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The Isaacs–Navarro conjecture for covering groups of the symmetric and alternating groups in odd characteristic

Jean-Baptiste Gramain

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Abstract In this paper, we prove that a refinement of the Alperin–McKay Conjecture for p-blocks of finite groups, formulated by I.M. Isaacs and G. Navarro in 2002, holds for all covering groups of the symmetric and alternating groups, whenever p is an odd prime.

Keywords Representation theory · Symmetric group · Covering groups · Bar-partitions

1 Introduction

In order to understand properties of the p-modular representation theory of a finite group G, one often tries to reduce to a problem about the p-local subgroups of G, i.e., the normalizers of its p-subgroups. This is illustrated by many results, such as Brauer's three Main Theorems, and several conjectures, such as Broué's Abelian Defect Conjecture, Dade's Conjecture or the Alperin–McKay Conjecture.

I.M. Isaacs and G. Navarro have formulated in [5] some refinements of the McKay and Alperin–McKay Conjectures for arbitrary finite groups. Consider a finite group G and a prime p. Let B be a p-block of G, with defect group D, and let b be the Brauer correspondent of B in $N_G(D)$. Throughout this paper, we will use a p-valuation ν on $\mathbb{Z}$, given by $\nu(n) = a$ if $n = p^a q$ with $(p, q) = 1$. The height $\mathfrak{h}(\chi) \in \mathbb{Z}_{\geq 0}$ of an irreducible (complex) character $\chi \in B$ is then defined by the equality $\nu(\chi(1)) = \nu(|G|) - \nu(|D|) + \mathfrak{h}(\chi)$. We denote by $M(B)$ and $M(b)$ the sets of characters of height 0 of B and b, respectively. The Alperin–McKay Conjecture then asserts that $|M(B)| = |M(b)|$ (while the McKay Conjecture states that

J.-B. Gramain (✉)
Institut de Mathématiques de Jussieu, UFR de Mathématiques, Université Denis Diderot, Paris VII,
2 place Jussieu, 75251 Paris Cedex 05, France
e-mail: gramain@math.jussieu.fr

$|M(G)| = |M(N_G(P))|$, where $P \in \mathrm{Syl}_p(G)$, and $M(G)$ and $M(N_G(P))$ denote the sets of irreducible characters of p'-degree of G and $N_G(P)$, respectively).

In [5], Isaacs and Navarro predicted that something stronger must happen, namely that this equality can be refined when considering the p'-parts of the character degrees. For any $n \in \mathbb{N}$, we write $n = n_p n_{p'}$, with $n_p = p^{\nu(n)}$. For any $1 \le k \le p-1$, we define subsets $M_k(B)$ and $M_k(b)$ of $M(B)$ and $M(b)$, respectively, by letting $M_k(B) = \{\chi \in M(B); \chi(1)_{p'} \equiv \pm k \pmod p\}$ and $M_k(b) = \{\varphi \in M(b); \varphi(1)_{p'} \equiv \pm k \pmod p\}$. We then have the following

Conjecture 1.1 [5, Conjecture B] *For* $1 \le k \le p-1$, *we have* $|M_{ck}(B)| = |M_k(b)|$, *where* $c = [G : N_G(D)]_{p'}$.

Note that Conjecture 1.1 obviously implies the Alperin–McKay Conjecture (by letting k run through $\{1, \dots, p-1\}$), but also implies another refinement of the McKay Conjecture; if we let $M_k(G) = \{\chi \in \mathrm{Irr}(G); \chi(1) \equiv \pm k \pmod p\}$ then, by considering all blocks of G with defect group $P \in \mathrm{Syl}_p(G)$, we obtain $|M_k(G)| = |M_k(N_G(P))|$, since $[G : N_G(P)] \equiv 1 \pmod p$ (see [5, Conjecture B]).

Isaacs and Navarro proved Conjecture 1.1 whenever D is cyclic, or G is p-solvable or sporadic. P. Fong proved it for symmetric groups $S(n)$ in [2], and R. Nath for alternating groups $A(n)$ in [8]. In this paper, we prove that Conjecture 1.1 holds in all the covering groups of the symmetric and alternating groups, provided p is odd (Theorem 5.1). The proof makes heavy use of the powerful combinatorics underlying the representation theory of these groups. In particular, Conjecture 1.1 comes from an explicit bijection, given in terms of the bar-partitions used to parametrize the irreducible characters.

In Sect. 2, we present the covering groups $S^+(n)$ and $S^-(n)$ and their irreducible characters, first studied by I. Schur in [11], as well as their p-blocks. It turns out that the main work to be done is on so-called spin blocks. We also give various results on the degrees of spin characters, generalizing the methods used by Fong in [2]. Most of these results are of a combinatorial nature, and the concepts they involve are also presented here. Section 3 is devoted to proving Theorem 3.4 which reduces the problem to proving only that Conjecture 1.1 holds for the principal spin block of $S^+(pw)$. This reduction theorem is a refinement of [6, Theorem 2.2] that G.O. Michler and J.B. Olsson proved in order to establish that the Alperin–McKay Conjecture holds for covering groups. Finally, the case of the principal spin block of $S^+(pw)$ is treated in Sect. 4.

2 Covering groups

In this section, we introduce the objects and preliminary results we will need about covering groups and their characters. Unless stated otherwise, the following results can be found in [6].

2.1 Covering groups

For any integer $n \ge 1$, I. Schur has defined (by generators and relations) two central extensions $\hat{S}(n)$ and $\tilde{S}(n)$ of the symmetric group $S(n)$ (see [11], p. 164). We have

$\hat{S}(1) \cong \tilde{S}(1) \cong \mathbb{Z}/2\mathbb{Z}$, and, for $n \geq 2$, there is a nonsplit exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow \hat{S}(n) \xrightarrow{\pi} S(n) \longrightarrow 1,$$

where $\langle z \rangle = Z(\hat{S}(n)) \cong \mathbb{Z}/2\mathbb{Z}$.

Whenever $n \geq 4$, these two extensions are non-isomorphic, except when $n = 6$. However, they are isoclinic, so that their representation theory is virtually the same. Hence, for our purpose, it is sufficient to study one of them. Throughout this paper, we will write $S^+(n)$ for $\hat{S}(n)$.

If H is a subgroup of $S(n)$, we let $H^+ = \pi^{-1}(H)$ and $H^- = \pi^{-1}(H \cap A(n))$. In particular, H^- has index 1 or 2 in H^+, and $H^+ = H^-$ if and only if $H \subset A(n)$. We define $S^-(n) = A(n)^- = A(n)^+$. Hence $S^-(n)$ is a central extension of $A(n)$ of degree 2.

The groups $A(6)$ and $A(7)$ also have one 6-fold cover each, which, together with the above groups, give all the covering groups of $S(n)$ and $A(n)$.

2.2 Characters, blocks and twisted central product

From now on, we fix an odd prime p. For any $H \leq S(n)$, the irreducible complex characters of H^ε fall into two categories: those that have z in their kernel, and which can be identified with those of H (if $\varepsilon = 1$) or those of $H \cap A(n)$ (if $\varepsilon = -1$), and those that don't have z in their kernel. These (faithful) characters are called *spin characters*. We denote by $SI(H^\varepsilon)$ the set of spin characters of H^ε, and we let $SI_0(H^\varepsilon) = SI(H^\varepsilon) \cap M(H^\varepsilon)$ (with the notation of Sect. 1).

If B is a p-block of H^ε then, because p is odd, it is known that either $B \cap SI(H^\varepsilon) = \emptyset$ or $B \subset SI(H^\varepsilon)$, in which case we say that B is a *spin block* of H^ε.

Any two $\chi, \psi \in \mathrm{Irr}(H^\varepsilon)$ are called *associate* if $\chi \uparrow^{H^+} = \psi \uparrow^{H^+}$ (if $\varepsilon = -1$) or if $\chi \downarrow_{H^-} = \psi \downarrow_{H^-}$ (if $\varepsilon = 1$). Then each irreducible character of H^ε has exactly 1 or 2 associate characters. If χ is itself its only associate, we say that χ is *self-associate* (written s.a.), we put $\chi^a = \chi$ and let $\sigma(\chi) = 1$. Otherwise, χ has a unique associate $\psi \neq \chi$; we say that χ is *non-self-associate* (written n.s.a.), we put $\chi^a = \psi$ and we let $\sigma(\chi) = -1$.

If $H^+ \neq H^-$, then $\chi \in \mathrm{Irr}(H^+)$ and $\varphi \in \mathrm{Irr}(H^-)$ are said to *correspond* if $\langle \chi, \varphi \uparrow^{H^+} \rangle_{H^+} \neq 0$. In this case, Clifford's theory implies that $\sigma(\chi) = -\sigma(\varphi)$.

If $H_1, H_2, \ldots, H_k \leq S(n)$ act (non-trivially) on disjoint subsets of $\{1, \ldots, n\}$, then one can define the *twisted central product* $H^+ = H_1^+ \hat{\times} \cdots \hat{\times} H_k^+ \leq S^+(n)$ (see [11] or [3]). Then $|H^+| = \frac{1}{2^{k-1}} |H_1^+| \cdot |H_2^+| \cdots |H_k^+| = 2|H_1| \cdot |H_2| \cdots |H_k|$. Also, one obtains $SI(H^+)$ from the $SI(H_i^+)$'s as follows:

Proposition 2.1 (See [11, §28]) *There is a surjective map*

$$\hat{\otimes}: \quad \begin{cases} SI(H_1^+) \times \cdots \times SI(H_k^+) \longrightarrow SI(H^+), \\ (\chi_1, \ldots, \chi_k) \longmapsto \chi_1 \hat{\otimes} \cdots \hat{\otimes} \chi_k, \end{cases}$$

which satisfies the following properties. Suppose $\chi_i, \psi_i \in SI(H_i^+)$ for $1 \leq i \leq k$. Then

(i) $\sigma(\chi_1\hat{\otimes}\cdots\hat{\otimes}\chi_k)=\sigma(\chi_1)\cdots\sigma(\chi_k)$, *and* $(\chi_1\hat{\otimes}\cdots\hat{\otimes}\chi_k)(1)=2^{\lfloor s/2\rfloor}\chi_1(1)\cdots\times\chi_k(1)$, *where s is the number of n.s.a. characters in* $\{\chi_1,\dots,\chi_k\}$ *and* $\lfloor\ \rfloor$ *denotes integral part.*

(ii) $\chi_1\hat{\otimes}\cdots\hat{\otimes}\chi_k$ *and* $\psi_1\hat{\otimes}\cdots\hat{\otimes}\psi_k$ *are associate if and only if* χ_i *and* ψ_i *are associate for all* i.

(iii) $\chi_1\hat{\otimes}\cdots\hat{\otimes}\chi_k=\psi_1\hat{\otimes}\cdots\hat{\otimes}\psi_k$ *if and only if* χ_i *and* ψ_i *are associate for all* i *and* $[\sigma(\chi_1)\cdots\sigma(\chi_k)=1]$ *or* $[\sigma(\chi_1)\cdots\sigma(\chi_k)=-1$ *and* $|\{i\mid\chi_i\neq\psi_i\}|$ *is even*$]$.

2.3 Partitions and bar-partitions

Just as the irreducible characters of $S(n)$ are parametrized by the partitions of n, the spin characters of $S^+(n)$ have a combinatorial description. We let $P(n)$ be the set of all partitions of n, and $P_0(n)$ be the subset of all partitions in distinct parts, also called *bar-partitions*. We write $\lambda\vdash n$ for $\lambda\in P(n)$, and $\lambda\succ n$ for $\lambda\in P_0(n)$. We also write, in both cases, $|\lambda|=n$.

It is well known that $\mathrm{Irr}(S(n))=\{\chi_\lambda,\lambda\vdash n\}$. For any $\lambda\vdash n$, we write $h(\lambda)$ for the product of all hook-lengths in λ. We then have $h(\lambda)=h_{\lambda,p}h_{\lambda,p'}$, where $h_{\lambda,p}$ (respectively, $h_{\lambda,p'}$) is the product of all hook-lengths divisible by p (prime to p, respectively) in λ. The Hook-Length Formula then gives $\chi_\lambda(1)=\frac{n!}{h(\lambda)}$.

If we remove all the hooks of length divisible by p in λ, we obtain its *p-core* $\lambda_{(p)}$. The information on p-hooks is stored in the *p-quotient* $\lambda^{(p)}$ of λ. If $n=pw+r$, with $\lambda_{(p)}\vdash r$, then $\lambda^{(p)}$ is a p-tuple of partitions of w, i.e., $\lambda^{(p)}=(\lambda^{(0)},\dots,\lambda^{(p-1)})$ and $|\lambda^{(0)}|+\cdots+|\lambda^{(p-1)}|=w$. The partition λ is uniquely determined by its p-core and p-quotient. Also, for any integer k, there exists a (canonical) bijection between the kp-hooks in λ and the k-hooks in $\lambda^{(p)}$ (i.e., in the $\lambda^{(i)}$'s).

Finally, the Nakayama Conjecture states that $\chi_\lambda,\chi_\mu\in\mathrm{Irr}(S(n))$ belong to the same p-block if and only if λ and μ have the same p-core.

We now present the analogue properties for bar-partitions and spin characters. For any bar-partition $\lambda=(a_1,\dots,a_m)$ of n, with $a_1>\cdots>a_m>0$, we let $m(\lambda)=m$, and define the *sign* of λ by $\sigma(\lambda)=(-1)^{n-m(\lambda)}$. We then have

Theorem 2.2 (See [11, §41]) *For each sign* $\varepsilon\in\{1,-1\}$, *there is a* (*canonical*) *surjective map* $f^\varepsilon:SI(S^\varepsilon(n))\longrightarrow P_0(n)$ *such that*:

(i) $\sigma(\chi)=\varepsilon\sigma(f^\varepsilon(\chi))$ *for all* $\chi\in SI(S^\varepsilon(n))$.

(ii) *For any* $\chi,\psi\in SI(S^\varepsilon(n))$, *we have* $f^\varepsilon(\chi)=f^\varepsilon(\psi)$ *if and only if* χ *and* ψ *are associate.*

(iii) *If* $\chi\in SI(S^+(n))$ *and* $\varphi\in SI(S^-(n))$, *then* $f^+(\chi)=f^-(\varphi)$ *if and only if* χ *and* φ *correspond.*

In particular, each $\lambda\succ n$ labels one s.a. character χ or two associate characters χ and χ^a. Throughout this paper, we will denote by $\langle\lambda\rangle$ the set of spin characters labeled by λ, and write (abusively) $\langle\lambda\rangle\in SI(S^\varepsilon(n))$, and $\langle\lambda\rangle(1)$ for the (common) degree of any spin character in $\langle\lambda\rangle$. We will also sometimes write $\langle\lambda\rangle_+$ to emphasize that $\langle\lambda\rangle\in SI(S^+(n))$ (and $\langle\lambda\rangle_-$ if $\langle\lambda\rangle\in SI(S^-(n))$).

For the following results on bars, cores and quotients, we refer to [9]. For any odd integer q, let $e=(q-1)/2$. We define a *$\bar{q}$-quotient of weight w* to be any tuple of partitions $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e)})$ such that $\lambda^{(0)} \in P_0(w_0)$, $\lambda^{(i)} \in P(w_i)$ for $1 \leq i \leq e$, and $w_0 + w_1 + \cdots + w_e = w$. We define its *sign* by $\sigma((\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e)})) = (-1)^{w-w_0}\sigma(\lambda^{(0)})$.

Now take any bar-partition $\lambda = (a_1, \dots, a_m)$ of n as above. The *bars* in λ can be read in the *shifted Young diagram* $S(\lambda)$ of λ. This is obtained from the usual Young diagram of λ by shifting the ith row $i-1$ positions to the right. The jth node in the ith row is called the (i, j)-node, and corresponds to the *bar* B_{ij}. The *bar-lengths* in the ith row are obtained by writing (from left to right in $S(\lambda)$) the elements of the following set in decreasing order: $\{1, 2, \dots, a_i\} \cup \{a_i + a_j \mid j > i\} \setminus \{a_i - a_j \mid j > i\}$. The bars are of three types:

- Type 1. These are bars B_{ij} with $i + j \geq m + 2$ (i.e., in the right part of $S(\lambda)$). They are ordinary hooks in $S(\lambda)$, and their lengths are the elements of $\{1, 2, \dots, a_i - 1\} \setminus \{a_i - a_j \mid j > i\}$.
- Type 2. These are bars B_{ij} with $i + j = m + 1$ (in particular, the corresponding nodes all belong to the same column of $S(\lambda)$). Their length is precisely a_i, and the bar is all of the ith row of $S(\lambda)$.
- Type 3. The lengths $\{a_i + a_j \mid j > i\}$ correspond to bars B_{ij} with $i + j \leq m$. The bar consists of the ith row together with the jth row of $S(\lambda)$.

Bars of type 1 and 2 are called *unmixed*, while those of type 3 are called *mixed*. The unmixed bars in λ correspond exactly to the hooks in the partition λ^*, which admits as a β-set the set of parts of λ.

For any $\lambda \succ n$, we write $\bar{h}(\lambda)$ for the product of all bar-lengths in λ. We then have $\bar{h}(\lambda) = \bar{h}_{\lambda,p}\bar{h}_{\lambda,p'}$, where $\bar{h}_{\lambda,p}$ (respectively, $\bar{h}_{\lambda,p'}$) is the product of all bar-lengths divisible by p (prime to p, respectively) in λ. We then have the following analogue of the Hook-Length Formula (proved by A.O. Morris [7, Theorem 1])

$$\langle\lambda\rangle(1) = 2^{\lfloor (n-m(\lambda))/2 \rfloor} \frac{n!}{\bar{h}(\lambda)}.$$

If we remove all the bars of length divisible by p in λ, we obtain its $\bar{p}$-*core* $\lambda_{(\bar{p})}$ (which is still a bar-partition), and its $\bar{p}$-*quotient* $\lambda^{(\bar{p})}$. If $n = pw + r$, with $\lambda_{(\bar{p})} \succ r$, then $\lambda^{(\bar{p})}$ is a $\bar{p}$-quotient of weight w in the sense defined above. The bar-partition λ is uniquely determined by its $\bar{p}$-core and $\bar{p}$-quotient. Also, for any integer k, there exists a canonical bijection between the set of kp-bars in λ and the set of k-bars in $\lambda^{(\bar{p})}$ (where a k-bar in $\lambda^{(\bar{p})} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{((p-1)/2)})$ is a k-bar in $\lambda^{(0)}$ or a k-hook in one of $\lambda^{(1)}, \dots, \lambda^{((p-1)/2)}$).

The distribution of the spin characters of $S^+(n)$ into spin blocks was first conjectured for p odd by Morris. It was first proved by J.F. Humphreys in [4], then differently by M. Cabanes, who also determined the structure of the defect groups of spin blocks (see [1]).

Proposition 2.3 *Let $\chi, \psi \in SI(S^\varepsilon(n))$ and p be an odd prime. Then χ is of p-defect* 0 *if and only if $f^\varepsilon(\chi)$ is a $\bar{p}$-core. If $f^\varepsilon(\chi)$ is not a $\bar{p}$-core, then χ and ψ belong to the same p-block if and only if $f^\varepsilon(\chi)_{(\bar{p})} = f^\varepsilon(\psi)_{(\bar{p})}$.*

One can therefore define the $\bar{p}$-core of a spin block B and its *weight* $w(B)$, as well as its *sign* $\delta(B) = \sigma(f^\varepsilon(\chi)_{(\bar{p})})$ (for any $\chi \in B$). We then have

Proposition 2.4 (See [1]) *If B is a spin block of $S^\varepsilon(n)$ of weight w, then a defect group X of B is a Sylow p-subgroup of $S^\varepsilon(pw)$.*

2.4 Removal of p-bars

The following result is the bar-analogue of [2, Lemma 3.2]; it describes how the removal of p-bars affects the product of p'-bar-lengths.

Proposition 2.5 *Suppose $\lambda \succ n$ has $\bar{p}$-core $\lambda_{(\bar{p})}$. Then*

$$\bar{h}_{\lambda,p'} \equiv \pm 2^{-a(\lambda)} \bar{h}_{\lambda_{(\bar{p})},p'} = \pm 2^{-a(\lambda)} \bar{h}(\lambda_{(\bar{p})}) \pmod p,$$

where $a(\lambda)$ is the number of p-bars of type 3 to remove from λ to get $\lambda_{(\bar{p})}$.

Proof Let B_{ij} be a p-bar in λ and $\lambda - B_{ij}$ be the bar-partition obtained from λ by removing B_{ij}. We distinguish two cases, depending on whether B_{ij} is unmixed or mixed.

First suppose that B_{ij} is unmixed (i.e., $i + j > m(\lambda)$). We start by examining the unmixed p'-bars in λ and $\lambda - B_{ij}$. These correspond, in the notation above, to the p'-hooks in λ^* and $(\lambda - B_{ij})^*$, respectively (considering λ and $\lambda - B_{ij}$ as β-sets). The set of parts of λ is $X = \{a_1, \dots, a_m\}$, and the set of non-zero parts of $\lambda - B_{ij}$ is $Y = \{a_1, \dots, a_{i-1}, a_i - p, a_{i+1}, \dots, a_m\}$ (or $Y = \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m\}$ if $a_i = p$). The p'-hooks in λ^* (resp., $(\lambda - B_{ij})^*$) therefore correspond to pairs (x, y) with $0 \le x < y$, $(y - x, p) = 1$, and $x \notin X$, $y \in X$ (resp., $x \notin Y$, $y \in Y$).

If B_{ij} is of type 1 (i.e., $i + j > m(\lambda) + 1$), then $a_i - p > 0$, so that $|Y| = |X|$ and $(\lambda - B_{ij})^* = \lambda^* - h$ for some p-hook h in λ. In this case, we are thus exactly in the same context as [2, Lemma 3.2], and we get $h_{\lambda^*,p'} \equiv \pm h_{\lambda^*-h,p'} = \pm h_{(\lambda-B_{ij})^*,p'} \pmod p$. Note that the result of [2, Lemma 3.2] is, in fact, incorrect, as the right hand side should be multiplied by $(-1)^{\mu/\kappa}$, where μ/κ is the *relative sign* associated to μ and κ. The mistake is to be found in the proof, where the leg-length L_h of the hook removed should appear (four lines before the end), yielding, in our case, $h_{\lambda^*,p'} \equiv (-1)^{L_h+1} h_{\lambda^*-h,p'} \pmod p$.

If, on the other hand, B_{ij} is of type 2 (i.e., $i + j = m(\lambda) + 1$), then $a_i - p = 0$, and $Y = X \setminus \{p\}$. Note that, in this case, Y is not a β-set for a partition of $|\lambda^*| - p$, while $Y \cup \{0\}$ is. The p'-hooks in $(\lambda - B_{ij})^*$ correspond to either pairs (x, y) with $y \ne a_i$, which also correspond to p'-hooks in λ^*, or to pairs (p, y), with $y > p$ and $y \in X$. These new hooks have lengths $(a_1 - p), \dots, (a_{i-1} - p)$. Finally, some hooks have disappeared: those corresponding to pairs (x, p) with $x < p$ and $x \notin X$. These have lengths $(p - x)$, for $0 \le x < p$ and $x \notin \{a_{i+1}, \dots, a_m\}$.

We now turn to the mixed p'-bars in λ and $\lambda - B_{ij}$. Suppose first that B_{ij} is of type 1. Then $m(\lambda) = m(\lambda - B_{ij})$. Suppose that

$$a_1 > \cdots > a_{i-1} > a_{i+1} > \cdots > a_k > a_i - p > a_{k+1} > \cdots > a_m.$$

To prove the result, we can simply ignore the bar-lengths which are common to λ and $\lambda - B_{ij}$. The mixed bars which disappear when going from λ to $\lambda - B_{ij}$ have lengths

$$(a_1 + a_i), (a_2 + a_i), \dots, (a_{i-1} + a_i) \quad \text{and}$$

$$(a_i + a_{i+1}), (a_i + a_{i+2}), \ldots, (a_i + a_m).$$

The mixed bars which appear have lengths

$$(a_1 + a_i - p), (a_2 + a_i - p), \ldots, (a_{i-1} + a_i - p),$$

$$(a_{i+1} + a_i - p), \ldots, (a_k + a_i - p) \quad \text{and} \quad (a_i - p + a_{k+1}), \ldots, (a_i - p + a_m).$$

If we then just consider the lengths not divisible by p, it is easy to see that we can pair the bars disappearing with those appearing. The pairs are of the form (b, b'), where b is a bar in λ and b' is a bar in $\lambda - B_{ij}$, and $|b'| = |b| - p$. We thus get, in this case,

$$\prod_{b \text{ mixed } p'\text{-bar in } \lambda} |b| \equiv \prod_{b' \text{ mixed } p'\text{-bar in } \lambda - B_{ij}} |b'| \pmod{p}.$$

Together with the equality obtained above for unmixed p'-bars, we obtain that, if B_{ij} is a p-bar of type 1 in λ, then $\bar{h}_{\lambda,p'} \equiv \pm \bar{h}_{\lambda - B_{ij},p'} \pmod{p}$.

Now suppose that B_{ij} is of type 2, i.e., $a_i = p$. Then the mixed bars which disappear when going from λ to $\lambda - B_{ij}$ have lengths $(a_1 + p), (a_2 + p), \ldots, (a_{i-1} + p)$ (call these A) and $(p + a_{i+1}), (p + a_{i+2}), \ldots, (p + a_m)$ (call these B), while no new mixed bar appears.

The bars disappearing in A are compensated for by the hooks appearing in $(\lambda - B_{ij})^*$ in the study of unmixed bars above (since $a_u + p \equiv a_u - p \pmod{p}$ for all $1 \le u \le i - 1$, the p'-parts are congruent mod p when these are not divisible by p).

On the other hand, since $0 < a_m < \cdots < a_{i+1} < a_i = p$, all the bar-lengths in B are coprime to p, and their product is

$$(p + a_{i+1})(p + a_{i+2}) \cdots (p + a_m) \equiv a_{i+1} a_{i+2} \cdots a_m \pmod{p}.$$

Now the hooks disappearing in the above discussion of unmixed bars all have length prime to p except one (corresponding to $x = 0$). The product of the lengths prime to p is thus

$$\prod_{0 < x < p, x \notin \{a_i+1, \ldots, a_m\}} (p - x) \equiv (-1)^{p-1-m+i} \prod_{0 < x < p, x \notin \{a_i+1, \ldots, a_m\}} x \pmod{p}.$$

Hence the product of the p'-hook-lengths disappearing and the p'-bar-lengths in B is congruent (mod p) to

$$(-1)^{p-1-m+i} \prod_{0<y<p} y = (-1)^{p-1-m+i}(p-1)! \equiv (-1)^{p-m+i} \pmod{p}$$

(by Wilson's Theorem). Finally, we obtain that, if B_{ij} is a p-bar of type 2 in λ, then $\bar{h}_{\lambda,p'} \equiv (-1)^{p-m+i} \bar{h}_{\lambda - B_{ij},p'} \pmod{p}$.

We now suppose that B_{ij} is a p-bar of type 3 in λ, i.e., $i < j$, $a_i > a_j$ and $a_i + a_j = p$. The set of parts of λ is $X = \{a_1, \ldots, a_m\}$ and the set of parts of $\lambda - B_{ij}$ is $Y = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m\}$. Ignoring as before the bars which are common to λ and $\lambda - B_{ij}$, we see that the unmixed bars which disappear from λ to $\lambda - B_{ij}$ have lengths

$$(a_i - x)\ \big(0 \le x < a_i, x \notin \{a_m, \dots, a_{i+1}\}\big) \quad \text{and}$$
$$(a_j - x)\ \big(0 \le x < a_j, x \notin \{a_m, \dots, a_{j+1}\}\big),$$

while those appearing have lengths

$$(a_1 - a_i), \dots, (a_{i-1} - a_i), (a_1 - a_j), \dots, (a_{i-1} - a_j) \quad \text{and}$$
$$(a_{i+1} - a_j), \dots, (a_{j-1} - a_j).$$

On the other hand, there is no mixed bar appearing, while the mixed bars disappearing have lengths

$$(a_1 + a_i), \dots, (a_{i-1} + a_i) \quad (\text{rows } 1, \dots, i-1, \text{column } i),$$

$$(a_i + a_{i+1}), \dots, (a_i + a_{j-1}), (a_i + a_j), \dots, (a_i + a_m) \quad (\text{row } i),$$

$$(a_1 + a_j), \dots, (a_{i-1} + a_j) \quad (\text{rows } 1, \dots, i-1, \text{column } j),$$

$$(a_{i+1} + a_j), \dots, (a_{j-1} + a_j) \quad (\text{rows } i+1, \dots, j-1, \text{column } j), \quad \text{and}$$

$$(a_j + a_{j+1}), \dots, (a_j + a_m) \quad (\text{row } j).$$

Now, since $a_i + a_j = p$, we have, for any $1 \le k \le m$,

$$a_k - a_i \equiv a_k + a_j \pmod p \quad \text{and} \quad a_k + a_i \equiv a_k - a_j \pmod p.$$

In particular, $a_k - a_i$ (resp., $a_k + a_i$) is coprime to p if and only if $a_k + a_j$ (resp., $a_k - a_j$) is coprime to p, and, in that case,

$$(a_k \pm a_i)_{p'} = a_k \pm a_i \equiv a_k \mp a_j \equiv (a_k \mp a_j)_{p'} \pmod p.$$

We thus have the following compensations between the appearing unmixed bars and the appearing mixed bars:

$$(a_1 - a_i), \dots, (a_{i-1} - a_i) \longleftrightarrow (a_1 + a_j), \dots, (a_{i-1} + a_j)$$
$$(a_1 - a_j), \dots, (a_{i-1} - a_j) \longleftrightarrow (a_1 + a_i), \dots, (a_{i-1} + a_i) \quad \text{and}$$
$$(a_{i+1} - a_j), \dots, (a_{j-1} - a_j) \longleftrightarrow (a_i + a_{i+1}), \dots, (a_i + a_{j-1}).$$

This accounts for all the appearing (unmixed) bars, and we're left exactly with the following disappearing bar-lengths:

$$(a_i - x)\ \big(0 \le x < a_i, x \notin \{a_m, \dots, a_{i+1}\}\big) \quad \text{unmixed of type 1},$$

$$(a_j - x)\ \big(0 \le x < a_j, x \notin \{a_m, \dots, a_{j+1}\}\big) \quad \text{unmixed of type 2},$$

$$(a_{i+1}+a_j),\dots,(a_{j-1}+a_j),(a_j+a_{j+1}),\dots,(a_j+a_m) \quad \text{mixed of type 1},$$

$$(a_i+a_{j+1}),\dots,(a_i+a_m) \quad \text{mixed of type 2},$$

and $(a_i+a_j)=p$ which can thus be ignored.

Now, for any $i+1\le k\le m$, $a_j+a_k=p-a_i+a_k\equiv-(a_i-a_k) \pmod p$, and, for $j+1\le k\le m$, $a_i+a_k\equiv-(a_j-a_k) \pmod p$. Hence, taking the product, we obtain (modulo p):

$$\prod_{0\le x<a_i, x\ne a_j}(a_i-x)=\frac{a_i!}{a_i-a_j}\ \text{(type 1)} \quad \text{and} \quad \prod_{0\le x<a_j}(a_j-x)=a_j!\ \text{(type 2)}.$$

Now $a_j!=1\cdot2\cdots a_j=(-1)^{a_j}(-1)\cdots(-a_j)\equiv(-1)^{a_j}(p-1)\cdots(p-a_j) \pmod p$, so that $a_j!\equiv(-1)^{a_j}(p-1)\cdots(a_i+1)a_i \pmod p$. We thus have, disappearing,

$$\pm\frac{a_i a_i!(a_i+1)\cdots(p-1)}{a_i-a_j}\equiv\pm\frac{a_i}{a_i-a_j}(p-1)!\equiv\mp\frac{a_i}{a_i-a_j} \pmod p$$

(this last equality being true by Wilson's Theorem).

Finally, $a_i-a_j=a_i-(p-a_i)\equiv-2a_i \pmod p$, yielding a total of $\pm2^{-1} \pmod p$ disappearing (since, p being odd, 2 is invertible (mod p), and $a_i<p$ so that we can simplify by a_i). We thus get that, if B_{ij} is a p-bar of type 3 in λ, then $\bar{h}_{\lambda,p'}\equiv\pm\frac{1}{2}\bar{h}_{\lambda-B_{ij},p'} \pmod p$.

Iterating the above results on all the p-bars to remove from λ to get to its $\bar{p}$-core $\lambda_{(\bar{p})}$, we finally obtain the desired equality, writing $a(\lambda)$ for the number of p-bars of type 3 to remove:

$$\bar{h}_{\lambda,p'}\equiv\pm2^{-a(\lambda)}\bar{h}_{\lambda_{(\bar{p})},p'}=\pm2^{-a(\lambda)}\bar{h}(\lambda_{(\bar{p})}) \pmod p$$

(since all the bars in $\lambda_{(\bar{p})}$ have length coprime to p). □

2.5 $\bar{p}$-Core tower, $\bar{p}$-quotient tower and characters of p'-degree

In this section, we want to obtain an expression for the (value modulo p of the) p'-part of the degree of a spin character. We start by describing the $\bar{p}$-core tower of a bar-partition, introduced by Olsson in [9].

Take any $\lambda\succ n$. the $\bar{p}$-core tower of λ has rows $R_0^\lambda, R_1^\lambda, R_2^\lambda,\dots$, where the ith row R_i^λ contains one $\bar{p}$-core and $(p^i-1)/2$ p-cores (in particular, one can consider R_i^λ as a $\bar{p}^i$-quotient). We have $R_0^\lambda=\{\lambda_{(\bar{p})}\}$ (the $\bar{p}$-core of λ). If the $\bar{p}$-quotient of λ is $\lambda^{(\bar{p})}=(\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(e)})$ (where $e=(p-1)/2$), then $R_1^\lambda=\{\lambda^{(0)}_{(\bar{p})},\lambda^{(1)}_{(p)},\dots,\lambda^{(e)}_{(p)}\}$. Writing $\lambda^{(0)(\bar{p})}=(\lambda^{(0,0)},\lambda^{(0,1)},\dots,\lambda^{(0,e)})$ the $\bar{p}$-quotient of $\lambda^{(0)}$ and $\lambda^{(i)(p)}=(\lambda^{(i,1)},\lambda^{(i,2)},\dots,\lambda^{(i,p)})$ the p-quotient of $\lambda^{(i)}$ $(1\le i\le e)$, and taking cores, we let

$$R_2^\lambda=\{\lambda^{(0,0)}_{(\bar{p})},\lambda^{(0,1)}_{(p)},\dots,\lambda^{(0,e)}_{(p)},\lambda^{(1,1)}_{(p)},\dots,\lambda^{(1,p)}_{(p)},\lambda^{(2,1)}_{(p)},\dots,\lambda^{(e,p)}_{(p)}\}.$$

Continuing in this way, we obtain the $\bar{p}$-core tower of λ. We define the $\bar{p}$-quotient tower of λ in a similar fashion: it has rows $Q_0^\lambda, Q_1^\lambda, Q_2^\lambda,\dots$, where the ith row

Q_i^λ contains one $\bar{p}$-quotient and $(p^i-1)/2$ p-quotients (in particular, Q_i^λ can be seen as a $\bar{p}^{i+1}$-quotient). With the above notation, we have $Q_0^\lambda = \{\lambda^{(\bar{p})}\}$, $Q_1^\lambda = \{\lambda^{(0)(\bar{p})}, \lambda^{(1)(p)}, \dots, \lambda^{(e)(p)}\}$ and

$$Q_2^\lambda = \{\lambda^{(0,0)(\bar{p})}, \lambda^{(0,1)(p)}, \dots, \lambda^{(0,e)(p)}, \lambda^{(1,1)(p)}, \dots, \lambda^{(1,p)(p)}, \lambda^{(2,1)(p)}, \dots, \lambda^{(e,p)(p)}\}.$$

The following result will be useful later.

Lemma 2.6 *If $\lambda \succ n$ has $\bar{p}$-core tower $(R_0^\lambda, R_1^\lambda, \dots, R_m^\lambda)$, then $\sigma(\lambda) = \prod_{i=0}^m \sigma(R_i^\lambda)$.*

Proof We have $\sigma(\lambda) = \sigma(\lambda_{(\bar{p})})\sigma(\lambda^{(\bar{p})})$, and $\sigma(\lambda_{(\bar{p})}) = \sigma(R_0^\lambda)$.

Also, $\sigma(\lambda^{(\bar{p})}) = \sigma(\lambda^{(0)})(-1)^{\sum_{i\geq 1}|\lambda^{(i)}|}$, and

$$\sigma(\lambda^{(0)}) = \sigma(\lambda^{(0)}_{(\bar{p})})\sigma(\lambda^{(0)(\bar{p})}) = \sigma(Q_0^\lambda) = \sigma(\lambda^{(0)}_{(\bar{p})})\sigma(\lambda^{(0,0)})(-1)^{\sum_{j\geq 1}|\lambda^{(0,j)}|}.$$

Now $\sigma(R_1^\lambda) = \sigma(\lambda^{(0)(\bar{p})})(-1)^{\sum_{i\geq 1}|\lambda^{(i)}_{(\bar{p})}|}$ and $\sigma(Q_1^\lambda) = \sigma(\lambda^{(0,0)})(-1)^{\sum_{i\geq 0, j\geq 1}|\lambda^{(i,j)}|}$, so that

$$\begin{aligned}\sigma(R_1^\lambda)\sigma(Q_1^\lambda) &= \sigma(\lambda^{(0)(\bar{p})})\sigma(\lambda^{(0,0)})(-1)^{\sum_{i\geq 1}|\lambda^{(i)}_{(\bar{p})}| + \sum_{i\geq 0, j\geq 1}|\lambda^{(i,j)}|} \\ &= \sigma(\lambda^{(0)})(-1)^{\sum_{i\geq 1}|\lambda^{(i)}_{(\bar{p})}| + \sum_{i,j\geq 1}|\lambda^{(i,j)}|}.\end{aligned}$$

However, for each $i \geq 1$, we have $|\lambda^{(i)}_{(\bar{p})}| + \sum_{j\geq 1}|\lambda^{(i,j)}| \equiv |\lambda^{(i)}_{(\bar{p})}| + p\sum_{j\geq 1}|\lambda^{(i,j)}|$ (mod 2) (since p is odd), and $|\lambda^{(i)}_{(\bar{p})}| + p\sum_{j\geq 1}|\lambda^{(i,j)}| = |\lambda^{(i)}|$. We therefore get

$$\sigma(R_1^\lambda)\sigma(Q_1^\lambda)\sigma(\lambda^{(0)})(-1)^{\sum_{i\geq 1}|\lambda^{(i)}|} = \sigma(\lambda^{(\bar{p})}) = \sigma(Q_0^\lambda).$$

Finally, we have $\sigma(\lambda) = \sigma(R_0^\lambda)\sigma(Q_0^\lambda)$, and $\sigma(Q_0^\lambda) = \sigma(R_1^\lambda)\sigma(Q_1^\lambda)$, whence $\sigma(\lambda) = \sigma(R_0^\lambda)\sigma(R_1^\lambda)\sigma(Q_1^\lambda)$. Iterating this process, we deduce the result. □

Now, writing $\beta_i(\lambda)$ for the sum of the cardinalities of the partitions in R_i^λ, one shows easily that $|\lambda| = \sum_{i\geq 0}\beta_i(\lambda)p^i$ (see [9]). Also, one gets the following bar-analogue of [2, Proposition 1.1]:

Proposition 2.7 [9, Proposition 3.1] *In the above notation,*

$$\nu_p(\bar{h}(\lambda)) = \frac{n - \sum_{i\geq 0}\beta_i(\lambda)}{p-1}.$$

In particular, $\langle\lambda\rangle$ has p'-degree if and only if $\sum_{i\geq 0}\beta_i(\lambda)p^i$ is the p-adic decomposition of n.

Let $n = \sum_{i=0}^k t_i p^i$ be the p-adic decomposition of n. For each $0 \leq i \leq k$, let $e_i = (p^i-1)/2$, and write $R_i^\lambda = \{\mu_i^{(0)}, \mu_i^{(1)}, \dots, \mu_i^{(e_i)}\}$ and $Q_i^\lambda = \{\lambda_i^{(0)}, \lambda_i^{(1)}, \dots, \lambda_i^{(e_{i+1})}\}$. Note that $Q_k^\lambda = \{\emptyset, \dots, \emptyset\}$.

We let $\bar{h}(R_i^\lambda)=\bar{h}(\mu_i^{(0)})\prod_{j=1}^{e_i}h(\mu_i^{(j)})$, and $\bar{h}(Q_i^\lambda)=\bar{h}(\lambda_i^{(0)})\prod_{j=1}^{e_{i+1}}h(\lambda_i^{(j)})$, and we let $m_i^{(0)}=m(\mu_i^{(0)})$ and $\beta_i=\beta_i(\lambda)$.

Proposition 2.8 *With the above notation, we have, for any $\lambda\succ n$,*

$$\frac{|S^+(n)|_{p'}}{\langle\lambda\rangle(1)_{p'}}\equiv\pm\frac{2}{2^{\lfloor\frac{S}{2}\rfloor}}\prod_{i=0}^{k}\frac{1}{2^{\lfloor(\beta_i-m_i^{(0)})/2\rfloor}}\bar{h}(R_i^\lambda)\ (\mathrm{mod}\ p),$$

where $S=|\{0\le i\le k;\ \beta_i-m_i^{(0)}\ odd\}|$.

Proof We have

$$\frac{|S^+(n)|_{p'}}{\langle\lambda\rangle(1)_{p'}}=\frac{2}{2^{\lfloor(n-m(\lambda))/2\rfloor}}\bar{h}(\lambda)_{p'}.$$

Now $\bar{h}(\lambda)_{p'}=\bar{h}_{\lambda,p'}(\bar{h}_{\lambda,p})_{p'}=\bar{h}_{\lambda,p'}(\bar{h}(Q_0^\lambda)_{p'})$ (since there is a bijection between the set of bars divisible by p in λ and the set of bars in the quotient Q_0^λ).

By Proposition 2.5, we have $\bar{h}_{\lambda,p'}\equiv\pm2^{-a(\lambda)}\bar{h}(\lambda_{(\bar{p})})\equiv\pm2^{-a(\lambda)}\bar{h}(R_0^\lambda)\ (\mathrm{mod}\ p)$. Also,

$$\begin{aligned}\bar{h}\big(Q_0^\lambda\big)_{p'}&=\bar{h}_{Q_0^\lambda,p'}(\bar{h}_{Q_0^\lambda,p})_{p'}\\&=\bar{h}_{\lambda_0^{(0)},p'}h_{\lambda_0^{(1)},p'}\cdots h_{\lambda_0^{(e_1)},p'}\bar{h}\big(Q_1^\lambda\big)_{p'}\\&\equiv\pm2^{-a(\lambda_0^{(0)})}\bar{h}_{\mu_1^{(0)}}h_{\mu_1^{(1)}}\cdots h_{\mu_1^{(e_1)}}\bar{h}\big(Q_1^\lambda\big)_{p'}\ (\mathrm{mod}\ p),\end{aligned}$$

this last equality holding by 2.5 (applied to $\lambda_0^{(0)}$) and by [2, Lemma 3.2] (applied to $\lambda_0^{(1)},\dots,\lambda_0^{(e_1)}$). We thus get $\bar{h}(Q_0^\lambda)_{p'}\equiv\pm2^{-a(\lambda_0^{(0)})}\bar{h}(R_1^\lambda)\bar{h}(Q_1^\lambda)_{p'}\ (\mathrm{mod}\ p)$, and

$$\bar{h}(\lambda)_{p'}\equiv\pm2^{-a(\lambda)-a(\lambda_0^{(0)})}\bar{h}\big(R_0^\lambda\big)\bar{h}\big(R_1^\lambda\big)\bar{h}\big(Q_1^\lambda\big)_{p'}\ (\mathrm{mod}\ p).$$

Iterating this, until we get to $Q_k^\lambda=\{\emptyset,\dots,\emptyset\}$, we obtain

$$\bar{h}(\lambda)_{p'}\equiv\pm2^{-\lfloor a(\lambda)+a(\lambda_0^{(0)})+a(\lambda_1^{(0)})+\cdots+a(\lambda_{k-1}^{(0)})\rfloor}\prod_{i=0}^{k}\bar{h}\big(R_i^\lambda\big)\ (\mathrm{mod}\ p).$$

On the other hand, repeated use of [9, Corollary 2.6] yields

$$\begin{aligned}m(\lambda)&=m_0^{(0)}+m\big(\lambda_0^{(0)}\big)+2a(\lambda)\\&=m_0^{(0)}+m_1^{(0)}+m\big(\lambda_1^{(0)}\big)+2a(\lambda)+2a\big(\lambda_0^{(0)}\big)\\&=(\cdots)\\&=m_0^{(0)}+m_1^{(0)}+\cdots+m_k^{(0)}+2\big(a(\lambda)+a\big(\lambda_0^{(0)}\big)+\cdots+a\big(\lambda_{k-1}^{(0)}\big)\big),\end{aligned}$$

so that

$$\left\lfloor \frac{(n-m(\lambda))}{2} \right\rfloor = \left\lfloor \frac{(n-m_1^{(0)}-\cdots-m_k^{(0)})}{2} - \left(a(\lambda)+a\left(\lambda_0^{(0)}\right)+\cdots+a\left(\lambda_{k-1}^{(0)}\right)\right) \right\rfloor$$
$$= \left\lfloor \frac{(n-(m_1^{(0)}+\cdots+m_k^{(0)})}{2} \right\rfloor - \left(a(\lambda)+a\left(\lambda_0^{(0)}\right)+\cdots+a\left(\lambda_{k-1}^{(0)}\right)\right).$$

Together with the expression we obtained for $\bar{h}(\lambda)_{p'}$, this gives

$$\frac{|S^+(n)|_{p'}}{\langle\lambda\rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor (n-m_0^{(0)}-\cdots-m_k^{(0)})/2\rfloor}} \prod_{i=0}^{k} \bar{h}\left(R_i^\lambda\right) \pmod p.$$

Now recall that $n=\sum_{i=0}^k \beta_i p^i$. Also, for any $1\le i\le k$, we have

$$\left\lfloor \frac{\beta_i p^i - m_i^{(0)}}{2} \right\rfloor = \left\lfloor \frac{\beta_i(p^i-1)}{2} + \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor$$
$$= \left\lfloor \frac{(p-1)\beta_i(1+p+\cdots+p^{i-1})}{2} + \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor$$
$$= \frac{(p-1)}{2}\beta_i\left(1+p+\cdots+p^{i-1}\right) + \left\lfloor \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor,$$

and

$$2^{\frac{(p-1)}{2}\beta_i(1+p+\cdots+p^{i-1})} = (2^{\frac{(p-1)}{2}})^{\beta_i(1+\cdots+p^{i-1})} \equiv (-1)^{\beta_i(1+\cdots+p^{i-1})} \equiv \pm 1 \pmod p.$$

Hence

$$2^{\lfloor \frac{n-(m_0^{(0)}+\cdots+m_k^{(0)})}{2} \rfloor} = 2^{\lfloor \frac{\sum_{i=0}^k(\beta_i p^i - m_i^{(0)})}{2} \rfloor}$$
$$= 2^{\lfloor \sum_{i=1}^k \frac{p-1}{2}\beta_i(1+\cdots+p^{i-1}) + \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor}$$
$$= 2^{\sum_{i=1}^k \frac{p-1}{2}\beta_i(1+\cdots+p^{i-1}) + \lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor}$$
$$\equiv \pm 2^{\lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor} \pmod p.$$

Now

$$\left\lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor = \left\lfloor \sum_{i=0,\beta_i - m_i^{(0)} \text{ even}}^{k} \frac{\beta_i - m_i^{(0)}}{2} + \sum_{i=0,\beta_i - m_i^{(0)} \text{ odd}}^{k} \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor$$
$$= \sum_{i=0,\beta_i - m_i^{(0)} \text{ even}}^{k} \left\lfloor \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor + \left\lfloor \sum_{i=0,\beta_i - m_i^{(0)} \text{ odd}}^{k} \frac{\beta_i - m_i^{(0)}}{2} \right\rfloor$$

and we have $\lfloor\sum_{i=0,\beta_i-m_i^{(0)}\text{ odd}}^k \frac{\beta_i-m_i^{(0)}}{2}\rfloor = \lfloor\frac{S}{2}\rfloor + \sum_{i=0,\beta_i-m_i^{(0)}\text{ odd}}^k \lfloor\frac{\beta_i-m_i^{(0)}}{2}\rfloor$, where $S = |\{0 \le i \le k;\, \beta_i - m_i^{(0)} \text{ odd}\}|$. We finally obtain

$$\frac{|S^+(n)|_{p'}}{\langle\lambda\rangle(1)_{p'}} \equiv \pm\frac{2}{2^{\lfloor\frac{S}{2}\rfloor}}\prod_{i=0}^{k}\frac{1}{2^{\lfloor(\beta_i-m_i^{(0)})/2\rfloor}}\bar{h}\big(R_i^\lambda\big) \pmod p. \qquad \square$$

3 Reduction theorem

In this section, we show that, in order to prove Conjecture 1.1 for any spin block B of $S^\varepsilon(n)$ of positive weight w, it is enough to prove it for the principal spin block of $S^+(pw)$ (i.e., that with empty $\bar{p}$-core). Our main tool to navigate between $S^+(n)$ and $S^-(n)$ is the strong duality that exists between their spin blocks.

Let $H \le S(n)$. A block of H^ε is called *proper* if it contains both an s.a. character and an n.s.a. character. By [10, 2.1], any spin block of $S^\varepsilon(n)$ of positive weight is proper. Now, if B is a proper block of H^ε, then $H^\varepsilon \ne H^{-\varepsilon}$, and there exists a unique block B^* of $H^{-\varepsilon}$ covering B (if $\varepsilon = -1$) or covered by B (if $\varepsilon = 1$), and B^* is also proper. We say that B and B^* are *(dual) corresponding blocks*. Finally, if B is proper, then it follows that B consists of s.a. characters and pairs of n.s.a. characters. In particular, we can still write (abusively) $\langle\lambda\rangle \in B$ or $\langle\lambda\rangle \in M(B)$. Also, for any sign ε, if $\langle\lambda\rangle_\varepsilon \in SI(S^\varepsilon(n))$, then we call $\langle\lambda\rangle_{-\varepsilon} \in SI(S^{-\varepsilon}(n))$ the *dual correspondent* of $\langle\lambda\rangle_\varepsilon$.

3.1 Preliminaries: the case $\varepsilon = 1$

Let B be a spin block of $S^+(n)$ of weight $w = w(B) > 0$ and sign $\delta = \delta(B)$, and let B_0 be the principal spin block of $S^\delta(pw)$. Let $r = n - wp$. Let μ be the $\bar{p}$-core of B, so that $\sigma(\mu) = \delta$. The characters in B are indexed by the $\bar{p}$-quotients of weight w. For any bar-partition λ with $\bar{p}$-core μ, we denote the $\bar{p}$-quotient of λ by $\lambda^{(\bar{p})}$ (so that $\sigma(\lambda) = \delta\sigma(\lambda^{(\bar{p})})$), and we let $\tilde{\lambda}$ be the bar-partition of wp with empty $\bar{p}$-core, and $\bar{p}$-quotient $\lambda^{(\bar{p})}$.

Lemma 3.1 *If $\delta = 1$, then, with the above notation, $\lambda \longmapsto \tilde{\lambda}$ induces a sign-preserving bijection $\mathcal{I}$ between B and B_0 which is also height-preserving. Furthermore,*

$$\langle\lambda\rangle(1)_{p'} \equiv \pm\frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}}\langle\mu\rangle(1)_{p'}\langle\tilde{\lambda}\rangle(1)_{p'} \pmod p.$$

Proof Since $\delta = 1$, we have $\langle\tilde{\lambda}\rangle \in B_0$ and $\sigma(\langle\lambda\rangle) = \sigma(\lambda) = \sigma(\tilde{\lambda}) = \sigma(\langle\tilde{\lambda}\rangle)$, so that $\mathcal{I} : \langle\lambda\rangle \longmapsto \langle\tilde{\lambda}\rangle$ is sign preserving, and therefore gives a bijection between B and B_0.

Now, for any $\langle\lambda\rangle \in B$, we have

$$\langle\lambda\rangle(1) = 2^{\lfloor(n-m(\lambda))/2\rfloor}\frac{n!}{\bar{h}(\lambda)} \quad\text{and}\quad \langle\tilde{\lambda}\rangle(1) = 2^{\lfloor(wp-m(\tilde{\lambda}))/2\rfloor}\frac{(wp)!}{\bar{h}(\tilde{\lambda})}.$$

In particular, since B and B_0 have a common defect group X (which is a Sylow p-subgroup of $S(pw)$ and $S^+(pw)$), the heights of $\langle\lambda\rangle$ and $\langle\tilde{\lambda}\rangle$ are

$$\mathfrak{h}\big(\langle\lambda\rangle\big)=\nu\big(|X|\big)-\nu\big(\bar{h}(\lambda)\big) \quad \text{and} \quad \mathfrak{h}\big(\langle\tilde{\lambda}\rangle\big)=\nu\big(|X|\big)-\nu\big(\bar{h}(\tilde{\lambda})\big), \text{ respectively.}$$

Now, using the notation of Sect. 2, we have $\bar{h}(\lambda)=\bar{h}_{\lambda,p}\bar{h}_{\lambda,p'}$ and $\bar{h}(\tilde{\lambda})=\bar{h}_{\tilde{\lambda},p}\bar{h}_{\tilde{\lambda},p'}$, so that $\nu(\bar{h}(\lambda))=\nu(\bar{h}_{\lambda,p})$ and $\nu(\bar{h}(\tilde{\lambda}))=\nu(\bar{h}_{\tilde{\lambda},p})$. However, because of the bijection between bars of length divisible by p in λ and bars in the $\bar{p}$-quotient $\lambda^{(\bar{p})}$, we have $\bar{h}_{\lambda,p}=p^w\bar{h}(\lambda^{(\bar{p})})=p^w\bar{h}(\tilde{\lambda}^{(\bar{p})})=\bar{h}_{\tilde{\lambda},p}$, whence $\nu(\bar{h}(\lambda))=\nu(\bar{h}(\tilde{\lambda}))$ and $\mathfrak{h}(\langle\lambda\rangle)=\mathfrak{h}(\langle\tilde{\lambda}\rangle)$. This proves that $\mathcal{I}$ is height-preserving. We also get

$$\begin{aligned}\frac{\langle\lambda\rangle(1)_{p'}}{\langle\tilde{\lambda}\rangle(1)_{p'}}&=\frac{2^{\lfloor(n-m(\lambda))/2\rfloor}}{2^{\lfloor(wp-m(\tilde{\lambda}))/2\rfloor}}\frac{(n!)_{p'}}{((wp)!)_{p'}}\frac{(\bar{h}_{\tilde{\lambda},p})_{p'}\bar{h}_{\tilde{\lambda},p'}}{(\bar{h}_{\lambda,p})_{p'}\bar{h}_{\lambda,p'}}\\&=\frac{2^{\lfloor(n-m(\lambda))/2\rfloor}}{2^{\lfloor(wp-m(\tilde{\lambda}))/2\rfloor}}\frac{(n!)_{p'}}{((wp)!)_{p'}}\frac{\bar{h}_{\tilde{\lambda},p'}}{\bar{h}_{\lambda,p'}}.\end{aligned}$$

If we write $\lambda^{(\bar{p})}=(\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{((p-1)/2)})$, then, by [9, Corollary (2.6)], we have $m(\lambda)=m(\lambda^{(0)})+m(\mu)+2a(\lambda)$ and $m(\tilde{\lambda})=m(\lambda^{(0)})+m(\emptyset)+2a(\tilde{\lambda})$. This implies that

$$\big\lfloor\big(n-m(\lambda)\big)/2\big\rfloor=\big\lfloor\big(n-m\big(\lambda^{(0)}\big)-m(\mu)\big)/2\big\rfloor-a(\lambda)$$

and

$$\big\lfloor\big(wp-m(\tilde{\lambda})\big)/2\big\rfloor=\big\lfloor\big(wp-m\big(\lambda^{(0)}\big)\big)/2\big\rfloor-a(\tilde{\lambda}).$$

Now Proposition 2.5 gives $\bar{h}_{\lambda,p'}\equiv\pm2^{-a(\lambda)}\bar{h}(\mu) \pmod p$ and $\bar{h}_{\tilde{\lambda},p'}\equiv\pm2^{-a(\tilde{\lambda})} \pmod p$. This yields

$$\frac{2^{\lfloor(n-m(\lambda))/2\rfloor}}{\bar{h}_{\lambda,p'}}\equiv\pm\frac{2^{\lfloor(n-m(\lambda^{(0)})-m(\mu))/2\rfloor}}{\bar{h}(\mu)} \pmod p$$

and

$$\frac{\bar{h}_{\tilde{\lambda},p'}}{2^{\lfloor(n-m(\tilde{\lambda}))/2\rfloor}}\equiv\pm\frac{1}{2^{\lfloor(wp-m(\lambda^{(0)}))/2\rfloor}} \pmod p.$$

By hypothesis, we have $\delta=\sigma(\mu)=(-1)^{r-m(\mu)}=1$, so that $r-m(\mu)$ is even. Thus $\lfloor(n-m(\lambda^{(0)})-m(\mu))/2\rfloor=\lfloor(wp-m(\lambda^{(0)}))/2\rfloor-\lfloor(r-m(\mu))/2\rfloor$, which in turn implies, together with the above,

$$\frac{\langle\lambda\rangle(1)_{p'}}{\langle\tilde{\lambda}\rangle(1)_{p'}}\equiv\pm\frac{(n!)_{p'}}{((wp)!)_{p'}}\frac{2^{\lfloor(r-m(\mu))/2\rfloor}}{\bar{h}(\mu)}=\pm\frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}}\langle\mu\rangle(1)_{p'} \pmod p. \qquad \square$$

The corresponding result when $\delta=-1$ is given by the following

Lemma 3.2 *If $\delta = -1$, then, with the above notation, $\lambda \longmapsto \tilde{\lambda}$ induces a sign-preserving bijection $\mathcal{I} : \langle\lambda\rangle \longmapsto \langle\tilde{\lambda}\rangle_-$ between B and B_0 which is also height-preserving. Furthermore, if $\langle\tilde{\lambda}\rangle_+ \in B_0^* \subset SI(S^+(pw))$ is the dual correspondent of $\langle\tilde{\lambda}\rangle_-$, then*

$$\langle\lambda\rangle(1)_{p'} \equiv \pm\frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}}\langle\mu\rangle(1)_{p'}\langle\tilde{\lambda}\rangle_+(1)_{p'}2^{s(\lambda)} \pmod p,$$

where $s(\lambda) = 1$ if $\sigma(\tilde{\lambda}) = -1$ and $s(\lambda) = 0$ if $\sigma(\tilde{\lambda}) = 1$.

Proof Since $\delta = -1$, we have $\sigma(\tilde{\lambda}) = -\sigma(\lambda)$, and $\sigma(\langle\tilde{\lambda}\rangle_-) = -\sigma(f^-(\langle\tilde{\lambda}\rangle_-)) = -\sigma(\tilde{\lambda}) = \sigma(\langle\lambda\rangle)$, so that $\mathcal{I} : \langle\lambda\rangle \longmapsto \langle\tilde{\lambda}\rangle_-$ is sign preserving, and therefore gives a bijection between B and B_0.

Now, for any $\langle\lambda\rangle \in B$, we have

$$\langle\lambda\rangle(1) = 2^{\lfloor(n-m(\lambda))/2\rfloor}\frac{n!}{\bar{h}(\lambda)} \quad \text{and} \quad \langle\tilde{\lambda}\rangle_+(1) = 2^{\lfloor(wp-m(\tilde{\lambda}))/2\rfloor}\frac{(wp)!}{\bar{h}(\tilde{\lambda})},$$

and, by duality,

$$\langle\tilde{\lambda}\rangle_-(1) = \begin{cases}\langle\tilde{\lambda}\rangle_+(1) & \text{if } \sigma(\langle\tilde{\lambda}\rangle_-) = 1,\\ \langle\tilde{\lambda}\rangle_+(1)/2 & \text{if } \sigma(\langle\tilde{\lambda}\rangle_-) = -1.\end{cases}$$

As in the proof of Lemma 3.1, this implies that $\mathcal{I}$ is height-preserving.

As above, we have

$$\lfloor(n - m(\lambda))/2\rfloor = \lfloor(n - m(\lambda^{(0)}) - m(\mu))/2\rfloor - a(\lambda)$$

and

$$\lfloor(wp - m(\tilde{\lambda}))/2\rfloor = \lfloor(wp - m(\lambda^{(0)}))/2\rfloor - a(\tilde{\lambda}),$$

so that Proposition 2.5 yields

$$\frac{2^{\lfloor(n-m(\lambda))/2\rfloor}}{\bar{h}_{\lambda,p'}} \equiv \pm\frac{2^{\lfloor(n-m(\lambda^{(0)})-m(\mu))/2\rfloor}}{\bar{h}(\mu)} \pmod p$$

and

$$\frac{\bar{h}_{\tilde{\lambda},p'}}{2^{\lfloor(n-m(\tilde{\lambda}))/2\rfloor}} \equiv \pm\frac{1}{2^{\lfloor(wp-m(\lambda^{(0)}))/2\rfloor}} \pmod p.$$

However, this time, we have $\delta = \sigma(\mu) = (-1)^{r-m(\mu)} = -1$, so that $r - m(\mu)$ is odd. Thus

$$\begin{aligned}\lfloor(n - m(\lambda^{(0)}) - m(\mu))/2\rfloor &= \lfloor(wp - m(\lambda^{(0)}) + 1)/2\rfloor - \lfloor(r - m(\mu) - 1)/2\rfloor\\ &= \lfloor(wp - m(\lambda^{(0)}) + 1)/2\rfloor - \lfloor(r - m(\mu))/2\rfloor.\end{aligned}$$

Now

$$\lfloor (wp - m(\lambda^{(0)}) + 1)/2 \rfloor - \lfloor (wp - m(\lambda^{(0)}))/2 \rfloor = \begin{cases} 1 & \text{if } wp - m(\lambda^{(0)}) \text{ is odd,} \\ 0 & \text{if } wp - m(\lambda^{(0)}) \text{ is even.} \end{cases}$$

But $(-1)^{wp-m(\lambda^{(0)})} = (-1)^{wp-|\lambda^{(0)}|}(-1)^{|\lambda^{(0)}|-m(\lambda^{(0)})} = (-1)^{wp-|\lambda^{(0)}|}\sigma(\lambda^{(0)})$, and, since p is odd, $(-1)^{wp-|\lambda^{(0)}|} = (-1)^{w-|\lambda^{(0)}|}$, so that $(-1)^{wp-m(\lambda^{(0)})} = \sigma(\lambda^{(\bar{p})}) = -\sigma(\lambda) = \sigma(\tilde{\lambda})$. This implies the result. □

We now turn to the p-local situation. As mentioned above, the defect group X of the block B can be chosen to be a Sylow p-subgroup of $S(pw)$ and of $S^+(pw)$. If we let $N_0(X) = N_{S(pw)}(X)$, then we have

$$N_{S(n)}(X) = N_0(X) \times S(r) \quad \text{and} \quad N := N_{S^+(n)}(X) = N_0(X)^+ \hat{\times} S^+(r).$$

In particular, we have, writing $\mathcal{N} = |N_0(X)^+| = |N_{S(pw)}(X)^+| = |N_{S^+(pw)}(X)|$,

$$\left[S^+(n) \colon N_{S^+(n)}(X)\right] = \frac{2n!}{|N_0(X)^+ \hat{\times} S^+(r)|} = \frac{2n!}{(\mathcal{N}2r!)/2} = \frac{2n!}{\mathcal{N}r!}$$

and

$$\left[S^+(pw) \colon N_{S^+(pw)}(X)\right] = \frac{2(pw)!}{\mathcal{N}}.$$

The following result will also be useful later.

Lemma 3.3 *If X is a Sylow p-subgroup of $S^-(pw)$, then*

$$\left[S^-(pw) \colon N_{S^-(pw)}(X)\right] = \left[S^+(pw) \colon N_{S^+(pw)}(X)\right]$$

and

$$\left[S^-(n) \colon N_{S^-(n)}(X)\right] = \left[S^+(n) \colon N_{S^+(n)}(X)\right].$$

Proof Both assertions will be consequences of the following result, which is easily derived from the Orbit-Stabilizer Theorem: if H is a subgroup of the finite group G, and if P is a subgroup of H such that H acts transitively by conjugation on the G-conjugates of P, then $[G \colon N_G(P)] = [H \colon N_H(P)]$.

Take any Sylow p-subgroup X of $S^-(pw)$. Then, since $S^-(pw)$ is of index 2 in $S^+(pw)$ (while p is odd), X is also a Sylow p-subgroup of $S^+(pw)$. Also, all the $S^+(pw)$-conjugates of X are in $S^-(pw)$ (which is normal in $S^+(pw)$), hence are Sylow p-subgroups of $S^-(pw)$, and thus are $S^-(pw)$-conjugate to X (note that the same argument shows that the Sylow p-subgroups of $S^+(n)$ and $S^-(n)$ are the same). This proves that $S^-(pw)$ acts transitively (by conjugation) on the $S^+(pw)$-conjugates of X. By the above, this proves the first assertion.

Now take any subgroup Q of $S^+(n)$ isomorphic to $S^-(pw)$, and X a Sylow p-subgroup of Q, and take any $g \in S^+(n)$. Then the conjugate X^g of X is a Sylow

p-subgroup of Q^g. However, $S^-(n)$ acts transitively by conjugation on the $S^+(n)$-conjugates of Q, so that there exists $h \in S^-(n)$ such that $Q^{gh} = Q$, and X^{gh} is a Sylow p-subgroup of Q. Hence there exists $h' \in Q$ such that $X^{ghh'} = X$, and thus $X^g = X^{(hh')^{-1}}$. This proves that all $S^+(n)$-conjugates of X are $S^-(n)$-conjugate to X, which implies the second assertion. □

If $\mu \succ r$ is the $\bar{p}$-core of B, we choose $\gamma \in SI(S^+(r))$ such that $f^+(\gamma) = \mu$. We have $\gamma = \gamma^a$ if and only if $\delta = 1$.

If we then denote by b the Brauer correspondent of B in N, we have (see the proof of [6, Theorem 2.2])

$$b = \{\chi \hat{\otimes} \gamma, \chi \hat{\otimes} \gamma^a \mid \chi \in SI(N_0(X)^+)\} = \{\chi \hat{\otimes} \gamma, \chi \hat{\otimes} \gamma^a \mid \chi \in \beta_0\},$$

where β_0 is *the* spin block of $N_0(X)^+$, and thus the Brauer correspondent of the principal spin block of $S^+(pw)$. In particular, $\beta_0 = b_0$ if $\delta = 1$ and $\beta_0 = b_0^*$ if $\delta = -1$.

For any $\chi \in \beta_0$, we have $(\chi \hat{\otimes} \gamma)(1) = (\chi \hat{\otimes} \gamma^a)(1) = 2^{\lfloor s/2 \rfloor} \chi(1)\gamma(1)$, where s is the number of n.s.a. characters in $\{\chi, \gamma\}$. If $\delta = 1$, we therefore get $s = 0$ (if $\chi^a = \chi$) or $s = 1$ (if $\chi^a \neq \chi$), so that $\lfloor s/2 \rfloor = 0$ and $(\chi \hat{\otimes} \gamma)(1) = \chi(1)\gamma(1)$. If $\delta = -1$, we have $s = 1$ and $\lfloor s/2 \rfloor = 0$ if $\chi^a = \chi$, and $s = 2$ and $\lfloor s/2 \rfloor = 1$ if $\chi^a \neq \chi$, so that

$$(\chi \hat{\otimes} \gamma)(1) = (\chi \hat{\otimes} \gamma^a)(1) = \begin{cases} \chi(1)\gamma(1) & \text{if } \chi^a = \chi, \\ 2\chi(1)\gamma(1) & \text{if } \chi^a \neq \chi. \end{cases}$$

3.2 Reduction theorem

We can now prove the main result of this section:

Theorem 3.4 *Let B be a spin block of $S^\varepsilon(n)$ of weight $w = w(B) > 0$ and sign $\delta = \delta(B)$ and let b be its Brauer correspondent in $N_{S^\varepsilon(n)}(X)$, where X is a defect group of B. Suppose the Isaacs–Navarro Conjecture holds for the principal spin block of $S^+(pw)$ via a sign-preserving bijection. Then it also holds for B.*

Proof We first suppose $\varepsilon = \mathbf{1}$.

We use the same notation as in Sect. 3.1. Let B_0 be the principal spin block of $S^\delta(pw)$ and b_0 be its Brauer correspondent. Let $\mu \succ r = n - wp$ be the $\bar{p}$-core of B, and $\gamma \in SI(S^+(r))$ such that $f^+(\gamma) = \mu$. If λ is a bar-partition of n with $\bar{p}$-core μ and $\bar{p}$-quotient $\lambda^{(\bar{p})}$, let $\tilde{\lambda}$ be the bar-partition of wp with empty $\bar{p}$-core and $\bar{p}$-quotient $\lambda^{(\bar{p})}$.

Suppose furthermore that $\delta = 1$. Then, by Lemma 3.1, $\lambda \longmapsto \tilde{\lambda}$ induces a sign-preserving bijection $\mathcal{I}$ between B and B_0 which is also height-preserving, and

$$\langle\lambda\rangle(1)_{p'} \equiv \pm \frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}} \gamma(1)_{p'} \langle\tilde{\lambda}\rangle(1)_{p'} \pmod{p}.$$

Now let $c = [S^+(pw) \colon N_{S^+(pw)}(X)]_{p'}$, and let $\varphi \colon M(B_0) \longrightarrow M(b_0)$ be a bijection such that, for each k such that $(p, k) = 1$, we have $M_k(b_0) = \varphi(M_{ck}(B_0))$ (such a φ exists by hypothesis). We have $b = \{\chi \hat{\otimes} \gamma \mid \chi \in b_0\}$, and, by [6, Proposition 1.2],

$\chi\hat{\otimes}\gamma=\psi\hat{\otimes}\gamma$ if and only if $\psi\in\{\chi,\chi^a\}$ and $(\sigma(\chi)\sigma(\gamma)=\sigma(\chi)=1)$ or $(\sigma(\chi)=-1$ and $\chi=\psi)$, i.e., $\chi\hat{\otimes}\gamma=\psi\hat{\otimes}\gamma$ if and only if $\psi=\chi$. Thus, by the results of Sect. 3.1, $\chi\longmapsto\chi\hat{\otimes}\gamma$ is a height-preserving bijection between b_0 and b. Hence

$$\Phi\colon\quad \begin{cases} M(B)\longrightarrow M(b),\\ \langle\lambda\rangle\longmapsto\varphi(\langle\tilde{\lambda}\rangle)\hat{\otimes}\gamma\end{cases}$$

is a (height-preserving) sign-preserving bijection.

Now, if $\langle\lambda\rangle\in M(B)$, then $\Phi(\langle\lambda\rangle)(1)=(\varphi(\langle\tilde{\lambda}\rangle))(1)\gamma(1)$, so that

$$\begin{aligned}\Phi\big(\langle\lambda\rangle\big)(1)_{p'}&=\big(\varphi\big(\langle\tilde{\lambda}\rangle\big)\big)(1)_{p'}\gamma(1)_{p'}\\ &\equiv\pm\frac{\langle\tilde{\lambda}\rangle(1)_{p'}}{c}\gamma(1)_{p'}\ (\mathrm{mod}\ p)\quad\text{(by definition of }\varphi)\\ &\equiv\pm\frac{1}{c}\left(\frac{r!(pw)!}{n!}\right)_{p'}\langle\lambda\rangle(1)_{p'}\ (\mathrm{mod}\ p),\end{aligned}$$

and, writing $\mathcal{N}=|N_{S(pw)}(X)^+|=|N_{S^+(pw)}(X)|$,

$$\begin{aligned}\frac{1}{c}\left(\frac{r!(pw)!}{n!}\right)_{p'}&=\frac{1}{[S^+(pw)\colon N_{S^+(pw)}(X)]_{p'}}\left(\frac{r!(pw)!}{n!}\right)_{p'}\\ &=\left(\frac{\mathcal{N}}{2(pw)!}\right)_{p'}\left(\frac{r!(pw)!}{n!}\right)_{p'}=\left(\frac{\mathcal{N}r!}{2n!}\right)_{p'}\\ &=\frac{1}{[S^+(n)\colon N_{S^+(n)}(X)]_{p'}},\end{aligned}$$

whence we finally get

$$\Phi\big(\langle\lambda\rangle\big)(1)_{p'}\equiv\pm\frac{\langle\lambda\rangle(1)_{p'}}{[S^+(n)\colon N_{S^+(n)}(X)]_{p'}}\ (\mathrm{mod}\ p),$$

i.e., Φ is an Isaacs–Navarro bijection between B and b.

Suppose now that $\delta=-1$. Then B_0 is the principal spin block of $S^-(pw)$, its dual B_0^* is the principal spin block of $S^+(pw)$, and b_0^* is the Brauer correspondent of B_0^*. Writing D_+ for the set of s.a. characters in D and D_- for the set of pairs of n.s.a. characters in D (so that $|M_k(B_0)|=|M_k(B_0)_+|+2|M_k(B_0)_-|$), we thus have, for each k such that $(p,k)=1$, the following equalities:

$$\big|M_k(B_0)_+\big|=\big|M_k\big(B_0^*\big)_-\big|=\big|M_{k/c}\big(b_0^*\big)_-\big|$$

and

$$\big|M_k(B_0)_-\big|=\big|M_{2k}\big(B_0^*\big)_+\big|=\big|M_{2k/c}\big(b_0^*\big)_+\big|,$$

where $c=[S^+(pw)\colon N_{S^+(pw)}(X)]_{p'}=[S^-(pw)\colon N_{S^-(pw)}(X)]_{p'}$ (by Lemma 3.3).

On the other hand, we have $b=\{\chi\hat{\otimes}\gamma,\chi\hat{\otimes}\gamma^a\mid\chi\in\beta_0=b_0^*\}$. For any $\chi,\psi\in\beta_0$ and $\gamma_1,\gamma_2\in\{\gamma,\gamma^a\}$, we have $\chi\hat{\otimes}\gamma_1=\psi\hat{\otimes}\gamma_2$ if and only if χ and ψ are associate and

$$\begin{cases}\sigma(\chi)\sigma(\gamma_1) = -\sigma(\chi) = 1\\ \text{or}\\ \sigma(\chi)\sigma(\gamma_1) = -\sigma(\chi) = -1 \quad \text{and} \quad [(\chi = \psi, \gamma_1 = \gamma_2) \quad \text{or} \quad (\chi \neq \psi, \gamma_1 \neq \gamma_2)].\end{cases}$$

Hence, if $\chi \in \beta_{0+}$, then we get two irreducible characters, $\chi\hat{\otimes}\gamma$ and $\chi\hat{\otimes}\gamma^a$, while, if $\chi \in \beta_{0-}$, then we get one irreducible character, $\chi\hat{\otimes}\gamma = \chi^a\hat{\otimes}\gamma = \chi^a\hat{\otimes}\gamma^a = \chi\hat{\otimes}\gamma^a$. Note that $\chi \longmapsto \chi\hat{\otimes}\gamma$ and $\chi \longmapsto \chi\hat{\otimes}\gamma^a$ are height preserving. Using the equalities above, as well as Lemma 3.2, we obtain the following height-preserving and sign-preserving bijection:

$$\Phi\colon \quad \begin{cases} M(B) \xrightarrow{+} M(B_0) \xrightarrow{-} M(B_0^*) \xrightarrow{+} M(b_0^*) \xrightarrow{-} M(b),\\ \langle\lambda\rangle \mapsto \langle\tilde{\lambda}\rangle_- \mapsto \langle\tilde{\lambda}\rangle_+ \mapsto \varphi(\langle\tilde{\lambda}\rangle_+) \mapsto \varphi(\langle\tilde{\lambda}\rangle_+)\hat{\otimes}\langle\mu\rangle,\end{cases}$$

where, as before, φ is the (sign-preserving) Isaacs–Navarro bijection we supposed exists between $M(B_0^*)$ and $M(b_0^*)$, and $\xrightarrow{+}$ (respectively, $\xrightarrow{-}$) denotes a sign-preserving (respectively, sign-inverting) bijection.

Now, by hypothesis, $\langle\tilde{\lambda}\rangle_+(1)_{p'} \equiv c\varphi(\langle\tilde{\lambda}\rangle_+)(1)_{p'} \pmod p$, so that, by Lemma 3.2, we obtain

$$\langle\lambda\rangle(1)_{p'} \equiv \left(\frac{n!}{(wp)!r!}\right)_{p'} c\varphi\big(\langle\tilde{\lambda}\rangle_+\big)(1)_{p'}\langle\mu\rangle(1)_{p'}2^{s(\lambda)} \pmod p,$$

and, as in the case $\delta = 1$, we have $(\frac{n!}{(wp)!r!})_{p'}c = [S^+(n)\colon N_{S^+(n)}(X)]_{p'}$. Finally, since $\sigma(\tilde{\lambda}) = 1 \Longleftrightarrow \sigma(\langle\tilde{\lambda}\rangle_+) = 1 \Longleftrightarrow \langle\tilde{\lambda}\rangle_+$ is s.a. $\Longleftrightarrow \varphi(\langle\tilde{\lambda}\rangle_+)$ is s.a., we get

$$\begin{aligned}\varphi\big(\langle\tilde{\lambda}\rangle_+\big)(1)_{p'}\langle\mu\rangle(1)_{p'}2^{s(\lambda)} &= \begin{cases}2\varphi(\langle\tilde{\lambda}\rangle_+)(1)_{p'}\langle\mu\rangle(1)_{p'} & \text{if } \sigma(\tilde{\lambda}) = -1,\\ \varphi(\langle\tilde{\lambda}\rangle_+)(1)_{p'}\langle\mu\rangle(1)_{p'} & \text{if } \sigma(\tilde{\lambda}) = 1,\end{cases}\\ &= \big(\varphi\big(\langle\tilde{\lambda}\rangle_+\big)\hat{\otimes}\langle\mu\rangle\big)(1)_{p'}\\ &= \Phi\big(\langle\lambda\rangle\big)(1)_{p'},\end{aligned}$$

whence $\langle\lambda\rangle(1)_{p'} \equiv [S^+(n)\colon N_{S^+(n)}(X)]_{p'}\Phi(\langle\lambda\rangle)(1)_{p'} \pmod p$, i.e., Φ is a (sign-preserving) Isaacs–Navarro bijection between $M(B)$ and $M(b)$.

We now suppose $\boldsymbol{\varepsilon = -1}$.

In this case, B is a spin block of $S^-(n)$ and b is its Brauer correspondent in $N_{S^-(n)}(X)$. Thus B^* is a spin block of $S^+(n)$, and, by [6, Lemma 2.3] (which is due to H. Blau), the dual b^* of b is the Brauer correspondent of B^*. By the case $\varepsilon = 1$, there exists a sign-preserving Isaacs–Navarro bijection $\varphi\colon M(B^*) \longrightarrow M(b^*)$. We define the sign-preserving bijection

$$\Phi\colon \quad \begin{cases} M(B) \longrightarrow M(b),\\ \langle\lambda\rangle \longmapsto (\varphi(\langle\lambda\rangle^*))^*.\end{cases}$$

By Lemma 3.3, we have $c = [S^-(n)\colon N_{S^-(n)}(X)]_{p'} = [S^+(n)\colon N_{S^+(n)}(X)]_{p'}$, and, for each k such that $(p,k)=1$, we have

$$\left|M_{ck}(B)_+\right| = \left|M_{ck}(B^*)_-\right| = \left|M_k(b^*)_-\right| = \left|M_k(b)_+\right|$$

and

$$\left|M_{ck}(B)_-\right| = \left|M_{2ck}(B^*)_+\right| = \left|M_{2k}(b^*)_+\right| = \left|M_k(b)_-\right|,$$

whence $|M_{ck}(B)| = |M_{ck}(B)_+| + 2|M_{ck}(B)_-| = |M_k(b)|$. □

4 Principal block

By Theorem 3.4, it is now sufficient to prove that the Isaacs–Navarro Conjecture holds for the principal spin block of $S^+(pw)$ via a sign-preserving bijection. Throughout this section, we therefore consider the following situation. We take $G = S^+(pw)$ (where $w \geq 1$ is an integer), B the principal spin block of G, and b the Brauer correspondent of B. Hence b is the principal spin block of $N_G(X)$ for some $X \in \mathrm{Syl}_p(G)$.

4.1 Spin characters of height 0 of the normalizer

The normalizer $N^+ = N_G(X)$ and its irreducible spin characters are described in Sects. 3 and 4 of [6]. Let $pw = \sum_{i=1}^k t_i p^i$ be the p-adic decomposition of pw. We then have $N^+ = [N_1 \wr S(t_1)]^+ \hat{\times} \cdots \hat{\times} [N_k \wr S(t_k)]^+$, where, for each $1 \leq i \leq k$, $N_i = N_{S(p^i)}(X_i)$ for some $X_i \in \mathrm{Syl}_p(S(p^i))$.

Now fix $1 \leq i \leq k$, and let $e_i = (p^i - 1)/2$. Then $H_i^+ = (N_i \wr S(t_i))^+ = M_i^+ S_{t_i}^+$, where $M_i^+ = N_i^{(1)+} \hat{\times} \cdots \hat{\times} N_i^{(t_i)+} \lhd H_i^+$ and $S_{t_i} \cong \Delta_{p^i} S(t_i) \subset S(p^i t_i)$, and where

$$S_{t_i}^+ \cong \begin{cases} \hat{S}(t_i) & \text{if } p^i \equiv 1 \pmod 4, \\ \tilde{S}(t_i) & \text{if } p^i \equiv -1 \pmod 4). \end{cases}$$

By [6, Proposition 3.9], N_i^+ has one s.a. spin character ζ_0 of degree $(p-1)^i$, and $e_i = (p^i - 1)/2$ pairs of n.s.a. spin characters $\{\zeta_1, \zeta_1^a, \ldots, \zeta_{e_i}, \zeta_{e_i}^a\}$ of degree 1.

Let $\mathcal{A}_i = \{(t_i^{(0)}, t_i^{(1)}, \ldots, t_i^{(e_i)}) \mid t_i^{(j)} \in \mathbb{N} \cup \{0\}, \sum_{j=0}^{e_i} t_i^{(j)} = t_i\}$. Then, by [6, Proposition 3.12], a complete set of representatives for the $S_{t_i}^+$-conjugacy classes in $SI_0(M_i^+)$ is given by

$$\mathcal{R} = \{\theta_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{A}_i\} \cup \left\{\theta_{\mathbf{s}}^a \mid \mathbf{s} = \left(t_i^{(0)}, t_i^{(1)}, \ldots, t_i^{(e_i)}\right) \in \mathcal{A}_i, t_i - t_i^{(0)} \text{ odd},\ t_i^{(0)} \leq 1\right\},$$

where $\theta_{\mathbf{s}} = \theta_0 \hat{\otimes} \theta_1 \hat{\otimes} \cdots \hat{\otimes} \theta_{e_i}$, with $\theta_j = \zeta_j \hat{\otimes} \cdots \hat{\otimes} \zeta_j$ ($t_i^{(j)}$ factors). Also, the inertial subgroup $T_i^+ = I_{H_i^+}(\theta_{\mathbf{s}})$ of $\theta_{\mathbf{s}}$ in H_i^+ satisfies

$$T_i^+/M_i^+ \cong \begin{cases} A(t_i^{(0)}) \times S(t_i^{(1)}) \times \cdots \times S(t_i^{(e_i)}) & \text{if } t_i - t_i^{(0)} \text{ is odd}, \\ S(t_i^{(0)}) \times S(t_i^{(1)}) \times \cdots \times S(t_i^{(e_i)}) & \text{if } t_i - t_i^{(0)} \text{ is even}. \end{cases}$$

We can now describe how to induce each θ_j from $M_i^{(j)+} = (N_i^+)^{\hat{\times} t_i^{(j)}}$ to the corresponding factor $T_i^{(j)+}$ of its inertial subgroup.

Proposition 4.1 (See [6, Proposition 4.4]) *If ζ is an n.s.a. linear representation of N_i^+, then $\theta_j = \zeta^{t_i^{(j)}} = \zeta \hat{\otimes} \cdots \hat{\otimes} \zeta \in \mathrm{Irr}(M_i^{(j)+})$ can be extended to a negative representation $D_\zeta \in \mathrm{Irr}(T_i^{(j)+})$, and every irreducible constituent V of $\theta \uparrow^{T_i^{(j)+}}$ is of the form $V = D_\zeta \otimes R$, where R is an irreducible representation of $T_i^{(j)+}/M_i^{(j)+} \cong S(t_i^{(j)})$. If $t_i^{(j)}$ is odd, then every irreducible constituent V of $\theta \uparrow^{T_i^{(j)+}}$ is n.s.a., and, if $t_i^{(j)}$ is even, then every irreducible constituent V of $\theta \uparrow^{T_i^{(j)+}}$ is s.a.*

In the above notation, if ψ is the character of $V = D_\zeta \otimes R$, and if R has character $\chi_\lambda \in \mathrm{Irr}(S(t_i^{(j)}))$, then $\psi(1) = \zeta^{t_i^{(j)}}(1)\chi_\lambda(1)$. Also, since ζ is n.s.a., we have $\zeta^{t_i^{(j)}}(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor}\zeta(1)^{t_i^{(j)}} = 2^{\lfloor t_i^{(j)}/2 \rfloor}$, and $\psi(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor}\chi_\lambda(1)$. Finally, ψ is s.a. if and only if $t_i^{(j)}$ is even.

Proposition 4.2 (See [6, Proposition 4.8]) *Let $t_i^{(0)} \geq 4$, and let D be the s.a. spin representation of N_i^+ with degree $(p-1)^i$. Then $D^{t_i^{(0)}} = D \hat{\otimes} \cdots \hat{\otimes} D \in \mathrm{Irr}(M_i^{(0)+})$ can neither be extended to an irreducible representation of $T_i^{(0)-} = M_i^{(0)+} A^+_{t_i^{(0)}}$ nor to one of $T_i^{(0)+} = M_i^{(0)+} S^+_{t_i^{(0)}}$. Furthermore, every irreducible constituent V of $D^{t_i^{(0)}} \uparrow^{T_i^{(0)+}}$ is of the form $V = D^{t_i^{(0)}} \otimes S$, where S is an irreducible spin representation of $S^+_{t_i^{(0)}}$, and every irreducible constituent V of $D^{t_i^{(0)}} \uparrow^{T_i^{(0)-}}$ is of the form $V = D^{t_i^{(0)}} \otimes S$, where S is an irreducible spin representation of $A^+_{t_i^{(0)}}$. In each case, V is s.a. if and only if S is s.a.*

In this notation, if ψ is the character of V, and if S has character χ_S, then $\psi(1) = \zeta_0^{t_i^{(0)}}(1)\chi_S(1)$. And, since ζ_0 is s.a., we have $\psi(1) = (p-1)^{it_i^{(0)}}\chi_S(1)$.

We can now describe all the characters of height 0 in b. Recall that these are exactly the spin characters with p'-degree in N^+. Still writing $pw = \sum_{i=1}^k t_i p^i$ the p-adic decomposition of pw, Olsson proved in [10] that, for any sign σ,

$$\left|M(b)_\sigma\right| = \sum_{\{(\sigma_1,\ldots,\sigma_k)\}} \prod_{i=1}^k q^{\sigma_i}\left(\bar{p}^i, t_i\right),$$

where $(\sigma_1, \ldots, \sigma_k)$ runs through all k-tuples of signs satisfying $\sigma_1 \cdots \sigma_k = \sigma$, and where $q^{\sigma_i}(\bar{p}^i, t_i)$ denotes the number of all $\bar{p}^i$-quotients with sign σ_i and weight t_i.

The correspondence goes as follows. For each $1 \leq i \leq k$, pick $\mathbf{s_i} \in \mathcal{A}_i$ and the corresponding $\theta_{\mathbf{s_i}} \in SI_0(M^+_{\mathbf{s_i}})$ (where, if $\mathbf{s_i} = (t_i^{(0)}, t_i^{(1)}, \ldots, t_i^{(e_i)})$, then $M^+_{\mathbf{s_i}} = (N_i^{\hat{\times} t_i^{(0)}})^+ \hat{\times} \cdots \hat{\times} (N_i^{\hat{\times} t_i^{(e_i)}})^+$). Inducing $\theta_{\mathbf{s_i}}$ (or $\theta_{\mathbf{s_i}} + \theta^a_{\mathbf{s_i}}$ if $t_i - t_i^{(0)}$ is odd and $t_i^{(0)} \leq 1$)

to its inertial subgroup T_i^+, we obtain s.a. irreducible constituents and pairs of n.s.a. irreducible constituents described by Propositions 4.1 and 4.2 and labeled by the $\bar{p}^i$-quotients of weight t_i: if $Q_i = (\lambda_i^{(0)}, \lambda_i^{(1)}, \dots, \lambda_i^{(e_i)})$ is a $\bar{p}^i$-quotient of weight t_i, then $\langle \Psi_{Q_i} \rangle = \langle \psi_i^{(0)} \rangle \hat{\times} \langle \psi_i^{(1)} \rangle \hat{\times} \cdots \hat{\times} \langle \psi_i^{(e_i)} \rangle \in SI_0(T_i^+)$. Also, by Propositions 4.1 and 4.2,

- For $1 \le j \le e_i$, $\psi_i^{(j)}(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor} \chi_{\lambda_i^{(j)}}(1)$ (with $\chi_{\lambda_i^{(j)}} \in \mathrm{Irr}(S(t_i^{(j)}))$) and $\psi_i^{(j)}$ is s.a. if and only if $t_i^{(j)}$ is even.
- $\psi_i^{(0)}(1) = (p-1)^{it_i^{(0)}} \chi_{\lambda_i^{(0)}}(1)$ (with $\chi_{\lambda_i^{(0)}} \in SI(S^+_{t_i^{(0)}})$ if $t_i - t_i^{(0)}$ is even and $\chi_{\lambda_i^{(0)}} \in SI(A^+_{t_i^{(0)}})$ if $t_i - t_i^{(0)}$ is odd) and $\psi_i^{(0)}$ is s.a. if and only if $\chi_{\lambda_i^{(0)}}$ is s.a.

Finally, $\langle \Psi_{Q_i} \rangle(1) = 2^{\lfloor S_i/2 \rfloor} \langle \psi_i^{(0)} \rangle(1) \langle \psi_i^{(1)} \rangle(1) \cdots \langle \psi_i^{(e_i)} \rangle(1)$, where S_i is the number of (pairs of) n.s.a. characters in $\{\langle \psi_i^{(0)} \rangle, \langle \psi_i^{(1)} \rangle, \dots, \langle \psi_i^{(e_i)} \rangle\}$.

Inducing to $H_i^+ = [N_i \wr S(t_i)]^+$, we obtain a s.a. irreducible spin character, or a pair of associate (n.s.a.) spin characters, $\langle Q_i \rangle$, labeled by Q_i.

Given the structure of T_i^+, we see that $\sigma(\langle Q_i \rangle) = (-1)^{t_i - t_i^{(0)}} \sigma(\Psi_{Q_i})$. However, we have $\sigma(\Psi_{Q_i}) = \sigma(\psi_i^{(0)}) \sigma(\psi_i^{(1)}) \cdots \sigma(\psi_i^{(e_i)})$. Also, for $1 \le j \le e_i$, we have $\sigma(\psi_i^{(j)}) = (-1)^{t_i^{(j)}}$, and $\sigma(\psi_i^{(0)}) = \sigma(\chi_{\lambda_i^{(0)}}) = \sigma(\lambda_i^{(0)})(-1)^{t_i - t_i^{(0)}}$, so that $\sigma(\Psi_{Q_i}) = \sigma(\lambda_i^{(0)})$ (since $\sum_{j=1}^{e_i} t_i^{(j)} = t_i - t_i^{(0)}$) and $\sigma(\langle Q_i \rangle) = (-1)^{t_i - t_i^{(0)}} \sigma(\lambda_i^{(0)}) = \sigma(Q_i)$. Note that, writing $m_i^{(0)}$ for the number of (non-zero) parts in $\lambda_i^{(0)}$, we have $\sigma(\lambda_i^{(0)}) = (-1)^{t_i^{(0)} - m_i^{(0)}}$, so that $\sigma(\langle Q_i \rangle) = (-1)^{t_i - m_i^{(0)}}$, and $\langle Q_i \rangle$ is s.a. if and only if $t_i - m_i^{(0)}$ is even.

Also, we have $\langle Q_i \rangle(1) = (|H_i^+|/|T_i^+|) \langle \Psi_{Q_i} \rangle(1)$, unless $\chi_{\lambda_i^{(0)}}$ is an s.a. irreducible spin character of $A^+_{t_i^{(0)}}$ (i.e., $t_i - t_i^{(0)}$ is odd and $\chi_{\lambda_i^{(0)}}$ is s.a.), in which case $\langle Q_i \rangle(1) = (|H_i^+|/|T_i^+|) \langle \Psi_{Q_i} \rangle(1)/2$.

Finally, the irreducible characters of height 0 in b are parametrized by the sequences $(Q_1, \dots, Q_k)$, where Q_i is a $\bar{p}^i$-quotient of weight t_i. We have $\langle (Q_1, \dots, Q_k) \rangle = \langle Q_1 \rangle \hat{\otimes} \cdots \hat{\otimes} \langle Q_k \rangle$, and $\langle (Q_1, \dots, Q_k) \rangle(1) = 2^{\lfloor s/2 \rfloor} \prod_{i=1}^k \langle Q_i \rangle(1)$, where s is the number of (pairs of) n.s.a. characters in $\{\langle Q_1 \rangle, \dots, \langle Q_k \rangle\}$. By the above remark on the sign of $\langle Q_i \rangle$, we see that $s = |\{1 \le i \le k; t_i - m_i^{(0)} \text{ odd}\}|$.

Proposition 4.3 *With the above notation, we have*

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor s/2 \rfloor}} \prod_{i=1}^k \frac{1}{2^{\lfloor (t_i - m_i^{(0)})/2 \rfloor}} \bar{h}(Q_i) \pmod{p},$$

where $s = |\{1 \le i \le k; t_i - m_i^{(0)} \text{ odd}\}|$, and, for each $1 \le i \le k$, $\bar{h}(Q_i)$ is the product of all bar-lengths in Q_i.

Proof We have

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1,\dots,Q_k)\rangle(1)_{p'}}=\frac{\prod_{i=1}^k |H_i^+|}{2^{k-1}\langle (Q_1,\dots,Q_k)\rangle(1)_{p'}}=\frac{\prod_{i=1}^k |H_i^+|}{2^{k-1}2^{\lfloor s/2\rfloor}\prod_{i=1}^k\langle Q_i\rangle(1)_{p'}}.$$

This gives

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1,\dots,Q_k)\rangle(1)_{p'}}=\frac{1}{2^{k-1}}\frac{1}{2^{\lfloor s/2\rfloor}}D_1^{(+1)}D_0^{(+1)}D_1^{(-1)}D_0^{(-1)},$$

where, for $\varepsilon\in\{+1,-1\}$ and $a\in\{0,1\}$,

$$D_a^{(\varepsilon)}=\prod_{\substack{1\le i\le k\\ t_i-t_i^{(0)}\equiv a \pmod 2\\ \sigma(\chi_{\lambda_i^{(0)}})=\varepsilon}}\frac{|H_i^+|}{\langle Q_i\rangle(1)_{p'}}.$$

Now we have, whenever $(\varepsilon,a)\in\{(+1,0),(-1,1),(-1,0)\}$,

$$D_a^{(\varepsilon)}=\prod_{\substack{1\le i\le k\\ t_i-t_i^{(0)}\equiv a \pmod 2\\ \sigma(\chi_{\lambda_i^{(0)}})=\varepsilon}}\frac{|T_i^+|}{\langle \Psi_{Q_i}\rangle(1)_{p'}}=\prod_{\substack{1\le i\le k\\ t_i-t_i^{(0)}\equiv a \pmod 2\\ \sigma(\chi_{\lambda_i^{(0)}})=\varepsilon}}\frac{|T_i^+|}{2^{\lfloor S_i/2\rfloor}\prod_{j=0}^{e_i}\psi_i^{(j)}(1)},$$

while

$$D_1^{(+1)}=\prod_{\substack{1\le i\le k\\ t_i-t_i^{(0)}\ \text{odd}\\ \sigma(\chi_{\lambda_i^{(0)}})=1}}\frac{|T_i^+|}{\langle \Psi_{Q_i}\rangle(1)_{p'}/2}=\prod_{\substack{1\le i\le k\\ t_i-t_i^{(0)}\ \text{odd}\\ \sigma(\chi_{\lambda_i^{(0)}})=1}}\frac{|T_i^+|}{2^{\lfloor S_i/2\rfloor-1}\prod_{j=0}^{e_i}\psi_i^{(j)}(1)},$$

where S_i is the number of (pairs of) n.s.a. characters in $\{\langle\psi_i^{(0)}\rangle,\dots,\langle\psi_i^{(e_i)}\rangle\}$.

For each $1\le i\le k$, we have $|T_i^+|_{p'}=|T_i^+/M_i^+|_{p'}|M_i^+|_{p'}$. Also, $M_i^+\cong(N_i^+)^{\hat{\times}t_i}$, so that $|M_i^+|=\frac{|N_i^+|^{t_i}}{2^{t_i-1}}=2|N_i|^{t_i}$. But $N_i=N_{S(p^i)}(X_i)$ for some $X_i\in \mathrm{Syl}_p(S(p^i))$; thus $|N_i|=|X_i|\cdot|N_i/X_i|$, and we have $N_i/X_i=K_i\cong(\mathbb{Z}/(p-1)\mathbb{Z})^i$ (see [6, page 89]). Hence $|N_i|_{p'}=(p-1)^i$, $|M_i^+|_{p'}=2(p-1)^{it_i}$, and

$$|T_i^+|_{p'}\equiv\begin{cases}(-1)^{it_i}t_i^{(0)}!t_i^{(1)}!\dots t_i^{(e_i)}! \pmod p & \text{if } t_i-t_i^{(0)} \text{ is odd,}\\ 2(-1)^{it_i}t_i^{(0)}!t_i^{(1)}!\dots t_i^{(e_i)}! \pmod p & \text{if } t_i-t_i^{(0)} \text{ is even.}\end{cases}$$

Now fix $1\le i\le k$, and let $\{1,\dots,e_i\}=I_1^{(i)}\cup I_2^{(i)}$, where

$$I_1^{(i)}=\bigl\{j\in\{1,\dots,e_i\}\,|\,t_i^{(j)}=2k_i^{(j)}+1\bigl(k_i^{(j)}\in\mathbb{N}\cup\{0\}\bigr)\bigr\}$$

and

$$I_2^{(i)}=\bigl\{j\in\{1,\dots,e_i\}\,|\,t_i^{(j)}=2k_i^{(j)}\bigl(k_i^{(j)}\in\mathbb{N}\bigr)\bigr\}.$$

We obtain

$$\prod_{j=1}^{e_i}\psi_i^{(j)}(1)=\prod_{j\in I_1^{(i)}}2^{k_i^{(j)}}\chi_{\lambda_i^{(j)}}(1)\prod_{j\in I_2^{(i)}}2^{k_i^{(j)}}\chi_{\lambda_i^{(j)}}(1)=2^{\sum_{j=1}^{e_i}k_i^{(j)}}\prod_{j=1}^{e_i}\chi_{\lambda_i^{(j)}}(1).$$

Note that $S_i=|I_1^{(i)}|$ if $\psi_i^{(0)}$ is s.a., while $S_i=|I_1^{(i)}|+1$ if $\psi_i^{(0)}$ is n.s.a. We thus have

$$2^{\lfloor S_i/2\rfloor}\prod_{j=1}^{e_i}\psi_i^{(j)}(1)=2^{\sum_{j=1}^{e_i}k_i^{(j)}+\lfloor S_i/2\rfloor}\prod_{j=1}^{e_i}\chi_{\lambda_i^{(j)}}(1)=2^{\lfloor\sum_{j=1}^{e_i}k_i^{(j)}+S_i/2\rfloor}\prod_{j=1}^{e_i}\chi_{\lambda_i^{(j)}}(1),$$

and

$$\begin{aligned}\left\lfloor\sum_{j=1}^{e_i}k_i^{(j)}+\frac{S_i}{2}\right\rfloor&=\left\lfloor\sum_{j\in I_1^{(i)}}\frac{t_i^{(j)}-1}{2}+\sum_{j\in I_2^{(i)}}\frac{t_i^{(j)}}{2}+\frac{S_i}{2}\right\rfloor\\&=\begin{cases}\left\lfloor\sum_{j\in I_1^{(i)}}\frac{t_i^{(j)}}{2}+\sum_{j\in I_2^{(i)}}\frac{t_i^{(j)}}{2}\right\rfloor=\left\lfloor\frac{t_i-t_i^{(0)}}{2}\right\rfloor\\\quad\text{if }\sigma(\psi_i^{(0)})=1,\\\left\lfloor\sum_{j\in I_1^{(i)}}\frac{t_i^{(j)}}{2}+\sum_{j\in I_2^{(i)}}\frac{t_i^{(j)}}{2}+\frac{1}{2}\right\rfloor=\left\lfloor\frac{t_i-t_i^{(0)}+1}{2}\right\rfloor\\\quad\text{if }\sigma(\psi_i^{(0)})=-1.\end{cases}\end{aligned}$$

We can now compute $D_1^{(+1)}$, $D_0^{(+1)}$, $D_1^{(-1)}$ and $D_0^{(-1)}$. Take any $1\le i\le k$, and first suppose that $t_i-t_i^{(0)}$ is odd and $\sigma(\chi_{\lambda_i^{(0)}})=1$. Then $\psi_i^{(0)}$ is n.s.a., $S_i=|I_1^{(i)}|+1$, $t_i-m_i^{(0)}$ is even and

$$\chi_{\lambda_i^{(0)}}(1)=2^{\lfloor\frac{t_i^{(0)}-m_i^{(0)}}{2}\rfloor}\frac{t_i^{(0)}!}{\bar h(\lambda_i^{(0)})}.$$

Finally, $\lfloor\frac{t_i-t_i^{(0)}+1}{2}\rfloor=\lfloor\frac{t_i-t_i^{(0)}}{2}\rfloor+1$ (since $t_i-t_i^{(0)}$ is odd). We therefore get

$$\begin{aligned}\frac{|T_i^+|}{2^{\lfloor S_i/2\rfloor-1}\prod_{j=0}^{e_i}\psi_i^{(j)}(1)}&\equiv\pm\frac{\bar h(\lambda_i^{(0)})\prod_{j=1}^{e_i}h(\lambda_i^{(j)})}{2^{\lfloor\frac{t_i-t_i^{(0)}+1}{2}\rfloor+\lfloor\frac{t_i^{(0)}-m_i^{(0)}}{2}\rfloor-1}}\pmod p\\&\equiv\pm\frac{1}{2^{\lfloor\frac{t_i-m_i^{(0)}+1}{2}\rfloor-1}}\bar h(Q_i)\pmod p\\&\equiv\pm\frac{2}{2^{\lfloor\frac{t_i-m_i^{(0)}}{2}\rfloor}}\bar h(Q_i)\pmod p\end{aligned}$$

(since $t_i - m_i^{(0)}$ is even). By similar arguments, we obtain, in all other cases,

$$\frac{|T_i^+|}{2^{\lfloor S_i/2\rfloor}\prod_{j=0}^{e_i}\psi_i^{(j)}(1)} \equiv \pm\frac{2}{2^{\lfloor\frac{t_i-m_i^{(0)}}{2}\rfloor}}\bar{h}(Q_i) \pmod p.$$

Finally, we get

$$\frac{|N_G(X)|_{p'}}{\langle(Q_1,\dots,Q_k)\rangle(1)_{p'}} \equiv \pm\frac{2^k}{2^{k-1}2^{\lfloor s/2\rfloor}}\prod_{i=1}^{k}\frac{1}{2^{\lfloor(t_i-m_i^{(0)})/2\rfloor}}\bar{h}(Q_i) \pmod p,$$

as announced. □

4.2 Isaacs–Navarro Conjecture

We can now prove the main result of this section.

Theorem 4.4 *The Isaacs–Navarro Conjecture holds for the principal spin block of $S^+(pw)$ via a sign-preserving bijection.*

Proof Let B be the principal spin block of $G = S^+(pw)$, and b its Brauer correspondent in $N_G(X)$. Let $pw = \sum_{i=1}^{k} t_i p^i$ be the p-adic decomposition of pw. By Proposition 2.7, $\lambda \succ pw$ labels a spin character of B of p'-degree if and only λ has $\bar{p}$-core tower $(R_1^\lambda,\dots,R_k^\lambda)$ with $|R_i^\lambda| = t_i$ for each $1 \le i \le k$. Also, for any such λ, we have, by Lemma 2.6, $\sigma(\langle\lambda\rangle) = \sigma(\lambda) = \sigma(R_1^\lambda)\cdots\sigma(R_k^\lambda)$. By the above description of $M(b)$, this implies that

$$\Phi\colon\quad \begin{cases} M(B) \longrightarrow M(b),\\ \langle\lambda\rangle \longmapsto \langle(R_1^\lambda,\dots,R_k^\lambda)\rangle \end{cases}$$

is a sign-preserving bijection. Furthermore, it is immediate from Propositions 2.8 and 4.3 that, for any $\langle\lambda\rangle \in M(B)$,

$$\frac{|G|_{p'}}{\langle\lambda\rangle(1)_{p'}} \equiv \pm\frac{|N_G(X)|_{p'}}{\langle(Q_1,\dots,Q_k)\rangle(1)_{p'}} \pmod p.$$

This proves the result. □

5 Main theorem

We can now finally give our main theorem:

Theorem 5.1 *The Isaacs–Navarro conjecture holds for all covering groups of the symmetric and alternating groups, whenever p is an odd prime.*

Proof First, let G be any central extension of degree 2 of $S(n)$ or $A(n)$, and B be a p-block of G. If B is an unfaithful block, then the Isaacs–Navarro Conjecture holds for B by the results of Fong ([2]) and Nath ([8]). If B is a spin-block of G of weight $w > 0$, then the Isaacs–Navarro Conjecture holds for B by Theorems 3.4 and 4.4. If $w = 0$, then B contains a unique spin character (of p-defect 0), and the result is immediate.

Finally, the case of the exceptional 6-fold covers of $A(6)$ and $A(7)$ can easily be checked using the character tables given in [6, Appendix], or with a computer. □

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References

1. Cabanes, M.: Local structure of the p-blocks of $\tilde{S}_n$. Math. Z. **198**, 519–543 (1988)
2. Fong, P.: The Isaacs–Navarro Conjecture for symmetric groups. J. Algebra **260**(1), 154–161 (2003)
3. Humphreys, J.F.: On certain projective modular representations of direct products. J. Lond. Math. Soc. (2) **32**, 449–459 (1985)
4. Humphreys, J.F.: Blocks of projective representations of the symmetric groups. J. Lond. Math. Soc. (2) **32**, 441–452 (1986)
5. Isaacs, I.M., Navarro, G.: New refinements of the McKay Conjecture for arbitrary finite groups. Ann. Math. **156**(1), 333–344 (2002)
6. Michler, G.O., Olsson, J.B.: The Alperin–McKay Conjecture holds in the covering groups of symmetric and alternating groups, $p \neq 2$. J. Reine Angew. Math. **405**, 78–111 (1990)
7. Morris, A.O.: The spin representation of the symmetric group. Can. J. Math. **17**, 543–549 (1965)
8. Nath, R.: The Isaacs–Navarro Conjecture for the alternating groups. J. Algebra **321**, 1632–1642 (2009)
9. Olsson, J.B.: Frobenius symbols for partitions and degrees of spin characters. Math. Scand. **61**, 223–247 (1987)
10. Olsson, J.B.: On the p-blocks of symmetric and alternating groups and their covering groups. J. Algebra **128**, 188–213 (1990)
11. Schur, I.: Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen. J. Reine Angew. Math. **139**, 155–250 (1911)