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Roots of Ehrhart polynomials arising from graphs

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Abstract Several polytopes arise from finite graphs. For edge and symmetric edge polytopes, in particular, exhaustive computation of the Ehrhart polynomials not merely supports the conjecture of Beck et al. that all roots α of Ehrhart polynomials of polytopes of dimension D satisfy $-D \leq \mathrm{Re}(\alpha) \leq D-1$, but also reveals some interesting phenomena for each type of polytope. Here we present two new conjectures: (1) the roots of the Ehrhart polynomial of an edge polytope for a complete multipartite graph of order d lie in the circle $|z + \frac{d}{4}| \leq \frac{d}{4}$ or are negative integers, and (2) a Gorenstein Fano polytope of dimension D has the roots of its Ehrhart polynomial in the narrower strip $-\frac{D}{2} \leq \mathrm{Re}(\alpha) \leq \frac{D}{2} - 1$. Some rigorous results to sup-

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port them are obtained as well as for the original conjecture. The root distribution of Ehrhart polynomials of each type of polytope is plotted in figures.

Keywords Ehrhart polynomial · Edge polytope · Fano polytope · Smooth polytope

1 Introduction

The root distribution of Ehrhart polynomials is one of the current topics in computational commutative algebra. It is well-known that the coefficients of an Ehrhart polynomial reflect combinatorial and geometric properties such as the volume of the polytope in the leading coefficient, gathered information about its faces in the second coefficient, etc. The roots of an Ehrhart polynomial should also reflect properties of a polytope that are hard to elicit just from the coefficients. Among the many papers on the topic, including [4–6, 13, 24], Beck et al. [3] conjecture that

Conjecture 1.1 *All roots α of Ehrhart polynomials of lattice D-polytopes satisfy* $-D \leq \mathrm{Re}(\alpha) \leq D-1$.

Compared with the norm bound, which is $O(D^2)$ in general [5], the strip in the conjecture puts a tight restriction on the distribution of roots for any Ehrhart polynomial.

This paper investigates the roots of Ehrhart polynomials of polytopes arising from graphs, namely, edge polytopes and symmetric edge polytopes. The results obtained not merely support Conjecture 1.1, but also reveal some interesting phenomena. Regarding the scope of the paper, note that both kinds of polytopes are "small" in a sense: That is, each edge polytope from a graph without loops is contained in a unit hypercube, and one from a graph with loops, in twice a unit hypercube; whereas each symmetric edge polytope is contained in twice a unit hypercube.

In Sect. 2, the distribution of roots of Ehrhart polynomials of edge polytopes is computed, and as a special case, that of complete multipartite graphs is studied. We observed from exhaustive computation that all roots have a negative real part and they are in the range of Conjecture 1.1. Moreover, for complete multipartite graphs of order d, the roots lie in the circle $|z+\frac{d}{4}| \leq \frac{d}{4}$ or are negative integers greater than $-(d-1)$. And we conjecture its validity beyond the computed range of d (Conjecture 2.4).

Simple edge polytopes constructed from graphs with possible loops are studied in Sect. 3. Roots of the Ehrhart polynomials are determined in some cases. Let G be a graph of order d with loops and G' its subgraph of order p induced by vertices without a loop attached. Then, Theorem 3.5 proves that the real roots are in the interval $[-(d-2), 0)$, especially all integers in $\{-(d-p), \dots, -1\}$ are roots of the polynomial; Theorem 3.6 determines that if $d-2p+2 \geq 0$, there are $p-1$ real non-integer roots each of which is unique in one of ranges $(-k, -k+1)$ for $k = 1, \dots, p-1$; and Theorem 3.7 proves that if $d > p \geq 2$, all the integers $-\lfloor \frac{d-1}{2} \rfloor, \dots, -1$ are roots of the polynomial. We observed that all roots have a negative real part and are in the range of Conjecture 1.1.

The symmetric edge polytopes in Sect. 4 are Gorenstein Fano polytopes. A unimodular equivalence condition for two symmetric edge polytopes is also described in the language of graphs (Theorem 4.5). The polytopes have Ehrhart polynomials with an interesting root distribution: the roots are distributed symmetrically with respect to the vertical line $\mathrm{Re}(z) = -\frac{1}{2}$. We not only observe that all roots are in the range of Conjecture 1.1, but also conjecture that all roots are in $-\frac{D}{2} \leq \mathrm{Re}(\alpha) \leq \frac{D}{2} - 1$ for Gorenstein Fano polytopes of dimension D (Conjecture 4.10).

Before starting the discussion, let us summarize the definitions of edge polytopes, symmetric edge polytopes, etc.

Throughout this paper, graphs are always finite, and so we usually omit the adjective "finite." Let G be a graph having no multiple edges on the vertex set $V(G) = \{1, \dots, d\}$ and the edge set $E(G) = \{e_1, \dots, e_n\} \subset V(G)^2$. Graphs may have loops in their edge sets unless explicitly excluded; in which case the graphs are called *simple* graphs. A *walk* of G of length q is a sequence $(e_{i_1}, e_{i_2}, \dots, e_{i_q})$ of the edges of G, where $e_{i_k} = \{u_k, u_{k+1}\}$ for $k = 1, \dots, q$. If, moreover, $u_{q+1} = u_1$ holds, then the walk is a *closed* walk. Such a closed walk is called a *cycle* of length q if $u_k \neq u_{k'}$ for all $1 \leq k < k' \leq q$. In particular, a loop is a cycle of length 1. Another notation, $(u_1, u_2, \dots, u_q)$, will be also used for the same cycle with $(\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_q, u_1\})$. Two vertices u and v of G are *connected* if $u = v$ or there exists a walk $(e_{i_1}, e_{i_2}, \dots, e_{i_q})$ of G such that $e_{i_1} = \{u, v_1\}$ and $e_{i_q} = \{u_q, v\}$. The connectedness is an equivalence relation and the equivalence classes are called the *components* of G. If G itself is the only component, then G is a *connected graph*. For further information on graph theory, we refer the reader to e.g. [11, 33].

If $e = \{i, j\}$ is an edge of G between $i \in V(G)$ and $j \in V(G)$, then we define $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here, $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^d$. In particular, for a loop $e = \{i, i\}$ at $i \in V(G)$, one has $\rho(e) = 2\mathbf{e}_i$. The *edge polytope* of G is the convex polytope $\mathcal{P}_G$ ($\subset \mathbb{R}^d$), which is the convex hull of the finite set $\{\rho(e_1), \dots, \rho(e_n)\}$. The dimension of $\mathcal{P}_G$ equals $d - 2$ if the graph G is a connected bipartite graph, or $d - 1$, for other connected graphs [21]. The edge polytopes of complete multipartite graphs are studied in [22]. Note that if the graph G is a complete graph, the edge polytope $\mathcal{P}_G$ is also called the second hypersimplex in [31, Sect. 9].

Similarly, we define $\sigma(e) = \mathbf{e}_i - \mathbf{e}_j$ for an edge $e = \{i, j\}$ of a simple graph G. Then, the *symmetric edge polytope* of G is the convex polytope $\mathcal{P}_G^{\pm}$ ($\subset \mathbb{R}^d$), which is the convex hull of the finite set $\{\pm\sigma(e_1), \dots, \pm\sigma(e_n)\}$. Note that if G is the complete graph K_d, the symmetric edge polytope $\mathcal{P}_{K_d}^{\pm}$ coincides with the root polytope of the lattice A_d defined in [1].

If $\mathcal{P} \subset \mathbb{R}^N$ is an integral convex polytope, then we define $i(\mathcal{P}, m)$ by

$$i(\mathcal{P}, m) = \left| m\mathcal{P} \cap \mathbb{Z}^N \right|.$$

We call $i(\mathcal{P}, m)$ the *Ehrhart polynomial* of $\mathcal{P}$ after Ehrhart, who succeeded in proving that $i(\mathcal{P}, m)$ is a polynomial in m of degree $\dim \mathcal{P}$ with $i(\mathcal{P}, 0) = 1$. If $\mathrm{vol}(\mathcal{P})$ is the normalized volume of $\mathcal{P}$, then the leading coefficient of $i(\mathcal{P}, m)$ is $\frac{\mathrm{vol}(\mathcal{P})}{(\dim \mathcal{P})!}$.

An Ehrhart polynomial $i(\mathcal{P}, m)$ of $\mathcal{P}$ is related to a sequence of integers called the *δ-vector*, $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_D)$, of $\mathcal{P}$ by

$$\sum_{m=0}^{\infty} i(\mathcal{P}, m)t^m = \frac{\sum_{j=0}^{D} \delta_j t^j}{(1-t)^{D+1}},$$

where D is the degree of $i(\mathcal{P}, m)$. We call the polynomial in the numerator on the right-hand side of the equation above $\delta_{\mathcal{P}}(t)$, the *δ-polynomial* of $\mathcal{P}$. Note that the δ-vectors and δ-polynomials are referred to by other names in the literature: e.g., in [29, 30], h^*-vector or i-Eulerian numbers are synonyms of δ-vector, and h^*-polynomial or i-Eulerian polynomial, of δ-polynomial. It follows from the definition that $\delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (D+1)$, etc. It is known that each δ_i is nonnegative [28]. If $\delta_D \neq 0$, then $\delta_1 \leq \delta_i$ for every $1 \leq i < D$ [16]. Though the roots of the polynomial are the focus of this paper, the δ-vector is also a very important research subject. For the detailed discussion on Ehrhart polynomials of convex polytopes, we refer the reader to [14].

2 Edge polytopes of simple graphs

The aim in this section is to provide evidence for Conjecture 1.1 for the Ehrhart polynomials of edge polytopes constructed from connected simple graphs, mainly by computational means.

2.1 Exhaustive computation for small graphs

Let $\mathbb{C}[X]$ denote the polynomial ring in one variable over the field of complex numbers. Given a polynomial $f = f(X) \in \mathbb{C}[X]$, we write $\mathbf{V}(f)$ for the set of roots of f, i.e.,

$$\mathbf{V}(f) = \{a \in \mathbb{C} \mid f(a) = 0\}.$$

We computed the Ehrhart polynomial $i(\mathcal{P}_G, m)$ of each edge polytope $\mathcal{P}_G$ for connected simple graphs G of orders up to nine; there are $1, 2, \ldots, 261080$ connected simple graphs of orders $2, 3, \ldots, 9$.[1] Then, we solved each equation $i(\mathcal{P}_G, X) = 0$ in the field of complex numbers. For the readers interested in our method of computation, see the small note in Appendix.

Let $\mathbf{V}_d^{\mathrm{cs}}$ denote $\bigcup \mathbf{V}(i(\mathcal{P}_G, m))$, where the union runs over all connected simple graphs G of order d. Figure 1 plots points of $\mathbf{V}_9^{\mathrm{cs}}$, as a representative of all results. For all connected simple graphs of order 2–9, Conjecture 1.1 holds.

Since an edge polytope is a kind of 0/1-polytope, the points in Fig. 1 for $\mathbf{V}_9^{\mathrm{cs}}$ are similar to those in Fig. 6 of [3]. However, the former has many more points, which form three clusters: one on the real axis, and other two being complex conjugates of each other and located nearer to the imaginary axis than the first cluster. The

[1] These numbers of such graphs are known; see, e.g., [12, Chap. 4] or `A001349` of the On-Line Encyclopedia of Integer Sequences.

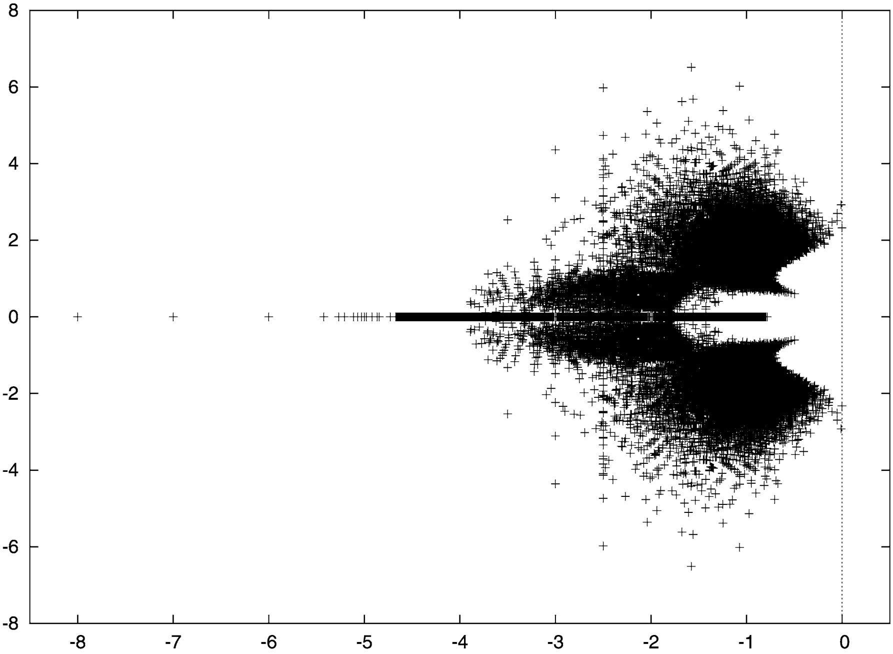


Fig. 1 $\mathbf{V}_9^{\mathrm{cs}}$

interesting thing is that no roots appear in the right half plane of the figure. The closest points to the imaginary axis are approximately $-0.583002 \pm 0.645775i \in \mathbf{V}_7^{\mathrm{cs}}$, $-0.213574 \pm 2.469065i \in \mathbf{V}_8^{\mathrm{cs}}$, and $-0.001610 \pm 2.324505i \in \mathbf{V}_9^{\mathrm{cs}}$. A polynomial with roots only in the left half plane is called a *stable* polynomial. This observation raises an open question:

Question 2.1 For any d and any connected simple graph G of order d, is $i(\mathcal{P}_G, m)$ always a stable polynomial?

For a few infinite families of graphs, rigorous proofs are known: see Proposition 2.2 just below and Examples in the next subsection.

Proposition 2.2 *A root α of the Ehrhart polynomial $i(\mathcal{P}_{K_d}, m)$ of the complete graph K_d satisfies*

(1) $\alpha \in \{-1, -2\}$ *if* $d = 3$ *or*
(2) $-\frac{d}{2} < \mathrm{Re}(\alpha) < 0$ *if* $d \geq 4$.

Proof The Ehrhart polynomial $i(\mathcal{P}_{K_d}, m)$ of the complete graph K_d is given in [31, Corollary 9.6]:

$$i(\mathcal{P}_{K_d}, m) = \binom{d + 2m - 1}{d - 1} - d\binom{m + d - 2}{d - 1}.$$

In cases where $d = 2$ or 3, the Ehrhart polynomials are binomial coefficients, since the edge polytopes are simplices. Actually, they are

$$i(\mathcal{P}_{K_2}, m) = 1 \quad \text{and} \quad i(\mathcal{P}_{K_3}, m) = \binom{m+2}{2}.$$

Thus, there are no roots for $d = 2$, whereas $\{-1, -2\}$ are the roots for $d = 3$.

Hereafter, we assume $d \geq 4$. It is easy to see that $\{-1, -2, \dots, -\lfloor \frac{d-1}{2} \rfloor\}$ are included in $\mathbf{V}(i(\mathcal{P}_{K_d}, m))$.

We shall first prove that $\mathrm{Re}(\alpha) < 0$. Let $q_d^{(1)}(m) = (2m + d - 1) \cdots (2m + 1)$ and $q_d^{(2)}(m) = d(m + d - 2) \cdots m$. Then for a complex number z, $i(\mathcal{P}_{K_d}, z) = 0$ if and only if $q_d^{(1)}(z) = q_d^{(2)}(z)$, since $q_d^{(1)}(z) - q_d^{(2)}(z)$ is $(d - 1)!\, i(\mathcal{P}_{K_d}, z)$. Let us prove $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ for any complex number z with a nonnegative real part by mathematical induction on $d \geq 4$.

If $d = 4$,

$$\begin{aligned} \left|q_4^{(1)}(z)\right| = \left|(2z+3)(2z+2)(2z+1)\right| &= |2z+3||z+1||4z+2| \\ &> |z+2||z+1||4z| = \left|q_4^{(2)}(z)\right| \end{aligned}$$

holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Assume for d that $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ is true for any complex number z with $\mathrm{Re}(z) \geq 0$.

Then, by

$$\begin{aligned} \left|q_{d+1}^{(1)}(z)\right| &= |2z+d|\left|q_d^{(1)}(z)\right|, \\ \left|q_{d+1}^{(2)}(z)\right| &= \frac{d+1}{d}|z+d-1|\left|q_d^{(2)}(z)\right| \end{aligned}$$

and

$$\left|2dz + d^2\right| > \left|(d+1)z + d^2 - 1\right|$$

from $2d > d + 1$ and $d^2 > d^2 - 1$, one can deduce

$$\begin{aligned} d\left|q_{d+1}^{(1)}(z)\right| = \left|2dz+d^2\right|\left|q_d^{(1)}(z)\right| &> \left|(d+1)z+d^2-1\right|\left|q_d^{(2)}(z)\right| \\ &= (d+1)|z+d-1|\left|q_d^{(2)}(z)\right| \\ &= d\frac{d+1}{d}|z+d-1|\left|q_d^{(2)}(z)\right| = d\left|q_{d+1}^{(2)}(z)\right|. \end{aligned}$$

Thus, $|q_{d+1}^{(1)}(z)| > |q_{d+1}^{(2)}(z)|$ holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Therefore, for any $d \geq 4$, the inequality $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ holds for any complex number z with a nonnegative real part. This implies that the real part of any complex root of $i(\mathcal{P}_{K_d}, m)$ is negative.

We shall also prove the other half, that $-\frac{d}{2} < \mathrm{Re}(\alpha)$. To this end, it suffices to show that all roots of $j_d(l) = i(\mathcal{P}_{K_d}, -l - \frac{d}{2})$ have negative real parts. Let $r_d^{(1)}(l)$ and $r_d^{(2)}(l)$ be

$$r_d^{(1)}(l) = (-1)^{d-1} q_d^{(1)}\left(-l - \frac{d}{2}\right) = (2l+1)\cdots(2l+d-1),$$

$$r_d^{(2)}(l) = (-1)^{d-1} q_d^{(2)}\left(-l - \frac{d}{2}\right) = d\left(l - \frac{d-4}{2}\right)\cdots\left(l + \frac{d}{2}\right).$$

Then for a complex number z, it holds that

$$j_d(z) = 0 \quad \Longleftrightarrow \quad r_d^{(1)}(z) = r_d^{(2)}(z).$$

Let us prove $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ for any complex number z with a nonnegative real part by mathematical induction on $d \geq 4$.

For $d = 4$, it immediately follows from the inequality between $q_4^{(1)}$ and $q_4^{(2)}$:

$$\left|r_4^{(1)}(z)\right| = \left|q_4^{(1)}(z)\right| > \left|q_4^{(2)}(z)\right| = \left|r_4^{(2)}(z)\right|.$$

And so we need $d = 5$ also as a base case:

$$\begin{aligned}
\left|r_5^{(1)}(z)\right| &= |2z+1||2z+2||2z+3||2z+4| \\
&> \frac{5}{4}|z+1||2z+1||2z+3||2z+4| \\
&> \frac{5}{4}\left|z - \frac{1}{2}\right||2z+1||2z+3|\left|z + \frac{5}{2}\right| \\
&= 5\left|z - \frac{1}{2}\right|\left|z + \frac{1}{2}\right|\left|z + \frac{3}{2}\right|\left|z + \frac{5}{2}\right| = \left|r_5^{(2)}(z)\right|.
\end{aligned}$$

Assume for d the validity of $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ for any complex number z with $\mathrm{Re}(z) \geq 0$.

Then, from the fact that

$$\begin{aligned}
\left|r_{d+2}^{(1)}(z)\right| &= |2z+d||2z+d+1|\left|r_d^{(1)}(z)\right| \\
\left|r_{d+2}^{(2)}(z)\right| &= \frac{d+2}{d}\left|z - \frac{d}{2} + 1\right|\left|z + \frac{d}{2} + 1\right|\left|r_d^{(2)}(z)\right|,
\end{aligned}$$

it follows that

$$\begin{aligned}
d\left|r_{d+2}^{(1)}(z)\right| &= d|2z+d||2z+d+1|\left|r_d^{(1)}(z)\right| \\
&> d|2z+d|\left|z + \frac{d}{2} + 1\right|\left|r_d^{(2)}(z)\right| \\
&= \left|2dz + d^2\right|\left|z + \frac{d}{2} + 1\right|\left|r_d^{(2)}(z)\right|
\end{aligned}$$

$$> \left|(d+2)z+d^2-4\right|\left|z+\frac{d}{2}+1\right|\left|r_d^{(2)}(z)\right|$$

$$> (d+2)\left|z-\frac{d-2}{2}\right|\left|z+\frac{d}{2}+1\right|\left|r_d^{(2)}(z)\right| = d\left|r_{d+2}^{(2)}(z)\right|.$$

Thus, $|r_{d+2}^{(1)}(z)| > |r_{d+2}^{(2)}(z)|$ holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Therefore, for any $d \geq 4$, the inequality $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ holds for any complex number z with a nonnegative real part. This implies that any complex root of $j_d(l)$ has a negative real part. □

2.2 Complete multipartite graphs

We computed the roots of the Ehrhart polynomials $i(\mathcal{P}_G, m)$ of complete multipartite graphs G as well. Since complete multipartite graphs are a special subclass of connected simple graphs, our interest is mainly in the cases where the general method could not complete the computation, i.e., complete multipartite graphs of orders $d \geq 10$.

A complete multipartite graph of type $(q_1, \ldots, q_t)$, denoted by $K_{q_1,\ldots,q_t}$, is constructed as follows. Let $V(K_{q_1,\ldots,q_t}) = \bigcup_{i=1}^t V_i$ be a disjoint union of vertices with $|V_i| = q_i$ for each i and the edge set $E(K_{q_1,\ldots,q_t})$ be $\{\{u, v\} \mid u \in V_i,\ v \in V_j\ (i \neq j)\}$. The graph $K_{q_1,\ldots,q_t}$ is unique up to isomorphism.

The Ehrhart polynomials for complete multipartite graphs are explicitly given in [22]:

$$i(\mathcal{P}_G, m) = \binom{d+2m-1}{d-1} - \sum_{k=1}^{t} \sum_{1\leq i\leq j\leq q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j}, \tag{1}$$

where $d = \sum_{k=1}^t q_k$ is a partition of d and $G = K_{q_1,\ldots,q_t}$.

Another simpler formula is newly obtained.

Proposition 2.3 *The Ehrhart polynomial $i(\mathcal{P}_G, m)$ of the edge polytope of a complete multipartite graph $G = K_{q_1,\ldots,q_t}$ is*

$$i(\mathcal{P}_G, m) = f(m; d, d) - \sum_{k=1}^{t} f(m; d, q_k),$$

where $d = \sum_{k=1}^t q_k$ and

$$f(m; d, j) = \sum_{k=1}^{j} p(m; d, k)$$

with

$$p(m; d, j) = \binom{j+m-1}{j-1}\binom{d-j+m-1}{d-j}.$$

Proof Let G denote a complete multipartite graph $K_{q_1,\dots,q_t}$. We start from the formula (1).

First, it holds that

$$\binom{d+2m-1}{d-1} = f(m; d, d).$$

On one hand, $\binom{d+2m-1}{d-1}$ is the number of combinations with repetitions choosing $2m$ elements from a set of cardinality d. On the other hand,

$$f(m; d, d) = \sum_{j=1}^{d} \binom{j+m-1}{j-1}\binom{d-j+m-1}{d-j}$$

counts the same number of combinations as the sum of the number of combinations in which the $(m+1)$th smallest number is j.

Second, it holds that

$$\sum_{k=1}^{t} \sum_{1\le i\le j\le q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j} = \sum_{k=1}^{t} f(m; d, q_k).$$

Since the outermost summations are the same on both sides, it suffices to show that

$$\sum_{1\le i\le j\le q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j} = f(m; d, q_k).$$

The summation of the left-hand side can be transformed as follows:

$$\begin{aligned}
&\sum_{1\le i\le j\le q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j} \\
&\quad= \sum_{j=1}^{q_k}\sum_{i=1}^{j} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j} \\
&\quad= \sum_{j=1}^{q_k} \binom{d-j+m-1}{d-j} \sum_{i=1}^{j} \binom{j-i+m-1}{j-i} \\
&\quad= \sum_{j=1}^{q_k} \binom{d-j+m-1}{d-j}\binom{m+j-1}{j-1} \\
&\quad= \sum_{j=1}^{q_k} p(m; d, j) = f(m; d, q_k).
\end{aligned}$$

Finally, substituting these transformed terms into the original formula (1) gives the desired result. □

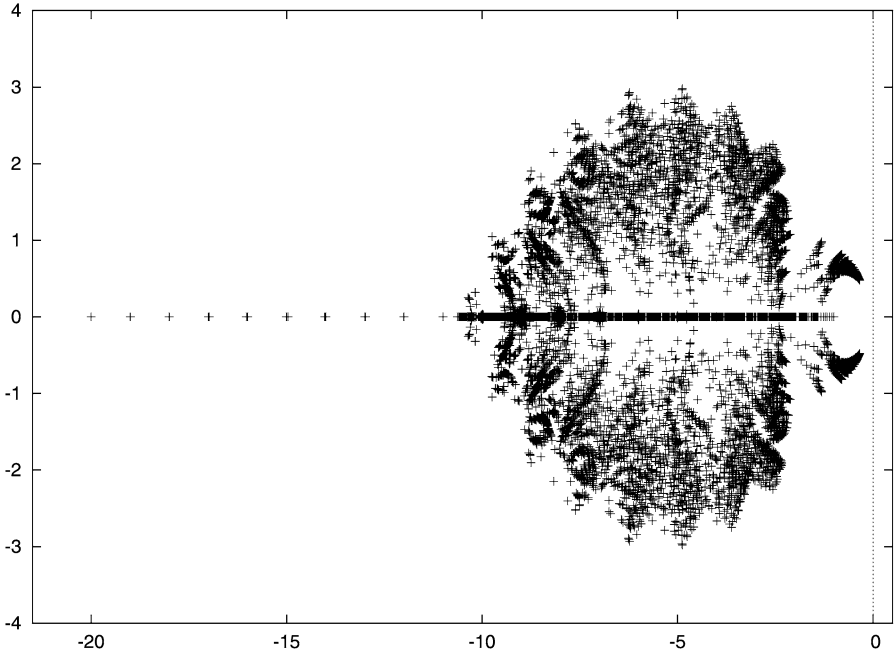


Fig. 2 $\mathbf{V}_{22}^{\mathrm{mp}}$

By the new formula above, we computed the roots of Ehrhart polynomials. Let $\mathbf{V}_d^{\mathrm{mp}}$ denote $\bigcup \mathbf{V}(i(\mathcal{P}_G, m))$, where the union runs over all complete multipartite graphs G of order d. Figure 2 plots the points of $\mathbf{V}_{22}^{\mathrm{mp}}$. For all complete multipartite graphs of order 10–22, Conjecture 1.1 holds.

Figure 2, for $\mathbf{V}_{22}^{\mathrm{mp}}$, shows that the non-integer roots lie in the circle $|z + \frac{11}{2}| \le \frac{11}{2}$. This fact is not exclusive to 22 alone, but similar conditions hold for all $d \le 22$. We conjecture:

Conjecture 2.4 *For any $d \ge 3$,*

$$\mathbf{V}_d^{\mathrm{mp}} \subset \left\{ z \in \mathbb{C} \,\middle|\, \left| z + \frac{d}{4} \right| \le \frac{d}{4} \right\} \cup \left\{ -(d-1), \ldots, -2, -1 \right\}.$$

Remark 2.5 (1) The leftmost point $-(d-1)$ can only be attained by K_3; this is shown in Proposition 2.9. Therefore, if we choose $d \ge 4$, the set of negative integers in the statement can be replaced with the set $\{-(d-2), \ldots, -2, -1\}$. However, $-(d-2)$ can be attained by the tree $K_{d-1,1}$ for any d; see Example 2.6 below.

(2) Since 0 can never be a root of an Ehrhart polynomial, Conjecture 2.4 answers Question 2.1 in the affirmative for complete multipartite graphs. Moreover, if Conjecture 2.4 holds, then Conjecture 1.1 holds for those graphs.

(3) The method of Pfeifle [24] might be useful if the δ-vector can be determined for edge polytopes of complete multipartite graphs.

Example 2.6 The Ehrhart polynomial for complete bipartite graph $K_{p,q}$ is given in, e.g., [22, Corollary 2.7(b)]:

$$i(\mathcal{P}_{K_{p,q}}, m) = \binom{m+p-1}{p-1}\binom{m+q-1}{q-1},$$

and thus the roots are

$$\mathbf{V}\bigl(i(\mathcal{P}_{K_{p,q}}, m)\bigr) = \bigl\{-1, \dots, -\max(p-1, q-1)\bigr\}$$

and all of them are negative integers satisfying the condition in Conjecture 2.4.

Example 2.7 The edge polytope of a complete 3-partite graph $\mathcal{P}_{K_{n,1,1}}$ for $n \geq 2$ can be obtained as a pyramid from $\mathcal{P}_{K_{n,2}}$ by adjoining a vertex. Therefore, its Ehrhart polynomial is the following:

$$i(\mathcal{P}_{K_{n,1,1}}, m) = \sum_{j=0}^{m} i(\mathcal{P}_{K_{n,2}}, j).$$

Each term on the right-hand side is given in Example 2.6 above. By some elementary algebraic manipulations of binomial coefficients, it becomes

$$i(\mathcal{P}_{K_{n,1,1}}, m) = \binom{m+n}{n}\frac{nm+n+1}{n+1}.$$

The non-integer root $\frac{-(n+1)}{n}$ is a real number in the circle of Conjecture 2.4.

Now we prepare the following lemma for proving Proposition 2.9.

Lemma 2.8 *For any integer* $1 \leq j \leq \frac{d}{2}$, *the polynomial* $p(m; d, j)$ *in Proposition* 2.3 *satisfies*:

$$p(m; d, d-j) = \left(\frac{d}{j} - 1\right) p(m; d, j).$$

Proof It is an easy transformation:

$$\begin{aligned} p(m; d, d-j) &= \binom{(d-j)+m-1}{(d-j)-1}\binom{d-(d-j)+m-1}{d-(d-j)} \\ &= \binom{d-j+m-1}{d-j-1}\binom{j+m-1}{j} \\ &= \frac{d-j}{j}\binom{d-j+m-1}{d-j}\binom{j+m-1}{j-1} \\ &= \left(\frac{d}{j} - 1\right) p(m; d, j). \end{aligned}$$

□

Proposition 2.9 *Let* $(q_1, \dots, q_t)$ *be a partition of* $d \geq 3$, *satisfying* $q_1 \geq q_2 \geq \cdots \geq q_t$. *The Ehrhart polynomial* $i(\mathcal{P}_G, m)$ *of the edge polytope of the complete multipartite graph* $G = K_{q_1,\dots,q_t}$ *does not have a root at* $-(d-1)$ *except when the graph is* K_3.

Proof From Proposition 2.3, the Ehrhart polynomial of the edge polytope of $G = K_{q_1,\dots,q_t}$ is

$$\begin{aligned} i(\mathcal{P}_G, m) &= f(m; d, d) - \sum_{k=1}^{t} f(m; d, q_k) \\ &= p(m; d, d) + \sum_{j=1}^{d-1} p(m; d, j) - \sum_{k=1}^{t}\sum_{j=1}^{q_k} p(m; d, j). \end{aligned}$$

Since $p(m; d, d)$ has $-(d-1)$ as one of its roots, it suffices to show that the rest of the expression does not have $-(d-1)$ as one of its roots.

We evaluate $p(m; d, j)$ at $-(d-1)$ for j from 1 to $d-1$:

$$p\bigl(-(d-1); d, j\bigr) = \binom{j-d}{j-1}\binom{-j}{d-j}$$

by the definition of $p(m; d, j)$. If $j > 1$, its sign is $(-1)^{j-1+d-j} = (-1)^{d-1}$ since $j - d < 0$ and $-j < 0$. In case where $j = 1$, since $j - 1$ is zero,

$$p\bigl(-(d-1); d, 1\bigr) = \binom{-1}{d-1} = (-1)^{d-1}$$

gives the same sign with other values of j.

By the conjugate partition $(q'_1, \dots, q'_{t'})$ of $(q_1, \dots, q_t)$, which is given by $q'_j = |\{i \le t \mid q_i \ge j\}|$, we obtain

$$\sum_{j=1}^{d-1} p(m; d, j) - \sum_{k=1}^{t}\sum_{j=1}^{q_k} p(m; d, j) = \sum_{j=1}^{d-1}\bigl(1 - q'_j\bigr) p(m; d, j), \tag{2}$$

where we set, for simplicity, $q'_j = 0$ for $j > t'$.

We show that all the coefficients of $p(m; d, j)$ are nonnegative for any j from 1 to $d-1$ and there is at least one positive coefficient among them.

(I) $q_1 \ge \frac{d}{2}$:
The coefficients of $p(m; d, j)$ are zero for $q_1 \ge j \ge d - q_1$, unless $d = q_1 + q_2$, i.e., when the graph is a complete bipartite graph; the exceptional case will be discussed later. We assume, therefore, $q_2 < d - q_1$ for a while. Though (2) gives the coefficient of $p(m; d, j)$ as 1 for $d > j > q_1$, by using Lemma 2.8, we are able to let them be zero and the coefficient of $p(m; d, j)$ be $\frac{d}{j} - q'_j$ for $d - q_1 > j > 0$. Then all the coefficients of $p(m; d, j)$'s are positive, since the occurrence of integers greater than or equal to j in a partition of $d - q_1$ cannot be greater than $\frac{d-q_1}{j}$.

(II) $q_1 < \frac{d}{2}$:
Each coefficient of $p(m; d, j)$ in (2) is 1 for $d > j > \frac{d}{2}$. By Lemma 2.8, we transfer them to lower j terms so as to make the coefficients for $\frac{d}{2} > j > 0$

be $\frac{d}{j} - q'_j$. Then all the coefficients of $p(m; d, j)$'s are nonnegative, since the occurrence of integers greater than or equal to j in a partition of d cannot be greater than $\frac{d}{j}$. Moreover, the coefficient is zero for at most one j, less than $\frac{d}{2}$. If $d = 3$ and $q_1 = q_2 = q_3 = 1$, i.e., in case of K_3, there does not remain a positive coefficient. This exceptional case will be discussed later.

For both (I) and (II), ignoring the exceptional cases, the terms on the right-hand side of equation (2) are all nonnegative when $d \equiv 1 \pmod 2$, or nonpositive otherwise, and there is at least one nonzero term. That is, $-(d-1)$ is not a root of

$$\sum_{j=1}^{d-1} p(m; d, j) - \sum_{k=1}^{t} \sum_{j=1}^{q_k} p(m; d, j).$$

The Ehrhart polynomial $i(\mathcal{P}_G, m)$ is a sum of a polynomial whose roots include $-(d-1)$ and another polynomial whose roots do not include $-(d-1)$. Therefore, $-(d-1)$ is not a root of $i(\mathcal{P}_G, m)$.

Finally, we discuss the exceptional cases. The complete bipartite graphs are treated in Example 2.6. In these cases, $-(d-1)$ is not a root of the Ehrhart polynomials. However, $-(d-1) = -2$ is actually a root of the Ehrhart polynomial of the edge polytope constructed from the complete graph K_3, as shown in Proposition 2.2(1). □

3 Edge polytopes of graphs with loops

A convex polytope $\mathcal{P}$ of dimension D is *simple* if each vertex of $\mathcal{P}$ belongs to exactly D edges of $\mathcal{P}$. A simple polytope $\mathcal{P}$ is *smooth* if at each vertex of $\mathcal{P}$, the primitive edge directions form a lattice basis.

Now, if $e = \{i, j\}$ is an edge of G, then $\rho(e)$ cannot be a vertex of $\mathcal{P}_G$ if and only if $i \neq j$ and G has a loop at each of the vertices i and j. Suppose that G has a loop at $i \in V(G)$ and $j \in V(G)$ and that $\{i, j\}$ is not an edge of G. Then $\mathcal{P}_G = \mathcal{P}_{G'}$ for the graph G' defined by $E(G') = E(G) \cup \{\{i, j\}\}$. Considering this fact, throughout this section, we assume that G satisfies the following condition:

($*$) If $i, j \in V(G)$ and if G has a loop at each of i and j, then the edge $\{i, j\}$ belongs to G.

The graphs G (allowing loops) whose edge polytope $\mathcal{P}_G$ is simple are completely classified by the following:

Theorem 3.1 [23, Theorem 1.8] *Let W denote the set of vertices $i \in V(G)$ such that G has no loop at i and let G' denote the induced subgraph of G on W. Then the following conditions are equivalent*:

(i) *$\mathcal{P}_G$ is simple, but not a simplex*;
(ii) *$\mathcal{P}_G$ is smooth, but not a simplex*;
(iii) *$W \neq \emptyset$ and G is one of the following graphs*:
  (α) *G is a complete bipartite graph with at least one cycle of length* 4;

(β) *G has exactly one loop, G' is a complete bipartite graph and if G has a loop at i, then $\{i, j\} \in E(G)$ for all $j \in W$;*
(γ) *G has at least two loops, G' has no edge and if G has a loop at i, then $\{i, j\} \in E(G)$ for all $j \in W$.*

From the theory of Gröbner bases, we obtain the Ehrhart polynomial $i(\mathcal{P}_G, m)$ of the edge polytope $\mathcal{P}_G$ above. In fact,

Theorem 3.2 [23, Theorem 3.1] *Let G be a graph as in Theorem 3.1*(iii). *Let W denote the set of vertices $i \in V(G)$ such that G has no loop at i and let G' denote the induced subgraph of G on W. Then the Ehrhart polynomial $i(\mathcal{P}_G, m)$ of the edge polytope $\mathcal{P}_G$ are as follows*:

(α) *If G is the complete bipartite graph on the vertex set $V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$, then we have*

$$i(\mathcal{P}_G, m) = \binom{p+m-1}{p-1}\binom{q+m-1}{q-1};$$

(β) *If G' is the complete bipartite graph on the vertex set $V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$, then we have*

$$i(\mathcal{P}_G, m) = \binom{p+m}{p}\binom{q+m}{q};$$

(γ) *If G possesses p loops and $|V(G)| = d$, then we have*

$$i(\mathcal{P}_G, m) = \sum_{j=1}^{p} \binom{j+m-2}{j-1}\binom{d-j+m}{d-j}.$$

The goal of this section is to discuss the roots of Ehrhart polynomials of simple edge polytopes in Theorem 3.1 (Theorems 3.5, 3.6, and 3.7).

3.1 Roots of Ehrhart polynomials

The consequences of the theorems above support Conjecture 1.1. Recall that $\mathbf{V}(f)$ denotes the set of roots of given polynomial f.

Example 3.3 The Ehrhart polynomial for a graph G, the induced subgraph G' of which is a complete bipartite graph $K_{p,q}$, is given in Theorem 3.2(β):

$$i(\mathcal{P}_G, n) = \binom{p+m}{p}\binom{q+m}{q},$$

and thus the roots are

$$\mathbf{V}\left(\binom{p+m}{p}\binom{q+m}{q}\right) = \{-1, -2, \dots, -\max(p, q)\}.$$

Example 3.4 Explicit computation of the roots of the Ehrhart polynomials obtained in Theorem 3.2(γ) seems, in general, to be rather difficult.

Let $p = 2$. Then

$$\begin{aligned}
&\binom{m-1}{0}\binom{d-1+m}{d-1} + \binom{m}{1}\binom{d-2+m}{d-2} \\
&\quad = \binom{d-1+m}{d-1} + m\binom{d-2+m}{d-2} \\
&\quad = \left(\frac{d-1+m}{d-1} + m\right)\binom{d-2+m}{d-2} \\
&\quad = \frac{dm+d-1}{d-1}\binom{d-2+m}{d-2}.
\end{aligned}$$

Thus,

$$\mathbf{V}\big(i(\mathcal{P}_G, m)\big) = \left\{-1, -2, \ldots, -(d-2), -\frac{d-1}{d}\right\}.$$

Let $p = 3$. Then

$$\begin{aligned}
&\binom{m-1}{0}\binom{d-1+m}{d-1} + \binom{m}{1}\binom{d-2+m}{d-2} + \binom{m+1}{2}\binom{d-3+m}{d-3} \\
&\quad = \binom{d-1+m}{d-1} + m\binom{d-2+m}{d-2} + \frac{m(m+1)}{2}\binom{d-3+m}{d-3} \\
&\quad = \left(\frac{(d-1+m)(d-2+m)}{(d-1)(d-2)} + m\frac{d-2+m}{d-2} + \frac{m(m+1)}{2}\right)\binom{d-3+m}{d-3}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(d-1+m)(d-2+m)}{(d-1)(d-2)} + m\frac{d-2+m}{d-2} + \frac{m(m+1)}{2} \\
&\quad = \frac{2(d-1+m)(d-2+m) + 2(d-1)m(d-2+m) + (d-1)(d-2)m(m+1)}{2(d-1)(d-2)} \\
&\quad = \frac{(d^2-d+2)m^2 + (3d^2-5d)m + (2d^2-6d+4)}{2(d-1)(d-2)}.
\end{aligned}$$

Let

$$f(m) = \big(d^2-d+2\big)m^2 + \big(3d^2-5d\big)m + \big(2d^2-6d+4\big).$$

Since $d > p = 3$, one has

$$\begin{aligned}
f(0) &= 2d^2 - 6d + 4 = 2(d-1)(d-2) > 0; \\
f(-1) &= \big(d^2-d+2\big) - \big(3d^2-5d\big) + \big(2d^2-6d+4\big) = -2d + 6 < 0; \\
f(-2) &= 4\big(d^2-d+2\big) - 2\big(3d^2-5d\big) + \big(2d^2-6d+4\big) = 12 > 0.
\end{aligned}$$

Hence,

$$\mathbf{V}\big(i(\mathcal{P}_G, m)\big) = \big\{-1, -2, \ldots, -(d-3), \alpha, \beta\big\}$$

where $-2 < \alpha < -1 < \beta < 0$.

We try to find information about the roots of the Ehrhart polynomials obtained in Theorem 3.2(γ) with $d > p \ge 2$.

Theorem 3.5 *Let d and p be integers with $d > p \ge 2$ and let*

$$f_{d,p}(m) = \sum_{j=1}^{p} \binom{j+m-2}{j-1}\binom{d-j+m}{d-j}$$

be a polynomial of degree $d-1$ in the variable m. Then

$$\{-1,-2,\dots,-(d-p)\} \subset \mathbf{V}(f_{d,p}) \cap \mathbb{R} \subset [-(d-2),0).$$

Proof It is easy to see that $f_{d,p}(0) = 1$ and $f_{d,p}(m) > 0$ for all $m > 0$.

From Example 3.4, we may assume that $4 \le p < d$. Then

$$\begin{aligned} &f_{d,p}(m)\\ &\quad= \binom{d-1+m}{d-1} + m\binom{d-2+m}{d-2} + \sum_{j=3}^{p}\binom{j+m-2}{j-1}\binom{d-j+m}{d-j}\\ &\quad= \left(\frac{d-1+m}{d-1} + m\right)\binom{d-2+m}{d-2} + \sum_{j=3}^{p}\binom{j+m-2}{j-1}\binom{d-j+m}{d-j}\\ &\quad= \frac{md+d-1}{d-1}\binom{d-2+m}{d-2} + \sum_{j=3}^{p}\binom{j+m-2}{j-1}\binom{d-j+m}{d-j}. \end{aligned}$$

If $m < -(d-2)$, then $m+d-2 < 0$, $md+d-1 < -(d-2)d+d-1 = -(d-3)d-1 < 0$,

$$\begin{aligned} &m+d-j \le m+d-3 < 0,\\ &m+j-2 \le m+p-2 \le m+d-3 < 0 \end{aligned}$$

for each $j = 3,4,\dots,p$. Hence, we have $(-1)^{d-1} f_{d,p}(m) > 0$ for all $m < -(d-2)$. Thus, we have $\mathbf{V}(f_{d,p}) \cap \mathbb{R} \subset [-(d-2),0)$.

Since

$$f_{d,p}(m) = \binom{d-p+m}{d-p}\sum_{j=1}^{p}\binom{j+m-2}{j-1}\frac{(d-j+m)\cdots(d-p+1+m)}{(d-j)\cdots(d-p+1)},$$

it follows that

$$\mathbf{V}\left(\binom{d-p+m}{d-p}\right) = \{-1,-2,\dots,-(d-p)\} \subset \mathbf{V}(f_{d,p}).$$

□

Theorem 3.6 *Let d and p be integers with $d > p \ge 2$ and let $f_{d,p}(m)$ be the polynomial defined above. If $d-2p+2 \ge 0$, then*

$$\mathbf{V}(f_{d,p}) = \{-1,-2,\dots,-(d-p),\alpha_1,\alpha_2,\dots,\alpha_{p-1}\}$$

where

$$-(p-1) < \alpha_{p-1} < -(p-2) < \alpha_{p-2} < -(p-3) < \cdots < -1 < \alpha_1 < 0.$$

Proof Let

$$g_{d,p}(m) = \frac{f_{d,p}(m)}{\binom{d-p+m}{d-p}} = \sum_{j=1}^{p} \binom{j+m-2}{j-1} \frac{(d-j+m)\cdots(d-p+1+m)}{(d-j)\cdots(d-p+1)}.$$

It is enough to show that

$$(-1)^k g_{d,p}(k) > 0$$

for $k = 0, -1, -2, \dots, -(p-1)$.

(*First Step*) We claim that $(-1)^{-(p-1)} g_{d,p}(-(p-1)) > 0$. A routine computation on binomial coefficients yields the equalities

$$\begin{aligned} & g_{d,p}\bigl(-(p-1)\bigr) \\ &\quad = \frac{\sum_{j=1}^{p} (-1)^{j-1} \binom{p-1}{j-1} \prod_{i=1}^{j-1} (d-i) \prod_{k=j}^{p-1} (d-k-(p-1))}{(d-1)\cdots(d-p+1)} \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=1}^{p} (-1)^{j-1} \binom{p-1}{j-1} \prod_{i=1}^{j-1} (d-i) \prod_{k=j}^{p-1} \bigl(d-k-(p-1)\bigr) \\ &\quad = (-1)^{p-1} (p-1)p\cdots(2p-3). \end{aligned}$$

Hence,

$$(-1)^{p-1} g_{d,p}\bigl(-(p-1)\bigr) = \frac{(p-1)p\cdots(2p-3)}{(d-1)\cdots(d-p+1)} > 0.$$

(*Second Step*) Working by induction on p, we now show that

$$(-1)^k g_{d,p}(k) > 0$$

for $k = 0, -1, -2, \dots, -(p-2)$. Again, a routine computation on binomial coefficients yields

$$g_{d,p}(m) = \binom{p+m-2}{p-1} + \frac{d-p+1+m}{d-p+1} g_{d,p-1}(m).$$

Hence,

$$(-1)^k g_{d,p}(k) = \frac{d-p+1+k}{d-p+1} (-1)^k g_{d,p-1}(k).$$

Since $d - 2p + 2 \geq 0$, one has

$$d - p + 1 + k \geq d - p + 1 - (p-2) = d - 2p + 3 > 0.$$

By virtue of $(-1)^{-(p-1)} g_{d,p}(-(p-1)) > 0$, together with the induction hypothesis, it follows that

$$(-1)^k g_{d,p-1}(k) > 0.$$

Thus,

$$(-1)^k g_{d,p}(k) > 0,$$

as desired. □

If $d - 2p + 2 \geq 0$, then it follows that

$$\left\lfloor \frac{d-1}{2} \right\rfloor \leq d - p.$$

In this case, around half of the elements of $\mathbf{V}(f_{d,p})$ are negative integers. This fact remains true even if $d - 2p + 2 < 0$.

Theorem 3.7 *Let d and p be integers with $d > p \geq 2$ and let $f_{d,p}(m)$ be the polynomial defined above. Then*

$$\left\{-1, -2, \ldots, -\left\lfloor \frac{d-1}{2} \right\rfloor\right\} \subset \mathbf{V}(f_{d,p}).$$

Proof If $d - 2p + 2 \geq 0$, then it follows from Theorem 3.5. (Note that if $p = 2$, then $d - 2p + 2 = d - 2 > 0$.)

Work with induction on p. Let $d - 2p + 2 < 0$. By Theorem 3.5, it is enough to show that $g_{d,p}(k) = 0$ for all $k = -(d-p+1), \ldots, -\lfloor \frac{d-1}{2} \rfloor$. As in the proof of Theorem 3.6, we have

$$g_{d,p}(m) = \binom{p+m-2}{p-1} + \frac{d-p+1+m}{d-p+1} g_{d,p-1}(m).$$

Since $d - 2p + 2 < 0$, it follows that $\lfloor \frac{d-1}{2} \rfloor \leq p - 2$. Thus,

$$g_{d,p}(k) = \frac{d-p+1+k}{d-p+1} g_{d,p-1}(k).$$

By virtue of

$$g_{d,p}\big(-(d-p+1)\big) = \frac{0}{d-p+1} g_{d,p-1}\big(-(d-p+1)\big) = 0$$

together with the induction hypothesis, it follows that $g_{d,p}(k) = 0$ for all $k = -(d-p+1), \ldots, -\lfloor \frac{d-1}{2} \rfloor$. □

Example 3.8 Let $d = 12$. Then $d - 2p + 2 \geq 0$ if and only if $p \leq 7$. For $p = 2, 3, \ldots, 7$, the roots of the Ehrhart polynomials are $-1, -2, \ldots, -(d-p) = p - 12$, together with the real numbers listed as follows:

$p=2$	-0.92					
$p=3$	-1.92	-0.85				
$p=4$	-2.90	-1.83	-0.80			
$p=5$	-3.83	-2.77	-1.74	-0.76		
$p=6$	-4.67	-3.65	-2.65	-1.66	-0.72	
$p=7$	-5.31	-4.42	-3.47	-2.53	-1.58	-0.69

For $p=8, 9, 10, 11$, the roots of the Ehrhart polynomials are $-1, -2, -3, -4, -5 = -\lfloor\frac{d-1}{2}\rfloor$, together with the following complex numbers:

$p=8$	-5.56	-4.19	-3.31	-2.41	-1.51	-0.65
$p=9$	-5.47	-4.79	-3.16	-2.29	-1.43	-0.62
$p=10$	-5.51	$-4.16+0.18i$	$-4.16-0.18i$	-2.16	-1.34	-0.59
$p=11$	-5.50	-4.53	$-3.08+0.06i$	$-3.08-0.06i$	-1.24	-0.55

(Computed by `Maxima` [19].) Thus, in particular, the real parts of all roots are negative.

4 Symmetric edge polytopes

Among the many topics explored in recent papers on the roots of Ehrhart polynomials of convex polytopes, one of the most fascinating is the Gorenstein Fano polytope.

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d.

- We say that $\mathcal{P}$ is a *Fano polytope* if the origin of $\mathbb{R}^d$ is the unique integer point belonging to the interior of $\mathcal{P}$.
- A Fano polytope is said to be *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is a convex polytope that consists of those $x \in \mathbb{R}^d$ such that $\langle x, y\rangle \le 1$ for all $y \in \mathcal{P}$, where $\langle x, y\rangle$ is the usual inner product on $\mathbb{R}^d$.)

In this section, we will prove that symmetric edge polytopes arising from connected simple graphs are Gorenstein Fano polytopes (Proposition 4.2). Moreover, we will consider the condition of unimodular equivalence (Theorem 4.5). In addition, we will compute the Ehrhart polynomials of symmetric edge polytopes and discuss their roots.

4.1 Fano polytopes arising from graphs

Throughout this section, let G denote a simple graph on the vertex set $V(G) = \{1, \dots, d\}$ with $E(G) = \{e_1, \dots, e_n\}$ being the edge set. Moreover, let $\mathcal{P}_G^{\pm} \subset \mathbb{R}^d$ denote a symmetric edge polytope constructed from G.

Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $x_1 + x_2 + \cdots + x_d = 0$. Now, since the integral points $\pm\sigma(e_1), \dots, \pm\sigma(e_n)$ lie on the hyperplane $\mathcal{H}$, we have $\dim(\mathcal{P}_G^{\pm}) \le d-1$.

Proposition 4.1 *One has* $\dim(\mathcal{P}_G^{\pm}) = d-1$ *if and only if* G *is connected.*

Proof Suppose that G is not connected. Let $G_1, \dots, G_m$ with $m > 1$ denote the connected components of G. Let, say, $\{1, \dots, d_1\}$ be the vertex set of G_1 and $\{d_1+1, \dots, d_2\}$ the vertex set of G_2. Then $\mathcal{P}_G^{\pm}$ lies on two hyperplanes defined by the equations $x_1 + \cdots + x_{d_1} = 0$ and $x_{d_1+1} + \cdots + x_{d_2} = 0$. Thus, $\dim(\mathcal{P}_G^{\pm}) < d-1$.

Next, we assume that G is connected. Suppose that $\mathcal{P}_G^{\pm}$ lies on the hyperplane defined by the equation $a_1x_1 + \cdots + a_dx_d = b$ with $a_1, \dots, a_d, b \in \mathbb{Z}$. Let $e = \{i, j\}$ be an edge of G. Then because $\sigma(e)$ lies on this hyperplane together with $-\sigma(e)$, we obtain

$$a_i - a_j = -(a_i - a_j) = b.$$

Thus $a_i = a_j$ and $b = 0$. For all edges of G, since G is connected, we have $a_1 = a_2 = \cdots = a_d$ and $b = 0$. Therefore, $\mathcal{P}_G^{\pm}$ lies only on the hyperplane $x_1 + x_2 + \cdots + x_d = 0$. □

For the rest of this section, we assume that G is connected.

Proposition 4.2 *Let* $\mathcal{P}_G^{\pm}$ *be a symmetric edge polytope of a graph* G. *Then* $\mathcal{P}_G^{\pm} \subset \mathcal{H}$ *is a Gorenstein Fano polytope of dimension* $d-1$.

Proof Let $\varphi : \mathbb{R}^{d-1} \to \mathcal{H}$ be the bijective homomorphism with

$$\varphi(y_1, \dots, y_{d-1}) = \big(y_1, \dots, y_{d-1}, -(y_1 + \cdots + y_{d-1})\big).$$

Thus, we can identify $\mathcal{H}$ with $\mathbb{R}^{d-1}$. Therefore, $\varphi^{-1}(\mathcal{P}_G^{\pm})$ is isomorphic to $\mathcal{P}_G^{\pm}$.

Since one has

$$\frac{1}{2n}\sum_{j=1}^{n}\sigma(e_j) + \frac{1}{2n}\sum_{j=1}^{n}\big(-\sigma(e_j)\big) = (0, \dots, 0) \in \mathbb{R}^d,$$

the origin of $\mathbb{R}^d$ is contained in the relative interior of $\mathcal{P}_G^{\pm} \subset \mathcal{H}$. Moreover, since

$$\mathcal{P}_G^{\pm} \subset \big\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid -1 \le x_i \le 1, i = 1, \dots, d\big\},$$

it is not possible for an integral point to exist anywhere in the interior of $\mathcal{P}_G^{\pm}$ except at the origin. Thus, $\mathcal{P}_G^{\pm} \subset \mathcal{H}$ is a Fano polytope of dimension $d-1$.

Next, we prove that $\mathcal{P}_G^{\pm}$ is Gorenstein. Let M be an integer matrix whose row vectors are $\sigma(e)$ or $-\sigma(e)$ with $e \in E(G)$. Then M is a totally unimodular matrix. From the theory of totally unimodular matrices ([27, Chap. 9]), it follows that a system of equations $yA = (1, \dots, 1)$ has integral solutions, where A is a submatrix of M. This implies that the equation of each supporting hyperplane of $\mathcal{P}_G^{\pm}$ is of the form $a_1x_1 + \cdots + a_dx_d = 1$ with each $a_i \in \mathbb{Z}$. In other words, the dual polytope of $\mathcal{P}_G^{\pm}$ is integral. Hence, $\mathcal{P}_G^{\pm}$ is Gorenstein, as required. □

4.2 When is $\mathcal{P}_G^{\pm}$ unimodular equivalent?

In this subsection, we consider the conditions under which $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$ for graphs G and G'.

Recall that for a connected graph G, we call G a *2-connected* graph if the induced subgraph with the vertex set $V(G)\backslash\{i\}$ is still connected for any vertex i of G.

Let us say a Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ *splits* into $\mathcal{P}_1$ and $\mathcal{P}_2$ if $\mathcal{P}$ is the convex hull of the two Fano polytopes $\mathcal{P}_1 \subset \mathbb{R}^{d_1}$ and $\mathcal{P}_2 \subset \mathbb{R}^{d_2}$ with $d = d_1 + d_2$. That is, by arranging the numbering of coordinates, we have

$$\mathcal{P} = \mathrm{conv}\big(\big\{(\alpha_1, \mathbf{0}) \in \mathbb{R}^d \mid \alpha_1 \in \mathcal{P}_1\big\} \cup \big\{(\mathbf{0}, \alpha_2) \in \mathbb{R}^d \mid \alpha_2 \in \mathcal{P}_2\big\}\big).$$

Lemma 4.3 *$\mathcal{P}_G^{\pm}$ cannot split if and only if G is 2-connected.*

Proof ("*Only if*") Suppose that G is not 2-connected, i.e., there is a vertex i of G such that the induced subgraph G' of G with the vertex set $V(G)\backslash\{i\}$ is not connected. For a matrix

$$\begin{pmatrix} \sigma(e_1) \\ -\sigma(e_1) \\ \vdots \\ \sigma(e_n) \\ -\sigma(e_n) \end{pmatrix} \tag{3}$$

whose row vectors are the vertices of $\mathcal{P}_G^{\pm}$, we add all the columns of (3) except the ith column to the ith column. Then the ith column vector becomes equal to the zero vector. Let, say, $\{1, \dots, i-1\}$ and $\{i+1, \dots, d\}$ denote the vertex set of the connected components of G'. Then, by arranging the row vectors of (3) if necessary, the matrix (3) can be transformed into

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

This means that $\mathcal{P}_G^{\pm}$ splits into $\mathcal{P}_1$ and $\mathcal{P}_2$, where the vertex set of $\mathcal{P}_1$ (respectively $\mathcal{P}_2$) constitutes the row vectors of M_1 (respectively M_2).

("*If*") We assume that G is 2-connected. Suppose that $\mathcal{P}_G^{\pm}$ splits into $\mathcal{P}_1, \dots, \mathcal{P}_m$ and each $\mathcal{P}_i$ cannot split, where $m > 1$. Then by arranging the row vectors if necessary, the matrix (3) can be transformed into

$$\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_m \end{pmatrix}.$$

Now, for a row vector v of each matrix M_i, $-v$ is also a row vector of M_i. Let

$$v_{i_1}, \dots, v_{i_{k_i}}, -v_{i_1}, \dots, -v_{i_{k_i}}$$

denote the row vectors of M_i, where $e_{i_1}, \dots, e_{i_{k_i}}$ are the edges of G with $v_{i_j} = \sigma(e_{i_j})$ or $v_{i_j} = -\sigma(e_{i_j})$, and G_i denote the subgraph of G with the edge set $\{e_{i_1}, \dots, e_{i_{k_i}}\}$. Then for the subgraphs $G_1, \dots, G_m$ of G, one has

$$\bigl|V(G_1)\bigr| + \cdots + \bigl|V(G_m)\bigr| \geq d + 2(m-1), \tag{4}$$

where $V(G_i)$ is the vertex set of G_i.

(In fact, the inequality (4) follows by induction on m. When $m = 2$, since G is 2-connected, G_1 and G_2 share at least two vertices. Thus, one has $|V(G_1)| + |V(G_2)| \geq d + 2$. When $m = k + 1$, since G is 2-connected, one has

$$\left|\left(\bigcup_{i=1}^{k} V(G_i)\right) \cap V(G_{k+1})\right| \geq 2.$$

Let d' be the sum of the numbers of the columns of $M_1, \dots, M_{k-1}$ and M_k and d'' be the number of the columns of M_{k+1}, where $d' + d'' = d$. Then one has

$$\begin{aligned} \bigl|V(G_1)\bigr| + \cdots + \bigl|V(G_k)\bigr| + \bigl|V(G_{k+1})\bigr| &\geq d' + 2(k-1) + \bigl|V(G_{k+1})\bigr| \\ &\geq d' + d'' + 2(k-1) + 2 = d + 2k \end{aligned}$$

by the hypothesis of induction.)

In addition, each $\mathcal{P}^{\pm}_{G_i}$ cannot split. Thus one has $\dim(\mathcal{P}^{\pm}_{G_i}) = |V(G_i)| - 1$ since each G_i is connected by the proof of the "only if" part. It then follows from this equality and the inequality (4) that

$$\begin{aligned} d - 1 = \dim\bigl(\mathcal{P}^{\pm}_{G_1}\bigr) + \cdots + \dim\bigl(\mathcal{P}^{\pm}_{G_m}\bigr) &= \bigl|V(G_1)\bigr| + \cdots + \bigl|V(G_m)\bigr| - m \\ &\geq d + 2m - 2 - m = d + m - 2 \geq d \quad (m \geq 2), \end{aligned}$$

a contradiction. Therefore, $\mathcal{P}^{\pm}_{G}$ cannot split. □

Lemma 4.4 *Let G be a 2-connected graph. Then, for a graph G', $\mathcal{P}^{\pm}_{G}$ is unimodular equivalent with $\mathcal{P}^{\pm}_{G'}$ as an integral convex polytope if and only if G is isomorphic to G' as a graph.*

Proof If $|V(G)| = 2$, the statement is obvious. Thus, we assume that $|V(G)| > 2$.

("*Only if*") Suppose that $\mathcal{P}^{\pm}_{G}$ is unimodular equivalent with $\mathcal{P}^{\pm}_{G'}$. Let M_G (respectively $M_{G'}$) denote the matrix whose row vectors are the vertices of $\mathcal{P}^{\pm}_{G}$ (respectively $\mathcal{P}^{\pm}_{G'}$). Then there is a unimodular transformation U such that one has

$$M_G U = M_{G'}. \tag{5}$$

Thus, each row vector of M_G, i.e., each edge of G, one-to-one corresponds to each edge of G'. Hence, G and G' have the same number of edges. Moreover, since G is 2-connected, $\mathcal{P}^{\pm}_{G}$ cannot split by Lemma 4.3. Thus, $\mathcal{P}^{\pm}_{G'}$ also cannot split; that is to say, G' is also 2-connected. In addition, if we suppose that G and G' do not have the same number of vertices, then $\dim(\mathcal{P}^{\pm}_{G}) \neq \dim(\mathcal{P}^{\pm}_{G'})$ since G and G' are connected, a contradiction. Thus, the number of the vertices of G is equal to that of G'.

Now an arbitrary 2-connected graph with $|V(G)| > 2$ can be obtained by the following method: start from a cycle and repeatedly append an H-path to a graph H that has been already constructed. (Consult, e.g., [33].) In other words, there is one cycle C_1 and $(m-1)$ paths $\Gamma_2, \ldots, \Gamma_m$ such that

$$G = C_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m. \tag{6}$$

Under the assumption that G is 2-connected and one has the equality (5), we show that G is isomorphic to G' by induction on m.

If $m = 1$, i.e., G is a cycle, then G has d edges. Let $a_i, i = 1, \ldots, d$ denote the degree of each vertex i of G'. Then one has

$$a_1 + a_2 + \cdots + a_d = 2d.$$

If there is i with $a_i = 1$, then G' is not 2-connected. Thus, $a_i \geq 2$ for $i = 1, \ldots, d$. Hence, $a_1 = \cdots = a_d = 2$. It then follows that G' is also a cycle of the same length as G, which implies that G is isomorphic to G'.

When $m = k + 1$, we assume (6). Let $\tilde{G}$ denote the subgraph of G with

$$\tilde{G} = C_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k.$$

Then $\tilde{G}$ is a 2-connected graph. Since each edge of G has one-to-one correspondence with each edge of G', there is a subgraph $\tilde{G}'$ of G' each of whose edges corresponds to those of $\tilde{G}$. Then one has $M_{\tilde{G}} U = M_{\tilde{G}'}$, where $M_{\tilde{G}}$ (respectively $M_{\tilde{G}'}$) is a submatrix of M_G (respectively $M_{G'}$) whose row vectors are the vertices of $\mathcal{P}^{\pm}_{\tilde{G}}$ (respectively $\mathcal{P}^{\pm}_{\tilde{G}'}$). Thus, $\tilde{G}$ is isomorphic to $\tilde{G}'$ by the induction hypothesis. Let $\Gamma_{k+1} = (i_0, i_1, \ldots, i_p)$ with $i_0 < i_1 < \cdots < i_p$ and $e_{i_l} = \{i_{l-1}, i_l\}, l = 1, \ldots, p$ denote the edges of Γ_{k+1}. In addition, let $e'_{i_1}, \ldots, e'_{i_p}$ denote the edges of G' corresponding to the edges $e_{i_1}, \ldots, e_{i_p}$ of G. Here, the edges $e'_{i_1}, \ldots, e'_{i_p}$ of G' are not the edges of $\tilde{G}'$. Since i_0 and i_p are distinct vertices of $\tilde{G}$ and $\tilde{G}$ is connected, there is a path $\Gamma = (i_0, j_1, j_2, \ldots, j_{q-1}, i_p)$ with $i_0 = j_0 < j_1 < j_2 < \cdots < j_{q-1} < j_q = i_p$ in $\tilde{G}$. Let $e_{j_l} = \{j_{l-1}, j_l\}, l = 1, \ldots, q$ denote the edges of Γ. Then by renumbering the vertices of $\tilde{G}'$ if necessary, there is a path $\Gamma' = (i'_0, j'_1, j'_2, \ldots, j'_{q-1}, i'_p)$ with $i'_0 = j'_0 < j'_1 < j'_2 < \cdots < j'_{q-1} < j'_q = i'_p$ in $\tilde{G}'$ since $\tilde{G}$ is isomorphic to $\tilde{G}'$. Let $e'_{j_l} = \{j'_{l-1}, j'_l\}, l = 1, \ldots, q$ denote the edges of Γ'. However, by (5), each edge e_{j_l} of $\tilde{G}$ has one-to-one correspondence with each edge e''_{j_l} of $\tilde{G}'$. Thus, each edge e'_{j_l} of $\tilde{G}'$ has one-to-one correspondence with each edge e''_{j_l} of $\tilde{G}'$. In other words, one has

$$\{e'_{j_l} \mid l = 1, \ldots, q\} = \{e''_{j_l} \mid l = 1, \ldots, q\}.$$

Since there are Γ_{k+1} and Γ that are paths from i_0 to i_p, one has

$$\sum_{l=1}^{p} \sigma(e_{i_l}) = \sum_{l=1}^{q} \sigma(e_{j_l}). \tag{7}$$

On the one hand, if we multiply the left-hand side of (7) with U, then we have

$$\sum_{l=1}^{p}\sigma(e_{i_l})U=\sum_{l=1}^{p}\sigma(e'_{i_l}).$$

On the other hand, if we multiply the right-hand side of (7) with U, then we have

$$\sum_{l=1}^{q}\sigma(e_{j_l})U=\sum_{l=1}^{q}\sigma(e''_{j_l})=\sum_{l=1}^{q}\sigma(e'_{j_l})=\mathbf{e}_{i'_0}-\mathbf{e}_{i'_p}.$$

Hence, we have $\sum_{l=1}^{p}\sigma(e'_{i_l})=\mathbf{e}_{i'_0}-\mathbf{e}_{i'_p}$. This means that the edges $e'_{i_1},\ldots,e'_{i_p}$ of G' construct a path from the vertex i'_0 to i'_p, which is isomorphic to Γ_{k+1}. Therefore, G is isomorphic to G'.

("*If*") Suppose that G is isomorphic to G'. Then by renumbering the vertices if necessary, it can be easily verified that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. □

Theorem 4.5 *For a connected simple graph G (respectively G'), let $G_1,\ldots,G_m$ (respectively $G'_1,\ldots,G'_{m'}$) denote the 2-connected components of G (respectively G'). Then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$ if and only if $m=m'$ and G_i is isomorphic to G'_i by renumbering if necessary.*

Proof It is clear from Lemmas 4.3 and 4.4. If G_i is isomorphic to G'_i for $i=1,\ldots,m$, by virtue of Lemmas 4.3 and 4.4, then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. On the contrary, suppose that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. If $m\neq m'$, one has a contradiction by Lemma 4.3. Thus, $m=m'$. Moreover, by our assumption, G_i is isomorphic to G'_i by Lemma 4.4. □

4.3 Roots of the Ehrhart polynomials of $\mathcal{P}_G^{\pm}$

In this subsection, we study the Ehrhart polynomials of $\mathcal{P}_G^{\pm}$ and their roots.

Let $\mathcal{P}\subset\mathbb{R}^D$ be a Fano polytope with $\delta(\mathcal{P})=(\delta_0,\delta_1,\ldots,\delta_D)$ being its δ-vector. It follows from [2] and [15] that the following conditions are equivalent:

- $\mathcal{P}$ is Gorenstein;
- $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_i=\delta_{D-i}$ for every $0\le i\le D$;
- $i(\mathcal{P},m)=(-1)^D i(\mathcal{P},-m-1)$.

Since $i(\mathcal{P},m)=(-1)^D i(\mathcal{P},-m-1)$, the roots of $i(\mathcal{P},m)$ locate symmetrically in the complex plane with respect to the line $\mathrm{Re}(z)=-\frac{1}{2}$.

Proposition 4.6 *If G is a tree, then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with*

$$\mathrm{conv}\big(\{\pm\mathbf{e}_1,\ldots,\pm\mathbf{e}_{d-1}\}\big). \tag{8}$$

Proof If G is a tree, then any 2-connected component of G consists of one edge and G possesses $(d-1)$ 2-connected components. Thus, by Theorem 4.5, for any tree

G, $\mathcal{P}_G^{\pm}$ is unimodular equivalent. Hence we should prove only the case where G is a path, i.e., the edge set of G is $\{\{i, i+1\} \mid i = 1, \dots, d-1\}$.

Let

$$\begin{pmatrix} \sigma(e_1) \\ -\sigma(e_1) \\ \vdots \\ \sigma(e_{d-1}) \\ -\sigma(e_{d-1}) \end{pmatrix}$$

denote the matrix whose row vectors are the vertices of $\mathcal{P}_G^{\pm}$, where $e_i = \{i, i+1\}$, $i = 1, \dots, d-1$ are the edges of G. If we add the dth column to the $(d-1)$th column, the $(d-1)$th column to the $(d-2)$th column, $\dots$, and the second column to the first column, then the above matrix is transformed into

$$\begin{pmatrix} 0 & M & & \mathbf{0} \\ \vdots & & \ddots & \\ 0 & \mathbf{0} & & M \end{pmatrix},$$

where M is the 2×1 matrix $\binom{-1}{1}$. This implies that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with (8). $\square$

Let $(\delta_0, \delta_1, \dots, \delta_{d-1}) \in \mathbb{Z}^d$ be the δ-vector of (8). Then it can be calculated that

$$\delta_i = \binom{d-1}{i}, \quad i = 0, 1, \dots, d-1.$$

It then follows from the well-known theorem [26] that if G is tree, the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, m)$ are equal to $-\frac{1}{2}$. That is to say, all the roots z of $i(\mathcal{P}_G^{\pm}, m)$ lie on the vertical line $\mathrm{Re}(z) = -\frac{1}{2}$, which is the bisector of the vertical strip $-(d-1) \leq \mathrm{Re}(z) \leq d-2$.

We consider the other two classes of graphs. Let G be a complete bipartite graph of type $(2, d-2)$, i.e., the edges of G are either $\{1, j\}$ or $\{2, j\}$ with $3 \leq j \leq d$. Then the δ-polynomial of $\mathcal{P}_G^{\pm}$ coincides with

$$(1+t)^{d-3}\bigl(1 + 2(d-2)t + t^2\bigr).$$

Using computational evidence, we propose the following:

Conjecture 4.7 *Let G be a complete bipartite graph of type $(2, d-2)$. Then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, m)$ are equal to $-\frac{1}{2}$.*

Let G be a complete graph with d vertices and $\delta(\mathcal{P}_G^{\pm}) = (\delta_0, \delta_1, \dots, \delta_{d-1})$ be its δ-vector. In [1, Theorem 13], the $\delta(\mathcal{P}_G^{\pm})$ is calculated; that is,

$$\delta_i = \binom{d-1}{i}^2, \quad i = 0, 1, \dots, d-1.$$

Using computational evidence, we also propose the following:

Conjecture 4.8 *Let G be a complete graph. Then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, m)$ are equal to $-\frac{1}{2}$.*

In addition, by computational results, we can say the following:

Proposition 4.9 *If $d \leq 6$, then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, m)$ are equal to $-\frac{1}{2}$ for any graph with d vertices.*

However, it is not true for $d = 7$ or $d = 8$. In fact, there are some counterexamples. The following Figs. 3 and 4 illustrate how the roots are distanced from the line $\mathrm{Re}(z) = -\frac{1}{2}$. (They are computed by `CoCoA` [7] and `Maple` [32].)

Let G be a cycle of length d. When $d \leq 6$, although the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, m)$ are equal to $-\frac{1}{2}$, there are also some counterexamples when $d \geq 7$. The following Fig. 5 illustrates the behavior of the roots for $7 \leq d \leq 30$.

However, in the range of graphs which we computed, all the roots z of $i(\mathcal{P}_G^{\pm}, m)$ whose real parts are not equal to $-\frac{1}{2}$ satisfy $-(d-1) \leq \mathrm{Re}(z) \leq d-2$. In more detail, they satisfy $-\frac{d-1}{2} \leq \mathrm{Re}(z) \leq \frac{d-1}{2} - 1$, though we do not know the general case. Then we propose the following:

Conjecture 4.10 *All roots α of the Ehrhart polynomials of Gorenstein Fano polytopes of dimension D satisfy $-\frac{D}{2} \leq \mathrm{Re}(\alpha) \leq \frac{D}{2} - 1$.*

In the table drawn below, in the second row, the number of connected simple graphs with $d(\leq 8)$ vertices, up to isomorphism, is written. In the third row, among

Fig. 3 $d = 7$

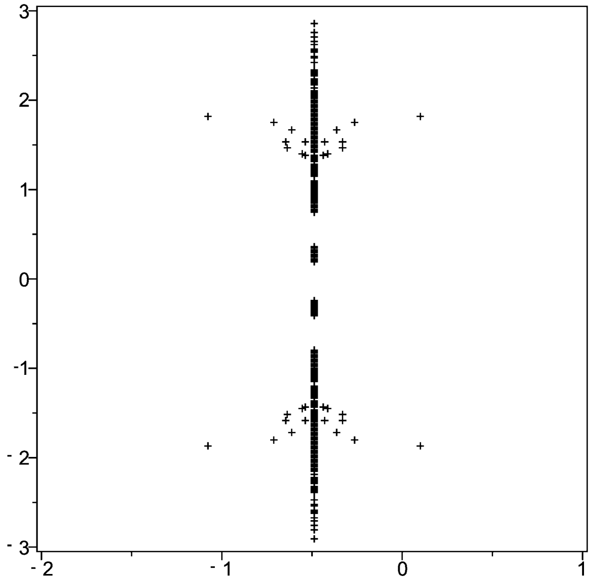


Fig. 4 $d = 8$

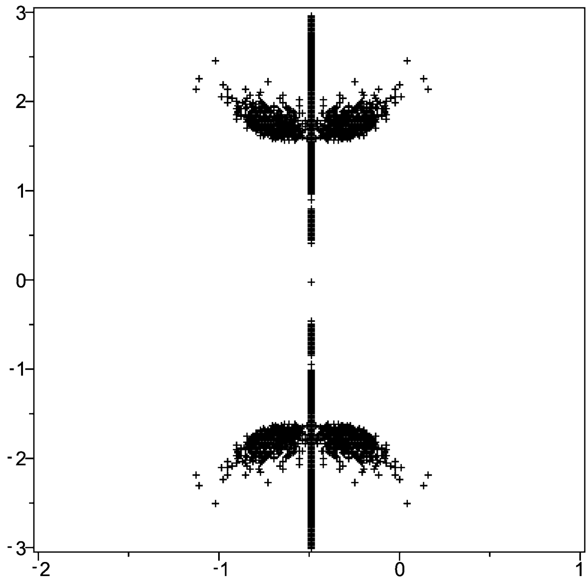
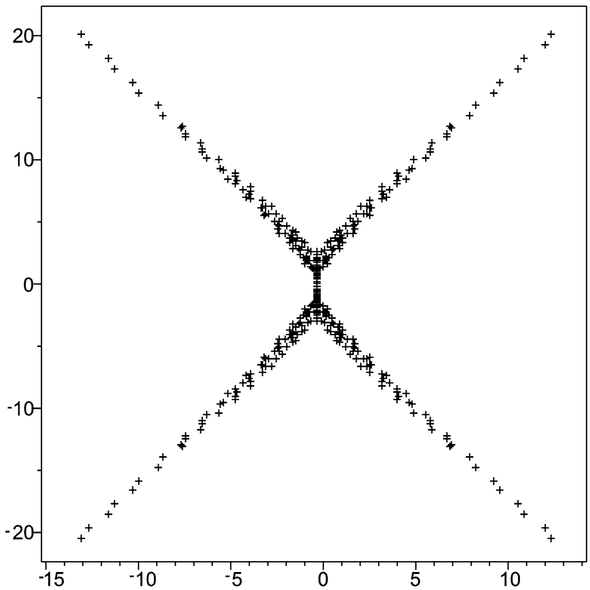


Fig. 5 All cycles $7 \le d \le 30$



these, the number of graphs, up to unimodular equivalence, i.e., satisfying the condition in Theorem 4.5, is written. In the fourth row, among these, in turn, the number of graphs that are counterexamples, i.e., there is a root of $i(\mathcal{P}_G^{\pm}, m)$ whose real part is not equal to $-\frac{1}{2}$, is written.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
Connected graphs	1	2	6	21	112	853	11117
Non equivalent	1	2	5	16	75	560	7772
Counterexamples	0	0	0	0	0	12	1092

Appendix: Method of computation

This appendix presents an outline of the procedure used to compute the roots of the Ehrhart polynomials of edge or symmetric edge polytopes in Sects. 2 and 4. Both polytopes are constructed from connected simple graphs. For each number of vertices d, steps below are taken.

(1) Construct the set of connected simple graphs of order d.
(2) Obtain a facet representation of a polytope for each graph.
(3) Compute the Hilbert series for a facet representation.
(4) Build the Ehrhart polynomial from the series and solve it.

The program for step 1 was written by the authors in the `Python` programming language with an aid of `NZMATH` [18, 20]. The source code is available at:

https://bitbucket.org/mft/csg/.

Step 2 is performed with `Polymake` [10, 25]. Then, `LattE` [9] (or `LattE macchiato` [17]) computes the series for step 3. The final step uses `Maxima` [19] or `Maple` [32].

A small remark has to be made on the interface between steps 3 and 4. If one uses `LattE`'s rational function as the input to `Maxima`, memory consumption becomes very high. `LattE` can send it to `Maple` by itself if you specify "simplify," but this still presents the same problem for the user. Instead, it is preferable to use the coefficient of the first several terms of the Taylor expansion for interpolation.

Finally, it should be mentioned that there is a package of `Macaulay2` for the computations for graphs, which is called `Nauty` [8]. This might be helpful to readers who are interested in running experiments of their own. This package is available at:

http://www.ms.uky.edu/~dcook/files/.

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