, \mu) > 1$. By the previous paragraph it suffices treat the case of $\kappa \notin X_\mu$. In this case, there exists $H' \neq H''$ such that $\mu_{H'} \neq \kappa_{H'}$ and $\mu_{H''} \neq \kappa_{H''}$. Let us define $\kappa', \kappa'', \kappa'''$ by

$$\kappa'_H = \begin{cases} \kappa_H & \text{if } H \neq H', \\ \kappa_{H'} - 1 & \text{if } H = H' \text{ and } \kappa_{H'} > \mu_{H'}, \\ \kappa_{H'} + 1 & \text{if } H = H' \text{ and } \kappa_{H'} < \mu_{H'}, \end{cases}$$

$$\kappa''_H = \begin{cases} \kappa_H & \text{if } H \neq H'', \\ \kappa_{H''} - 1 & \text{if } H = H'' \text{ and } \kappa_{H''} > \mu_{H''}, \\ \kappa_{H''} + 1 & \text{if } H = H'' \text{ and } \kappa_{H''} < \mu_{H''}, \end{cases}$$

$$\kappa'''_H = \begin{cases} \kappa_H & \text{if } H' \neq H \neq H'', \\ \kappa'_{H'} & \text{if } H = H', \\ \kappa''_{H''} & \text{if } H = H''. \end{cases}$$

Then $d(\kappa, \mu) - 1 = d(\kappa', \mu) = d(\kappa'', \mu) = d(\kappa''', \mu) + 1$. By the induction hypothesis, $\Delta(\kappa') = \Delta(\kappa'') = \Delta(\kappa''') - 1 = \Delta(\mu) - d(\kappa, \mu) + 1 > 0$. It follows from Lemma 4.10 that $\Delta(\kappa) - \Delta(\kappa') = \Delta(\kappa'') - \Delta(\kappa''') = -1$. Hence $\Delta(\kappa) = \Delta(\mu) - d(\kappa, \mu)$. □

Proof of Theorem 3.2 Let $C \in \mathrm{cc}_0(\Lambda')$ and $\mu \in P(C)$. Then it follows from Lemma 4.5 that $\Delta|_{C_{\mu,H}}$ is unimodal for all $H \in \mathcal{A}$. Hence Lemma 4.11 completes the proof. □

Lemma 4.12 *Let $H \in \mathcal{A}$ and $\mu, \mu' \in C \in \mathrm{cc}_\infty(\Lambda')$ satisfy $\mu \dot{\subset} \mu'$ with $\mu_H + 1 = \mu'_H$. If $|C_{\mu,H'}| < \infty$ for some $H' \in \mathcal{A} \setminus \{H\}$, then $|C_{\mu',H'}| < \infty$. Moreover, for $H' \in \mathcal{A} \setminus \{H\}$, μ is the peak element with respect to $C_{\mu,H'}$ if and only if μ' is the peak element with respect to $C_{\mu',H'}$.*

Proof First consider the case when $|C_{\mu,H'}| = 1$. In this case, $\Delta(\mu) = 1$ and $\Delta(\mu') = 2$. Define $\mu^{(1)}$ by $\mu \dot{\subset} \mu^{(1)}$ with $\mu_{H'} + 1 = \mu^{(1)}_{H'}$. Then $\mu^{(1)} \vee \mu' \in C_{\mu',H'}$. By the assumption $\Delta(\mu^{(1)}) = 0$. So Lemma 4.10 implies $\Delta(\mu^{(1)} \vee \mu') = 1$. Define $\mu'^{(-1)}$ by $\mu'^{(-1)} \dot{\subset} \mu'$ with $\mu'^{(-1)}_{H'} + 1 = \mu'_{H'}$. Then $\mu'^{(-1)} \in C_{\mu',H'}$. By the assumption $\Delta(\mu \wedge \mu'^{(-1)}) = 0$. So Lemma 4.10 implies $\Delta(\mu'^{(-1)}) = 1$. Since $\mu'^{(-1)} \dot{\subset} \mu' \dot{\subset} \mu^{(1)} \vee \mu'$ and $\Delta(\mu^{(1)} \vee \mu') < \Delta(\mu') > \Delta(\mu'^{(-1)})$, by Lemma 4.5, μ' is the peak element with respect to $C_{\mu',H'}$, and $|C_{\mu',H'}| = 3 < \infty$.

Next consider the case when $|C_{\mu,H'}| > 1$. Let $\mu^{(0)}$ be the peak element with respect to $C_{\mu,H'}$, and

$$C_{\mu,H'} = \left\{ \mu^{(i)} \,\middle|\, \cdots \dot{\subset} \mu^{(-1)} \dot{\subset} \mu^{(0)} \dot{\subset} \mu^{(1)} \dot{\subset} \cdots \right\}.$$

Then $\Delta(\mu^{(-i)}) = \Delta(\mu^{(i)})$. Let us define $\mu'^{(i)}$ by $\mu^{(i)} \dot{\subset} \mu'^{(i)}$ and $\mu^{(i)}_H + 1 = \mu'^{(i)}_H$. If $\mu = \mu^{(j)}$, then $\mu' = \mu'^{(j)}$. By direct calculation, we have $\mu'^{(i)} \vee \mu^{(i+1)} = \mu'^{(i+1)}$ and $\mu'^{(i)} \wedge \mu^{(i+1)} = \mu^{(i)}$. For $i < 0$, $\Delta(\mu^{(i+1)}) > \Delta(\mu^{(i)}) > 0$. Hence $\Delta(\mu'^{(i+1)}) > 0$. It follows from Lemma 4.10 that

$$\Delta\bigl(\mu'^{(i+1)}\bigr) - \Delta\bigl(\mu'^{(i)}\bigr) = \Delta\bigl(\mu^{(i+1)}\bigr) - \Delta\bigl(\mu^{(i)}\bigr) = 1.$$

On the other hand, for $i > 0$, the same argument implies that

$$\Delta\bigl(\mu'^{(i-1)}\bigr) - \Delta\bigl(\mu'^{(i)}\bigr) = \Delta\bigl(\mu^{(i-1)}\bigr) - \Delta\bigl(\mu^{(i)}\bigr) = 1.$$

Hence, by Lemma 4.5, $\mu'^{(k)}$ is the peak element with respect to $C_{\mu',H'}$, and $|C_{\mu',H'}| < \infty$. The same proof is valid if μ is replaced by μ'. □

Lemma 4.13 *Let $C \in \mathrm{cc}(\Lambda')$. If there exists $\mu \in C$ satisfying $|C_{\mu,H}| < \infty$ for any $H \in \mathcal{A}$, then $C \in \mathrm{cc}_0(\Lambda')$. Hence, for $\mu \in C \in \mathrm{cc}_\infty(\Lambda')$, there exists $H \in \mathcal{A}$ such that $|C_{\mu,H}| = \infty$.*

Proof For $H \in \mathcal{A}$, $C \in \mathrm{cc}(\Lambda')$ and $\mu \in C$, define $m_{\mu,H}$ and $\mathcal{B}_\mu$ by $m_{\mu,H} := \max\{\Delta(\mu') | \mu' \in C_{\mu,H}\}$ and $\mathcal{B}_\mu := \{H \in \mathcal{A} | \Delta(\mu) = m_{\mu,H}\}$. Assume that $|C_{\mu,H}| < \infty$ for all $H \in \mathcal{A}$. Let us construct ν as follows:

(1) Let ν be μ.
(2) Repeat the following until $\mathcal{A} = \mathcal{B}_\nu$:
   (a) Choose $H_0 \in \mathcal{A} \setminus \mathcal{B}_\nu$ and the peak element ν' with respect to C_{ν,H_0}.
   (b) Let ν be ν'.

By the assumption and Lemma 4.12, $|C_{\nu',H}| < \infty$ for all $H \in \mathcal{A}$ and $\Delta(\nu') = m_{\nu',H}$ for all $H \in \mathcal{B}_\nu$. Hence, by Lemma 4.12, $\mathcal{B}_{\nu'} = \mathcal{B}_\nu \cup \{H_0\}$. Since $|\mathcal{A}| < \infty$, we can always find $\nu \in C$ such that $\Delta(\nu) = m_{\nu,H}$ for all $H \in \mathcal{A}$. Hence Lemma 4.11 implies that $C \in \mathrm{cc}_0(\Lambda')$. □

Lemma 4.14 $\mathrm{cc}_\infty(\Lambda') = \{\Lambda_H | H \in \mathcal{A}\}$.

Proof Lemma 4.13 implies that, for $\mu \in C \in \mathrm{cc}_\infty(\Lambda')$, there exists H such that $|C_{\mu,H}| = \infty$. Hence if $\nu \in \Lambda$ satisfies

$$\nu_{H'} = \begin{cases} \mu_H + |\mu| & (H = H'), \\ \mu_{H'} & (H \neq H'), \end{cases}$$

then $\nu \in C_{\mu,H}$. By definition, $\nu \in \Lambda_H$. Since μ and ν belong to the same component C, μ is also in Λ_H. On the other hand, $\Lambda_H \in \mathrm{cc}_\infty(\Lambda')$. Since Λ_H is connected, $C = \Lambda_H$. □

Proof of Theorem 3.1 Apply Lemmas 4.2 and 4.14. □

4.2 Proof of Theorem 3.4

In this subsection we prove Theorem 3.4. Roughly speaking, the proof is based on the observation of θ_μ for μ in some finite balls in Theorem 3.2.

Lemma 4.15 *Let $C \in \mathrm{cc}_0(\Lambda')$, $\kappa \in C$ and $\mu \in P(C)$. Then we can construct θ_μ from θ_κ, and vice versa.*

Proof By Theorem 3.2, $\mu \wedge \kappa \in C$. It follows from Lemma 4.4 that

$$\theta_\mu = \alpha^{(\rho)} \theta_{\mu\wedge\kappa},$$
$$\theta_\kappa = \alpha^{(\rho')} \theta_{\mu\wedge\kappa}$$

for some saturated chains ρ and ρ'. Hence we have

$$\theta_\mu = \frac{\alpha^{(\rho)}}{\alpha^{(\rho')}} \theta_\kappa,$$
$$\theta_\kappa = \frac{\alpha^{(\rho')}}{\alpha^{(\rho)}} \theta_\mu.$$ □

Lemma 4.16 *Let $C \in \mathrm{cc}_0(\Lambda')$ and $\mu, \nu \in C$. Then $\{\theta_\mu, \theta_\nu\}$ is S-linearly dependent.*

Proof The lemma follows from Lemma 4.15. □

Lemma 4.17 *Let $\mu, \nu \in \Lambda'$ satisfy $d(\mu, \nu) = 2$. If $\Delta(\kappa) = 0$ for all $\kappa \in \Lambda$ such that $d(\mu, \kappa) = d(\nu, \kappa) = 1$, then $\{\theta_\mu, \theta_\nu\}$ is S-linearly independent.*

Proof First assume that $\mu_H + 2 = \nu_H$ for some $H \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H\}$. Let $\kappa \in \Lambda$ be the element such that $\mu \dot{\subset} \kappa \dot{\subset} \nu$. Since $\Delta(\kappa) = 0$, $\theta_\mu \notin D(\mathcal{A}, \kappa)$. Hence $\alpha_H \theta_\mu \in D(\mathcal{A}, \kappa)$ and is a part of basis. Let $\{\alpha_H \theta_\mu, \theta'\}$ be a basis for the S-module $D(\mathcal{A}, \kappa)$. Since $D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\alpha_H \theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S-linearly dependent, then $b = 0$, i.e., $\theta_\nu = a\alpha_H \theta_\mu$. Since $\alpha_H \theta_\mu \in D(\mathcal{A}, \nu)$, $\alpha_H \theta_\mu(\alpha_H) \in S \cdot \alpha_H^{\nu_H} = S \cdot \alpha_H^{\kappa_H + 1}$. Hence $\theta_\mu(\alpha_H) \in S \cdot \alpha_H^{\kappa_H}$ and $\theta_\mu \in D(\mathcal{A}, \kappa)$, which is a contradiction.

Next assume that $\mu_H + 1 = \nu_H$ and $\mu_{H'} + 1 = \nu_{H'}$ for some $H, H' \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H, H'\}$. Let $\kappa \in \Lambda$ be the element such that $\kappa_H = \mu_H + 1$ and $\kappa_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H\}$, and $\kappa' \in \Lambda$ such that $\kappa_{H'} = \mu_{H'} + 1$ and $\kappa_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H'\}$. By the assumption, $\Delta(\kappa) = \Delta(\kappa') = 0$. Hence $\theta_\mu \notin D(\mathcal{A}, \kappa)$ and $\theta_\mu \notin D(\mathcal{A}, \kappa')$. Let $\{\alpha_H \theta_\mu, \theta'\}$ be a basis for the S-module $D(\mathcal{A}, \kappa)$. Since $\theta_\nu \in D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\alpha_H \theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S-linearly dependent, then $\theta_\nu = a\alpha_H \theta_\mu$. Since $\theta_\nu(\alpha_{H'}) = a\alpha_H \theta_\mu(\alpha_{H'}) \in S \cdot \alpha_{H'}^{\nu_{H'}} = S \cdot \alpha_{H'}^{\kappa'_{H'}}$, $\theta_\mu(\alpha_{H'}) \in S \cdot \alpha_{H'}^{\kappa'_{H'}}$. Hence $\theta_\mu \in D(\mathcal{A}, \kappa')$, which is contradiction.

Finally assume that $\mu_H + 1 = \nu_H$, $\mu_{H'} = \nu_{H'} + 1$ for some $H, H' \in \mathcal{A}$ and $\mu_{H''} = \nu_{H''}$ for $H'' \in \mathcal{A} \setminus \{H, H'\}$. Let $\kappa = \mu \wedge \nu$ and $\kappa' = \mu \vee \nu$. By the assumption, $\Delta(\kappa') = \Delta(\kappa) = 0$. Hence $\theta_\mu, \theta_\nu \notin D(\mathcal{A}, \kappa')$. We may choose a basis $\{\theta_\mu, \theta'\}$ for $D(\mathcal{A}, \kappa)$ such that $\{\theta_\mu, \alpha_{H'} \theta'\}$ is a basis for $D(\mathcal{A}, \mu)$. Since $D(\mathcal{A}, \nu) \subset D(\mathcal{A}, \kappa)$, $\theta_\nu = a\theta_\mu + b\theta'$ for some $a, b \in \mathbb{K}$. If $\{\theta_\mu, \theta_\nu\}$ is S-linearly dependent, then $\theta_\nu = a\theta_\mu$. Since $\theta_\nu = a\theta_\mu \in D(\mathcal{A}, \mu) \cap D(\mathcal{A}, \nu)$, $\theta_\nu \in D(\mathcal{A}, \kappa')$ which is a contradiction. □

Lemma 4.18 *If $\mu, \nu \in P(\Lambda')$ satisfy $d(\mu, \nu) = \Delta(\mu) + \Delta(\nu)$, then $\{\theta_\mu, \theta_\nu\}$ is S-linearly independent.*

Proof By the assumption, there exist some $\mu', \nu' \in \Lambda'$ such that

- $d(\mu', \nu') = 2$,
- $\Delta(\kappa) = 0$ for all $\kappa \in \Lambda$ such that $d(\mu', \kappa) = d(\nu', \kappa) = 1$,
- $\mu, \mu' \in C \in \mathrm{cc}_0(\Lambda')$, and
- $\nu, \nu' \in C' \in \mathrm{cc}_0(\Lambda')$.

By Lemma 4.17 $\{\theta_{\mu'}, \theta_{\nu'}\}$ is S-linearly independent. Hence Lemma 4.15 completes the proof. □

Proof of Theorem 3.4 Apply Lemmas 4.16 and 4.18. □

4.3 Proof of Theorem 3.9

Lemma 4.19 *Assume that $\mu, \nu \in \Lambda'$ satisfy $\Delta(\mu) + \Delta(\nu) = d(\mu, \nu)$, and that $\{\theta_\mu, \theta_\nu\}$ is S-linearly independent. Then $\{\theta_\mu, \theta_\nu\}$ is a basis for $D(\mathcal{A}, \mu \wedge \nu)$.*

Proof Since $(\mu \wedge \nu)_H = \min\{\mu_H, \nu_H\}$ for $H \in \mathcal{A}$,

$$|\mu \wedge \nu| = \sum_{H \in \mathcal{A}} \min\{\mu_H, \nu_H\}.$$

On the other hand,

$$\begin{aligned}\deg(\theta_\mu)+\deg(\theta_\nu) &= \frac{|\mu|-\Delta(\mu)}{2}+\frac{|\nu|-\Delta(\nu)}{2}\\ &= \frac{|\mu|+|\nu|-\Delta(\mu)-\Delta(\nu)}{2}\\ &= \frac{|\mu|+|\nu|-d(\mu,\nu)}{2}\\ &= \sum_{H\in\mathcal{A}}\frac{\mu_H+\nu_H-|\mu_H-\nu_H|}{2}\\ &= \sum_{H\in\mathcal{A}}\min\{\mu_H,\nu_H\}=|\mu\wedge\nu|.\end{aligned}$$

Since $\mu\wedge\nu\subset\mu,\nu$, it follows from Saito's criterion that $\{\theta_\mu,\theta_\nu\}$ is a basis for $D(\mathcal{A},\mu\wedge\nu)$. □

Lemma 4.20 *Assume that $\mu,\nu\in\Lambda'$ satisfy $\Delta(\mu)+\Delta(\nu)=d(\mu,\nu)$ and that $\{\theta_\mu,\theta_\nu\}$ is S-linearly independent. For $\kappa\in\Lambda$ such that $\mu\wedge\nu\subset\kappa\subset\mu\vee\nu$, let us define*

$$\begin{aligned}\alpha_{\mu,\kappa} &= \prod_{H\in\mathcal{A}}\alpha_H^{\max\{\kappa_H-\mu_H,0\}},\\ \alpha_{\nu,\kappa} &= \prod_{H\in\mathcal{A}}\alpha_H^{\max\{\kappa_H-\nu_H,0\}}.\end{aligned}$$

Then $\{\alpha_{\mu,\kappa}\theta_\mu,\alpha_{\nu,\kappa}\theta_\nu\}$ is a basis for $D(\mathcal{A},\kappa)$.

Proof Note that $\deg(\alpha_{\mu,\kappa})+\deg(\alpha_{\nu,\kappa})=d(\kappa,\mu\wedge\nu)$ and that $\alpha_{\mu,\kappa}\theta_\mu$, $\alpha_{\nu,\kappa}\theta_\nu\in D(\mathcal{A},\kappa)$. Thus Saito's criterion and Lemma 4.19 completes the proof. □

Proof of Theorem 3.9 By Theorem 3.4, $\{\theta_\mu,\theta_\nu\}$ is S-linearly independent. Hence Lemma 4.20 completes the proof. □

5 Application

In this section, we consider the case when a group acts on V. Let W be a group acting on V from the left. Canonically, this action induces actions on S and $\mathrm{Der}_{\mathbb{K}}(S)$, i.e., W acts on S and $\mathrm{Der}_{\mathbb{K}}(S)$ by $(\sigma f)(v)=f(\sigma^{-1}v)$ and $(\sigma\delta)(f)=\sigma(\delta(\sigma^{-1}f))$ for $\sigma\in W$, $f\in S$, $\delta\in\mathrm{Der}_{\mathbb{K}}(S)$ and $v\in V$. For each $\sigma\in W$, we assume $\mathcal{A}=\sigma\mathcal{A}$. In this case, W also acts on $\mathcal{A}$ as a subgroup of the symmetric group of $\mathcal{A}$. Hence W also acts on Λ by $(\sigma\mu)_H=\mu_{\sigma^{-1}H}$.

Lemma 5.1 *For $\mu\in\Lambda$ and $\sigma\in W$, $\Delta(\mu)=\Delta(\sigma\mu)$.*

Proof If $\{\theta, \theta'\}$ is a homogeneous basis for $D(\mathcal{A}, \mu)$, then $\{\sigma\theta, \sigma\theta'\}$ is a homogeneous basis for $D(\mathcal{A}, \sigma\mu)$. □

Next we assume that $\mathcal{A}^W = \emptyset$, i.e., for each $H \in \mathcal{A}$, there exists $\sigma_H \in W$ such that $\sigma_H H \neq H$.

Lemma 5.2 *Let $\mu \in \Lambda'$ satisfy $\sigma\mu = \mu$ for all $\sigma \in W$. If there exist ν and κ satisfying the following, then $\mu \in P(\Lambda')$: $\mu' \subset \nu$ for all $\mu \dot{\subset} \mu'$; $\Delta(\mu) - \Delta(\nu) > d(\mu, \nu) - 4$; $\kappa \subset \mu'$ for all $\mu' \dot{\subset} \mu$; and $\Delta(\mu) - \Delta(\kappa) > d(\kappa, \mu) - 4$.*

Proof It suffices to show that $\Delta(\mu') < \Delta(\mu)$ if $\mu \dot{\subset} \mu'$ or $\mu' \dot{\subset} \mu$. First let us assume $\mu \dot{\subset} \mu'$, $\Delta(\mu') > \Delta(\mu)$ and $\mu'_H \neq \mu_H$. Since $\mathcal{A}^W = \emptyset$, $H \neq \sigma H$ for some $\sigma \in W$. For such σ, it holds that $(\sigma\mu')_H = \mu'_{\sigma^{-1}H} = \mu_{\sigma^{-1}H} = \mu_H \neq \mu'_H = \mu_H + 1$, where the second equality holds because $\sigma^{-1}H \neq H$ and because of the definition of μ', the third because of the W-invariance of μ. Hence $\sigma\mu' \neq \mu'$. By the same computation, we can show that

$$(\sigma\mu')_{\sigma H} = \mu'_{\sigma H} + 1 \quad \text{and}$$
$$(\sigma\mu')_{H'} = \mu'_{H'} \ \big(H' \in \mathcal{A} \setminus \{H, \sigma H\}\big).$$

Hence $d(\sigma\mu', \mu') = 2$ and $\mu = \mu' \wedge \sigma\mu'$. By the assumption $\Delta(\mu') = \Delta(\sigma\mu') > \Delta(\mu) > 0$. Hence, by Lemma 4.10,

$$\Delta(\mu' \vee \sigma\mu') = \Delta(\mu') + 1 = \Delta(\mu) + 2.$$

By Lemma 4.2, $\Delta(\mu' \vee \sigma\mu') = \Delta(\mu) + 2 \leq d(\nu, \mu' \vee \sigma\mu') + \Delta(\nu)$. Since $d(\nu, \mu' \vee \sigma\mu') + \Delta(\nu) = d(\nu, \mu) - 2 + \Delta(\nu)$, we have $\Delta(\mu) - \Delta(\nu) \leq d(\nu, \mu) - 4$, which is a contradiction.

The same argument is valid for the case where $\mu' \dot{\subset} \mu$, $\Delta(\mu') > \Delta(\mu)$ and $\mu'_H \neq \mu_H$. Hence we have the lemma. □

As an application of the results above, we consider the exponents of Coxeter arrangements, which is a set of all reflecting hyperplanes of a finite irreducible Coxeter group. Since A_2-type is investigated in [10], let us consider Coxeter arrangements of type $I_2(n)$ $(n \geq 4)$.

It is shown by Terao in [9] that the constant multiplicity on the Coxeter arrangement is free and the exponents are also determined. We give the meaning of Terao's result from our point of view, i.e., the role of constant multiplicity in the multiplicity lattice.

Proposition 5.3 *Let $\mathcal{A}$ be a Coxeter arrangement of type $I_2(n)$ $(n \geq 4)$. Then $\mu = (2k+1, \dots, 2k+1) \in P(\Lambda')$.*

Proof Let $\mathcal{A}$ be the Coxeter arrangement of type $I_2(n)$. Then we can take the Coxeter group of type $I_2(n)$ as W. Let $\nu = (2k+2, \dots, 2k+2)$ and $\kappa = (2k, \dots, 2k)$. Then $d(\mu, \nu) = d(\mu, \kappa) = n$. Since $\Delta(\mu) = n - 2$ and $\Delta(\nu) = \Delta(\kappa) = 0$ by [9], it follows from Lemma 5.2 that $\mu \in P(\Lambda')$. □

Now we can determine the basis and exponents of multiplicities on Coxeter arrangements when they are near the constant one, which is based on the primitive derivation methods in [9] and [13].

Corollary 5.4 *Let $\mathcal{A}$ be a Coxeter arrangement of type $I_2(n)$ $(n \geq 4)$, $\mu = (2k+1, \dots, 2k+1) \in \Lambda$ and $i \in \mathbb{Z}^{|\mathcal{A}|}$ such that $|I| := \sum_H |i_H| < |\mathcal{A}| = n$. If $\nu \in \Lambda$ is defined by $\nu_H = \mu_H + i_H$ and $I := \sum_H i_H$, then*

$$\exp(\mathcal{A}, \nu) = \left(kn + 1 + \frac{I + |I|}{2}, (k+1)n - 1 + \frac{I - |I|}{2}\right).$$

The proof of above corollary is completed by applying Corollary 3.3 and Proposition 5.3.

Remark 5.5 Recently in [4], by using the results in this article, the first author proved that $\Delta(\mu) \leq |\mathcal{A}| - 2$ for $\mu \in P(\Lambda')$ in the case when a two-dimensional arrangement $\mathcal{A}$ is defined over a field of characteristic zero.

Remark 5.6 In [11] it is proved that for the Coxeter multiarrangement $(\mathcal{A}, \mu)$ of type B_2 defined by

$$x^{2k+1} y^{2k+1} (x-y)^{2j+1} (x+y)^{2j+1} = 0,$$

it holds that $\Delta(\mu) = 2$. However, to determine explicitly which multiplicity makes $\Delta = 2$ is difficult even for B_2-type.

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