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Centerpole sets for colorings of abelian groups

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Abstract A subset $C \subset G$ of a group G is called *k-centerpole* if for each k-coloring of G there is an infinite monochromatic subset G, which is symmetric with respect to a point $c \in C$ in the sense that $S = cS^{-1}c$. By $c_k(G)$ we denote the smallest cardinality $c_k(G)$ of a k-centerpole subset in G. We prove that $c_k(G) = c_k(\mathbb{Z}^m)$ if G is an abelian group of free rank $m \geq k$. Also we prove that $c_1(\mathbb{Z}^{n+1}) = 1$, $c_2(\mathbb{Z}^{n+2}) = 3$, $c_3(\mathbb{Z}^{n+3}) = 6$, $8 \leq c_4(\mathbb{Z}^{n+4}) \leq c_4(\mathbb{Z}^4) = 12$ for all $n \in \omega$, and $\frac{1}{2}(k^2 + 3k - 4) \leq c_k(\mathbb{Z}^n) \leq 2^k - 1 - \max_{s \leq k-2} \binom{k-1}{s-1}$ for all $n \geq k \geq 4$.

Keywords Abelian group · Centerpole set · Coloring · Symmetric subset · Monochromatic subset

1 Introduction

Answering a problem posed in [11], T. Banakh and I. Protasov [4] proved that for any k-coloring $\chi : \mathbb{Z}^k \to k = \{0, \dots, k-1\}$ of the abelian group $\mathbb{Z}^k$ there is an infinite monochromatic subset $S \subset \mathbb{Z}^k$ such that $S - c = c - S$ for some point $c \in \{0, 1\}^k$. The equality $S - c = c - S$ means that the set S is symmetric with respect to the point c. On the other hand, a suitable partition of $\mathbb{R}^k$ into $k + 1$ convex cones determines a Borel $(k + 1)$-coloring of $\mathbb{R}^k$ without unbounded monochromatic symmetric subsets. These two results motivate the following definition, cf. [1, 3].

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Definition 1 A subset C of a topological group G is called *k-centerpole*[1] *for (Borel) colorings of* G if for any (Borel) k-coloring $\chi : G \to k$ of G there is an unbounded monochromatic subset $S \subset G$, symmetric with respect to some point $c \in C$ in the sense that $Sc^{-1} = cS^{-1}$.

The smallest cardinality $|C|$ of such a k-centerpole set $C \subset G$ is denoted by $c_k(G)$ (resp. $c_k^B(G)$). If no k-centerpole set $C \subset G$ exists, then we write $c_k(G) = \infty$ (resp. $c_k^B(G) = \infty$) and assume that ∞ is greater than any cardinal that appears in our considerations.

Now we explain some terminology that appears in this definition. A subset B of a topological group G is called *totally bounded* if B can be covered by finitely many left shifts of any neighborhood U of the neutral element of X. In the opposite case B is called *unbounded*. A subset of a discrete topological group is unbounded if and only if it is infinite.

A cardinal number k is identified with the set $\{\alpha : |\alpha| < \kappa\}$ of ordinals of smaller cardinality and endowed with the discrete topology. By a (*Borel*) *k-coloring* of a topological space X we mean a (Borel) function $\chi : X \to k$. A function $\chi : X \to k$ is *Borel* if for every color $i \in k$ the set $\chi^{-1}(i)$ of points of color i in X is Borel.

The definition of the numbers $c_k(G)$ and $c_k^B(G)$ implies that

$$c_k^B(G) \le c_k(G)$$

for any topological group G and any cardinal number k. If the topological group G is discrete, then each coloring of G is Borel, so $c_k^B(G) = c_k(G)$ for all k. In general, the cardinal numbers $c_k(G)$ and $c_k^B(G)$ are different. For example, $c_\omega^B(\mathbb{R}^\omega) = \omega$ while $c_\omega(\mathbb{R}^\omega) = \infty$, see Theorem 2.

It follows from the definition that $c_k(G)$ and $c_k^B(G)$ considered as functions of k and G are non-decreasing with respect to k and non-increasing with respect to G. More precisely, for a number $k \in \mathbb{N}$, a topological group G and its subgroup H we have the inequalities

$$\begin{aligned} c_k(H) &\ge c_k(G), \qquad c_k(G) \le c_{k+1}(G) \quad \text{and} \\ c_k^B(H) &\ge c_k^B(G), \qquad c_k^B(G) \le c_{k+1}^B(G). \end{aligned}$$

In the sequel we shall use these monotonicity properties of $c_k(G)$ and $c_k^B(G)$ without any special reference.

In this paper we investigate the problem of calculating the numbers $c_k(G)$ and $c_k^B(G)$ for an abelian topological group G and show that in many cases this problem reduces to calculating the numbers $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ and $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ where $n = r_{\mathbb{R}}(G)$ is the $\mathbb{R}$-rank and $m = r_{\mathbb{Z}}(G)$ is the $\mathbb{Z}$-rank of the group G.

For topological groups G and H the *H-rank* $r_H(G)$ of G is defined as

$$r_H(G) = \sup\{k \in \omega : H^k \hookrightarrow G\}$$

[1] So, a centerpole set can be thought as a set of poles of central symmetries that detects unbounded monochromatic symmetric subsets.

where $H^k \hookrightarrow G$ means that H^k is topologically isomorphic to a subgroup of the topological group G. It is clear that $r_{\mathbb{R}}(G) \le r_{\mathbb{Z}}(G)$ for each topological group G.

It is interesting to remark that the $\mathbb{Z}$-rank appears in the formula for calculating the value of the function

$$\nu(G) = \min\{\kappa : c_k(G) = \infty\}$$

introduced and studied in [12] and [4]. By [4], for any discrete abelian group G

$$\nu(G) = \begin{cases} \max\{|G[2]|, \log|G|\} & \text{if } G \text{ is uncountable or } G[2] \text{ is infinite,} \\ r_{\mathbb{Z}}(G) + 1 & \text{if } G \text{ is finitely generated,} \\ r_{\mathbb{Z}}(G) + 2 & \text{otherwise.} \end{cases}$$

Here $G[2] = \{x \in G : 2x = 0\}$ is the *Boolean subgroup* of G and $\log|G| = \min\{\kappa : |G| \le 2^\kappa\}$.

A topological group G is called *inductively locally compact* (briefly, an ILC-*group*) if each finitely generated subgroup $H \subset G$ has locally compact closure in G. The class of ILC-groups includes all locally compact groups and all closed subgroups of topological vector spaces.

Our aim is to calculate the numbers $c_k(G)$ and $c_k^B(G)$ for an abelian ILC-group G. First, let us exclude two cases in which these numbers can be found in a trivial way. One of them happens if the number of colors is 1. In this case

$$c_1^B(G) = c_1(G) = \begin{cases} 1 & \text{if } G \text{ is not totally bounded,} \\ \infty & \text{if } G \text{ is totally bounded.} \end{cases}$$

The other trivial case happens if the Boolean subgroup $G[2] = \{x \in G : 2x = 0\} \subset G$ is unbounded in G. In this case, for each finite coloring $\chi : G \to k$ there is a color $i \in k$ such that the set $S = G[2] \cap \chi^{-1}(i)$ is unbounded. Since $S = -S$, we conclude that S is an unbounded monochromatic symmetric subset with respect to 0, which means that the singleton $\{0\}$ is k-centerpole in G and thus

$$c_k(G) = c_k^B(G) = 1 \quad \text{for all } k \in \mathbb{N}.$$

It remains to calculate the values of the cardinal numbers $c_k(G)$ and $c_k^B(G)$ for $k \ge 2$ and an abelian topological group G with totally bounded Boolean subgroup $G[2]$.

The following theorem reduces this problem of calculation of $c_k(G)$ to the case of the group $\mathbb{R}^n \oplus \mathbb{Z}^{m-n}$ where $n = r_{\mathbb{R}}(G)$ and $m = r_{\mathbb{Z}}(G)$.

Theorem 1 *Let $k \in \mathbb{N}$ and G be an abelian* ILC-*group G with totally bounded Boolean subgroup $G[2]$ and ranks $n = r_{\mathbb{R}}(G)$ and $m = r_{\mathbb{Z}}(G)$. Then*

(1) $c_k(G) = c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ *if $k \le m$, and*
(2) $c_k(G) \ge \omega$ *if $k > m$.*

If the topological group G is metrizable, then

(3) $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ *if* $k \le m$, *and*
(4) $c_k^B(G) \ge \omega$ *if* $k > m$.

Here we assume that $\omega - \omega = 0$ and $\omega - n = \omega$ for each $n \in \omega$.

Theorem 1 will be proved in Sect. 10. It reduces the problem of calculation of the numbers $c_k(G)$ and $c_k^B(G)$ to calculating these numbers for the groups $\mathbb{R}^n \times \mathbb{Z}^{m-n}$ where $n \le m$. The latter problem turned out to be highly non-trivial. In the following theorem we collect all the available information on the precise values of the numbers $c_k(\mathbb{R}^n \times \mathbb{Z}^m)$ and $c_k^B(\mathbb{R}^n \times \mathbb{Z}^m)$.

Theorem 2 *Let* k, n, m *be cardinal numbers.*

(1) *If* $n + m \ge 1$, *then* $c_1^B(\mathbb{R}^n \times \mathbb{Z}^m) = c_1(\mathbb{R}^n \times \mathbb{Z}^m) = 1$.
(2) *If* $n + m \ge 2$, *then* $c_2^B(\mathbb{R}^n \times \mathbb{Z}^m) = c_2(\mathbb{R}^n \times \mathbb{Z}^m) = 3$.
(3) *If* $n + m \ge 3$, *then* $c_3^B(\mathbb{R}^n \times \mathbb{Z}^m) = c_3(\mathbb{R}^n \times \mathbb{Z}^m) = 6$.
(4) *If* $n + m = 4$, *then* $c_4^B(\mathbb{R}^n \times \mathbb{Z}^m) = c_4(\mathbb{R}^n \times \mathbb{Z}^m) = 12$.
(5) *If* $k \ge n + m + 1 < \omega$, *then* $c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) = \infty$.
(6) *If* $k \ge n + m + 1$, *then* $c_k(\mathbb{R}^n \times \mathbb{Z}^m) = \infty$.
(7) *If* $n + m \ge \omega$ *and* $\omega \le k < \mathrm{cov}(\mathcal{M})$, *then* $c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) = \omega$.

In the last item by $\mathrm{cov}(\mathcal{M})$ we denote the smallest cardinality of the cover of the real line by meager subsets. It is known that $\aleph_1 \le \mathrm{cov}(\mathcal{M}) \le \mathfrak{c}$ and the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is equivalent to the Martin Axiom for countable posets, see [9, 19.9].

The equality $c_4(\mathbb{Z}^4) = 12$ from the statement (4) of Theorem 2 answers the problem of the calculation of $c_4(\mathbb{Z}^4)$ posed in [1] and then repeated in [5, Problem 2.4], [6, Problem 12], and [2, Question 4.5].

Theorem 2 presents all cases in which the exact values of the cardinals $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ and $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ are known. In the remaining cases we have some upper and lower bounds for these numbers. Because of the inequalities

$$c_k^B\big(\mathbb{R}^m\big) \le c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^{m-n}\big) \le c_k\big(\mathbb{R}^n \times \mathbb{Z}^{m-n}\big) \le c_k\big(\mathbb{Z}^m\big),$$

we see that the upper bounds for the numbers $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ and $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ would follow from the upper bounds for the numbers $c_k(\mathbb{Z}^m)$ while lower bounds from lower bounds on $c_k^B(\mathbb{R}^m)$.

Theorem 3 *For any numbers* $k \in \mathbb{N}$ *and* $n, m \in \mathbb{N} \cup \{\omega\}$, *we get*:

(1) $c_k(\mathbb{Z}^m) \le 2^k - 1 - \max_{s \le k-2} \binom{k-1}{s-1}$ *if* $k \le m$,
(2) $c_k^B(\mathbb{R}^k) \ge \frac{1}{2}(k^2 + 3k - 4)$ *if* $k \ge 4$,
(3) $c_k^B(\mathbb{R}^m) \ge k + 4$ *if* $m \ge k \ge 4$,
(4) $c_k^B(\mathbb{R}^n) < c_{k+1}^B(\mathbb{R}^{n+1})$ *and* $c_k(\mathbb{R}^n) < c_{k+1}(\mathbb{R}^{n+1})$ *if* $k \le n$,
(5) $c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}^B(\mathbb{R}^n \times \mathbb{Z}^{m+1})$ *and* $c_k(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1})$ *if* $k \le n + m$.

The binomial coefficient $\binom{k}{i}$ in statement (1) equals $\frac{k!}{i!(k-i)!}$ if $i \in \{0, \dots, k\}$ and zero otherwise. The upper bound from this statement improves the previously known upper bound $c_k(\mathbb{Z}^n) \le 2^k - 1$ proved in [1]. For $k = m \le 4$ it yields the upper bounds which coincide with the values of $c_k(\mathbb{Z}^m)$ given in Theorem 2.

The lower bound $c_n^B(\mathbb{R}^n) \ge \frac{1}{2}(n^2 + 3n - 4)$ from the item (2) improves the previously known lower bound $c_n^B(\mathbb{R}^n) \ge \frac{1}{2}(n^2 + n)$, proved in [1]. For $n = 4$ it gives the lower bound $12 \le c_4^B(\mathbb{R}^4)$, which coincides with the value of $c_4^B(\mathbb{R}^4) = c_4(\mathbb{Z}^4)$.

The statement (5) implies that the sequence $(c_k(\mathbb{Z}^k))_{k=1}^\infty$ is strictly increasing, which answers Question 2 posed in [1]. Theorem 3 will be proved in Sect. 8 after some preparatory work done in Sect. 2.

For every $k \in \mathbb{N}$ the sequence $(c_k(\mathbb{Z}^n))_{n=k}^\infty$ is non-increasing and thus it stabilizes starting from some n. The value of this number n is upper bounded by the cardinal number $rc_k^B(\mathbb{Z}^n)$ defined as follows.

For a topological group G and a number $k \in \mathbb{N}$ let $rc_k^B(G)$ be the minimal possible $\mathbb{Z}$-rank $r_\mathbb{Z}(\langle C\rangle)$ of a subgroup $\langle C\rangle$ of G generated by a k-centerpole subset $C \subset G$ of cardinality $|C| = c_k^B(G)$. If such a set C does not exist (which happens if $c_k^B(G) = \infty$), then we put $rc_k^B(G) = \infty$.

Theorem 4 (Stabilization) *Let $k \ge 2$ be an integer and G be an abelian* ILC*-group with totally bounded Boolean subgroup $G[2]$ and $\mathbb{R}$-rank $n = r_\mathbb{R}(G)$. Then*

(1) $c_k(G) = c_k^B(\mathbb{Z}^\omega)$ *if* $r_\mathbb{Z}(G) \ge rc_k^B(\mathbb{Z}^\omega)$,
(2) $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$ *if G is metrizable and* $r_\mathbb{Z}(G) \ge rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$,
(3) $c_k^B(G) = c_k^B(\mathbb{R}^\omega)$ *if G is metrizable and* $r_\mathbb{R}(\mathbb{R}) \ge rc_k^B(\mathbb{R}^\omega)$.

In light of Theorem 4 it is important to have lower and upper bounds for the numbers $rc_k(G)$.

Proposition 1 *For any metrizable abelian* ILC*-group G with totally bounded Boolean subgroup $G[2]$, and a natural number $2 \le k \le r_\mathbb{Z}(G)$ we get*

(1) $rc_k^B(G) = k$ *if* $k \le 3$, *and*
(2) $k \le rc_k^B(G) \le c_k^B(G) - 3$ *if* $k \ge 3$.

Finally, let us present the $(k + 1)$-centerpole subset Ξ_s^k of $\mathbb{R}^{1+k}$ that contains $2^k - 1 - \binom{k}{s}$ elements and gives the upper bound from Theorem 3(1). This $(k + 1)$-centerpole set Ξ_k is called the $\binom{k}{s}$-*sandwich.*

Definition 2 Let k be a non-negative integer and s be a real number. The subsets

$$\mathbf{2}^k_{<s} = \left\{(x_i) \in \mathbf{2}^k : \sum_{i=1}^k x_i < s\right\} \quad \text{and} \quad \mathbf{2}^k_{>s} = \left\{(x_i) \in \mathbf{2}^k : \sum_{i=1}^k x_i > s\right\}$$

are called the s-*slices* of the k-cube $\mathbf{2}^k$ where $\mathbf{2} = \{0, 1\}$ is the doubleton. For $s \in \{0, \dots, k\}$ the union of such slices has cardinality

$$\left|2^k_{<s} \cup 2^k_{>s}\right| = 2^k - \binom{k}{s} = 2^k - \frac{k!}{s!\,(k-s)!}.$$

The subset

$$\Xi^k_s = \big(\{-1\} \times \mathbf{2}^k_{<s}\big) \cup \big(\{0\} \times \mathbf{2}^k_{<k}\big) \cup \big(\{1\} \times \mathbf{2}^k_{>s}\big)$$

of the group $\mathbb{Z} \times \mathbb{Z}^k$ is called the $\binom{k}{s}$-*sandwich*. For $s \in \{0, \dots, k\}$ it has cardinality

$$\big|\Xi^k_s\big| = \big|\mathbf{2}^k_{<k}\big| \cup \big|\mathbf{2}^k_{<s} \cup \mathbf{2}^k_{>s}\big| = 2^{k+1} - 1 - \binom{k}{s}.$$

The following theorem implies the upper bound in Theorem 3(1). The proof of this theorem (given in Sect. 3) is not trivial and uses some elements of Algebraic Topology.

Theorem 5 *For every $k \in \mathbb{N}$ and $s \le k - 2$ the $\binom{k}{s}$-sandwich Ξ^k_s is a $(k+1)$-centerpole set in the group $\mathbb{Z} \times \mathbb{Z}^k$.*

In light of this theorem it is important to know the geometric structure of $\binom{k}{s}$-sandwiches Ξ^k_s for $s \le k-2$. For $k \le 3$ those sandwiches are written below:

- $\Xi^0_{-2} = \{(1,0)\}$ is a singleton in $\mathbb{Z} \times \mathbb{Z}^0 = \mathbb{Z} \times \{0\}$;
- $\Xi^1_{-1} = \{(0,1), (1,0), (1,1)\}$ is the unit square without a vertex in $\mathbb{Z}^2$;
- $\Xi^2_0 = \{(0,0,0), (0,0,1), (0,1,0), (1,0,1), (1,1,0), (1,1,1)\}$ is the unit cube without two opposite vertices in $\mathbb{Z}^3$;
- Ξ^3_0 is the unit cube without two opposite vertices in $\mathbb{Z}^4$, so $|\Xi^3_0| = 14$;
- Ξ^3_1 is a 12-element subset in $\mathbb{Z}^4$ whose slices $\{-1\} \times \mathbf{2}^3_{<1}$, $\{0\} \times \mathbf{2}^3_{<3}$, and $\{1\} \times \mathbf{2}^3_{>1}$ have one, seven, and four points, respectively.

By a *triangle* (*centered at the origin*) we shall understand any affinely independent subset $\{a, b, c\}$ in $\mathbb{R}^n$ (such that $a + b + c = 0$). A *tetrahedron* (*centered at the origin*) is any affinely independent subset $\{a, b, c, d\} \subset \mathbb{R}^n$ (with $a + b + c + d = 0$).

Let us observe that the sandwich

- Ξ^0_{-2} has cardinality $c_1(\mathbb{R}^1) = 1$ and is affinely equivalent to any singleton $\{a\}$ in $\mathbb{R}^1$;
- Ξ^1_{-1} has cardinality $c_2(\mathbb{R}^2) = 3$ and is affinely equivalent to any triangle $\Delta = \{a, b, c\}$ in $\mathbb{R}^2$;
- Ξ^2_0 has cardinality $c_3(\mathbb{R}^3) = 6$ and is affinely equivalent to $\Delta \cup (x - \Delta)$ where $\Delta \subset \mathbb{R}^3$ is a triangle centered at zero and $x \in \mathbb{R}^3$ does not belong to the linear span of Δ;
- Ξ^3_1 has cardinality $c_4(\mathbb{R}^4) = 12$ and is affinely equivalent to $(x - \Delta) \cup \Delta \cup (-x - \Delta)$ where $\Delta \subset \mathbb{R}^4$ is a tetrahedron centered at zero and $x \in \mathbb{R}^4$ does not belong to the linear span of Δ.

To see that Ξ^3_1 is of this form, observe that $c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the barycenter of Ξ^3_1 and $\Xi^3_1 - c = (x - \Delta) \cup \Delta \cup (-x - \Delta)$ for the tetrahedron

$$\Delta = \big\{(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,1,1,1)\big\} - c$$

and the point $x = (\frac{1}{2}, 0, 0, 0)$.

Now we briefly describe the structure of this paper. In Sect. 2 we establish a covering property of sandwiches, which will be essentially used in the proof of Theorem 5, given in Sect. 3. Section 4 is devoted to T-shaped sets which will give us lower bounds for the numbers $c_k^B(\mathbb{R}^k)$. In Sect. 5 we prove some lemmas that will help us to analyze the geometric structure of centerpole sets in Euclidean spaces. In Sect. 6 we study the interplay between centerpole properties of subsets in a group and those of its subgroups. In Sect. 7 we prove a particular case of the Stability Theorem 4 for the groups $\mathbb{R}^n \times \mathbb{Z}^{m-n}$. In Sects. 8, 9, and 10 we give the proofs of Theorems 3, 2, and 1, respectively. Sections 11 and 12 are devoted to the proofs of Proposition 1 and Theorem 4. The final Sect. 13 contains selected open problems.

2 Covering Σ_0-sets by shifts of the sandwich Ξ_s^k

In this section we shall prove a crucial covering property of the $\binom{k}{s}$-sandwich Ξ_s^k. In the next section this property will be used in the proof of Theorem 5. We assume that $k \in \omega$ and $s \le k-2$ is an integer.

First we introduce the notion of a Σ_0-subset of the cube $\mathbf{2}^{k+1} = \{0,1\}^{k+1}$. For $i \in \{0,\dots,k\}$ consider the ith coordinate projection

$$\mathrm{pr}_i : \mathbb{R}^{k+1} \to \mathbb{R}, \qquad \mathrm{pr}_i : (x_j)_{j=0}^k \mapsto x_i.$$

The subsets of the form $\mathbf{2}^{k+1} \cap \mathrm{pr}_i^{-1}(l)$ for $l \in \{0,1\}$ are called the *facets* of the cube $\mathbf{2}^{k+1}$.

Next, consider the function

$$\Sigma : \mathbb{R}^{k+1} \to \mathbb{R}, \qquad \Sigma : (x_i)_{i=0}^k \mapsto \sum_{i=1}^k x_i,$$

and observe that $\Sigma(\mathbf{2}^{k+1}) = \{0,\dots,k\}$.

Taking the diagonal product of the functions pr_0 and Σ, we obtain the linear operator

$$\Sigma_0 : \mathbb{R}^{k+1} \to \mathbb{R}^2, \qquad \Sigma_0 : (x_i)_{i=0}^k \mapsto \left(x_0, \sum_{i=1}^k x_i\right).$$

Definition 3 A subset $\tau \subset \mathbf{2}^{k+1}$ will be called a Σ_0-*set* if

- τ lies in a facet of $\mathbf{2}^{k+1}$;
- there exists $a \in \{0,\dots,k-1\}$ such that $\Sigma_0(\tau) \subset \{(0,a),(0,a+1),(1,a+1)\}$ or $\Sigma_0(\tau) \subset \{(0,a),(1,a),(1,a+1)\}$.

Lemma 1 *Each Σ_0-set $\tau \subset \mathbf{2}^{k+1}$ is covered by a suitable shift $x + \Xi_s^k$ of the $\binom{k}{s}$-sandwich Ξ_s^k.*

Proof Decompose the Σ_0-set τ into the union $\tau = \tau_0 \cup \tau_1$ where $\tau_i = \tau \cap \mathrm{pr}_0^{-1}(i)$ for $i \in \{0, 1\}$. By our hypothesis τ lies in a facet of the cube $\mathbf{2}^{k+1}$. Consequently, there are numbers $\gamma \in \{0, \dots, k\}$ and $l \in \{0, 1\}$ such that $\tau \subset \mathrm{pr}_\gamma^{-1}(l)$. If τ_0 or τ_1 is empty, then we can change the facet and assume that $\gamma = 0$.

Since τ is a Σ_0-set, the image $\Sigma_0(\tau)$ lies in one of the triangles: $\{(0, a), (0, a+1), (1, a+1)\}$ or $(\{(0, a), (1, a), (1, a+1)\}$ for some $a \in \{0, \dots, k-1\}$. This implies that $\Sigma(\tau) \subset \{a, a+1\}$.

Identify the cube $\mathbf{2}^k$ with the subcube $\{0\} \times \mathbf{2}^k$ of Ξ_s^k and let $\mathbf{e}_0 = (1, 0, \dots, 0) \in \mathbf{2}^{k+1}$. Then

$$\Xi_s^k = \mathbf{2}^k_{<k} \cup \left(\mathbf{e}_0 + \mathbf{2}^k_{>s}\right) \cup \left(-\mathbf{e}_0 + \mathbf{2}^k_{<s}\right).$$

Depending on the value of γ, two cases are possible.

0. $\gamma = 0$. This case has four subcases.
   0.1. If $l = 0$ and $a < k-1$ then $\Sigma_0(\tau) \subset \{(0, a), (0, a+1)\} \subset \{0, \dots, k-1\}$ and $\tau \subset \mathbf{2}^k_{<k} \subset \Xi_s^k$.
   0.2. If $l = 0$ and $a \ge k-1$, then $a > k-2 \ge s$ and $\tau \subset \mathbf{2}^k_{>s} \subset -\mathbf{e}_0 + \Xi_s^k$.
   0.3. If $l = 1$ and $a < k-1$, then $\Sigma_0(\tau) \subset \{(1, a), (1, a+1)\} \subset \{0, \dots, k-1\}$ and hence $\tau \subset \mathbf{e}_0 + \mathbf{2}^k_{<k} \subset \mathbf{e}_0 + \Xi_s^k$.
   0.4. If $l = 1$ and $a \ge k-1$, then $a > k-2 \ge s$ and then $\tau \subset \mathbf{e}_0 + \mathbf{2}^k_{>s} \subset \Xi_s^k$.

I. $\gamma \ne 0$. In this case τ_0 and τ_1 are not empty. Let $\mathbf{e}_\gamma$ be the basic vector whose γth coordinate is 1 and the others are zero. By our assumption, $\Sigma_0(\tau) \subset \{(0, a), (1, a), (1, a+1)\}$ or $\Sigma_0(\tau) \subset \{(0, a), (0, a+1), (1, a+1)\}$ for some $a \in \{0, \dots, k-1\}$. So, we consider two subcases.
   I.1. $\Sigma_0(\tau) \subset \{(0, a), (1, a), (1, a+1)\}$. This case has two subcases.
      I.1.0. $l = 0$. In this subcase $\Sigma(\tau) = \Sigma(\tau_0) \cup \Sigma(\tau_1) = \{a, a+1\} \subset \{0, \dots, k-1\}$ and hence $a \le k-2$. Depending on the value of a, we have three possibilities.
         If $a > s$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<k} \cup (\mathbf{e}_0 + \mathbf{2}^k_{>s}) \subset \Xi_s^k$.
         If $a = s$, then for the shifted set $\mathbf{e}_\gamma + \tau$ we get

$$\Sigma_0(\mathbf{e}_\gamma + \tau) \subset \left\{(0, a+1), (1, a+1), (1, a+2)\right\}.$$

Since $a = s \le k-2$, we conclude that $\mathbf{e}_\gamma + \tau_0 \subset \mathbf{2}^k_{<k} \subset \Xi_s^k$. On the other hand, $\mathbf{e}_\gamma + \tau_1 \subset \mathbf{e}_1 + \mathbf{2}^k_{>s} \subset \Xi_s^k$. Then $\tau \subset -\mathbf{e}_\gamma + \Xi_s^k$.
         If $a < s$, then $a+1 \le s \le k-2$ and hence $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) \subset \mathbf{e}_0 + \Xi_s^k$.
      I.1.1. $l = 1$. In this subcase three possibilities can occur:
         If $a > s$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<k} + (\mathbf{e}_0 + \mathbf{2}^k_{>s}) \subset \Xi_s^k$;
         If $a < s$, then $a+1 \le s \le k-2$ and then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) \subset \mathbf{e}_0 + \Xi_s^k$.
         If $a = s$, then for the shift $-\mathbf{e}_\gamma + \tau$ we get $\Sigma_0(-\mathbf{e}_\gamma + \tau) \subset \{(0, a-1), (1, a-1), (1, a)\}$ and hence $-\mathbf{e}_\gamma + \tau \subset \mathbf{2}^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) \subset \mathbf{e}_0 + \Xi_s^k$. Consequently, $\tau \subset \mathbf{e}_\gamma + \mathbf{e}_0 + \Xi_s^k$.
   I.2. $\Sigma_0(\tau) \subset \{(0, a), (0, a+1), (1, a+1)\}$. Depending on the value of $l \in \{0, 1\}$, consider two subcases.

I.2.0. $l = 0$. In this case $\{0, \dots, k-1\} \supset \Sigma(\tau) = \Sigma(\tau_0) \cup \Sigma(\tau_1) = \{a, a+1\} \cup \{a+1\}$ and consequently, $a+1 \le k-1$.

If $a \ge s$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<k} \cup (\mathbf{e}_0 + \mathbf{2}^k_{>s}) \subset \Xi^k_s$.

If $a = s-1$, then we can consider the shift $\mathbf{e}_\gamma + \tau$ and repeating the preceding argument, show that $\mathbf{e}_\gamma + \tau \subset \Xi^k_s$. Consequently, $\tau \subset -\mathbf{e}_\gamma + \Xi^k_s$.

If $a < s-1$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) \subset \mathbf{e}_0 + \Xi^k_s$.

I.2.1. $l = 1$. In this case we have four subcases.

If $a = k-1$, then for the shifted set $-\mathbf{e}_\gamma + \tau$ we get $\Sigma_0(-\mathbf{e}_\gamma + \tau) \subset \{(0, a-1), (0, a), (1, a)\}$ and $-\mathbf{e}_\gamma + \tau \subset \mathbf{2}^k_{<k} \cup (\mathbf{e}_0 + \mathbf{2}^k_{>s}) = \Xi^k_s$. Then $\tau \subset \mathbf{e}_\gamma + \Xi^k_s$.

If $s \le a < k-1$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<k} \cup (\mathbf{e}_0 + \mathbf{2}^k_{>s}) = \Xi^k_s$.

If $a = s-1$, then for the shifted set $-\mathbf{e}_\gamma + \tau$ we get $\Sigma_0(-\mathbf{e}_\gamma + \tau) \subset \{(0, a-1), (0, a), (1, a)\}$ and then $-\mathbf{e}_\gamma + \tau \subset 2^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) = \mathbf{e}_0 + \Xi^k_s$ and $\tau \subset \mathbf{e}_\gamma + \mathbf{e}_0 + \Xi^k_s$.

If $a < s-1$, then $\tau = \tau_0 \cup \tau_1 \subset \mathbf{2}^k_{<s} \cup (\mathbf{e}_0 + \mathbf{2}^k_{<k}) = \mathbf{e}_0 + \Xi^k_s$.

This was the last of the 17 cases we have considered.

□

3 Proof of Theorem 5

The proof of Theorem 5 uses the idea of the proof of Lemma 6 in [1] (which established the upper bound $c_3(\mathbb{Z}^3) \le 6$).

We need to prove that for every $k \le n$ and $s \le k-2$ the $\binom{k}{s}$-sandwich Ξ^k_s is $(k+1)$-centerpole in $\mathbb{Z} \times \mathbb{Z}^k = \mathbb{Z}^{1+k}$. Assuming that this is not true, find a coloring $\chi : \mathbb{Z}^{1+k} \to k+1 = \{0, \dots, k\}$ such that $\mathbb{Z}^{1+k}$ contains no unbounded monochromatic subset, symmetric with respect to some point $c \in \Xi^k_s$. Observe that for each color $i \in \{0, \dots, k\}$ the intersection $A_i \cap (2c - A_i)$ is the largest subset of A_i, symmetric with respect to the point c. By our assumption, the (maximal i-colored c-symmetric) set $A_i \cap (2c - A_i)$ is bounded and so is the union

$$B = \bigcup_{i=0}^{k} \bigcup_{c \in \Xi^k_s} A_i \cap (2c - A_i)$$

of all such maximal symmetric monochromatic subsets.

Claim 1 $\chi(x) \notin \chi(-x + 2\Xi^k_s)$ *for any* $x \notin B$.

Proof Assuming conversely that $\chi(x) = \chi(-x + 2c)$ for some $c \in \Xi^k_s$, we get $\frac{1}{2}(x + (-x + 2c)) = c$ and hence x and $-x + 2c$ are two points symmetric with respect to the center $c \in \Xi^k_s$ and colored by the same color. Consequently, $x \in B$ by the definition of B. □

Fix a number $n \in \mathbb{N}$ so big that the cube $K = [-2n, 2n]^{1+k} \subset \mathbb{R}^{1+k}$ contains the bounded set B in its interior and let ∂K be the topological boundary ∂K of the cube K in $\mathbb{R}^{1+k}$. Observe that Claim 1 implies:

Claim 2 *$\chi(-x) \notin \chi(x + 2\,\Xi^k_s)$ for each point $x \in \mathbb{Z}^{1+k} \cap \partial K$.*

We recall that for every $i \in k+1 = \{0, \dots, k\}$

$$\mathrm{pr}_i : \mathbb{R}^{1+k} \to \mathbb{R}, \qquad \mathrm{pr}_i : (x_j)_{j=0}^k \mapsto x_i,$$

denotes the ith coordinate projection and $\mathbf{e}_i$ is the unit vector along the ith coordinate axis, that is, $\mathrm{pr}_j(\mathbf{e}_i) = 1$ if $i = j$, and 0 otherwise.

For a subset $J \subset \{0, \dots, k\}$ let $\mathbf{e}_J = \sum_{j \in J} \mathbf{e}_j \in \mathbb{R}^{1+k}$ be the vector of the principal diagonal of the cube $\mathbf{2}^J = \{(x_i)_{i=0}^k \in \mathbf{2}^{1+k} : \forall i \notin J\ (x_i = 0)\} \subset \mathbf{2}^{1+k}$.

For a point $x \in \mathbb{R}^{1+k}$ let $J_x = \{i \in k+1 : x_i \notin 2\mathbb{Z}\}$ and let $\lfloor x \rfloor$ be the unique point in $(2\mathbb{Z})^{1+k}$ such that $x \in \lfloor x \rfloor + 2 \cdot \mathbf{2}^{J_x}$. So, $\lfloor x \rfloor \le x \le \lfloor x \rfloor + 2\,\mathbf{e}_{J_x}$.

Consider the function $\Sigma : \mathbb{R}^{k+1} \to \mathbb{R}$ assigning to each sequence $x = (x_i)_{i=0}^k$ the sum $\Sigma(x) = \sum_{i=1}^k x_i$. The map Σ combined with the 0th coordinate projection pr_0 compose the linear operator

$$\Sigma_0 : \mathbb{R}^{1+k} \to \mathbb{R}^2, \qquad \Sigma_0 : (x_i)_{i=0}^k \mapsto \bigl(x_0, \Sigma(x)\bigr) = \left(x_0, \sum_{i=1}^k x_i\right).$$

Choose a triangulation T of the boundary ∂K of the cube $K = [-2n, 2n]^{1+k}$ such that for each simplex τ of the triangulation there is a point $\dot\tau \in (2\mathbb{Z})^{1+k}$ such that $\frac{1}{2}(\tau - \dot\tau)$ is a Σ_0-subset of $\mathbf{2}^{1+k}$. The reader can easily check that such a triangulation T always exists. The choice of the triangulation T combined with Lemma 1 implies

Claim 3 *Each simplex τ of the triangulation T is covered by a suitable shift $x + 2\Xi^k_s$ of the homothetic copy $2\Xi^k_s$ of the $\binom{k}{s}$-sandwich Ξ^k_s.*

Let Δ be (the geometric realization of) a simplex in $\mathbb{R}^k$ with vertices $w_0, \dots, w_k$ such that $w_0 + \dots + w_k = 0$. The latter equality means that Δ is centered at the origin (which lies in the interior of Δ). By $\Delta^{(0)} = \{w_0, \dots, w_k\}$ we denote the set of vertices of the simplex Δ.

Each point $y \in \Delta$ can be uniquely written as the convex combination $y = \sum_{i=0}^k y_i w_i$ for some non-negative real numbers $y_0, \dots, y_k$ with $\sum_{i=0}^k y_i = 1$. The set

$$\mathrm{supp}(y) = \bigl\{i \in \{0, \dots, k\} : y_i \ne 0\bigr\}$$

is called the *support* of y. It is clear that $\mathrm{supp}(y)$ is the smallest subset of $\Delta^{(0)}$ whose convex hull contains the point y.

Identifying each number $i \in \{0, \dots, k\}$ with the vertex w_i of Δ, we can think of the coloring $\chi : \mathbb{Z}^{1+k} \to \{0, \dots, k\}$ as a function $\chi : \mathbb{Z}^{1+k} \to \Delta^{(0)} = \{w_0, \dots, w_k\}$.

Now extend the restriction $\chi|\partial K \cap (2\mathbb{Z})^{1+k}$ of χ to a simplicial map $f : \partial K \to \Delta$ (which is affine on the convex hull of each simplex $\tau \in T$). The simpliciality of f implies

Claim 4 *For each simplex $\tau \in T$ and a point $x \in \mathrm{conv}(\tau)$*

$$\mathrm{supp}\bigl(f(x)\bigr) \subset \chi(\tau) \subset \chi\bigl(\lfloor x \rfloor + 2 \cdot \mathbf{2}^{J_x}\bigr).$$

This claim has the following corollary.

Claim 5 $f(\partial K)\subset\partial\Delta$.

Proof Given any point $x\in\partial K$, find a simplex $\tau\in T$ whose convex hull contains x. By the choice of the triangulation T and Lemma 1, $\tau\subset -y+2\Xi^k_s$ for some point $y\in\mathbb{Z}^{1+k}$. By Claim 2, $\chi(-y)\notin\chi(\tau)$ and thus

$$f(x)\in\operatorname{conv}\bigl(f(\tau)\bigr)=\operatorname{conv}\bigl(\chi(\tau)\bigr)\subset\operatorname{conv}\bigl(\Delta^{(0)}\setminus\chi(-y)\bigr)\subset\partial\Delta. \qquad \square$$

Now consider the intersection $K_0=\{0\}\times[-2n,2n]^k$ of the cube K with the hyperplane $\{0\}\times\mathbb{R}^k$, which will be identified with the space $\mathbb{R}^k$, and let $\partial K_0=\partial K\cap\mathbb{R}^k$ be the boundary of K_0.

For each subset $J\subset k+1=\{0,\dots,k\}$ consider the map

$$p_J:\mathbb{R}^{1+k}\to\mathbb{R},\qquad p_J:(x_i)_{i=0}^k\mapsto 1\cdot\prod_{j\in J}x_j.$$

Here we assume that $p_\emptyset(x)=1$. It follows that $\sum_{J\subset k+1}p_J(x)>0$ for all $x\in[0,2]^{k+1}$.

We remind that for a point $x\in\mathbb{R}^{1+k}$, $J_x=\{i\in\{0,\dots,k\}:x_i\notin 2\mathbb{Z}\}$ and $\lfloor x\rfloor$ stands for the unique point in $(2\mathbb{Z})^{1+k}$ such that $x\in\lfloor x\rfloor+\mathbf{2}^{J_x}$ where $\mathbf{2}^J=\{(x_i)_{i=0}^k\in\mathbf{2}^{k+1}:\forall i\notin J\ (x_i=0)\}$.

Now consider the map $\varphi:\partial K_0\to\Delta$ defined by the formula

$$\varphi(x)=\frac{\sum_{J\subset k+1}p_J(x-\lfloor x\rfloor)\cdot\chi(\lfloor x\rfloor+\mathbf{e}_J)}{\sum_{J\subset k+1}p_J(x-\lfloor x\rfloor)}.$$

It can be shown that the map φ is well-defined and continuous.

Claim 6 $\operatorname{supp}(\varphi(x))=\chi(\lfloor x\rfloor+2\cdot\mathbf{2}^{J_x})\subset\chi(\lfloor x\rfloor+2\Xi^k_s)$ *for all* $x\in\partial K_0$.

Proof Let $x\in\partial K_0$ be any point. The definition of φ implies that $\operatorname{supp}(\varphi(x))=\chi(\lfloor x\rfloor+\mathbf{2}^{J_x})$. The inclusion $x\in\partial K_0$ implies that the set $J_x=\{j\in\{0,\dots,k\}:\operatorname{pr}_j(x)\notin 2\mathbb{Z}\}$ has cardinality $|J_x|<k$ and thus $\mathbf{2}^{J_x}\subset\{0\}\times\mathbf{2}^k_{<k}\subset\Xi^k_s$. Consequently, $\lfloor x\rfloor+2\cdot\mathbf{2}^{J_x}\subset\lfloor x\rfloor+2\Xi^k_s$ and $\chi(\lfloor x\rfloor+2\cdot\mathbf{2}^{J_x})\subset\chi(\lfloor x\rfloor+2\Xi^k_s)$. $\square$

Claim 7 $\varphi(x)\neq\varphi(-x)$ *for all* $x\in\partial K_0$.

Proof Observe that $J_x=J_{-x}$ and $\lfloor -x\rfloor=-\lfloor x\rfloor-2\mathbf{e}_{J_x}$. By Claim 6,

$$\chi\bigl(-\lfloor x\rfloor\bigr)=\chi\bigl(\lfloor -x\rfloor+2\cdot\mathbf{e}_{-J_x}\bigr)\in\chi\bigl([-x]+2\cdot\mathbf{2}^{J_{-x}}\bigr)=\operatorname{supp}\bigl(\varphi(-x)\bigr).$$

On the other hand, Claim 1 guarantees that

$$\chi\bigl(-\lfloor x\rfloor\bigr)\not\ni\chi\bigl(\lfloor x\rfloor+2\Xi^k_s\bigr)\supset\chi\bigl(\lfloor x\rfloor+2\cdot\mathbf{2}^{J_x}\bigr)=\operatorname{supp}\bigl(\varphi(x)\bigr).$$

Consequently, $\operatorname{supp}(\varphi(-x))\neq\operatorname{supp}(\varphi(x))$ and $\varphi(x)\neq\varphi(-x)$.

Finally, consider the homotopy

$$(f_t) : \partial K_0 \times [0,1] \to \Delta, \qquad f_t : x \mapsto t\varphi(x) + (1-t)f(x),$$

connecting the map $f = f_0$ with the map $\varphi = f_1$. □

Claim 8 $\operatorname{supp}(f_t(x)) \subset \chi(\lfloor x \rfloor + 2 \cdot \mathbf{2}^{J_x}) \subset \partial\Delta$ *for all* $x \in \partial K_0$ *and* $t \in [0,1]$.

Proof The inclusion $\operatorname{supp}(f_t(x)) \subset \chi(\lfloor x \rfloor + 2 \cdot \mathbf{2}^{J_x})$ follows from Claims 4 and 6.

The inclusion $x \in \partial K_0$ implies that the set $J_x = \{j \in \{0,\dots,k\} : \operatorname{pr}_j(x) \notin 2\mathbb{Z}\}$ has cardinality $|J_x| < k$ and thus $\mathbf{2}^{J_x} \subset \{0\} \times \mathbf{2}^k_{<k} \subset \Xi^k_s$. By Claim 1, $\chi(-\lfloor x \rfloor) \notin \chi(\lfloor x \rfloor + 2\Xi^k_s)$ and then

$$\begin{aligned} f_t(x) &\in \operatorname{conv}\bigl(\operatorname{supp}\bigl(f_t(x)\bigr) \subset \operatorname{conv}\bigl(\chi\bigl(\lfloor x \rfloor + 2 \cdot \mathbf{2}^{J_x}\bigr)\bigr) \\ &\subset \operatorname{conv}\bigl(\chi\bigl(\lfloor x \rfloor + 2\Xi^k_s\bigr)\bigr) \subset \operatorname{conv}\bigl(\Delta^{(0)} \setminus \chi\bigl(-\lfloor x \rfloor\bigr)\bigr) \subset \partial\Delta. \end{aligned}$$ □

Let $S^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^k$ with respect to the Euclidean norm $\|\cdot\|$ and $r : \mathbb{R}^k \setminus \{0\} \to S^{k-1}$, $r : x \mapsto x/\|x\|$, be the radial retraction. Observe that its restriction $r|\partial\Delta$ to the boundary of the geometric simplex Δ is a homeomorphism.

By Claim 5, $f(\partial K) \subset \partial\Delta \subset \mathbb{R}^k \setminus \{0\}$, so we can consider the map $g_0 : \partial K \to S^{k-1}$ defined by $g_0(x) \mapsto r \circ f(x) = f(x)/\|f(x)\|$. By Claim 8, the map $g_0|\partial K_0$ is homotopic to the map

$$g_1 : \partial K_0 \to S^{k-1}, \qquad g_1(x) \mapsto r \circ f_1(x) = r \circ \varphi(x).$$

It follows from Claim 7 that $g_1(x) \neq g_1(-x)$ for all $x \in \partial K_0$. This implies that the formula

$$h_t(x) = \frac{g_1(x) - t g_1(-x)}{\|g_1(x) - t g_1(-x)\|}, \quad x \in \partial K_0,\ t \in [0,1],$$

determines a well-defined homotopy $(h_t) : \partial K_0 \to S^{k-1}$ connecting the map g_1 with the map

$$h_1(x) = \frac{g_1(x) - g_1(-x)}{\|g_1(x) - g_1(-x)\|},$$

which is antipodal in the sense that $h_1(-x) = -h_1(x)$. By [13, Chap. 4, Sect. 7.10], each antipodal map between spheres of the same dimension is not homotopically trivial. Consequently, the antipodal map $h_1 : \partial K_0 \to S^{k-1}$ is not homotopically trivial. On the other hand, h_1 is homotopic to the map $h_0 = g_1$, which is homotopic to $g_0|\partial K_0$ and the latter map is homotopically trivial since the boundary ∂K_0 of the cube K_0 is contractible in the boundary ∂K of K. This contradiction completes the proof of Theorem 5. □

4 T-shaped sets in $\mathbb{R}^n$

Theorem 5, proved in the preceding section, yields an upper bound for the numbers $c_k(\mathbb{Z}^k)$. A lower bound for the numbers $c_k^B(\mathbb{R}^k)$ will be obtained by the technique of T-shaped sets created in [1].

Let $\mathbb{R}_+ = [0, \infty)$ be the closed half-line. For every $n \geq 0$ consider the subset $T_0 \subset \mathbb{R}^0$ defined inductively:

$$T_0 = \emptyset \subset \mathbb{R}^0 = \{0\}, \qquad T_1 = \{0\} \subset \mathbb{R}^1, \quad \text{and}$$
$$T_n = \big(\mathbb{R}^{n-1} \times \{0\}\big) \cup (T_{n-1} \times \mathbb{R}_+) \subset \mathbb{R}^n$$

for $n > 1$.

Definition 4 A subset $C \subset \mathbb{R}^n$ is called T-*shaped* if $f(C) \subset \mathbb{R} \times T_{n-1}$ for some affine transformation $f : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^{n-1}$. The smallest cardinality of a subset $A \subset \mathbb{R}^n$, which is not T-shaped is denoted by $t(\mathbb{R}^n)$.

Let us describe the geometric structure of T-shaped sets.

We say that for $k \leq n$, hyperplanes $H_1, \dots, H_k$ in $\mathbb{R}^n$ are in *general position* if they are pairwise distinct and their normal vectors are linearly independent. This happens if and only if there is an affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ that maps the ith hyperplane onto the hyperplane $\mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{n-i}$ for all $i \in \{1, \dots, k\}$.

We shall say that a hyperplane $H \subset \mathbb{R}^n$ *does not separate* a subset $S \subset \mathbb{R}^{n+1}$ if S lies in one of two closed half-spaces bounded by the hyperplane H. Such a hyperplane H will be called *non-separating* for S. A hyperplane H is called a *support hyperplane* for S if $H \cap S \neq \emptyset$ and H does not separate S.

Proposition 2 *Let $n \in \mathbb{N}$. A subset $S \subset \mathbb{R}^{n+1}$ is T-shaped if and only if*

$$S \subset H_1 \cup \cdots \cup H_n$$

for some hyperplanes $H_1, \dots, H_n$ in general position such that each hyperplane H_i, $1 \leq i \leq n$, does not separate the set $S \setminus (H_1 \cup \cdots \cup H_{i-1})$.

Proof This proposition can be easily derived from the equality

$$\mathbb{R} \times T_n = \bigcup_{i=0}^{n-1} \mathbb{R}^{n-i} \times \{0\} \times \mathbb{R}_+^i$$

that can be easily proved by induction on n. □

By Lemma 7 of [1], T-shaped subsets of Euclidean spaces $\mathbb{R}^k$ are k-centerpole for Borel colorings. Consequently, $t(\mathbb{R}^n) \leq c_n^B(\mathbb{R}^n)$. This gives us a lower bound for the numbers $c_k^B(\mathbb{R}^n)$ and $c_k(\mathbb{R}^n)$:

Proposition 3 $t(\mathbb{R}^k) \leq c_k^B(\mathbb{R}^k) \leq c_k^B(\mathbb{R}^n) \leq c_k(\mathbb{R}^n)$ *for any finite $k \leq n$.*

In the following theorem we collect all the available information on the numbers $t(\mathbb{R}^n)$.

Theorem 6

1. $t(\mathbb{R}^1)=1$,
2. $t(\mathbb{R}^2)=3$,
3. $t(\mathbb{R}^3)=6$,
4. $t(\mathbb{R}^4)=12$,
5. $t(\mathbb{R}^n)\le n^2-n+1$ *for every* $n\ge 1$,
6. $t(\mathbb{R}^n)\ge t(\mathbb{R}^{n-1})+n+1$ *for any* $n\ge 4$,
7. $t(\mathbb{R}^n)\ge \frac{1}{2}(n^2+3n-4)$ *for any* $n\ge 4$.

Proof

1. Since $T_0=\emptyset$, a subset of $\mathbb{R}^1$ is T-shaped if and only if it is empty. Consequently, $t(\mathbb{R}^1)=1$.
2. Since $T_1=\{0\}\subset\mathbb{R}^1$, a subset $C\subset\mathbb{R}^2$ is T-shaped if and only if C lies in an affine line. Consequently, $t(\mathbb{R}^2)=3$.
3. By Theorem 5, the 6-element $\binom{2}{0}$-sandwich Ξ^2_0 is 3-centerpole in $\mathbb{R}^3$. Consequently, $c_3(\mathbb{R}^3)\le 6$. By Proposition 3, $t(\mathbb{R}^3)\le c_3(\mathbb{R}^3)\le 6$. To see that $t(\mathbb{R}^3)\ge 6$, we need to check that a subset $C\subset\mathbb{R}^3$ of cardinality $|C|\le 5$ is T-shaped, which means that after a suitable affine transformation of $\mathbb{R}^3$, C can be embedded into $\mathbb{R}\times T_2$. By the definition, $T_2=\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}_+$.

   Consider the convex hull $\mathrm{conv}(C)$ of C in $\mathbb{R}^3$. If C lies in an affine plane H, then applying to $\mathbb{R}^3$ a suitable affine transformation, we can assume that $C\subset H=\mathbb{R}\times\mathbb{R}\times\{0\}\subset\mathbb{R}\times T_2$. If C does not lie in a plane, then the convex polyhedron $\mathrm{conv}(C)$ has a supporting plane H_1 such that $|H_1\cap C|\ge 3$. So, $C\setminus H_1$ lies in one of the closed half-spaces with respect to the plane H_1. Denote this subspace by H_1^+. The set $C\setminus H_1$ has cardinality $|C\setminus H_1|\le 2$ and hence it lies in an affine plane $H_2\subset\mathbb{R}^3$ that meets H_1. Find an affine transformation $f:\mathbb{R}^3\to\mathbb{R}^3$ such that $f(H_1)=\mathbb{R}\times\mathbb{R}\times\{0\}$, $f(H_1^+)=\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ and $f(H_2)=\{\mathbb{R}\}\times\{0\}\times\{\mathbb{R}\}$. Then

   $$f(C)\subset\mathbb{R}\times\mathbb{R}\times\{0\}\cup\mathbb{R}\times\{0\}\times\mathbb{R}_+=\mathbb{R}\times T_2$$

   and hence C is T-shaped.
4. By Theorem 5, the $\binom{3}{1}$-sandwich Ξ^3_1 is 4-centerpole in $\mathbb{Z}^4$. Consequently,

   $$t(\mathbb{R}^4)\le c_4(\mathbb{R}^4)\le c_4(\mathbb{Z}^4)\le\left|\Xi^3_1\right|=2^4-1-\binom{3}{1}=12.$$

   The reverse inequality $t(\mathbb{R}^4)\ge 12$ will be proved in Lemma 2 below.
5. Let $C\subset\mathbb{R}^n$ be a set consisting of $n^2-n+1=n(n-1)+1$ points in general position. This means that no $(n+1)$-element subset of C lies in a hyperplane. Then C cannot be covered by less than n hyperplanes and consequently C is not T-shaped (because the set $\mathbb{R}\times T_{n-1}$ lies in the union of $(n-1)$ hyperplanes). Then $t(\mathbb{R}^n)\le|C|=n^2-n+1$.

6. First we prove the inequality

$$t(\mathbb{R}^n) \geq \min\{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\} \tag{1}$$

for every $n \geq 2$. Take any subset $C \subset \mathbb{R}^n$ of cardinality $|C| < \min\{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\}$. We need to show that C is T-shaped.

Consider the convex hull $\mathrm{conv}(C)$ of C in $\mathbb{R}^n$. If $\mathrm{conv}(C)$ lies in some hyperplane, then C is T-shaped by the definition. So, we assume that $\mathrm{conv}(C)$ does not lie in a hyperplane and then $\mathrm{conv}(C)$ is a compact convex body in $\mathbb{R}^n$. Let H be a supporting hyperplane of $\mathrm{conv}(C)$ having maximal possible cardinality of the intersection $C \cap H$. It is clear that $|C \cap H| \geq n$.

Now two cases are possible:

(a) The set $C \setminus H$ lies in a hyperplane H_1, parallel to H. Then H_1 is a supporting hyperplane of $\mathrm{conv}(C)$ and then $|C \cap H_1| \leq |C \cap H|$ by the choice of H. Now we see that $|C \cap H_1| \leq \frac{1}{2}|C| < t(\mathbb{R}^{n-1})$.

Applying to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ a suitable affine transformation, we can assume that $H = \mathbb{R}^{n-1} \times \{0\}$ and $C \setminus H \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. Let $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection. Since $|\mathrm{pr}_n(C \setminus H)| < t(\mathbb{R}^{n-1})$, the set $C' = \mathrm{pr}_n(C \setminus H)$ is T-shaped. This means that there is an affine transformation $f : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $f(C') \subset \mathbb{R} \times T_{n-2}$. This affine transformation f induces the affine transformation

$$\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}, \qquad \Phi(x, y) = \bigl(f(x), y\bigr),$$

such that

$$\begin{aligned}\Phi(C) &= \Phi(C \cap H) \cup \Phi(C \setminus H) \subset \bigl(\mathbb{R} \times \mathbb{R}^{n-2} \times \{0\}\bigr) \cup (\mathbb{R} \times T_{n-2} \times \mathbb{R}_+) \\ &= \mathbb{R} \times T_{n-1}.\end{aligned}$$

The affine transformation Φ witnesses that the set C is T-shaped.

(b) The set $C \setminus H$ does not lie in a hyperplane parallel to H. Then $C \setminus H$ contains two distinct points x, y such that the vector $\vec{xy}$ is not parallel to H. Applying to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ a suitable affine transformation, we can assume that $H = \mathbb{R}^{n-1} \times \{0\}$, $C \setminus H \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$, and under the projection $\mathrm{pr} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ the images of the points x and y coincide. Then the projection $C' = \mathrm{pr}(C \setminus H)$ has cardinality $|C'| \leq |C \setminus H| - 1 < |C| - |C \cap H| - 1 < t(\mathbb{R}^{n-1}) + n + 1 - n - 1 = t(\mathbb{R}^{n-1})$. Continuing as in the preceding case, we can find an affine transformation Φ, witnessing that C is a T-shaped set in $\mathbb{R}^n$.

This proves the inequality (1). By analogy we can prove that $t(\mathbb{R}^n) \geq t(\mathbb{R}^{n-1}) + n$. Since $t(\mathbb{R}^1) = 1$, by induction we can show that $t(\mathbb{R}^n) \geq \frac{1}{2}n(n+1)$. In particular, $t(\mathbb{R}^{n-1}) \geq \frac{1}{2}n(n-1) \geq n+1$ for all $n \geq 4$. In this case

$$t(\mathbb{R}^n) \geq \min\{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\} = t(\mathbb{R}^{n-1}) + n + 1.$$

7. The lower bound $t(\mathbb{R}^n) \geq \frac{1}{2}(n^2 + 3n - 4)$, $n \geq 4$, will be proved by induction. For $n = 4$ it is true according to the statement (4). Assuming that it is true for some

$n > 4$ and applying the lower bound (6), we get

$$t(\mathbb{R}^{n+1}) \geq t(\mathbb{R}^n) + (n+1) + 1 \geq \frac{1}{2}(n^2 + 3n - 4) + n + 2$$
$$= \frac{1}{2}((n+1)^2 + 3(n+1) - 4).$$

To finish the proof of Theorem 6, it remains to prove the promised:

Lemma 2 *Each subset $C \subset \mathbb{R}^4$ of cardinality $|C| < 12$ is T-shaped.*

Proof Assume that some subset $C \subset \mathbb{R}^4$ of cardinality $|C| < 12$ is not T-shaped. Without loss of generality, $|C| = 11$.

We recall that a hyperplane $H \subset \mathbb{R}^4$ is called a *support hyperplane* for C if $C \cap H \neq \emptyset$ and H does not separate C (which means that C lies in a closed half-space H^+ bounded by the hyperplane). □

Claim 9 *Each support hyperplane $H \subset \mathbb{R}^4$ for C has at most five common points with C.*

Proof Assume that H is a support hyperplane for C with $|H \cap C| > 5$. After a suitable affine transformation of $\mathbb{R}^4$, we can assume that $H = \mathbb{R}^3 \times \{0\}$ and $C \subset \mathbb{R}^3 \times \mathbb{R}_+$. Let $\mathrm{pr} : \mathbb{R}^4 \to \mathbb{R}^3$ be the coordinate projection. Since $|C \setminus H| = |C| - |C \cap H| < 11 - 5 = 6$ and $t(\mathbb{R}^3) = 6$ (by Theorem 6(3)), $\mathrm{pr}(C \setminus H)$ is T-shaped in H and so C is T-shaped $\mathbb{R}^4$. □

Claim 10 *For any two parallel hyperplanes H_1 and H_2 in $\mathbb{R}^4$ the set $C \setminus (H_1 \cup H_2)$ is non-empty.*

Proof Otherwise one of these hyperplanes contains more than six points, which contradicts Claim 9. □

Claim 11 *Each support hyperplane H for the set C has less than five common points with C.*

Proof Previous claim guarantees the existence of two distinct points $a, b \in C$ that lie in an affine line L that meets H. After a suitable affine transformation of $\mathbb{R}^4$, we can assume that $H = \mathbb{R}^3 \times \{0\}$, $C \subset \mathbb{R}^3 \times \mathbb{R}_+$, and $L = \{0\}^3 \times \mathbb{R}$. Let $\mathrm{pr} : \mathbb{R}^4 \to \mathbb{R}^3$ be the coordinate projection. Assuming that $|H \cap C| \geq 5$ and taking into account that $\mathrm{pr}(a) = \mathrm{pr}(b)$, we conclude that

$$\left|\mathrm{pr}(C \setminus H)\right| \leq |C \setminus H| - 1 = |C| - |C \cap H| - 1 \leq 5 < 6 = t(\mathbb{R}^3).$$

It follows that $\mathrm{pr}(C \setminus H)$ is T-shaped in $\mathbb{R}^3$ and then C is T-shaped in $\mathbb{R}^4$. □

The characterization of T-shaped sets given in Proposition 2 implies:

Claim 12 *If H_1 is a support hyperplane for C, H_2 is a support hyperplane for $C \setminus H_1$ and H_1, H_2 are not parallel, then $|C \setminus (H_1 \cup H_2)| \geq 3$ and if $|C \setminus (H_1 \cup H_2)| = 3$, then the set $C \setminus (H_1 \cup H_2)$ does not lie in a line but lies in a plane, parallel to $H_1 \cap H_2$.*

Claim 13 *If H_1 and P_2 are parallel support hyperplanes for C and $|H_1 \cap C| = 4$, then $|P_2 \cap C| = 1$.*

Proof By Claim 11, $C \setminus H_1$ does not lie in a hyperplane. Now consider four cases.

(1) $|P_2 \cap C| > 4$. In this case C is T-shaped by Claim 11.
(2) $|P_2 \cap C| = 4$. We claim that the set $P_2 \cap C$ does not lie in a plane P. Otherwise P can be enlarged to a support hyperplane that contains ≥ 5 points of C, which is forbidden by Claim 11. Therefore, the convex hull of $P_2 \cap C$ is a convex body in P_2 and we can find a support hyperplane H_2 for $C \setminus H_1$ that meets H_1, has at least four common points with $C \setminus H_1$ and exactly three common points with the set $C \cap P_2$. In this case the unique point c_2 of the set $C \cap P_2 \setminus H_2$ lies in $C \setminus (H_1 \cup H_2)$. By Proposition 2, the set $C \setminus (H_1 \cup H_2)$ contains exactly three points that lie in a plane parallel to $H_1 \cap H_2$. Since this set contains the point $c_2 \in C \cap P_2$, we conclude that $C \setminus (H_1 \cup H_2) \subset P_2$ and hence $|C \cap P_2| = 6$, which is a contradiction.
(3) $|P_2 \cap C| = 3$. Let Pl be a plane which contains $P_2 \cap C$ and lies in the hyperplane P_2. We claim that the set $C \setminus (H_1 \cup Pl)$ lies in a plane Pl_1 that is parallel to Pl. Let S be the set of all points $x \in C \setminus (H_1 \cup Pl)$ that belong to a support hyperplane H_x to $C \setminus H_1$ that has at least four common points with $C \setminus H_1$ and contains the plane Pl. Claim 12 guarantees that the set $C \setminus (H_1 \cup H_x)$ contains exactly three elements and lies in a plane that is parallel to the intersection $H_1 \cap H_x$ (which is parallel to Pl). Since the set $C \setminus H_1$ does not lie in a hyperplane, the set S contains more that one point, which implies that the set $C \setminus (H_1 \cup Pl) = \bigcup_{x \in S} C \setminus (H_1 \cup H_x)$ lies in a plane Pl_1 that is parallel to the plane Pl. Let H_2 be the hyperplane that contains the parallel planes Pl and Pl_1. Since H_2 meets H_1, we see that $C \subset H_1 \cup H_2$ is T-shaped by Proposition 2 and this is a contradiction.
(4) $|P_2 \cap C| = 2$. Since $C \setminus H_1$ does not lie in a hyperplane, there is a support hyperplane H_2 to $C \setminus H_1$ such that $|H_2 \cap (C \setminus H_1)| \geq 4$ and $|H_2 \cap P_2 \cap C| = 1$. It follows that the hyperplane H_2 does not coincide with P_2 and hence meets the hyperplane H_1. By Claim 12, the complement $C \setminus (H_1 \cup H_2)$ contains exactly three points that lie in a plane, parallel to $H_1 \cap H_2$. Since $C \setminus (H_1 \cup H_2)$ meets the hyperplane P_2 we conclude that $C \setminus (H_1 \cup H_2) \subset P_2$ and $|C \cap P_2| \geq 4$, which is a contradiction.

□

Claim 14 *If P_1 and P_2 are parallel support hyperplanes for C and $|P_1 \cap C| = 4$, then the set $C \setminus (P_1 \cup P_2)$ lies in a hyperplane P_3 that is parallel to P_1 and P_2.*

Proof By Claim 13, $|P_2 \cap C| = 1$ and hence $|C \setminus (P_1 \cup P_2)| = 6$. Let x be the unique point of $P_2 \cap C$. Take any support hyperplane $H \ni x$ for the set $C \setminus P_1$ such that $|H \cap C| \geq 4$. Since H meets P_1, Proposition 2 guarantees that the set $C' = C \setminus (P_1 \cup H)$ contains exactly three points that lie in a plane parallel to the intersection

$P_1 \cap H$ and hence parallel to P_1. The hyperplane H' containing the set $C' \cup \{x\}$ is a support hyperplane for the set $C \setminus P_1$. Applying Proposition 2, we conclude that the set $C'' = C \setminus (P_1 \cup H') = C \cap H \setminus P_2$ contains exactly three points lying in a plane parallel to $P_1 \cap H'$. Thus $C \setminus (P_1 \cup P_2)$ lies in two planes parallel to P_1 and hence it lies in a hyperplane P_3. Proposition 2 implies that the hyperplane P_3 is parallel to P_1. □

By an *octahedron* in a linear space L we understand a set of the form

$$c + \{\mathbf{e}_i, -\mathbf{e}_i : 1 \le i \le 3\}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent vectors in L and $c \in L$ is the *center* of the octahedron. Up to an affine equivalence an octahedron is a unique 6-element set X with 3-dimensional affine hull A such that for each support plane $P \subset A$ of X with $|P \cap X| \ge 3$ the set $X \setminus P$ contains three points and lies in a plane P', parallel to P.

Claim 15 *If P_1 and P_2 are parallel support hyperplanes for X and $|P_1 \cap C| = 4$, then the set $C \setminus (P_1 \cup P_2)$ is an octahedron that lies in a hyperplane P_3, parallel to P_1.*

Proof By the preceding claim, the set $K = C \setminus (P_1 \cup P_2)$ lies in a hyperplane P_3, parallel to P_1. Let us show that K does not lie in a plane. In the opposite case, we could find a hyperplane H_2 that contains the set K and meets the hyperplane P_1. Then for each hyperplane H_3 that contains the unique point $C \cap P_2$ and has one-dimensional intersection with $P_1 \cap H_2$, we get $C \subset P_1 \cup H_2 \cup H_3$ witnessing that C is T-shaped.

Thus the affine hull of K is 3-dimensional. To see that K is an octahedron, it suffices to check that for each support plane $P \subset P_3$ of K with $|P \cap K| \ge 3$ the set $K \setminus P$ contains exactly three points and lies in a plane parallel to P.

Let x be the unique point of the set $C \cap P_2$ and H_2 be the hyperplane containing the plane P and passing through x. It follows that H_2 is a support hyperplane for the set $C \setminus P_1$. By Claim 12, the set $C \setminus (P_1 \cup H_2) = K \setminus P$ contains exactly three elements and lies in a plane P' parallel to the intersection $H_1 \cap H_2$.

Now let H_2' be the hyperplane that contains the support plane P' and passes through the point x. Since P' is a support plane for K in the hyperplane P_3, H_3 is a support hyperplane for $K \cup \{x\} = C \setminus P_1$ in $\mathbb{R}^4$. Since H_3' intersects P_1, Claim 12 guarantees that the set $C \setminus (P_1 \cup H_2') = K \setminus P'$ contains exactly three points and the plane P containing these three points is parallel to $P_1 \cap H_2'$ which is parallel to the plane P'. □

After this preparatory work we are ready to finish the proof of Lemma 2. As C is not T-shaped, it does not lie in a hyperplane. So, we can find a support hyperplane P_1 for C such that $|P_1 \cap C| \ge 4$. Let P_2 be a support hyperplane for C, which is parallel to P_1. By Claim 13, $|P_1 \cap C| = 4$ and $|P_2 \cap C| = 1$. Let p_2 be the unique point of the set $P_2 \cap C$. By Claim 15, $C \setminus (P_1 \cup P_2)$ is an octahedron that lies in a hyperplane P_3, parallel to the hyperplanes P_1 and P_2. Let c be the center of this octahedron and $2c - p_2$ be the point, symmetric to p_2 with respect to c.

Fix any 3-element subset F of $P_1 \cap C$ such that $2c - p_2 \in F$ if $2c - p_2 \in C \cap P_1$. Next, find a hyperplane H_1 for C that contains F and meets $C \setminus H_1$ at some point a. If $a = p_2$, then the set $C \subset H_1 \cup P_3 \cup (C \cap P_1 \setminus F)$ is T-shaped by Proposition 2.

Consequently, a is a point of the octahedron $C \cap P_3$ with center c. Let H_2 be a support hyperplane for C that is parallel to the hyperplane H_1. By Claims 13 and 15, $|C \cap H_1| = 4$, $|C \cap H_2| = 1$ and $C \setminus (H_1 \cup H_2)$ is an octahedron that lies in a hyperplane H_3, parallel to H_1 and H_2. If H_3 does not meet the octahedron $C \cap P_3$, then $(C \cap P_3) \cap (C \cap H_3) = (C \cap P_3) \setminus H_1 = C \cap P_3 \setminus \{a\}$. In this case the octahedra $C \cap P_3$ and $C \cap H_3$ have five common points and hence lie in the same hyperplane $P_3 = H_3$, which is not possible. So, the support hyperplane H_3 meets the octahedron $C \cap P_3$ at a single point and this point is $2c - a$. In this case the octahedra $C \cap P_3$ and $C \cap H_3$ have four common points which belong to the set $C \cap P_3 \setminus \{a, 2c - a\}$ and lie in the 2-dimensional plane $P_3 \cap H_3$. This implies that the octahedra $C \cap P_3$ and $C \cap H_3$ have the common center c. Since $p_2 \in C \cap H_3$, the point $2c - p_2$ belongs to the octahedron $C \cap H_3 \subset C$. It follows from $p_2 \in P_2$ and $c \in P_3$ that $2c - p_2 \in C \setminus (P_2 \cup P_3) = C \cap P_1$ and hence $2c - p_2 \subset F \subset H_1$ by the choice of the set F. On the other hand, $2c - p_2$ belongs to the hyperplane H_3, which is disjoint with H_1 and this is a desired contradiction. □

5 Enlarging non-centerpole sets

In this section we prove several lemmas on enlarging non-centerpole subsets. Namely, we show that under certain conditions, a non-k-centerpole subset C of a topological group X (possibly enlarged by one or two points) remains not k-centerpole in the direct sum $X \oplus \mathbb{R}$. The group $X \oplus \mathbb{R}$ can be identified with the direct product $X \times \mathbb{R}$ so that X is identified with the subgroup $X \times \{0\} \subset X \times \mathbb{R}$, while the real line $\mathbb{R}$ is identified with the subgroup $\{e\} \times \mathbb{R} \subset X \times \mathbb{R}$ where e is the neutral element of the group X.

Lemma 3 *If for $k \geq 2$ a subset $C \subset X$ of a topological group X is not k-centerpole (for Borel colorings), then set C is not k-centerpole in $X \oplus \mathbb{R}$.*

Proof Since the set $C \subset X$ is not k-centerpole (for Borel colorings), there exists a (Borel) coloring $\chi : X \to k$ such that X contains no monochromatic unbounded subset, which is symmetric with respect to a point $c \in C$. Extend χ to a (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \to k$ letting

$$\tilde{\chi}(x,t) = \begin{cases} \chi(x) & \text{if } t = 0, \\ 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases}$$

This coloring witnesses that C is not k-centerpole in $X \oplus \mathbb{R}$ (for Borel colorings). □

Lemma 4 *If for $k \geq 3$ a subset $C \subset X$ of a topological group X with $c_2^B(X) \geq 2$ is not k-centerpole (for Borel colorings), then for each $x \in X \times (0, \infty)$ the set $C \cup \{x\}$ is not k-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$.*

Proof Without loss of generality we may assume that $x = (e, 1)$ where e is the neutral element of topological group X. Fix a (Borel) coloring $\chi : X \to k$ witnessing that the subset $C \subset X$ is not k-centerpole (for Borel colorings).

This coloring induces a (Borel) 2-coloring $\chi_2 : X \to 2$ defined by

$$\chi_2(x) = \min\big(\{0, 1\} \setminus \chi\big(x^{-1}\big)\big) \quad \text{for } x \in X.$$

Since $c_2^B(X) \geq 2$, there exists a Borel coloring $\chi_1 : X \to 2$ witnessing that the singleton $\{e\}$ is not 2-centerpole for Borel colorings of X.

It is easy to see that the (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \to k$ defined by

$$\tilde{\chi}(x,t) = \begin{cases} \chi(x), & \text{if } t = 0, \\ \chi_1(x), & \text{if } t = 1, \\ \chi_2(x), & \text{if } t = 2, \\ 0, & \text{if } 1 < t \neq 2, \\ 1, & \text{if } 0 < t < 1, \\ 2 & \text{if } t < 0 \end{cases}$$

witnesses that the set $C \cup \{(e, 1)\}$ fails to be k-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$. □

Lemma 5 $c_3^B(\mathbb{R}^m) \geq 6$ *for all* $m \geq 3$.

Proof By Theorem 6(3) and Proposition 3, $c_3^B(\mathbb{R}^3) \geq t(\mathbb{R}^3) = 6$.

Next, we check that $c_3^B(\mathbb{R}^4) \geq 6$. Assuming that $c_3^B(\mathbb{R}^4) < 6$ find a subset $C \subset \mathbb{R}^4$ of cardinality $|C| \leq 5$, which is 3-centerpole for Borel colorings of $\mathbb{R}^4$.

Since $|C| \leq 5$, there is a 3-dimensional hyperplane $H_3 \subset \mathbb{R}^4$ such that $|C \setminus H_3| \leq 1$. Since $|C \cap H_3| \leq |C| < 6 = c_3^B(\mathbb{R}^3)$, the set $C \cap H_3$ is not 3-centerpole for Borel colorings of H_3. By (the proof of) Proposition 4.1 of [3], $c_2^B(\mathbb{R}^3) = 3 \geq 2$. By Lemma 4, the set C is not 3-centerpole for Borel colorings of $H_3 \oplus \mathbb{R}$ (which can be identified with $\mathbb{R}^4$).

Now assume that the inequality $c_3^B(\mathbb{R}^{m-1}) \geq 6$ has been proved for some $m \geq 4$. Assuming that $c_3^B(\mathbb{R}^m) \leq 5$ find a subset $C \subset \mathbb{R}^m$ of cardinality $|C| \leq 5$ which is 3-centerpole for Borel colorings of $\mathbb{R}^m$. This set lies in an $(m-1)$-dimensional hyperplane and according to Lemma 3, is 3-centerpole for Borel colorings of $\mathbb{R}^{m-1}$. Then $c_3^B(\mathbb{R}^{m-1}) \leq |C| \leq 5$, which contradicts the inductive assumption. □

Lemma 6 *If for* $k \geq 4$ *a subset* $C \subset X$ *of a topological group* X *with* $c_2^B(X) \geq 3$ *is not* k*-centerpole* (*for Borel colorings*), *then for any* 2*-element set* $A \subset X \times (0, \infty)$ *the set* $C \cup A$ *is not* k*-centerpole for* (*Borel*) *colorings of the topological group* $X \oplus \mathbb{R}$.

Proof Let (a, v) and (b, w) be the points of the 2-element set $A \subset X \times (0, \infty)$. We can assume that $v \leq w$. Let $\chi_0 : X \to k$ be a (Borel) coloring witnessing that the set C is not k-centerpole for (Borel) colorings of the group X.

Consider the Borel 4-coloring $\psi : \mathbb{R} \to 4$ of the real line defined by

$$\psi(t) = \begin{cases} 3 & \text{if } t \le 0 \\ 0 & \text{if } 0 < t \le v \\ 1 & \text{if } v < t \le w \\ 2 & \text{if } w < t \end{cases}$$

and observe that for each $c \in \{0, v, w\}$ and $t \in \mathbb{R} \setminus \{c\}$ we get $\psi(t) \neq \psi(2c - t)$.

We consider two cases.

(1) $v = w$. In this case we can assume that $v = w = 1$. Since $c_2^B(X) \ge 3$, there exists a Borel coloring $\chi_1 : X \to 2$ witnessing that the 2-element set $\{a, b\} \subset X$ is not 2-centerpole for Borel colorings of X. The (Borel) coloring χ_0 induces the (Borel) coloring $\chi_2 : X \to 3$ defined by the formula

$$\chi_2(x) = \min\big(\{0, 1, 2\} \setminus \big\{\chi_0\big(ax^{-1}a\big), \chi_0\big(bx^{-1}b\big)\big\}\big).$$

Now we see that the (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \to k$ defined by

$$\tilde{\chi}(x, t) = \begin{cases} \chi_t(x), & \text{if } t \in \{0, 1, 2\}, \\ \psi(t), & \text{otherwise} \end{cases}$$

witnesses that the set $C \cup A$ is not k-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$.

(2) The second case occurs when $v \neq w$. Without loss of generality, $v < w$ and $w - v = 1$. This case has three subcases.

(2a) $v = 1$ and $w = 2$. In this case we can assume that $b = e$ is the neutral element of the group X.

Since $c_2^B(X) \ge 3$, there is a Borel 2-coloring $\chi_1 : X \to 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole in X. By the same reason, there is a Borel 2-coloring $\phi : X \to 2$ witnessing that the singleton $\{b\} = \{e\}$ is not 2-centerpole for Borel colorings of X. Using the colorings ϕ and χ_0 one can define a (Borel) 3-coloring $\chi_2 : X \to 3$ such that $\chi_2(x) \neq \chi_0(ax^{-1}a)$ for all $x \in X$ and $\chi_2(x) \neq \chi_2(x^{-1})$ if and only if $\phi(x) \neq \phi(x^{-1})$.

Such a coloring $\chi_2 : X \to 3$ can be defined by the formula

$$\chi_2(x) = \begin{cases} \min(3 \setminus \{\chi_0(axa), \chi_0(ax^{-1}a)\}), \quad \text{if } \phi(x) = \phi(x^{-1}); \\ \phi(x), \\ \quad \text{if } \chi_0(ax^{-1}a) \neq \phi(x) \neq \phi(x^{-1}) \neq \chi_0(axa); \\ \min(3 \setminus \{\phi(x^{-1}), \chi_0(ax^{-1}a)\}), \\ \quad \text{if } \chi_0(ax^{-1}a) = \phi(x) \neq \phi(x^{-1}) \neq \chi_0(axa); \\ \phi(x), \\ \quad \text{if } \chi_0(ax^{-1}a) \neq \phi(x) \neq \phi(x^{-1}) = \chi_0(axa); \\ \phi(x^{-1}), \\ \quad \text{if } \chi_0(ax^{-1}a) = \phi(x) \neq \phi(x^{-1}) = \chi_0(axa). \end{cases}$$

Let $\chi_3 : X \to 2$ be the Borel 2-coloring defined by $\chi_3(x) = 1 - \chi_1(x^{-1})$ for $x \in X$. It is clear that $\chi_3(x^{-1}) \neq \chi_1(x)$ for all $x \in X$. Finally, consider the Borel 2-coloring $\chi_4 : X \to 2$ defined by

$$\chi_4(x) = \min\bigl(\{0,1\} \setminus \bigl\{\chi_0\bigl(x^{-1}\bigr)\bigr\}\bigr) \quad \text{for } x \in X.$$

The (Borel) colorings $\psi, \chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ compose a (Borel) k-coloring $\tilde{\chi} : X \times \mathbb{R} \to k$,

$$\tilde{\chi}(x,t) = \begin{cases} \chi_t(x), & \text{if } t \in \{0,1,2,3,4\}, \\ \psi(t), & \text{otherwise,} \end{cases}$$

witnessing that the set $C \cup A$ is not k-centerpole for (Borel) colorings of $X \oplus \mathbb{R}$.

(2b) $v = 2$ and $w = 3$. Since $c_2^B(X) \geq 3 > 1$, there is a Borel 2-coloring $\chi_2 : X \to 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole for Borel colorings of X. By the same reason, there is a Borel 2-coloring $\chi_3 : X \to 2$ witnessing that the singleton $\{b\}$ is not 2-centerpole for Borel colorings of X.

Next consider the (Borel) colorings $\chi_1 : X \to 2$, $\chi_4 : X \to 3$, and $\chi_6 : X \to 2$ defined by the formulas

$$\chi_1(x) = 1 - \chi_3\bigl(ax^{-1}a\bigr),$$
$$\chi_4(x) = \min\bigl(3 \setminus \bigl\{\chi_0\bigl(ax^{-1}a\bigr), \chi_2\bigl(bx^{-1}b\bigr)\bigr\}\bigr),$$
$$\chi_6(x) = \min\bigl(2 \setminus \bigl\{\chi_0\bigl(bx^{-1}b\bigr)\bigr\}\bigr).$$

The (Borel) colorings ψ and χ_t, $t \in \{0,1,2,3,4,6\}$, compose the (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \to k$ defined by

$$\tilde{\chi}(x,t) = \begin{cases} \chi_t(x), & \text{if } t \in \{0,1,2,3,4,6\}, \\ \psi(t), & \text{otherwise.} \end{cases}$$

This coloring $\tilde{\chi}$ witnesses that the set $C \cup A$ is not k-centerpole for (Borel) colorings of $X \oplus \mathbb{R}$.

(2c) $v \notin \{1,2\}$. Since $c_2^B(X) > 1$ there is a Borel 2-coloring $\chi_v : X \to 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole for Borel colorings of X. By the same reason, there is a Borel 2-coloring $\chi_w : X \to \{1,2\}$ witnessing that the singleton $\{b\}$ is not 2-centerpole for Borel colorings of X.

Next, define the (Borel) colorings $\chi_{2v}, \chi_{2w} : X \to 3$ by the formula

$$\chi_{2v}(x) = \min\bigl(3 \setminus \bigl\{\chi_0\bigl(ax^{-1}a\bigr), \psi(2)\bigr\}\bigr) \quad \text{and}$$
$$\chi_{2w}(x) = \min\bigl(2 \setminus \bigl\{\chi_0\bigl(bx^{-1}b\bigr)\bigr\}\bigr).$$

Here let us note that the points $2v$ and 2 are symmetric with respect to w in the group $\mathbb{R}$.

Finally, define a (Borel) k-coloring $\tilde{\chi} : X \oplus \mathbb{R} \to k$ letting

$$\tilde{\chi}(x,t) = \begin{cases} \chi_t(x) & \text{if } t \in \{0, v, w, 2v, 2w\} \\ \psi(t) & \text{otherwise.} \end{cases}$$

This coloring witnesses that the set $C \cup A$ is not k-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$.

□

Lemma 7 $c_4^B(\mathbb{R}^m) \geq 8$ *for all* $m \geq 4$.

Proof This lemma will be proved by induction on $m \geq 4$. For $m = 4$ the inequality $c_4^B(\mathbb{R}^4) \geq t(\mathbb{R}^4) = 12 \geq 8$ follows from Lemma 2. Assume that for some $m \geq 4$ we know that $c_4^B(\mathbb{R}^m) \geq 8$. The inequality $c_4^B(\mathbb{R}^{m+1}) \geq 8$ will follow as soon as we check that each 7-element subset $C \subset \mathbb{R}^{m+1}$ is not 4-centerpole for Borel colorings of $\mathbb{R}^{m+1}$.

Given a 7-element subset $C \subset \mathbb{R}^{m+1}$, find a support m-dimensional hyperplane $H \subset \mathbb{R}^{m+1}$ that has at least $\min\{m+1, |C|\} \geq 5$ common points with the set C. After a suitable shift, we can assume that the intersection $C \cap H$ contains the origin of $\mathbb{R}^{m+1}$. In this case H is a linear subspace of $\mathbb{R}^{m+1}$ and $\mathbb{R}^{m+1}$ can be written as the direct sum $\mathbb{R}^{m+1} = H \oplus \mathbb{R}$.

Since $|H \cap C| \leq |C| \leq 7$, the inductive assumption guarantees that $H \cap C$ is not 4-centerpole for Borel colorings of H. By Lemma 5, $c_3^B(\mathbb{R}^m) \geq 3$. Since $|C \setminus H| \leq 2$, we can apply Lemma 6 and conclude that C is not 4-centerpole for Borel colorings of the topological group $H \oplus \mathbb{R} = \mathbb{R}^{m+1}$. □

6 Centerpole sets in subgroups and groups

It is clear that each k-centerpole subset $C \subset H$ in a subgroup H of a topological group G is k-centerpole in G. In some cases the converse statement also is true.

Lemma 8 *If a subset C of an abelian topological group G is k-centerpole in G for some $k \geq 2$, then it is k-centerpole in the subgroup $H = \langle C\rangle + G[2]$.*

Proof Observe that for each $x \in G \setminus H$ the cosets $c + 2\langle C\rangle$ and $-x + 2\langle C\rangle$ are disjoint. Assuming the opposite, we would conclude that $2x \in 2\langle C\rangle$ and hence $x \in \langle C\rangle + G[2] = H$, which contradicts the choice of x.

Now we are able to prove that the set C is k-centerpole in the group H. Given any k-coloring $\chi : H \to k$, extend χ to a k-coloring $\tilde{\chi} : G \to k$ such that for each $x \in G \setminus H$ the coset $x + 2\langle C\rangle$ is monochromatic and its color is different from the color of the coset $-x + 2\langle C\rangle$.

Since C is k-centerpole in the group G, there is an unbounded monochromatic subset $S \subset G$ such that $S = 2c - S$ for some $c \in C$. We claim that $S \subset H$. Assuming the converse, we would find a point $x \in S \setminus H$ and conclude that the coset $x + 2\langle C\rangle$

has the same color as the coset $2c - x + 2\langle C\rangle = -x + 2\langle C\rangle$, which contradicts the choice of the coloring $\tilde{\chi}$. □

The Borel version of this result is a bit more difficult.

Lemma 9 *Let $k \geq 2$ and H be a Borel subgroup of an abelian topological group G such that $G[2] \subset H$. A subset $C \subset H$ is k-centerpole for Borel colorings of H if C is k-centerpole for Borel colorings of G, the subgroup $2H = \{2x : x \in H\}$ is closed in G, and the subspace $X = (G/2H) \setminus (H/2H)$ contains a Borel subset B that has one-point intersection with each set $\{x, -x\}$, $x \in X$. Such a Borel set $B \subset X$ exists if the space X is paracompact.*

Proof Given any Borel k-coloring $\chi : H \to k$, extend χ to a Borel k-coloring $\tilde{\chi} : G \to k$ defined by

$$\tilde{\chi}(x) = \begin{cases} \chi(x), & \text{if } x \in H, \\ 0, & \text{if } x \in G \setminus H \text{ and } x + 2H \in B, \\ 1, & \text{if } x \in G \setminus H \text{ and } x + 2H \notin B. \end{cases}$$

Since C is k-centerpole for Borel colorings of the group G, there is an unbounded monochromatic subset $S \subset G$, symmetric with respect to some point $c \in C$. We claim that $S \subset H$, witnessing that C is k-centerpole for Borel colorings of H.

Assuming conversely that $S \not\subset H$, find a point $x \in S \setminus H$. It follows that x and $2c - x$ have the same color. If this color is 0, then the cosets $x + 2H$ and $2c - x + 2H = -x + 2H = -(x + 2H)$ both belong to the set $B \subset G/2H$. By our hypothesis B has one-point intersection with the set $\{x + 2H, -(x + 2H)\}$. Consequently, $x + 2H = -(x + 2H)$ and hence $2x \in 2H$ and $x \in H + G[2] = H$, which contradicts the choice of the point x. If the color of the cosets $x + 2H$ and $2c - x + 2H = -(x + 2H)$ is 1, then $(x + 2H), -(x + 2H) \notin B$ and then $x + 2H = -(x + 2H)$ because B has one-point intersection with the set $\{x + 2H, -(x + 2H)\}$. This again leads to a contradiction. □

Claim 16 *If the space $X = (G/2H) \setminus (H/2H)$ is paracompact, then X contains a Borel subset $B \subset X$ that has one-point intersection with each set $\{x, -x\}$, $x \in X$.*

Consider the action

$$\alpha : C_2 \times X \to X, \qquad \alpha : (\varepsilon, x) \mapsto \varepsilon \cdot x,$$

of the cyclic group $C_2 = \{1, -1\}$ on the space X and let $X/C_2 = \{\{x, -x\} : x \in X\}$ be the orbit space of this action. It is easy to check that the orbit map $q : X \to X/C_2$ is closed and then the orbit space X/C_2 is paracompact as the image of a paracompact space under a closed map, see Michael, Theorem 5.1.33 in [7].

Since $H \supset 2H + G[2]$, for every $x \in G \setminus H$ the cosets $x + 2H$ and $-x + 2H$ are disjoint, which implies that each point $x \in X$ is distinct from $-x$. Then each point $x \in X$ has a neighborhood $U_x \subset X$ such that $U_x \cap -U_x = \emptyset$. Replacing U_x

by $U_x \cap (-U_{-x})$ we can additionally assume that $U_x = -U_{-x}$. Now consider the open neighborhood $U_{\pm x} = q(U_x) = q(U_{-x}) \subset X/C_2$ of the orbit $\{x, -x\} \in X/C_2$ of the point $x \in X$. By the paracompactness of X/C_2 the open cover $\{U_{\pm x} : x \in X\}$ of X/C_2 has a Σ-discrete refinement $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$. This means that each family $\mathcal{U}_n$, $n \in \omega$, is discrete in X/C_2. For each $U \in \mathcal{U}$ find a point $x_U \in X$ such that $U \subset U_{\pm x_U}$. For every $n \in \omega$ consider the open subset $W_n = \bigcup_{U\in\mathcal{U}_n} q^{-1}(U) \cap U_{x_U}$ of the space X and let $\pm W_n = -W_n \cup W_n$. One can check that the Borel subset

$$B = \bigcup_{n\in\omega} \left(W_n \setminus \bigcup_{i<n} \pm W_i \right)$$

of X has one-point intersection with each orbit $\{x, -x\}$, $x \in X$.

The following lemma will be helpful in the proof of the upper bound $rc_k^B(G) \le c_k^B(G) - 2$ from Proposition 1.

Lemma 10 *Let $k \ge 4$ and $C \subset \mathbb{R}^\omega$ be a finite k-centerpole subset for Borel colorings of $\mathbb{R}^\omega$. Then the affine hull of C in $\mathbb{R}^\omega$ has dimension $\le |C| - 3$.*

Proof This lemma will be proved by induction on the cardinality $|C|$.

First observe that $|C| \ge c_k^B(\mathbb{R}^\omega) \ge c_3^B(\mathbb{R}^\omega) \ge 6$ by Lemma 5. So, we start the induction with $|C| = 6$.

Suppose that either $m = 6$ or $m > 6$ and the lemma is true for all C with $6 \le |C| < m$. Fix a k-centerpole subset $C \subset \mathbb{R}^\omega$ for Borel colorings of cardinality $|C| = m$. We need to show that the affine hull A of C has dimension $\dim A \le m - 3$. Assuming the opposite, we can find a support hyperplane $H \subset A$ for C such that $|H \cap C| \ge \dim H + 1 = \dim A \ge |C| - 2$ and hence $0 < |C \setminus H| \le 2$. After a suitable shift, we can assume that H contains the origin of $\mathbb{R}^\omega$ and hence is a subgroup of $\mathbb{R}^\omega$. In this case the affine hull A is a linear subspace in $\mathbb{R}^\omega$ that can be identified with the direct sum $H \oplus \mathbb{R}$. It follows that $\dim H = \dim A - 1 \ge |C| - 2 - 1 \ge |C \cap H| - 2$.

We claim that the set $H \cap C$ is not k-centerpole for Borel colorings of the topological group H.

If $6 \le |C \cap H| < |C| = m$, then by the inductive assumption, the set $C \cap H$ is not k-centerpole for Borel colorings of $\mathbb{R}^\omega$ because its affine hull H has dimension $\dim H \ge |C \cap H| - 2$. If $|C \cap H| < 6$ (which happens for $m = 6$), then the inequalities $c_k^B(H) \ge c_3^B(H) \ge 6 = m = |C| > |H \cap C|$ given by Lemma 5 guarantee that $C \cap H$ is not k-centerpole for Borel colorings of $\mathbb{R}^\omega$.

By (the proof) of Proposition 1 in [3], $c_2^B(H) = 3$. Since H is a support hyperplane for C and $|C \setminus H| \le 2$, we can apply Lemma 6 and conclude that C is not k-centerpole for Borel colorings of $H \oplus \mathbb{R} = A$. Since the subgroup $2A$ is closed in the metrizable group $\mathbb{R}^\omega$, by Lemma 9, C is not k-centerpole for Borel colorings of $\mathbb{R}^\omega$ and this is a desired contradiction that completes the proof of the inductive step and base of the induction. □

7 Stability properties

In this section we shall prove some particular cases of the Stability Theorem 4.

Lemma 11 *For any numbers $k \geq 2$ and $n \leq m$*

$$c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^{m-n}\big) = \begin{cases} c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega), & \text{if } m \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega), \\ c_k^B(\mathbb{R}^\omega), & \text{if } n \geq rc_k^B(\mathbb{R}^\omega). \end{cases}$$

Proof First assume that $m \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$. By the definition of the number $r = rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$, the topological group $G = \mathbb{R}^n \times \mathbb{Z}^\omega$ contains a k-centerpole subset $C \subset G$ of cardinality $|C| = c_k^B(G)$ that generates a subgroup $\langle C\rangle \subset \mathbb{Z}^\omega$ of $\mathbb{Z}$-rank r. It follows that the linear subspace $L \subset \mathbb{R}^n \times \mathbb{R}^\omega$ generated by the set C has dimension r. Then $H = L \cap G$, being a closed subgroup of $\mathbb{Z}$-rank r in the r-dimensional vector space L is topologically isomorphic to $\mathbb{R}^s \times \mathbb{Z}^{r-s}$ for some $s \leq r \leq m$, see Theorem 6 in [10]. Taking into account that H is a closed subgroup of $G = \mathbb{R}^n \times \mathbb{Z}^\omega$, we conclude that $s \leq n$. By Lemma 9, the set C is k-centerpole in H for Borel colorings. Consequently,

$$\begin{aligned} c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^\omega\big) &\leq c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^{m-n}\big) \leq c_k^B\big(\mathbb{R}^s \times \mathbb{Z}^{r-s}\big) = c_k^B(H) \leq |C| = c_k^B(G) \\ &= c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^\omega\big) \end{aligned}$$

implies the desired equality $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$.

Now assume that $n \geq rc_k^B(\mathbb{R}^\omega)$. In this case we can repeat the above argument for a set $C \subset \mathbb{R}^\omega$ of cardinality $|C| = c_k^B(\mathbb{R}^\omega)$ that generates a subgroup $\langle C\rangle \subset \mathbb{R}^\omega$ of $\mathbb{Z}$-rank $r = rc_k^B(\mathbb{R}^\omega)$. Then the linear subspace $L \subset \mathbb{R}^\omega$ generated by the set C is topologically isomorphic to $\mathbb{R}^r$. By Lemma 9, the set C is k-centerpole for Borel colorings of L. Since $\mathbb{R}^r \hookrightarrow \mathbb{R}^n \times \mathbb{Z}^{m-n} \hookrightarrow \mathbb{R}^\omega$, we get

$$c_k^B\big(\mathbb{R}^\omega\big) \leq c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^{m-n}\big) \leq c_k^B\big(\mathbb{R}^r\big) = c_k^B(L) \leq |C| = c_k^B\big(\mathbb{R}^\omega\big)$$

and hence $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^\omega)$. □

Lemma 12 $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$ *for any numbers $k \in \mathbb{N}$ and $n \leq m$ with $m \geq c_k^B(\mathbb{Z}^\omega)$.*

Proof For $k = 1$ the equality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = 1 = c_k^B(\mathbb{Z}^\omega)$ is trivial. So we assume that $k \geq 2$.

We claim that $c_k^B(\mathbb{Z}^\omega) \leq c_k(\mathbb{R}^m)$. Indeed, take any k-centerpole subset $C \subset \mathbb{R}^\omega$ of cardinality $|C| = c_k(\mathbb{R}^m)$. By Lemma 8, the set C is k-centerpole in the subgroup $\langle C\rangle \subset \mathbb{R}^\omega$ generated by C. Being a torsion-free finitely generated abelian group, $\langle C\rangle$ is algebraically isomorphic to $\mathbb{Z}^r$ for some $r \in \omega$. Then

$$c_k\big(\mathbb{Z}^r\big) \leq c_k\big(\langle C\rangle\big) \leq |C| = c_k\big(\mathbb{R}^m\big).$$

On the other hand, Lemma 11 ensures that

$$c_k\big(\mathbb{R}^m\big) \leq c_k\big(\mathbb{Z}^m\big) = c_k^B\big(\mathbb{Z}^m\big) = c_k^B\big(\mathbb{Z}^\omega\big).$$

Unifying these inequalities we get

$$c_k^B(\mathbb{Z}^\omega) \le c_k^B(\mathbb{Z}^r) = c_k(\mathbb{Z}^r) \le c_k(\mathbb{R}^m) \le c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \le c_k(\mathbb{Z}^m) = c_k^B(\mathbb{Z}^m) = c_k^B(\mathbb{Z}^\omega),$$

which implies the desired equality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$. □

8 Proof of Theorem 3

1. The upper bound $c_k(\mathbb{Z}^n) \le c_k(\mathbb{Z}^k) \le 2^k - 1 - \max_{s \le k-2}\binom{k-1}{s-1}$ for $k \le n$ follows from Theorem 5.

2. By Proposition 3 and Theorem 6(7), $c_n(\mathbb{Z}^n) \ge c_n(\mathbb{R}^n) \ge c_n^B(\mathbb{R}^n) \ge t(\mathbb{R}^n) \ge \frac{1}{2}(n^2 + 3n - 4)$.

For technical reasons, first we prove the statement (4) of Theorem 3 and after that return back to the statement (3).

4. Let $1 \le k \le m \le \omega$ be two numbers. We need to prove that $c_k^B(\mathbb{R}^m) < c_{k+1}^B(\mathbb{R}^{m+1})$ and $c_k(\mathbb{R}^m) < c_{k+1}(\mathbb{R}^{m+1})$.

First we assume that m is finite. The strict inequality $c_k^B(\mathbb{R}^m) < c_{k+1}^B(\mathbb{R}^{m+1})$ will follow as soon as we show that any subset $C \subset \mathbb{R}^{m+1}$ of cardinality $|C| \le c_k^B(\mathbb{R}^m)$ fails to be $(k+1)$-centerpole for Borel colorings of $\mathbb{R}^{m+1}$. If C is a singleton, then it is not $(k+1)$-centerpole since $c_{k+1}^B(\mathbb{R}^{m+1}) \ge c_2^B(\mathbb{R}^{m+1}) \ge 3$ by (the proof of) Proposition 4.1 in [3]. So, C contains two distinct points a, b. Let $L = \mathbb{R} \cdot (a - b) \subset \mathbb{R}^{m+1}$ be the linear subspace generated by the vector $a - b$. Write the space $\mathbb{R}^{m+1}$ as the direct sum $\mathbb{R}^{m+1} = H \oplus L$ where H is a linear m-dimensional subspace of $\mathbb{R}^{m+1}$ and consider the projection $\mathrm{pr} : \mathbb{R}^{m+1} \to H$ whose kernel is equal to L. Since $\mathrm{pr}(a) = \mathrm{pr}(b)$, the projection of the set C onto the subspace H has cardinality $|\mathrm{pr}(C)| < |C| \le c_k^B(\mathbb{R}^m) = c_k^B(H)$ and hence $\mathrm{pr}_H(C)$ is not k-centerpole for Borel k-colorings of the group H. Consequently, there is a Borel k-coloring $\chi : H \to k$ such that no monochromatic unbounded subset of H is symmetric with respect to a point $c \in \mathrm{pr}(C)$.

For a real number $\gamma \in \mathbb{R}$, consider the half-line $L_\gamma^+ = \{t(a - b) : t \ge \gamma\}$ of L. Since the subset $C \subset \mathbb{R}^{m+1}$ is finite, there is $\gamma \in \mathbb{R}$ such that $C \subset H + L_\gamma^+$.

Now define a Borel $(k+1)$-coloring $\tilde{\chi} : H \oplus L \to k + 1 = \{0, \dots, k\}$ by the formula

$$\tilde{\chi}(x) = \begin{cases} \chi(\mathrm{pr}(x)), & \text{if } x \in H + L_\gamma^+, \\ k, & \text{otherwise.} \end{cases}$$

It can be shown that this coloring witnesses that C is not $(k+1)$-centerpole for Borel colorings of $\mathbb{R}^{m+1} = H \oplus L$.

Now assume that the number m is infinite. Then for the finite number $r = \max\{rc_k^B(\mathbb{R}^\omega), rc_{k+1}^B(\mathbb{R}^\omega)\}$ we get $c_k^B(\mathbb{R}^r) = c_k^B(\mathbb{R}^\omega)$ and $c_{k+1}^B(\mathbb{R}^{r+1}) = c_{k+1}^B(\mathbb{R}^\omega)$ by the stabilization Lemma 11. Since r is finite, the case considered above guarantees that

$$c_k^B(\mathbb{R}^m) = c_k^B(\mathbb{R}^m) = c_k^B(\mathbb{R}^r) < c_{k+1}^B(\mathbb{R}^{r+1}) = c_{k+1}^B(\mathbb{R}^\omega) = c_{k+1}(\mathbb{R}^{m+1}).$$

By analogy we can prove the strict inequality $c_k(\mathbb{R}^m) < c_k(\mathbb{R}^{m+1})$.

3. Now we are able to prove the lower bound $c^B_k(\mathbb{R}^\omega) \geq k+4$ from the statement (3) of Theorem 3. By the preceding item, $c^B_{k+1}(\mathbb{R}^\omega) \geq 1 + c^B_k(\mathbb{R}^\omega)$ for all $k \in \mathbb{N}$. By induction, we shall show that $c^B_k(\mathbb{R}^\omega) \geq k+4$ for all $k \geq 4$. For $k = 4$ the inequality $c^B_4(\mathbb{R}^\omega) \geq 8 \geq 4+4$ was proved in Lemma 7. Assuming that $c^B_k(\mathbb{R}^\omega) \geq k+4$ for some $k \geq 4$, we conclude that $c^B_{k+1}(\mathbb{R}^\omega) > c^B_k(\mathbb{R}^\omega) \geq k+4$ and hence $c^B_{k+1}(\mathbb{R}^\omega) \geq (k+1)+4$.

Now we see that for every $n \geq k \geq 4$ we have the desired lower bound:

$$c^B_k\left(\mathbb{R}^n\right) \geq c^B_k\left(\mathbb{R}^\omega\right) \geq k+4.$$

5. Let $k \in \mathbb{N}$ and $n, m \in \omega \cup \{\omega\}$ be numbers with $1 \leq k \leq n+m$. We need to prove that $c^B_k(\mathbb{R}^n \times \mathbb{Z}^m) < c^B_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1})$ and $c_k(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1})$. According to the Stabilization Lemma 11, it suffices to consider the case of finite numbers n, m.

First we prove the inequality $c^B_k(\mathbb{R}^n \times \mathbb{Z}^m) < c^B_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1})$. We need to show that each subset $C \subset \mathbb{R}^n \times \mathbb{Z}^{m+1}$ of cardinality $|C| \leq c^B_k(\mathbb{R}^n \times \mathbb{Z}^m)$ is not $(k+1)$-centerpole in $\mathbb{R}^n \times \mathbb{Z}^{m+1}$ for Borel colorings. We shall identify $\mathbb{R}^n \times \mathbb{Z}^{m+1}$ with the direct sum $\mathbb{R}^n \oplus \mathbb{Z}^{m+1}$. Since $k \leq n+m$, Theorem 5 implies that the numbers $|C| \leq c^B_k(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_k(\mathbb{Z}^{n+m}) \leq c_k(\mathbb{Z}^k)$ all are finite.

Three cases are possible.

(i) $|C| \leq 1$. In this case we can assume that $C = \{0\}$ and take any coloring $\chi : \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \to k+1$ such that the color of each non-zero element $x \in \mathbb{R}^n \times \mathbb{Z}^{m+1}$ differs from the color of $-x$. This coloring witnesses that C is not $(k+1)$-centerpole in $\mathbb{R}^n \times \mathbb{Z}^{m+1}$.

(ii) $|C| > 1$ and $C \subset z + \mathbb{R}^n$ for some $z \in \mathbb{Z}^{m+1}$. Without lose of generality, $z = 0$ and hence $C \subset \mathbb{R}^n$. Take two distinct points $a, b \in C$ and consider the 1-dimensional linear subspace $L = \mathbb{R} \cdot (a-b) \subset \mathbb{R}^n$ generated by the vector $a-b$. Write the space $\mathbb{R}^n$ as the direct sum $\mathbb{R}^n = L \oplus H$ where H is a linear $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and consider the projection $\mathrm{pr} : \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \to H \oplus \mathbb{Z}^{m+1}$ whose kernel is equal to L. Since $\mathrm{pr}(a) = \mathrm{pr}(b)$, the projection of the set C onto the subgroup $H \oplus \mathbb{Z}^{m+1}$ of $\mathbb{R}^n \oplus \mathbb{Z}^{m+1}$ has cardinality

$$\left|\mathrm{pr}(C)\right| < |C| \leq c^B_k\left(\mathbb{R}^n \times \mathbb{Z}^m\right) \leq c^B_k\left(\mathbb{R}^{n-1} \times \mathbb{Z}^{m+1}\right) = c^B_k\left(H \oplus \mathbb{Z}^{m+1}\right)$$

and hence $\mathrm{pr}_H(C)$ is not k-centerpole for Borel colorings of the group $H \oplus \mathbb{Z}^{m+1}$. Consequently, there is a Borel k-coloring $\chi : H \oplus \mathbb{Z}^{m+1} \to k$ such that no monochromatic unbounded subset of $H \oplus \mathbb{Z}^{m+1}$ is symmetric with respect to a point $c \in \mathrm{pr}(C)$.

For a real number $\gamma \in \mathbb{R}$, consider the half-line $L^+_\gamma = \{t(a-b) : t \geq \gamma\}$ of L. Since the subset $C \subset \mathbb{R}^n \oplus \mathbb{Z}^{m+1} = H \oplus L \oplus \mathbb{Z}^{m+1}$ is finite, there is $\gamma \in \mathbb{R}$ such that $C \subset H + L^+_\gamma + \mathbb{Z}^{m+1}$.

Now define a Borel $(k+1)$-coloring $\tilde{\chi} : H \oplus L \oplus \mathbb{Z}^{m+1} \to k+1 = \{0, \dots, k\}$ by the formula

$$\tilde{\chi}(x) = \begin{cases} \chi(\mathrm{pr}(x)), & \text{if } x \in H + L^+_\gamma + \mathbb{Z}^{m+1}, \\ k, & \text{otherwise.} \end{cases}$$

It can be shown that this coloring witnesses that C is not $(k+1)$-centerpole for Borel colorings of $\mathbb{R}^n \oplus \mathbb{Z}^{m+1} = H \oplus L \oplus \mathbb{Z}^{m+1}$.

(iii) The set $C \subset \mathbb{R}^n \oplus \mathbb{Z}^{m+1}$ contains two points a, b whose projections on the subspace $\mathbb{Z}^{m+1}$ are distinct. Without loss of generality, the projections of a, b on the last coordinate are distinct. Then the 1-dimensional subspace $L = \mathbb{R} \cdot (a-b)$ of $\mathbb{R}^n \times \mathbb{R}^{m+1}$ meets the subspace $\mathbb{R}^n \oplus \mathbb{R}^m$ and hence $\mathbb{R}^n \oplus \mathbb{R}^{m+1}$ can be identified with the direct sum $\mathbb{R}^n \oplus \mathbb{R}^m \oplus L$. Let $\mathrm{pr} : \mathbb{R}^n \times \mathbb{R}^{m+1} \to \mathbb{R}^n \times \mathbb{R}^m$ be the projection whose kernel coincides with L. Since pr is an open map, the image $H = \mathrm{pr}(\mathbb{R}^n \times \mathbb{Z}^{m+1})$ is a locally compact (and hence closed) subgroup of $\mathbb{R}^n \times \mathbb{R}^m$, which can be written as the countable union of shifted copies of the space $\mathbb{R}^n$. By Theorem 6 of [10], H is topologically isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$. It follows from the definition of H that $\mathbb{R}^n \oplus \mathbb{Z}^{m+1} \subset H \oplus L$.

Since $\mathrm{pr}(a) = \mathrm{pr}(b)$, the projection of the set C has cardinality $|\mathrm{pr}(C)| < |C| \le c_k^B(\mathbb{R}^n \oplus \mathbb{Z}^m) = c_k^B(H)$, which means that $\mathrm{pr}(C)$ is not k-centerpole for Borel colorings of H. Consequently, there is a Borel k-coloring $\chi : H \to k$ such that no monochromatic unbounded subset of H is symmetric with respect to a point $c \in \mathrm{pr}(C)$.

For a real number $\gamma \in \mathbb{R}$, consider the half-line $L_\gamma^+ = \{t(a-b) : t \ge \gamma\}$ of L. Since the subset $C \subset H \oplus L$ is finite, there is $\gamma \in \mathbb{R}$ such that $C \subset H + L_\gamma^+$.

Now define a Borel $(k+1)$-coloring $\tilde{\chi} : H \oplus L \to k+1$ by the formula

$$\tilde{\chi}(x) = \begin{cases} \chi(\mathrm{pr}(x)), & \text{if } x \in H + L_\gamma^+, \\ k, & \text{otherwise.} \end{cases}$$

It can be shown that this coloring witnesses that C is not $(k+1)$-centerpole for Borel colorings of $H \oplus L \supset \mathbb{R}^n \oplus \mathbb{Z}^{m+1}$.

After considering these three cases, we can conclude that $c_{k+1}^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) > c_k^B(\mathbb{R}^n \times \mathbb{Z}^m)$.

Deleting the adjective "Borel" from the above proof, we get the proof of the strict inequality

$$c_k\big(\mathbb{R}^n \times \mathbb{Z}^m\big) < c_{k+1}\big(\mathbb{R}^n \times \mathbb{Z}^{m+1}\big).$$

9 Proof of Theorem 2

In this section we prove Theorem 2. Let k, n, m be cardinals. We shall use known upper bounds for the numbers $c_k(\mathbb{Z}^n)$, lower bounds for $t(\mathbb{R}^n)$ and the inequality

$$t\big(\mathbb{R}^{n+m}\big) \le c_k^B\big(\mathbb{R}^{n+m}\big) \le c_k^B\big(\mathbb{R}^n \times \mathbb{Z}^m\big) \le c_k\big(\mathbb{R}^n \times \mathbb{Z}^m\big) \le c_k\big(\mathbb{Z}^m\big)$$

established in Proposition 3.

1. Assume that $n + m \ge 1$. Since each singleton is 1-centerpole for (Borel) colorings of the group $\mathbb{R}^n \times \mathbb{Z}^m$, we conclude that $c_1(\mathbb{R}^n \times \mathbb{Z}^m) = c_1^B(\mathbb{R}^n \times \mathbb{Z}^m) = 1$.

2. Assume that $n+m \geq 2$. The inequalities $3 \leq t(\mathbb{R}^2) \leq c_2^B(\mathbb{R}^2) \leq c_2(\mathbb{Z}^2) \leq 3$ follow from Theorem 5, 6(2) and Proposition 3.

   We claim that $c_2^B(\mathbb{R}^\omega) \geq 3$. Assuming that $c_2^B(\mathbb{R}^\omega) < 3$ we conclude that $rc_k^B(\mathbb{R}^\omega) \leq c_2^B(\mathbb{R}^\omega) - 1 \leq 1$. Then by the Stabilization Lemma 11, we get that $c_2(\mathbb{R}^1) = c_2(\mathbb{R}^\omega)$ is finite. On the other hand, the real line has the 2-coloring $\chi : \mathbb{R} \to 2$, $\chi^{-1}(1) = (0, \infty)$, without unbounded monochromatic symmetric subsets. This coloring witnesses that $c_2(\mathbb{R}^1) = \infty$ and this is a contradiction. Therefore,

$$3 \leq c_2^B(\mathbb{R}^\omega) \leq c_2^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_2^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_2(\mathbb{Z}^2) = 3.$$

3. Assume that $n+m \geq 3$. Lemma 5 and Theorem 5 imply the inequalities

$$6 \leq c_3^B(\mathbb{R}^m) \leq c_3^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_3^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_3(\mathbb{Z}^3) = 6$$

   that turn into equalities.
4. Assume that $n+m = 4$. Theorem 5, 6(4) and Proposition 3 imply the inequalities

$$12 \leq t(\mathbb{R}^4) \leq c_4^B(\mathbb{R}^4) \leq c_4^B(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_4(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_4(\mathbb{Z}^4) \leq 12,$$

   which actually are equalities.
5. We need to prove that $c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) = \infty$ if $k \geq n+m+1 < \omega$. This equality will follow as soon as we check that $c_k^B(\mathbb{R}^{n+m}) = \infty$. Let Δ be a simplex in $\mathbb{R}^{n+m}$ centered at the origin. Write the boundary $\partial\Delta$ as the union $\partial\Delta = \bigcup_{i=0}^{n+m} \Delta_i$ of its facets. Define a Borel k-coloring $\chi : \mathbb{R}^n \to \{0, \dots, n+m\} \subset k$ assigning to each point $x \in \mathbb{R}^n \setminus \{0\}$ the smallest number $i \leq n+m$ such that the ray $\mathbb{R}_+ \cdot x$ meets the facet Δ_i. Also put $\chi(0) = 0$. It is easy to check that the coloring χ witnesses that the set $\mathbb{R}^{n+m}$ is not k-centerpole for Borel colorings of $\mathbb{R}^{n+m}$ and consequently, $c_k^B(\mathbb{R}^{n+m}) = \infty$.
6. Assuming that $k \geq n+m+1$, we shall show that $c_k(\mathbb{R}^n \times \mathbb{Z}^m) = \infty$. If $n+m$ is finite, then this follows from the preceding item. So, we assume that $n+m$ is infinite. Then the group $G = \mathbb{R}^n \times \mathbb{Z}^m$ has cardinality 2^{n+m}. By Theorem 4 of [4], for the group G endowed with the discrete topology, we get $\nu(G) = \log|G| = \min\{\gamma : 2^\gamma \geq |G|\} \leq n+m \leq k$, which means that G admits a k-coloring without infinite monochromatic symmetric subset. This implies that the set G is not k-centerpole in G and thus $c_k(G) = \infty$.
7. Assume that $n+m \geq \omega$ and $\omega \leq k < \mathrm{cov}(\mathcal{M})$. The lower bound from Theorem 3(3) implies that $\omega \leq c_k^B(\mathbb{R}^\omega) \leq c_k^B(\mathbb{Z}^\omega)$. The upper bound $c_\kappa^B(\mathbb{Z}^\omega) \leq \omega$ will follow as soon as we check that each countable dense subset $C \subset \mathbb{Z}^\omega$ is κ-centerpole for Borel colorings of $\mathbb{Z}^\omega$. Let $\chi : \mathbb{Z}^\omega \to \kappa$ be a Borel κ-coloring of $\mathbb{Z}^\omega$. Taking into account that $\mathbb{Z}^\omega = \bigcup_{i\in\kappa} \chi^{-1}(i)$ is homeomorphic to a dense G_δ-subset of the real line, we conclude that for some color $i \in \kappa$ the preimage $A = \chi^{-1}(i)$ is not meager in $\mathbb{Z}^\omega$. Being a Borel subset of $\mathbb{Z}^\omega$, the set A has the Baire property, which means that for some open subset $U \subset \mathbb{Z}^\omega$ the symmetric difference $A \triangle U$ is meager in $\mathbb{Z}^\omega$. Since A is not meager, the set U is not empty. Take any point $c \in U \cap C$ and observe that $V = U \cap (2c - U)$ is an open symmetric

neighborhood of c. It follows that for the set $B = A \cap (2c - A)$ the symmetric difference $B \triangle V$ is meager. Since V is not meager in $\mathbb{Z}^\omega$, the set B is not meager and hence is unbounded in $\mathbb{Z}^\omega$ (since totally bounded subsets of $\mathbb{Z}^\omega$ are nowhere dense in $\mathbb{Z}^\omega$). Now we see that $B = A \cap (2c - A)$ is a monochromatic unbounded subset, symmetric with respect to the point c, witnessing that the set C is ω-centerpole for Borel coloring of $\mathbb{Z}^\omega$.

10 Proof of Theorem 1

Let $k \ge 2$ be a finite cardinal number and G be an abelian ILC-group with totally bounded Boolean subgroup $G[2]$ and ranks $n = r_{\mathbb{R}}(G)$ and $m = r_{\mathbb{Z}}(G)$. Let $\bar{G}$ be the completion of the group with respect to its (two-sided) uniformity.

We shall give the detailed proofs of the statements (3) and (4) of Theorem 1 holding under the additional assumption of the metrizability of the group G and indicate the changes which should be made for the proof of the statements (1) and (2).

Since $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) < \omega$ iff $k \le m$, the statements (3), (4) of Theorem 1 will follow as soon as we prove two inequalities:

(1) $c_k^B(G) \le c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ if $k \le m$, and
(2) $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \le c_k^B(G)$ if $c_k^B(G)$ is finite.

1. Assume that $k \le m$. If the $\mathbb{Z}$-rank $m = r_{\mathbb{Z}}(G)$ is finite, then so is the $\mathbb{R}$-rank $n = r_{\mathbb{R}}(G)$ and we can find copies of the topological groups $\mathbb{R}^n$ and $\mathbb{Z}^m$ in G. Now consider the closure H of the subgroup $\mathbb{R}^n + \mathbb{Z}^m$ in G. Since G is an ILC-group and $\mathbb{R}^n + \mathbb{Z}^m$ contains a dense finitely generated subgroup, the group H is locally compact. By the structure theorem of locally compact abelian groups [10, Theorem 25], H is topologically isomorphic to $\mathbb{R}^r \oplus Z$ for some $r \in \omega$ and a closed subgroup $Z \subset H$ that contains an open compact subgroup K. It follows from the inclusion $\mathbb{R}^n \subset H$ that $n \le r$. On the other hand, $r \le r_{\mathbb{Z}}(G) = n$. By the same reason, $r_{\mathbb{Z}}(H) = m = r_{\mathbb{Z}}(G)$. In particular, $r_{\mathbb{Z}}(Z) = m - n$ and hence H contains an isomorphic copy of the group $\mathbb{R}^n \times \mathbb{Z}^{m-n}$. Now we see that $r_k^B(G) \le r_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Next, assume that the $\mathbb{Z}$-rank $m = r_{\mathbb{Z}}(G)$ is infinite but $n = r_{\mathbb{R}}(G)$ is finite. By the Stabilization Lemma 11, $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{r-n})$ for $r = rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega) \le c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega) < \infty$. Repeating the above argument we can find a copy of the group $\mathbb{R}^n \oplus \mathbb{Z}^{s-n}$ in G for some finite $s \ge r$ and conclude that $c_k^B(G) \le c_k^B(\mathbb{R}^n \times \mathbb{Z}^{s-n}) \le c_k^B(\mathbb{R}^n \times \mathbb{Z}^{r-s}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Finally, assume that the $\mathbb{R}$-rank $n = r_{\mathbb{R}}(G)$ is infinite. Then $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^\omega) = c_k^B(\mathbb{R}^r)$ for $r = rc_k^B(\mathbb{R}^\omega) \le c_k^B(\mathbb{R}^\omega) < \omega$. By the definition of the $\mathbb{R}$-rank $r_{\mathbb{R}}(G) = n = \omega$, we can find a copy of the group $\mathbb{R}^r$ in G and conclude that $c_k^B(G) \le c_k^B(\mathbb{R}^r) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$. This completes the proof of the inequality $c_k^B(G) \le c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Deleting the adjective "Borel" from the above proof we get the proof of the inequality $c_k(G) \le c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ holding for each $k \le m$.

2. Now assuming that $c_k^B(G)$ is finite and the group G is metrizable, we prove the inequality $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \le c_k^B(G)$.

Fix a k-centerpole subset $C \subset G$ for Borel colorings of G with cardinality $|C| = c_k^B(G)$. The subgroup $G[2]$ is totally bounded and hence has compact closure K_2 in the completion $\bar{G}$ of the group G. It follows that $K_2 \subset \bar{G}[2]$. Since G is an ILC-group, the finitely generated subgroup $\langle C\rangle$ has locally compact closure $\overline{\langle C\rangle}$ in G. It follows from the compactness of the subgroup K_2 that the sum $H = \overline{\langle C\rangle} + K_2$ is a locally compact subgroup of $\bar{G}$. This subgroup is compactly generated because it contains a dense subgroup generated by the compact set $C + K_2$.

By the Structure Theorem for compactly generated locally compact abelian groups [10, Theorem 24], H is topologically isomorphic to $\mathbb{R}^r \oplus \mathbb{Z}^{s-r} \oplus K$ for some compact subgroup K that contains all torsion elements of H. In particular, $K_2 \subset K$. Now we see that the subgroup $2H = \{2x : x \in H\}$ is closed in H and consequently, the subgroup $2H \cap G$ is closed in G. The group G is metrizable and so is the quotient group $G/2H$. Then the subspace $X = (G/2H) \setminus (H/2H)$ is metrizable and thus paracompact. Since $H \supset G[2]$ we can apply Lemma 9 and conclude that the set C is k-centerpole for Borel colorings of the subgroup $H \cap G$. Since $H \cap G \subset H$, the set C is k-centerpole for Borel colorings of the group H.

The compactness of the subgroup $K \subset H$ implies that the image $q(C)$ of C under the quotient map $q : H \to H/K$ is a k-centerpole set for Borel colorings of the quotient group $H/K = \mathbb{R}^r \times \mathbb{Z}^{s-r}$. Since $H = \overline{\langle C\rangle} + K_2$ and $K_2 \subset K$, we conclude that $\overline{\langle C\rangle}/(\overline{\langle C\rangle} \cap K) = q(\overline{\langle C\rangle}) = H/K = \mathbb{R}^r \times \mathbb{Z}^{s-r}$ and hence $r \le n$ and $s \le m$. Consequently, $\mathbb{R}^r \times \mathbb{Z}^{s-r} \hookrightarrow \mathbb{R}^n \times \mathbb{Z}^{m-n}$ and

$$c_k^B\left(\mathbb{R}^n \times \mathbb{Z}^{m-n}\right) \subset c_k^B\left(\mathbb{R}^r \times \mathbb{Z}^{s-r}\right) = c_k^B(H/K) \le |C| = c_k^B(G).$$

This proves the statements (3) and (4) of Theorem 1. Deleting the adjective "Borel" from the above proof and applying Lemma 8 instead of Lemma 9, we get the proof of the inequality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \le c_k(G)$ under the assumption that the number $c_k(G)$ is finite. Since Lemma 8 does not require the metrizability of G, this upper bound holds without this assumption. In such a way, we prove the statements (1) and (2) of Theorem 1.

11 Proof of Proposition 1

Let G be a metrizable abelian ILC-group with totally bounded Boolean subgroup $G[2]$ and $k \in \mathbb{N}$ be such that $2 \le k \le r_{\mathbb{Z}}(G)$. Theorems 1 and 3 guarantee that $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) < \infty$ where $n = r_{\mathbb{Z}}(G)$ and $m = r_{\mathbb{Z}}(G)$.

Let $r = rc_k(G)$ and $C \subset G$ be a subset of cardinality $|C| = c_k^B(G)$ such that $r_{\mathbb{Z}}(\langle C\rangle) = r$. Without loss of generality, $0 \in C$. Since G is an ILC-group, the finitely generated subgroup $\langle C\rangle$ has locally compact closure in G.

The totally bounded Boolean subgroup $G[2]$ has compact closure K_2 in the completion $\bar{G}$ of the abelian topological group G. It follows that the subgroup $H = \overline{\langle C\rangle} + K_2$ of $\bar{G}$ is locally compact and compactly generated. Consequently, it contains a compact subgroup $K \supset K_2$ such that the quotient group H/K is topologically isomorphic to $\mathbb{R}^s \times \mathbb{Z}^{r-s}$ for some $r \le s$. It follows from Lemma 8 that the set C is k-centerpole for Borel colorings of the group H. The compactness of the subgroup

$K \subset H$ implies that the image $q(C) \subset H/K$ of C under the quotient homomorphism $q : H \to H/K$ is a k-centerpole set for Borel colorings of H/K. Consequently,

$$c_k^B(\mathbb{R}^r) \le c_k^B(\mathbb{R}^s \times \mathbb{Z}^{r-s}) = c_k^B(H/K) \le |q(C)| \le |C| = c_k^B(G) < \infty$$

and hence $r \ge k$ by Theorem 3(5).

Now assume that $k \ge 4$. Since the set $q(C)$ is k-centerpole for Borel colorings of $H/K = \mathbb{R}^s \times \mathbb{Z}^{r-s} \subset \mathbb{R}^r$, Lemma 10 implies that the affine hull of $q(C)$ in the linear space $\mathbb{R}^r$ has dimension $\le |q(C)| - 3$. Since $0 \in q(C)$, the affine hull of the set $q(C)$ coincides with its linear hull. Consequently, $r = r_{\mathbb{Z}}(\langle C\rangle) = r_{\mathbb{Z}}(\langle q(C)\rangle) \le |q(C)| - 3 \le |C| - 3 = c_k^B(G) - 3$. This completes the proof of the lower and upper bounds

$$k \le rc_k(G) \le c_k^B(G) - 3$$

for all $k \ge 3$.

Next, we show that $rc_k(G) = k$ for $k \in \{2, 3\}$. In this case $c_k^B(G) = c_k(\mathbb{Z}^k)$ by Theorems 1 and 2. Since $r_{\mathbb{Z}}(G) \ge k$, the group G contains an isomorphic copy of the group $\mathbb{Z}^k$. Then each k-centerpole subset $C \subset \mathbb{Z}^k \subset G$ with $|C| = c_k(\mathbb{Z}^k)$ is k-centerpole for Borel colorings of G and thus $k \le rc_k^B(G) \le r_{\mathbb{Z}}(\langle C\rangle) \le k$, which implies the desired equality $rc_k^B(G) = k$.

12 Proof of Stabilization Theorem 4

Let $k \ge 2$ and G be an abelian ILC-group with totally bounded Boolean subgroup $G[2]$. Let $n = r_{\mathbb{R}}(G)$ and $m = r_{\mathbb{Z}}(G)$.

1. Assume that $m = r_{\mathbb{Z}}(G) \ge rc_k^B(\mathbb{Z}^\omega)$. By Proposition 1, $k \le rc_k^B(\mathbb{Z}^\omega) \le r_{\mathbb{Z}}(G)$ and then $c_k(G) = c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ by Theorem 1. Since $m = r_{\mathbb{Z}}(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \ge rc_k^B(\mathbb{Z}^\omega)$, Lemma 12 guarantees that $c_k(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$.
2. Assume that the group G is metrizable and $r_{\mathbb{Z}}(G) \ge rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$. By Proposition 1, $k \le rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega) \le r_{\mathbb{Z}}(G) = m$ and hence $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ by Theorem 1. Since $m = r_{\mathbb{Z}}(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \ge rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$, Lemma 11 guarantees that $c_k^B(G) = c_{\mathbb{Z}}^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$.
3. By analogy with the preceding case we can prove that $c_k^B(G) = c_k^B(\mathbb{R}^\omega)$ if G is metrizable and $r_{\mathbb{R}}(G) \ge rc_k^B(\mathbb{R}^\omega)$.

13 Selected open problems

By Theorem 2, $c_k^B(\mathbb{R}^\omega) = c_k(\mathbb{Z}^\omega) = c_k(\mathbb{Z}^k)$ for all $k \le 4$.

Problem 1 Is $c_k(\mathbb{Z}^\omega) = c_k(\mathbb{Z}^k)$ for all $k \in \mathbb{N}$? In particular, is $c_4(\mathbb{Z}^n) = 12$ for every $n \ge 4$?

Problem 2 Is $c_k^B(\mathbb{R}^n) = c_k(\mathbb{R}^n)$ for every $k \le n$?

Theorem 3 gives upper and lower bounds for the numbers $c_k(\mathbb{Z}^k)$ that have exponential and polynomial growths, respectively.

Problem 3 Is the growth of the sequence $(c_n(\mathbb{Z}^n))_{n\in\mathbb{N}}$ exponential?

By [1], for every $k \in \{1, 2, 3\}$ any k-centerpole subset $C \subset \mathbb{Z}^k$ of cardinality $|C| = c_k(\mathbb{Z}^k)$ is affinely equivalent to the $\binom{k-1}{k-3}$-sandwich Ξ^{k-1}_{k-3}.

Problem 4 Is each 12-element 4-centerpole subset of $\mathbb{Z}^4$ affinely equivalent to the $\binom{3}{1}$-sandwich Ξ^3_1?

It follows from the proof of Theorem 1 in [8] that the free group F_2 with two generators and discrete topology has $c_2(F_2) \le 13$.

Problem 5 What is the value of the cardinal $c_2(F_2)$? Is $c_3(F_2)$ finite?

The last problem can be posed in a more general context.

Problem 6 Investigate the cardinal characteristics $c_k(G)$ and $c^B_k(G)$ for non-commutative topological groups G.

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