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The absolute order on the hyperoctahedral group

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Abstract The absolute order on the hyperoctahedral group B_n is investigated. Using a poset fiber theorem, it is proved that the order ideal of this poset generated by the Coxeter elements is homotopy Cohen–Macaulay. This method results in a new proof of Cohen–Macaulayness of the absolute order on the symmetric group. Moreover, it is shown that every closed interval in the absolute order on B_n is shellable and an example of a non-Cohen–Macaulay interval in the absolute order on D_4 is given. Finally, the closed intervals in the absolute order on B_n and D_n which are lattices are characterized and some of their important enumerative invariants are computed.

Keywords Hyperoctahedral group · Absolute order · Cohen–Macaulay poset

1 Introduction and results

Coxeter groups are fundamental combinatorial structures which appear in several areas of mathematics. Partial orders on Coxeter groups often provide an important tool for understanding the questions of interest. Examples of such partial orders are the Bruhat order and the weak order. We refer the reader to [8, 10, 19] for background on Coxeter groups and their orderings.

In this work we study the absolute order. Let W be a finite Coxeter group and let $\mathcal{T}$ be the set of *all* reflections in W. The absolute order on W is denoted by $\mathrm{Abs}(W)$ and defined as the partial order on W whose Hasse diagram is obtained from the Cayley graph of W with respect to $\mathcal{T}$ by directing its edges away from the identity (see Sect. 2.2 for a precise definition). The poset $\mathrm{Abs}(W)$ is locally self-dual and graded. It has a minimum element, the identity $e \in W$, but will typically not have a maximum since, for example, every Coxeter element of W is a maximal element of $\mathrm{Abs}(W)$. Its

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rank function is called the absolute length and is denoted by $\ell_{\mathcal{T}}$. The absolute length and order arise naturally in combinatorics [1], group theory [5, 14], statistics [16] and invariant theory [19]. For instance, $\ell_{\mathcal{T}}(w)$ can also be defined as the codimension of the fixed space of w, when W acts faithfully as a group generated by orthogonal reflections on a vector space V by its standard geometric representation. Moreover, the rank generating polynomial of $\mathrm{Abs}(W)$ satisfies

$$\sum_{w\in W} t^{\ell_{\mathcal{T}}(w)} = \prod_{i=1}^{\ell} (1+e_i t),$$

where $e_1, e_2, \dots, e_\ell$ are the exponents [19, Sect. 3.20] of W and ℓ is its rank. We refer to [1, Sect. 2.4] and [3, Sect. 1] for further discussion of the importance of the absolute order and related historical remarks.

We will be interested in the combinatorics and topology of $\mathrm{Abs}(W)$. These have been studied extensively for the interval $[e,c] := NC(W,c)$ of $\mathrm{Abs}(W)$, known as the poset of noncrossing partitions associated to W, where $c \in W$ denotes a Coxeter element. For instance, it was shown in [4] that $NC(W,c)$ is shellable for every finite Coxeter group W. In particular, $NC(W,c)$ is Cohen–Macaulay over $\mathbb{Z}$ and the order complex of $NC(W,c) \smallsetminus \{e,c\}$ has the homotopy type of a wedge of spheres.

The problem to study the topology of the poset $\mathrm{Abs}(W) \smallsetminus \{e\}$ and to decide whether $\mathrm{Abs}(W)$ is Cohen–Macaulay, or even shellable, was posed by Reiner [2, Problem 3.1] and Athanasiadis (unpublished); see also [29, Problem 3.3.7]. Computer calculations carried out by Reiner showed that the absolute order is not Cohen–Macaulay for the group D_4. This led Reiner to ask [2, Problem 3.1] whether the order ideal of $\mathrm{Abs}(W)$ generated by the set of Coxeter elements is Cohen–Macaulay (or shellable) for every finite Coxeter group W. In the case of the symmetric group S_n this ideal coincides with $\mathrm{Abs}(S_n)$, since every maximal element of S_n is a Coxeter element. Although it is not known whether $\mathrm{Abs}(S_n)$ is shellable, the following results were obtained in [3].

Theorem 1 [3, Theorem 1.1] *The partially ordered set* $\mathrm{Abs}(S_n)$ *is homotopy Cohen–Macaulay for every* $n \geq 1$. *In particular, the order complex of* $\mathrm{Abs}(S_n) \smallsetminus \{e\}$ *is homotopy equivalent to a wedge of* $(n-2)$*-dimensional spheres and Cohen–Macaulay over* $\mathbb{Z}$.

Theorem 2 [3, Theorem 1.2] *Let* $\bar{P}_n = \mathrm{Abs}(S_n) \smallsetminus \{e\}$. *The reduced Euler characteristic of the order complex* $\Delta(\bar{P}_n)$ *satisfies*

$$\sum_{n\geq 1} (-1)^n \tilde{\chi}\big(\Delta(\bar{P}_n)\big)\frac{t^n}{n!} = 1 - C(t)\exp\big\{-2tC(t)\big\},$$

where $C(t) = \frac{1}{2t}(1-\sqrt{1-4t})$ *is the ordinary generating function for the Catalan numbers.*

In the present paper we focus on the hyperoctahedral group B_n. We denote by $\mathcal{J}_n$ the order ideal of $\mathrm{Abs}(B_n)$ generated by the Coxeter elements of B_n and by $\bar{\mathcal{J}}_n$ its proper part $\mathcal{J}_n \setminus \{e\}$. Contrary to the case of the symmetric group, not every maximal element of $\mathrm{Abs}(B_n)$ is a Coxeter element. Our main results are as follows.

Theorem 3 *The poset $\mathcal{J}_n$ is homotopy Cohen–Macaulay for every $n \geq 2$.*

Theorem 4 *The reduced Euler characteristic of the order complex $\Delta(\bar{\mathcal{J}}_n)$ satisfies*

$$\sum_{n\geq 2}(-1)^n \tilde{\chi}\big(\Delta(\bar{\mathcal{J}}_n)\big)\frac{t^n}{n!} = 1 - \sqrt{C(2t)}\exp\big\{-2tC(2t)\big\}\left(1 + \sum_{n\geq 1} 2^{n-1}\binom{2n-1}{n}\frac{t^n}{n}\right),$$

where $C(t) = \frac{1}{2t}(1 - \sqrt{1-4t})$ is the ordinary generating function for the Catalan numbers.

The maximal (with respect to inclusion) intervals in $\mathrm{Abs}(B_n)$ include the posets $NC^B(n)$ of noncrossing partitions of type B, introduced and studied by Reiner [26], and $NC^B(p,q)$ of annular noncrossing partitions, studied recently by Krattenthaler [22] and by Nica and Oancea [24]. We have the following result concerning the intervals of $\mathrm{Abs}(B_n)$.

Theorem 5 *Every interval of* $\mathrm{Abs}(B_n)$ *is shellable.*

Furthermore, we consider the absolute order on the group D_n and give an example of a maximal element x of $\mathrm{Abs}(D_4)$ for which the interval $[e,x]$ is not Cohen–Macaulay over any field (Remark 2). This is in accordance with Reiner's computations, showing that $\mathrm{Abs}(D_4)$ is not Cohen–Macaulay and answers in the negative a question raised by Athanasiadis (personal communication), asking whether all intervals in the absolute order on Coxeter groups are shellable. Moreover, it shows that $\mathrm{Abs}(D_n)$ is not Cohen–Macaulay over any field for every $n \geq 4$. It is an open problem to decide whether $\mathrm{Abs}(B_n)$ is Cohen–Macaulay for every $n \geq 2$ and whether the order ideal of $\mathrm{Abs}(W)$ generated by the set of Coxeter elements is Cohen–Macaulay for every Coxeter group W [2, Problem 3.1].

This paper is organized as follows. In Sect. 2 we fix notation and terminology related to partially ordered sets and simplicial complexes and discuss the absolute order on the classical finite reflection groups. In Sect. 3 we prove Theorem 5 by showing that every closed interval of $\mathrm{Abs}(B_n)$ admits an EL-labeling (see Sect. 2 for the definition of EL-labeling). Theorems 3 and 4 are proved in Sect. 4. Our method to establish homotopy Cohen–Macaulayness is different from that of [3]. It is based on a poset fiber theorem due to Quillen [25, Corollary 9.7]. The same method gives an alternative proof of Theorem 1, which is also included in Sect. 4. Theorems 3 and 4 require some lemmas whose proofs are based on induction and use the notion of strong constructibility (see Sect. 2 for the definition of strongly constructible posets). Since these proofs are rather technical they appear in the Appendix. In Sect. 5 we characterize the closed intervals in $\mathrm{Abs}(B_n)$ and $\mathrm{Abs}(D_n)$ which are lattices. In Sect. 6 we study a special case of such an interval, namely the maximal interval $[e,x]$ of $\mathrm{Abs}(B_n)$, where $x = t_1 t_2 \cdots t_n$ and each t_i is a balanced reflection. Finally, in Sect. 7 we compute the zeta polynomial, cardinality and Möbius function of the intervals of $\mathrm{Abs}(B_n)$ which are lattices. These computations are based on results of Goulden, Nica and Oancea [18] concerning enumerative properties of the poset $NC^B(n-1,1)$.

2 Preliminaries

2.1 Partial orders and simplicial complexes

Let $(P, \leq)$ be a finite partially ordered set (poset for short) and $x, y \in P$. We say that y *covers* x, and write $x \to y$, if $x < y$ and there is no $z \in P$ such that $x < z < y$. The poset P is called *bounded* if there exist elements $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for every $x \in P$. The elements of P which cover $\hat{0}$ are called *atoms*. A subset C of a poset P is called a *chain* if any two elements of C are comparable in P. The length of a (finite) chain C is equal to $|C| - 1$. We say that P is *graded* if all maximal chains of P have the same length. In that case, the common length of all maximal chains of P is called the *rank* of P. Moreover, assuming P has a $\hat{0}$ element, there exists a unique function $\rho : P \to \mathbb{N}$, called the *rank function* of P, such that

$$\rho(y) = \begin{cases} 0 & \text{if } y = \hat{0}, \\ \rho(x) + 1 & \text{if } x \to y. \end{cases}$$

We say that x has *rank* i if $\rho(x) = i$. For $x \leq y$ in P we denote by $[x, y]$ the closed interval $\{z \in P : x \leq z \leq y\}$ of P, endowed with the partial order induced from P. If S is a subset of P, then the *order ideal* of P generated by S is the subposet $\langle S \rangle$ of P consisting of all $x \in P$ for which $x \preceq y$ holds for some $y \in S$. We will write $\langle y_1, y_2, \ldots, y_m \rangle$ for the order ideal of P generated by the set $\{y_1, y_2, \ldots, y_m\}$. In particular, $\langle x \rangle$ denotes the subposet of P consisting of all elements of P which are less than (or equal to) x. Given two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, a map $f : P \to Q$ is called a *poset map* if it is order preserving, i.e. $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for all $x, y \in P$. If, in addition, f is a bijection with order preserving inverse, then f is said to be a *poset isomorphism*. If there exists a poset isomorphism $f : P \to Q$, then the posets P and Q are said to be *isomorphic*, and we write $P \cong Q$. Assuming that P and Q are graded, the map $f : P \to Q$ is called *rank-preserving* if for every $x \in P$, the rank of $f(x)$ in Q is equal to the rank of x in P. The *direct product* of P and Q is the poset $P \times Q$ on the set $\{(x, y) : x \in P,\ y \in Q\}$ for which $(x, y) \leq (x', y')$ holds in $P \times Q$ if $x \leq_P x'$ and $y \leq_Q y'$. The *dual* of P is the poset P^* defined on the same ground set as P by letting $x \leq y$ in P^* if and only if $y \leq x$ in P. The poset P is called *self-dual* if P and P^* are isomorphic and *locally self-dual* if every closed interval of P is self-dual. For more information on partially ordered sets we refer the reader to [27, Chap. 3].

Let V be a nonempty finite set. An *abstract simplicial complex* Δ on the vertex set V is a collection of subsets of V such that $\{v\} \in \Delta$ for every $v \in V$ and such that $G \in \Delta$ and $F \subseteq G$ imply $F \in \Delta$. The elements of V and Δ are called *vertices* and *faces* of Δ, respectively. The maximal faces are called *facets*. The dimension of a face $F \in \Delta$ is equal to $|F| - 1$ and is denoted by $\dim F$. The *dimension* of Δ is defined as the maximum dimension of a face of Δ and is denoted by $\dim \Delta$. If all facets of Δ

have the same dimension, then Δ is said to be *pure*. The *link* of a face F of a simplicial complex Δ is defined as $\mathrm{link}_\Delta(F) = \{G \setminus F : G \in \Delta, F \subseteq G\}$. All topological properties of an abstract simplicial complex Δ we mention will refer to those of its geometric realization $\|\Delta\|$. The complex Δ is said to be *homotopy Cohen–Macaulay* if for all $F \in \Delta$ the link of F is topologically $(\dim \mathrm{link}_\Delta(F) - 1)$-connected. For a facet G of a simplicial complex Δ, we denote by $\bar{G}$ the Boolean interval $[\varnothing, G]$. A pure d-dimensional simplicial complex Δ is *shellable* if there exists a total ordering $G_1, G_2, \dots, G_m$ of the set of facets of Δ such that for all $1 < i \leq m$, the intersection of $\bar{G}_1 \cup \bar{G}_2 \cup \cdots \cup \bar{G}_{i-1}$ with $\bar{G}_i$ is pure of dimension $d-1$. For a d-dimensional simplicial complex we have the following implications: pure shellable $\Rightarrow$ homotopy Cohen–Macaulay $\Rightarrow$ homotopy equivalent to a wedge of d-dimensional spheres. For background concerning the topology of simplicial complexes we refer to [9] and [29].

To every poset P we associate an abstract simplicial complex $\Delta(P)$, called the *order complex* of P. The vertices of $\Delta(P)$ are the elements of P and its faces are the chains of P. If P is graded of rank n, then $\Delta(P)$ is pure of dimension n. All topological properties of a poset P we mention will refer to those of the geometric realization of $\Delta(P)$. We say that a poset P is *shellable* if its order complex $\Delta(P)$ is shellable.

We recall the notion of EL-shellability, defined by Björner [7]. Assume that P is bounded and graded and let $C(P) = \{(a, b) \in P \times P : \ a \to b\}$ be the set of covering relations of P. An *edge labeling* of P is a map $\lambda : C(P) \to \Lambda$, where Λ is some poset. Let $[x, y]$ be a closed interval of P of rank n. To each maximal chain $c : x \to x_1 \to \cdots \to x_{n-1} \to y$ of $[x, y]$ we associate the sequence $\lambda(c) = (\lambda(x, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{n-1}, y))$. We say that c is *strictly increasing* if the sequence $\lambda(c)$ is strictly increasing in the order of Λ. The maximal chains of $[x, y]$ can be totally ordered by using the lexicographic order on the corresponding sequences. An *edge-lexicographic labeling* (*EL-labeling*) of P is an edge labeling such that in each closed interval $[x, y]$ of P there is a unique strictly increasing maximal chain and this chain lexicographically precedes all other maximal chains of $[x, y]$. The poset P is called *EL-shellable* if it admits an EL-labeling. A finite poset P of rank d with a minimum element is called *strongly constructible* [3] if it is bounded and pure shellable, or it can be written as a union $P = I_1 \cup I_2$ of two strongly constructible proper ideals I_1, I_2 of rank n, such that $I_1 \cap I_2$ is strongly constructible of rank at least $n-1$.

Finally, we remark that every EL-shellable poset is shellable [7, Theorem 2.3]. We also recall the following lemmas.

Lemma 1 *Let P and Q be finite posets, each with a minimum element.*

(i) [3, *Lemma* 3.7] *If P and Q are strongly constructible, then so is the direct product $P \times Q$.*
(ii) [3, *Lemma* 3.8] *If P is the union of strongly constructible ideals $I_1, I_2, \dots, I_k$ of P of rank n and the intersection of any two or more of these ideals is strongly constructible of rank n or $n-1$, then P is also strongly constructible.*

Lemma 2 *Every strongly constructible poset is homotopy Cohen–Macaulay.*

Proof It follows from [3, Proposition 3.6] and [3, Corollary 3.3]. □

Lemma 3 *Let P and Q be finite posets, each with a minimum element.*

(i) *If P and Q are homotopy Cohen–Macaulay, then so is the direct product $P \times Q$.*
(ii) *If P is the union of homotopy Cohen–Macaulay ideals $I_1, I_2, \ldots, I_k$ of P of rank n and the intersection of any two or more of these ideals is homotopy Cohen–Macaulay of rank n or $n-1$, then P is also homotopy Cohen–Macaulay.*

Proof The first part follows from [12, Corollary 3.8]. The proof of the second part is similar to that of [3, Lemma 3.4]. □

2.2 The absolute length and absolute order

Let W be a finite Coxeter group and let $\mathcal{T}$ denote the set of all reflections in W. Given $w \in W$, the *absolute length* of w is defined as the smallest integer k such that w can be written as a product of k elements of $\mathcal{T}$; it is denoted by $\ell_{\mathcal{T}}(w)$. The *absolute order* $\mathrm{Abs}(W)$ is the partial order $\preceq$ on W defined by

$$u \preceq v \quad \text{if and only if} \quad \ell_{\mathcal{T}}(u) + \ell_{\mathcal{T}}\big(u^{-1}v\big) = \ell_{\mathcal{T}}(v)$$

for $u, v \in W$. Equivalently, $\preceq$ is the partial order on W with covering relations $w \to wt$, where $w \in W$ and $t \in \mathcal{T}$ are such that $\ell_{\mathcal{T}}(w) < \ell_{\mathcal{T}}(wt)$. In that case we write $w \xrightarrow{t} wt$. The poset $\mathrm{Abs}(W)$ is graded with rank function $\ell_{\mathcal{T}}$.

Every closed interval in W is isomorphic to one which contains the identity. Specifically, we have the following lemma (see also [4, Lemma 3.7]).

Lemma 4 *Let $u, v \in W$ with $u \preceq v$. The map $\phi : [u, v] \to [e, u^{-1}v]$ defined by $\phi(w) = u^{-1}w$ is a poset isomorphism.*

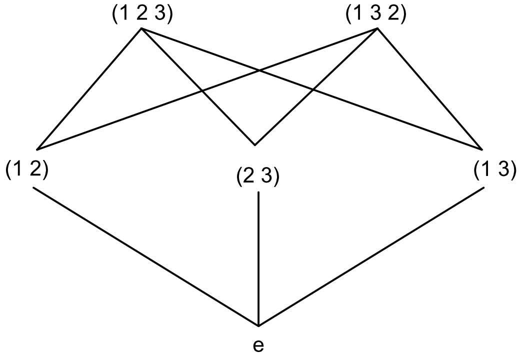
Proof It follows from [1, Lemma 2.5.4] by an argument similar to that in the proof of [1, Proposition 2.6.11]. □

For more information on the absolute order on W we refer the reader to [1, Sect. 2.4].

The absolute order on S_n

We view the group S_n as the group of permutations of the set $\{1, 2, \ldots, n\}$. The set $\mathcal{T}$ of reflections of S_n is equal to the set of all transpositions $(i\ j)$, where $1 \le i < j \le n$. The length $\ell_{\mathcal{T}}(w)$ of $w \in S_n$ is equal to $n - \gamma(w)$, where $\gamma(w)$ denotes the number of cycles in the cycle decomposition of w. Given a cycle $c = (i_1\, i_2 \cdots i_r)$ in S_n and indices $1 \le j_1 < j_2 < \cdots < j_s \le r$, we say that the cycle $(i_{j_1}\, i_{j_2} \cdots i_{j_s})$ can be obtained from c by deleting elements. Given two disjoint cycles a, b in S_n each of which can be obtained from c by deleting elements, we say that a and b are noncrossing with respect to c if there does not exist a cycle $(i\, j\, k\, l)$ of length four which can be obtained from c by deleting elements, such that i, k are elements of a

Fig. 1 The poset $\mathrm{Abs}(S_3)$



and j, l are elements of b. For instance, if $n = 9$ and $c = (3\,5\,1\,9\,2\,6\,4)$ then the cycles $(3\,6\,4)$ and $(5\,9\,2)$ are noncrossing with respect to c but $(3\,2\,4)$ and $(5\,9\,6)$ are not. It can be verified [13, Sect. 2] that for $u, v \in S_n$ we have $u \preceq v$ if and only if

– Every cycle in the cycle decomposition for u can be obtained from some cycle in the cycle decomposition for v by deleting elements, and
– Any two cycles of u which can be obtained from the same cycle c of v by deleting elements are noncrossing with respect to c

Clearly, the maximal elements of $\mathrm{Abs}(S_n)$ are precisely the n-cycles, which are the Coxeter elements of S_n. Figure 1 illustrates the Hasse diagram of the poset $\mathrm{Abs}(S_3)$.

The absolute order on B_n

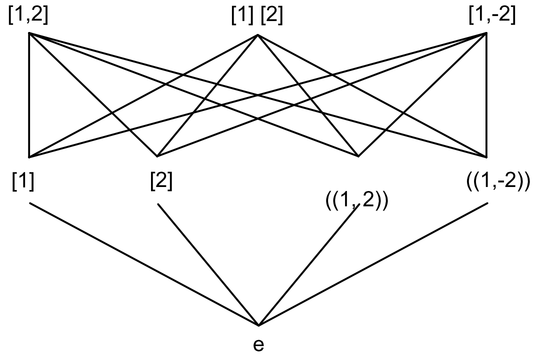
We view the hyperoctahedral group B_n as the group of permutations w of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $w(-i) = -w(i)$ for $1 \le i \le n$. Following [14], the permutation which has cycle form $(a_1\, a_2 \cdots a_k)(-a_1\ -a_2 \cdots -a_k)$ is denoted by $((a_1, a_2, \ldots, a_k))$ and is called a *paired* k-cycle, while the cycle $(a_1\, a_2 \cdots a_k\ -a_1\ -a_2 \cdots -a_k)$ is denoted by $[a_1, a_2, \ldots, a_k]$ and is called a *balanced* k-cycle. Every element $w \in B_n$ can be written as a product of disjoint paired or balanced cycles, called cycles of w. With this notation, the set $\mathcal{T}$ of reflections of B_n is equal to the union

$$\big\{[i] : 1 \le i \le n\big\} \cup \big\{((i, j)), ((i, -j)) : 1 \le i < j \le n\big\}. \tag{1}$$

The length $\ell_{\mathcal{T}}(w)$ of $w \in B_n$ is equal to $n - \gamma(w)$, where $\gamma(w)$ denotes the number of paired cycles in the cycle decomposition of w. An element $w \in B_n$ is maximal in $\mathrm{Abs}(B_n)$ if and only if it can be written as a product of disjoint balanced cycles whose lengths sum to n. The Coxeter elements of B_n are precisely the balanced n-cycles. The covering relations $w \xrightarrow{t} wt$ of $\mathrm{Abs}(B_n)$, when w and t are non-disjoint cycles, can be described as follows: For $1 \le i < j \le m \le n$, we have

(a) $((a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)) \xrightarrow{((a_{i-1}, a_i))} ((a_1, \ldots, a_m))$
(b) $((a_1, \ldots, a_m)) \xrightarrow{[a_i]} [a_1, \ldots, a_{i-1}, a_i, -a_{i+1}, \ldots, -a_m]$
(c) $((a_1, \ldots, a_m)) \xrightarrow{((a_i, -a_j))} [a_1, \ldots, a_i, -a_{j+1}, \ldots, -a_m][a_{i+1}, \ldots, a_j]$

Fig. 2 The poset $\mathrm{Abs}(B_2)$



(d) $[a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m] \xrightarrow{((a_{i-1}, a_i))} [a_1, \dots, a_m]$

(e) $[a_1, \dots, a_j]((a_{j+1}, \dots, a_m)) \xrightarrow{((a_j, a_m))} [a_1, \dots, a_m]$

(f) $((a_1, \dots, a_j))((a_{j+1}, \dots, a_m)) \xrightarrow{((a_j, a_m))} ((a_1, \dots, a_m))$

where $a_1, \dots, a_m$ are elements of $\{\pm 1, \dots, \pm n\}$ with pairwise distinct absolute values. Figure 2 illustrates the Hasse diagram of the poset $\mathrm{Abs}(B_2)$.

The absolute order on D_n

The Coxeter group D_n is the subgroup of index two of the group B_n, generated by the set of reflections

$$\{((i, j)), ((i, -j)) : 1 \le i < j \le n\} \tag{2}$$

(these are all reflections in D_n). An element $w \in B_n$ belongs to D_n if and only if w has an even number of balanced cycles in its cycle decomposition. The absolute length on D_n is the restriction of the absolute length of B_n on the set D_n and hence $\mathrm{Abs}(D_n)$ is a subposet of $\mathrm{Abs}(B_n)$. Every Coxeter element of D_n has the form $[a_1, a_2, \dots, a_{n-1}][a_n]$, where $a_1, \dots, a_n$ are elements of $\{\pm 1, \dots, \pm n\}$ with pairwise distinct absolute values.

Remark 1 Let $w = bp$ be an element in B_n or D_n, where b (respectively, p) is the product of all balanced (respectively, paired) cycles of w. The covering relations of $\mathrm{Abs}(B_n)$ imply the poset isomorphism $[e, w] \cong [e, b] \times [e, p]$. Moreover, if $p = p_1 \cdots p_k$ is written as a product of disjoint paired cycles, then

$$[e, w] \cong [e, b] \times [e, p_1] \times \cdots \times [e, p_k].$$

2.3 Projections

We recall that $\mathcal{J}_n$ denotes the order ideal of $\mathrm{Abs}(B_n)$ generated by the Coxeter elements of B_n. Let P_n be $\mathrm{Abs}(S_n)$ or $\mathcal{J}_n$ for some $n \ge 2$. For $i \in \{1, 2, \dots, n\}$ we define a map $\pi_i : P_n \to P_n$ by letting $\pi_i(w)$ be the permutation obtained when $\pm i$ is deleted from the cycle decomposition of w. For example, if $n = i = 5$ and $w = [1, -5, 2]((3, -4)) \in \mathcal{J}_5$, then $\pi_5(w) = [1, 2]((3, -4))$.

Lemma 5 *The following hold for the map* $\pi_i : P_n \to P_n$.

(i) $\pi_i(w) \preceq w$ *for every* $w \in P_n$.
(ii) π_i *is a poset map.*

Proof Let $w \in P_n$. If $w(i) = i$, then clearly $\pi_i(w) = w$. Suppose that $w(i) \neq i$. Then it follows from our description of $\mathrm{Abs}(S_n)$ and from the covering relations of types (a) and (d) of $\mathrm{Abs}(B_n)$ that $\pi_i(w)$ is covered by w. Hence $\pi_i(w) \preceq w$. This proves (i). To prove (ii), it suffices to show that for every covering relation $u \to v$ in P_n we have either $\pi_i(u) = \pi_i(v)$ or $\pi_i(u) \to \pi_i(v)$. Again, this follows from our discussion of $\mathrm{Abs}(S_n)$ and from our list of covering relations of $\mathrm{Abs}(B_n)$. □

Lemma 6 *Let* P_n *stand for either* $\mathrm{Abs}(S_n)$ *for every* $n \geq 1$, *or* $\mathcal{J}_n$ *for every* $n \geq 2$. *Let also* $w \in P_n$ *and* $u \in P_{n-1}$ *be such that* $\pi_n(w) \preceq u$. *Then there exists an element* $v \in P_n$ *which covers* u *and satisfies* $\pi_n(v) = u$ *and* $w \preceq v$.

Proof We may assume that w does not fix n, since otherwise the result is trivial. Suppose that $\pi_n(w) = w_1 \cdots w_l$ and $u = u_1 \cdots u_r$ are written as products of disjoint cycles in P_{n-1}.

Case 1: $P_n = \mathrm{Abs}(S_n)$ for $n \geq 1$. Then there is an index $i \in \{1, 2, \dots, l\}$ such that w is obtained from $\pi_n(w)$ by inserting n in the cycle w_i. Let y be the cycle of w containing n, so that $\pi_n(y) = w_i$. From the description of the absolute order on S_n given in this section, it follows that $w_i \preceq u_j$ for some $j \in \{1, 2, \dots, r\}$. We may insert n in the cycle u_j so that the resulting cycle v_j satisfies $y \preceq v_j$. Let v be the element of S_n obtained by replacing u_j in the cycle decomposition of u by v_j. Then u is covered by v, $\pi_n(v) = u$ and $w \preceq v$.

Case 2: $P_n = \mathcal{J}_n$ for $n \geq 2$. The result follows by a simple modification of the argument in the previous case, if $[n]$ is not a cycle of w. Assume the contrary, so that $w = \pi_n(w)[n]$ and all cycles of $\pi_n(w)$ are paired. If u has no balanced cycle, then $w \preceq u[n] \in \mathcal{J}_n$ and hence $v = u[n]$ has the desired properties. Suppose that u has a balanced cycle in its cycle decomposition, say $b = [a_1, \dots, a_k]$. We denote by p the product of all paired cycles of u, so that $u = bp$. If $\pi_n(w) \preceq p$, then the choice $v = [a_1, \dots, a_k, n]p$ works. Otherwise, we may assume that there is an index $m \in \{1, 2, \dots, l\}$ such that $w_1 \cdots w_m \preceq b$ and w_i and b are disjoint for every $i > m$. From the covering relations of $\mathrm{Abs}(B_n)$ of types (a), (b) and (f) it follows that there is a paired cycle c which is covered by b and satisfies $w_1 \cdots w_m \preceq c$. Thus $\pi_n(w) \preceq cp \preceq u$. More specifically, c has the form $((a_1, \dots, a_i, -a_{i+1}, \dots, -a_k))$ for some $i \in \{2, \dots, k\}$. We set $v = [a_1, \dots, a_i, n, a_{i+1}, \dots, a_l]p$. Then v covers u and $w \preceq cp[n] \preceq v$. This concludes the proof of the lemma. □

3 Shellability

In this section we prove Theorem 5 by showing that every closed interval of $\mathrm{Abs}(B_n)$ admits an EL-labeling. Let $C(B_n)$ be the set of covering relations of $\mathrm{Abs}(B_n)$ and $(a, b) \in C(B_n)$. Then $a^{-1}b$ is a reflection of B_n, thus either $a^{-1}b = [i]$ for some $i \in$

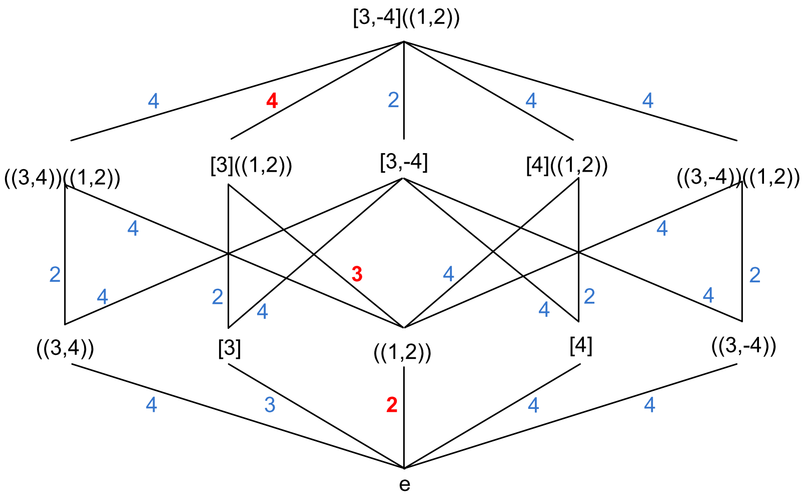


Fig. 3 The interval $[e, [e, [3, -4]((1, 2))]]$ in $\mathrm{Abs}(S_4)$

$\{1, 2, \dots, n\}$, or there exist $i, j \in \{1, 2, \dots, n\}$, with $i < j$, such that $a^{-1}b = ((i, j))$ or $a^{-1}b = ((i, -j))$. We define a map $\lambda : C(B_n) \to \{1, 2, \dots, n\}$ as follows:

$$\lambda(a, b) = \begin{cases} i, & \text{if } a^{-1}b = [i], \\ j, & \text{if } a^{-1}b = ((i, j)) \text{ or } ((i, -j)). \end{cases}$$

A similar labeling was used by Biane [6] in order to study the maximal chains of the poset $NC^B(n)$ of noncrossing B_n-partitions. Figure 3 illustrates the Hasse diagram of the interval $[e, [3, -4]((1, 2))]$, together with the corresponding labels.

Proposition 1 *Let $u, v \in B_n$ with $u \preceq v$. Then, the restriction of the map λ to the interval $[u, v]$ is an EL-labeling.*

Proof Let $u, v \in B_n$ with $u \preceq v$. We consider the poset isomorphism $\phi : [u, v] \to [e, u^{-1}v]$ from Lemma 4. Let $(a, b) \in C([u, v])$. Then we have

$$\phi(a)^{-1}\phi(b) = (u^{-1}a)^{-1}u^{-1}b = a^{-1}uu^{-1}b = a^{-1}b,$$

which implies that $\lambda(a, b) = \lambda(\phi(a), \phi(b))$. Thus, it suffices to show that $\lambda|_{[e,w]}$ is an EL-labeling for the interval $[e, w]$, where $w = u^{-1}v$.

Let $b_1 \cdots b_k\, p_1 \cdots p_l$ be the cycle decomposition of w, where $b_i = [b_i^1, \dots, b_i^{k_i}]$ for $i \le k$ and $p_j = ((p_j^1, \dots, p_j^{l_j}))$, with $p_j^1 = \min\{|p_j^m| : 1 \le m \le l_j\}$ for $j \le l$. We consider the sequence of positive integers obtained by placing the numbers $|b_i^h|$ and $|p_j^m|$, for $i, j, h \ge 1$ and $m > 1$, in increasing order. There are $r = \ell_{\mathcal{T}}(w)$ such integers. To simplify the notation, we denote by $c(w) = (c_1, c_2, \dots, c_r)$ this sequence and say that c_μ ($\mu = 1, 2, \dots, r$) belongs to a balanced (respectively, paired) cycle if it is equal to some $|b_i^h|$ (respectively, $|p_j^m|$). Clearly, we have

$$c_1 < c_2 < \cdots < c_r \tag{3}$$

and $\lambda(a,b) \in \{c_1, c_2, \ldots, c_r\}$ for all $(a,b) \in C([e,w])$. To the sequence (3) corresponds a unique maximal chain

$$\mathcal{C}_w : \; w_0 = e \xrightarrow{c_1} w_1 \xrightarrow{c_2} w_2 \xrightarrow{c_3} \cdots \xrightarrow{c_r} w_r = w,$$

which can be constructed inductively as follows (here, the integer κ in $a \xrightarrow{\kappa} b$ denotes the label $\lambda(a,b)$). If c_1 belongs to a balanced cycle, then $w_1 = [c_1]$. Otherwise, c_1 belongs to some p_i, say p_1, and we set w_1 to be either $((p_1^1, c_1))$ or $((p_1^1, -c_1))$, so that $w_1 \preceq p_1$ holds. Note that in both cases we have $\lambda(e, w_1) = c_1$ and $\lambda(e, w_1) < \lambda(e, w)$ for any other atom $t \in [e,w]$. Indeed, suppose that there is an atom $t \neq w_1$ such that $\lambda(e,t) = c_1$. We assume first that c_1 belongs to a balanced cycle, so $w_1 = [c_1]$. Then t is a reflection of the form $((c_0, \pm c_1))$, where $c_0 < c_1$ and, therefore, c_0 belongs to some paired cycle of w (if not then c_1 would not be minimum). However from the covering relations of $\mathrm{Abs}(B_n)$ written at the end of Sect. 2.2 it follows that $((c_0, \pm c_1)) \npreceq w$, thus $((c_0, \pm c_1)) \notin [e,w]$, a contradiction. Therefore c_1 belongs to a paired cycle of w, say p_1, and w_1, t are both paired reflections. Without loss of generality, let $w_1 = ((p_1^1, c_1))$ and $t = ((c_0, c_1))$, for some $c_0 < c_1$. By the first covering relation written at the end of Sect. 2.2 and the definition of λ, it follows that $c_0 = p_1^1$, thus $w_1 = t$, again a contradiction.

Suppose now that we have uniquely defined the elements $w_1, w_2, \ldots, w_j$, so that for every $i = 1, 2, \ldots, j$ we have $w_{i-1} \to w_i$ with $\lambda(w_{i-1}, w_i) = c_i$ and $\lambda(w_{i-1}, w_i) < \lambda(w_{i-1}, z)$ for every $z \in [e,w]$ such that $z \neq w_i$ and $w_{i-1} \to z$. We consider the number c_{j+1} and distinguish two cases.

Case 1: c_{j+1} belongs to a cycle whose elements have not been used. In this case, if c_{j+1} belongs to a balanced cycle, then we set $w_{j+1} = w_j[c_{j+1}]$, while if c_{j+1} belongs to p_s for some $s \in \{1, 2, \ldots, l\}$, then we set w_{j+1} to be either $w_j((p_s^1, c_{j+1}))$ or $w_j((p_s^1, -c_{j+1}))$, so that $w_j^{-1} w_{j+1} \preceq p_s$ holds.

Case 2: c_{j+1} belongs to a cycle some element of which has been used. Then there exists an index $i < j+1$ such that c_i belongs to the same cycle as c_{j+1}. If c_i, c_{j+1} belong to some b_s, then there is a balanced cycle of w_j, say a, that contains c_i. In this case we set w_{j+1} to be the permutation that we obtain from w_j if we add the number c_{j+1} in the cycle a in the same order and with the same sign that it appears in b_s. We proceed similarly if c_i, c_{j+1} belong to the same paired cycle.

In both cases we have $\lambda(w_j, w_{j+1}) = c_{j+1}$. This follows from the covering relations of $\mathrm{Abs}(B_n)$ given in the end of Sect. 2.2. Furthermore, we claim that if $z \in [e,w]$ with $z \neq w_{j+1}$ is such that $w_j \to z$, then $\lambda(w_j, w_{j+1}) < \lambda(w_j, z)$. Indeed, in view of the poset isomorphism $\phi : [u,v] \to [e, u^{-1}v]$ for $u = w_j$ and $v = w$, this follows from the special case $j = 0$ treated earlier. By definition of λ and the construction of $\mathcal{C}_u$, the sequence

$$\big(\lambda(e, w_1), \lambda(w_1, w_2), \ldots, \lambda(w_{r-1}, w)\big)$$

coincides with $c(w)$. Moreover, $\mathcal{C}_w$ is the unique maximal chain having this sequence of labels. This and the fact that the labels of any chain in $[e,w]$ are elements of the set $\{c_1, c_2, \ldots, c_r\}$ imply that $\mathcal{C}_w$ is the unique strictly increasing maximal chain. By what we have already shown, $\mathcal{C}_w$ lexicographically precedes all other maximal chains of $[e, w]$. Thus $\mathcal{C}_w$ is lexicographically first and the unique strictly increasing chain in $[e, w]$. Hence λ is an EL-labeling for the interval $[e, w]$ and Proposition 1 is proved. $\square$

Proof of Theorem 5 Let $[u, v]$ be an interval in $\mathrm{Abs}(B_n)$. It follows from Proposition 1 that $[u, v]$ is EL-shellable. However, [7, Theorem 2.3] implies that any EL-shellable poset is shellable. This concludes the proof of the theorem. $\square$

Examples 1

(i) *Let* $n = 7$ *and* $w = [1, -7][3]((2, -6, -5))((4)) \in B_7$. *Then* $c(w) = (1, 3, 5, 6, 7)$ *and*

$$\mathcal{C}_w : e \xrightarrow{1} [1] \xrightarrow{3} [1][3] \xrightarrow{5} [1][3]((2, -5)) \xrightarrow{6} [1][3]((2, -6, -5)) \xrightarrow{7} w.$$

(ii) *Let* $n = 4$ *and* $w = [3, -4]((1, 2))$. *Then* $c(w) = (2, 3, 4)$ *and*

$$\mathcal{C}_w : e \xrightarrow{2} ((1, 2)) \xrightarrow{3} ((1, 2))[3] \xrightarrow{4} w.$$

Remark 2 Figure 4 illustrates the Hasse diagram of the interval $I = [e, u]$ of $\mathrm{Abs}(D_4)$, where $u = [1][2][3][4]$ (in Fig. 4 some of the elements are written on two lines for reasons of space). Note that the Hasse diagram of the open interval (e, u) is disconnected and, therefore, I is not Cohen–Macaulay over any field. Since $\mathrm{Abs}(D_n)$ contains an interval which is isomorphic to I for any $n \geq 4$, it follows that $\mathrm{Abs}(D_n)$ is not Cohen–Macaulay over any field for $n \geq 4$ either (see [29, Corollary 3.1.9]).

4 Cohen–Macaulayness

In this section we prove Theorems 3 and 4. Our method to show that $\mathcal{J}_n$ is homotopy Cohen–Macaulay is based on the following theorem, due to Quillen [25, Corollary 9.7]; see also [11, Theorem 5.1]. The same method yields a new proof of Theorem 1, which we also include in this section.

Theorem 6 *Let P and Q be graded posets and let $f : P \to Q$ be a surjective rank-preserving poset map. Assume that for all $q \in Q$ the fiber $f^{-1}(\langle q \rangle)$ is homotopy Cohen–Macaulay. If Q is homotopy Cohen–Macaulay, then so is P.*

We recall (see Sect. 2.1) that by $\langle q \rangle$ we denote the order ideal of Q generated by the singleton $\{q\}$. For other poset fiber theorems of this type, see [11].

To prove Theorems 1 and 3, we need the following. Let $\{\hat{0}, \hat{1}\}$ be a two element chain, with $\hat{0} < \hat{1}$ and $i \in \{1, 2, \ldots, n\}$. We consider the map $\pi_i : P_n \to P_n$ of Sect. 2.3, where P_n is either $\mathrm{Abs}(S_n)$ or $\mathcal{J}_n$. We define the map

$$f_i : P_n \to \pi_i(P_n) \times \{\hat{0}, \hat{1}\}$$

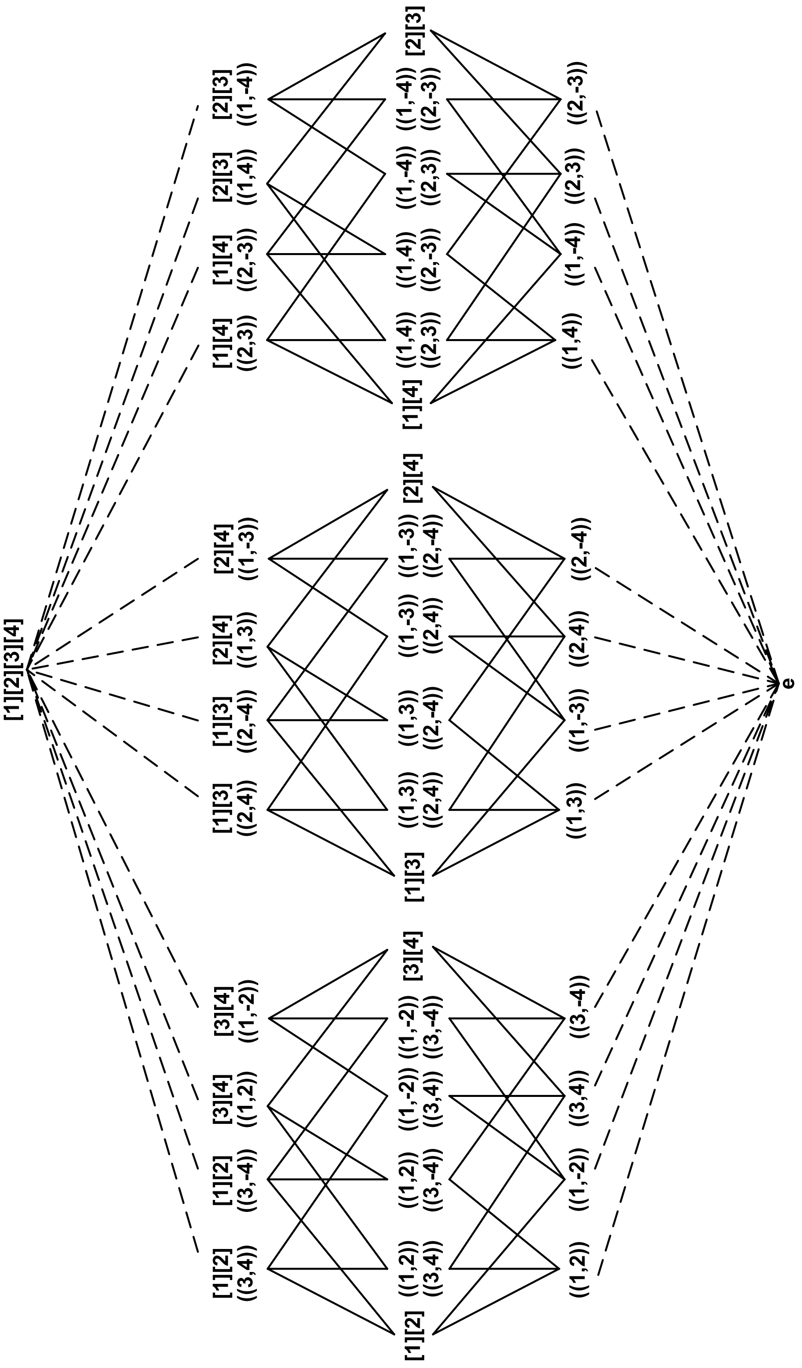


Fig. 4 The interval $[e, [1][2][3][4]]$ in $\mathrm{Abs}(D_4)$

by letting

$$f_i(w)=\begin{cases}(\pi_i(w),\hat{0}), & \text{if } w(i)=i,\\ (\pi_i(w),\hat{1}), & \text{if } w(i)\neq i\end{cases}$$

for $w\in P_n$. We first check that f_i is a surjective rank-preserving poset map. Indeed, by definition f_i is rank-preserving. Let $u,v\in P_n$ with $u\preceq v$. Lemma 5(ii) implies that $\pi_i(u)\preceq\pi_i(v)$. If $u(i)\neq i$, then $v(i)\neq i$ as well and hence $f_i(u)=(\pi_i(u),\hat{1})\le(\pi_i(v),\hat{1})=f_i(v)$. If $u(i)=i$, then $f_i(u)=(\pi_i(u),\hat{0})$ and hence $f_i(u)\le f_i(v)$. Thus f_i is a poset map. Moreover, if $w\in\pi_i(P_n)$, then $f_i^{-1}(\{(w,\hat{0})\})=\{w\}$ and any permutation obtained from w by inserting the element i in a cycle of w lies in $f_i^{-1}(\{(w,\hat{1})\})$. Thus $f_i^{-1}(\{q\})\neq\varnothing$ for every $q\in\pi_i(P_n)\times\{\hat{0},\hat{1}\}$, which means that f_i is surjective.

Given a map $f:P\to Q$, we abbreviate by $f^{-1}(q)$ the inverse image $f^{-1}(\{q\})$ of a singleton subset $\{q\}$ of Q. For subsets U and V of S_n (respectively, of B_n), we write $U\cdot V=\{uv:u\in U,v\in V\}$.

The following lemmas will be used in the proof of Theorem 1.

Lemma 7 *For every $q\in S_{n-1}\times\{\hat{0},\hat{1}\}$ we have $f_n^{-1}(\langle q\rangle)=\langle f_n^{-1}(q)\rangle$.*

Proof The result is trivial for $q=(u,\hat{0})\in S_{n-1}\times\{\hat{0},\hat{1}\}$, so suppose that $q=(u,\hat{1})$. Since f_n is a poset map, we have $\langle f_n^{-1}(q)\rangle\subseteq f_n^{-1}(\langle q\rangle)$. For the reverse inclusion consider any element $w\in f_n^{-1}(\langle q\rangle)$. Then $f_n(w)\le q$ and hence $\pi_n(w)\preceq u$. Lemma 6 implies that there exists an element $v\in S_n$ which covers u and satisfies $\pi_n(v)=u$ and $w\preceq v$. We then have $v\in f_n^{-1}(q)$ and hence $w\in\langle f_n^{-1}(q)\rangle$. This proves that $f_n^{-1}(\langle q\rangle)\subseteq\langle f_n^{-1}(q)\rangle$. □

Lemma 8 *For every $u\in S_{n-1}$, the order ideal*

$$M(u)=\big\langle v\in S_n:\pi_n(v)=u\big\rangle$$

of $\mathrm{Abs}(S_n)$ *is homotopy Cohen–Macaulay of rank $\ell_{\mathcal{T}}(u)+1$.*

Proof Let $u=u_1u_2\cdots u_l$ be written as a product of disjoint cycles in S_{n-1}. Then

$$M(u)=\bigcup_{i=1}^{l}C(u_i)\cdot\langle u_1\cdots\hat{u}_i\cdots u_l\rangle,$$

where $u_1\cdots\hat{u}_i\cdots u_l$ denotes the permutation obtained from u by deleting the cycle u_i and $C(u_i)$ denotes the order ideal of $\mathrm{Abs}(S_n)$ generated by the cycles v of S_n which cover u_i and satisfy $\pi_n(v)=u_i$. Lemma 11, proved in the Appendix, implies that $C(u_i)$ is homotopy Cohen–Macaulay of rank $\ell_{\mathcal{T}}(u_i)+1$ for every i. Each of the ideals $C(u_i)\cdot\langle u_1\cdots\hat{u}_i\cdots u_l\rangle$ is isomorphic to a direct product of homotopy Cohen–Macaulay posets and hence it is homotopy Cohen–Macaulay, by Lemma 3(i); their rank is equal to $\ell_{\mathcal{T}}(u)+1$. Moreover, the intersection of any two or more of the ideals $C(u_i)\cdot\langle u_1\cdots\hat{u}_i\cdots u_l\rangle$ is equal to $\langle u\rangle$, which is homotopy Cohen–Macaulay of rank $\ell_{\mathcal{T}}(u)$. Thus the result follows from Lemma 3(ii). □

Proof of Theorem 1 We proceed by induction on n. The result is trivial for $n \le 2$. Suppose that the poset $\mathrm{Abs}(S_{n-1})$ is homotopy Cohen–Macaulay. Then so is the direct product $\mathrm{Abs}(S_{n-1}) \times \{\hat{0}, \hat{1}\}$ by Lemma 3(i). We consider the map

$$f_n : \mathrm{Abs}(S_n) \to \mathrm{Abs}(S_{n-1}) \times \{\hat{0}, \hat{1}\}.$$

In view of Theorem 6 and Lemma 7, it suffices to show that for every $q \in S_{n-1} \times \{\hat{0}, \hat{1}\}$ the order ideal $\langle f_n^{-1}(q)\rangle$ of $\mathrm{Abs}(S_n)$ is homotopy Cohen–Macaulay. This is true in case $q = (u, \hat{0})$ for some $u \in S_{n-1}$, since then $\langle f_n^{-1}(q)\rangle = \langle u\rangle$ and every interval in $\mathrm{Abs}(S_n)$ is shellable. Suppose that $q = (u, \hat{1})$. Then $\langle f_n^{-1}(q)\rangle = M(u)$, which is homotopy Cohen–Macaulay by Lemma 8. This completes the induction and the proof of the theorem. □

We now focus on the hyperoctahedral group. The proof of Theorem 3 is based on the following lemmas.

Lemma 9 *For every $q \in \mathcal{J}_{n-1} \times \{\hat{0}, \hat{1}\}$ we have $f_n^{-1}(\langle q\rangle) = \langle f_n^{-1}(q)\rangle$.*

Proof The proof of Lemma 7 applies word by word, if one replaces S_{n-1} by the ideal $\mathcal{J}_{n-1}$. We thus omit the details. □

Lemma 10 *For every $u \in \mathcal{J}_{n-1}$ the order ideal*

$$M(u) = \big\langle v \in \mathcal{J}_n : \pi_n(v) = u\big\rangle$$

of $\mathrm{Abs}(B_n)$ is homotopy Cohen–Macaulay of rank $\ell_{\mathcal{T}}(u) + 1$.

Proof Let $u = u_1 u_2 \cdots u_l \in \mathcal{J}_{n-1}$ be written as a product of disjoint cycles. For $i \in \{1, \dots, l\}$, we denote by $C(u_i)$ the order ideal of $\mathcal{J}_n$ generated by all cycles $v \in \mathcal{J}_n$ which can be obtained by inserting either n or $-n$ at any place in the cycle u_i. The ideal $C(u_i)$ is graded of rank $\ell_{\mathcal{T}}(u_i) + 1$ and homotopy Cohen–Macaulay, by Lemma 12 proved in the Appendix. Let $u_1 \cdots \hat{u}_i \cdots u_l$ denote the permutation obtained from u by removing the cycle u_i. Suppose first that u has a balanced cycle in its cycle decomposition. Using Remark 1, we find that

$$M(u) = \bigcup_{i=1}^{l} C(u_i) \cdot \langle u_1 \cdots \hat{u}_i \cdots u_l\rangle.$$

Clearly, $M(u)$ is graded of rank $\ell_{\mathcal{T}}(u) + 1$. Each of the order ideals $C(u_i) \cdot \langle u_1 \cdots \hat{u}_i \cdots u_l\rangle$ is isomorphic to a direct product of homotopy Cohen–Macaulay posets and hence it is homotopy Cohen–Macaulay, by Lemma 3(i). Moreover, the intersection of any two or more of these ideals is equal to $\langle u\rangle$, which is homotopy Cohen–Macaulay of rank $\ell_{\mathcal{T}}(u)$, by Theorem 5. Suppose now that u has no balanced cycle in its cycle decomposition. Then

$$M(u) = \bigcup_{i=1}^{l} C(u_i) \cdot \langle u_1 \cdots \hat{u}_i \cdots u_l\rangle \cup \big\langle u[n]\big\rangle.$$

Again, $M(u)$ is graded of rank $\ell_{\mathcal{T}}(u)+1$, each of the ideals $C(u_i)\langle u_1\cdots\hat{u}_i\cdots u_l\rangle$ and $\langle u[n]\rangle$ is homotopy Cohen–Macaulay and the intersection of any two or more of these ideals is equal to $\langle u\rangle$. In either case, the result follows from Lemma 3(ii). □

Proof of Theorem 3 We proceed by induction on n. The result is trivial for $n\le 2$. Suppose that the poset $\mathcal{J}_{n-1}$ is homotopy Cohen–Macaulay. Then so is the direct product $\mathcal{J}_{n-1}\times\{\hat{0},\hat{1}\}$ by Lemma 3(i). We consider the map

$$f_n:\mathcal{J}_n\to\mathcal{J}_{n-1}\times\{\hat{0},\hat{1}\}.$$

In view of Theorem 6 and Lemma 9, it suffices to show that for every $q\in\mathcal{J}_{n-1}\times\{\hat{0},\hat{1}\}$ the order ideal $\langle f_n^{-1}(q)\rangle$ of $\mathrm{Abs}(B_n)$ is homotopy Cohen–Macaulay. This is true in case $q=(u,\hat{0})$ for some $u\in\mathcal{J}_{n-1}$, since then $\langle f_n^{-1}(q)\rangle=\langle u\rangle$ and every interval in $\mathrm{Abs}(B_n)$ is shellable by Theorem 5. Suppose that $q=(u,\hat{1})$. Then $\langle f_n^{-1}(q)\rangle=M(u)$, which is homotopy Cohen–Macaulay by Lemma 8. This completes the induction and the proof of the theorem. □

Proof of Theorem 4 Let us denote by $\hat{0}$ the minimum element of $\mathrm{Abs}(B_n)$. Let $\hat{\mathcal{J}}_n$ be the poset obtained from $\mathcal{J}_n$ by adding a maximum element $\hat{1}$ and let μ_n be the Möbius function of $\hat{\mathcal{J}}_n$. From Proposition 3.8.6 of [27] we have $\tilde{\chi}(\Delta(\bar{\mathcal{J}}_n))=\mu_n(\hat{0},\hat{1})$. Since $\mu_n(\hat{0},\hat{1})=-\sum_{x\in\mathcal{J}_n}\mu_n(\hat{0},x)$, we have

$$\tilde{\chi}\big(\Delta(\bar{\mathcal{J}}_n)\big)=-\sum_{x\in\mathcal{J}_n}\mu_n\big(\hat{0},x\big).\tag{4}$$

Suppose that $x\in B_n$ is a cycle. It is known [26] that

$$\mu\big(\hat{0},x\big)=\begin{cases}(-1)^m\binom{2m-1}{k}, & \text{if } x \text{ is a balanced } m\text{-cycle},\\ (-1)^{m-1}C_{m-1}, & \text{if } x \text{ is a paired } m\text{-cycle},\end{cases}$$

where $C_m=\frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number. We recall (Remark 1) that if $x\in\mathcal{J}_n$ has exactly $k+1$ paired cycles, say $p_1,\dots,p_{k+1}$, and one balanced cycle, say b, then $[\hat{0},x]\cong[\hat{0},b]\times[\hat{0},p_1]\times\cdots\times[\hat{0},p_k]$ and hence

$$\mu_n\big(\hat{0},x\big)=\mu_n\big(\hat{0},b\big)\prod_{i=1}^{k}\mu_n\big(\hat{0},p_i\big).$$

It follows that

$$\mu_n\big(\hat{0},x\big)=(-1)^{\ell_{\mathcal{T}}(b)}\binom{2\ell_{\mathcal{T}}(b)-1}{\ell_{\mathcal{T}}(b)}\prod_{i=1}^{k}(-1)^{\ell_{\mathcal{T}}(p_i)}C_{\ell_{\mathcal{T}}(p_i)}.\tag{5}$$

From (4), (5), [28, Proposition 5.1.1] and the exponential formula [28, Corollary 5.1.9], we conclude that

$$1-\sum_{n\ge 2}\tilde{\chi}\big(\Delta(\bar{\mathcal{J}}_n)\big)\frac{t^n}{n!}=\left(1+\sum_{n\ge 1}2^{n-1}\alpha_n\frac{t^n}{n}\right)\exp\left(\sum_{n\ge 1}2^{n-1}\beta_n\frac{t^n}{n}\right),\tag{6}$$

where $\alpha_n = (-1)^n \binom{2n-1}{n}$ is the Möbius function of a balanced n-cycle and $\beta_n = (-1)^{n-1} C_{n-1}$ is the Möbius function of a paired n-cycle. Thus it suffices to compute $\exp(\sum_{n\geq 1} 2^{n-1}\beta_n \frac{t^n}{n})$. From [3, Sect. 5] we have

$$\exp \sum_{n\geq 1} \beta_n \frac{t^n}{n} = \frac{\sqrt{1+4t}-1}{2t} \exp\bigl(\sqrt{1+4t}-1\bigr)$$

and hence, replacing t by $2t$,

$$\exp\left(\sum_{n\geq 1} 2^{n-1}\beta_n \frac{t^n}{n}\right) = \left(\frac{\sqrt{1+8t}-1}{4t}\right)^{1/2} \exp\left(\frac{\sqrt{1+8t}-1}{2}\right).$$

The right-hand side of (6) can now be written as

$$1 - \left(\frac{\sqrt{1+8t}-1}{4t}\right)^{1/2} \exp\left(\frac{\sqrt{1+8t}-1}{2}\right)\left(1 + \sum_{n\geq 1} 2^{n-1}\alpha_n \frac{t^n}{n}\right).$$

The result follows by switching t to $-t$. □

5 Intervals with the lattice property

Let W be a finite Coxeter group and $c \in W$ be a Coxeter element. It is known [5, 14, 15] that the interval $[e, c]$ in $\mathrm{Abs}(W)$ is a lattice. In this section we characterize the intervals in $\mathrm{Abs}(B_n)$ and $\mathrm{Abs}(D_n)$ which are lattices (Theorems 7 and 8, respectively). As we explain in the sequel, some partial results in this direction were obtained in [5, 14, 15, 18, 26].

To each $w \in B_n$ we associate the integer partition $\mu(w)$ whose parts are the absolute lengths of all balanced cycles of w, arranged in decreasing order. For example, if $n = 8$ and $w = [1, -5][2, 7][6]((3, 4))$, then $\mu(w) = (2, 2, 1)$. It follows from the results of [18, Sect. 6] that the interval $[e, w]$ in $\mathrm{Abs}(B_n)$ is a lattice if $\mu(w) = (n-1, 1)$ and that $[e, w]$ is not a lattice if $\mu(w) = (2, 2)$. Recall that a *hook partition* is an integer partition of the form $\mu = (k, 1, \dots, 1)$, also written as $\mu = (k, 1^r)$, where r is one less than the total number of parts of μ.

In the sequel we denote by $L(k, r)$ the lattice $[e, w] \subset \mathrm{Abs}(B_n)$, where $w = [1, 2, \dots, k][k+1] \cdots [k+r] \in B_n$. Clearly, $L(k, r)$ is isomorphic to any interval of the form $[e, u]$, where $u \in B_n$ has no paired cycles and satisfies $\mu(u) = (k, 1^r)$.

Our main results in this section are the following.

Theorem 7 *For $w \in B_n$, the interval $[e, w]$ in $\mathrm{Abs}(B_n)$ is a lattice if and only if $\mu(w)$ is a hook partition.*

Theorem 8 *For $w \in D_n$, the interval $[e, w]$ in $\mathrm{Abs}(D_n)$ is a lattice if and only if $\mu(w) = (k, 1)$ for some $k \leq n-1$, or $\mu(w) = (1, 1, 1, 1)$.*

We note that in view of Lemma 4, Theorems 7 and 8 characterize all closed intervals in $\mathrm{Abs}(B_n)$ and $\mathrm{Abs}(D_n)$ which are lattices. The following proposition provides one half of the first characterization.

Proposition 2 *Let $w \in B_n$. If $\mu(w)$ is a hook partition, then the interval $[e, w]$ in $\mathrm{Abs}(B_n)$ is a lattice.*

Proof Let us write $w = bp$, where b (respectively, p) is the product of all balanced (respectively, paired) cycles of w. We recall then that $[e, w] \cong [e, b] \times [e, p]$ (see Remark 1). Since $[e, p]$ is isomorphic to a direct product of noncrossing partition lattices, the interval $[e, w]$ is a lattice if and only if $[e, b]$ is a lattice. Thus we may assume that w is a product of disjoint balanced cycles. Since $\mu(w)$ is a hook partition, we may further assume that $w = [1, 2, \dots, k][k + 1] \cdots [k + r]$ with $k + r \leq n$. We will show that $L(k, r) = [e, w]$ is a lattice by induction on $k + r$. The result is trivial for $k + r = 2$. Suppose that $k + r \geq 3$ and that the poset $L(k, r)$ is a lattice whenever $k + r < \kappa + \rho \leq n$. We will show that $L(\kappa, \rho)$ is a lattice as well. For $\rho \leq 1$, this follows from [26, Proposition 2] and the result of [18] mentioned earlier. Thus we may assume that $\rho \geq 2$. Let $u, v \in L(\kappa, \rho)$. By [27, Proposition 3.3.1], it suffices to show that $[e, u] \cap [e, v] = [e, z]$ for some $z \in L(\kappa, \rho)$.

Suppose first that $u(i) = i$ for some $i \in \{1, 2, \dots, \kappa + \rho\}$ and let v' be the signed permutation obtained by deleting the element i from the cycle decomposition of v. We may assume that $u, v' \in L(\kappa_1, \rho_1)$, where either $\kappa_1 = \kappa - 1$ and $\rho_1 = \rho$, or $\kappa_1 = \kappa$ and $\rho_1 = \rho - 1$. We observe that $[e, u] \cap [e, v] = [e, u] \cap [e, v']$. Since $L(\kappa_1, \rho_1)$ is a lattice by induction, there exists an element $z \in L(\kappa_1, \rho_1)$ such that $[e, u] \cap [e, v'] = [e, z]$. We argue in a similar way if $v(i) = i$ for some $i \in \{1, 2, \dots, \kappa + \rho\}$.

Suppose that $u(i) \neq i$ and $v(i) \neq i$ for every $i \in \{1, 2, \dots, \kappa + \rho\}$. Since $\rho \geq 2$, each of u, v has at least one reflection in its cycle decomposition. Without loss of generality, we may assume that no cycle of u is comparable to a cycle of v in $\mathrm{Abs}(B_n)$ (otherwise the result follows by induction). Then at least one of the following holds:

- The reflection $[i]$ is a cycle of u or v for some $i \in \{\kappa + 1, \kappa + 2, \dots, \kappa + \rho\}$.
- There exist $i, j \in \{\kappa + 1, \kappa + 2, \dots, \kappa + \rho\}$ with $i < j$, such that either $((i, j))$ or $((i, -j))$ is a cycle of u and i and j belong to distinct cycles of v, or conversely.
- There exist $i, j \in \{\kappa + 1, \kappa + 2, \dots, \kappa + \rho\}$ with $i < j$, such that $((i, j))$ is a cycle of u and $((i, -j))$ is a cycle of v, or conversely.

In any of the previous cases, let u' and v' be the permutations obtained from u and v, respectively, by deleting the element i from their cycle decomposition. We may assume once again that $u', v' \in L(\kappa_1, \rho_1)$, where either $\kappa_1 = \kappa - 1$ and $\rho_1 = \rho$, or $\kappa_1 = \kappa$ and $\rho_1 = \rho - 1$. As before, $[e, u] \cap [e, v] = [e, u'] \cap [e, v']$. By the induction hypothesis, $L(\kappa_1, \rho_1)$ is a lattice and hence $[e, u'] \cap [e, v'] = [e, z]$ for some $z \in L(\kappa_1, \rho_1)$. This implies that $L(\kappa, \rho)$ is a lattice and completes the induction. □

Proof of Theorem 7 If $\mu(w)$ is a hook partition, then the result follows from Proposition 2. To prove the converse, assume that w has at least two balanced cycles, say w_1 and w_2, with $\ell_T(w_1), \ell_T(w_2) \geq 2$. Then there exist $i, j, l, m \in \{\pm 1, \pm 2, \dots, \pm n\}$ with $|i|, |j|, |l|, |m|$ pairwise distinct, such that $[i, j] \preceq w_1$ and $[l, m] \preceq w_2$. However, in [24, Sect. 5] it was shown that the poset $[e, [i, j][l, m]]$ is not a lattice. It follows that neither $[e, w]$ is a lattice. This completes the proof. $\square$

Proof of Theorem 8 The argument in the proof of Theorem 7 shows that the interval $[e, w]$ is not a lattice unless $\mu(w)$ is a hook partition. Moreover, it is known [5, 14] that $[e, w]$ is a lattice if $\mu(w) = (k, 1)$ for some $k \geq 1$. Suppose that $\mu(w) = (k, 1^r)$, where $r > 1$ and $r + k \leq n$. If $k \geq 2$, then there exist distinct elements of $[e, w]$ of the form $u = [a_1, a_2][a_3]$ and $v = [a_1, a_2][a_4]$. The intersection $[e, u] \cap [e, v] \subset \mathrm{Abs}(D_n)$ has two maximal elements, namely the paired reflections $((a_1, a_2))$ and $((a_1, -a_2))$. This implies that u and v do not have a meet and therefore the interval $[e, w]$ is not a lattice. Suppose that $k = 1$. Without loss of generality, we may assume that $[1][2]\cdots[r + 1] \preceq w$. Suppose that $r + 1 \geq 5$. We consider the elements $u = [1][2][3][4]$ and $v = [1][2][3][5]$ of $[e, w]$ and note that the intersection $[e, u] \cap [e, v]$ has three maximal elements, namely $[1][2]$, $[1][3]$ and $[2][3]$. This implies that the interval $[e, w]$ is not a lattice. Finally, if $r + 1 = 4$, then $\mu(w) = (1, 1, 1, 1)$ and $[e, w] = [e, [1][2][3][4]] \times [e, p]$, where p is a product of disjoint paired cycles which fixes each $i \in \{1, 2, 3, 4\}$. Figure 4 shows that the interval $[e, [1][2][3][4]]$ is a lattice and hence, so is $[e, w]$. This completes the proof. $\square$

6 The lattice $\mathcal{L}_n$

The poset $L(n, 0)$ is the interval $[e, c]$ of $\mathrm{Abs}(B_n)$, where c is the Coxeter element $[1, 2, \ldots, n]$ of B_n. This poset is isomorphic to the lattice $NC^B(n)$ of noncrossing partitions of type B. Reiner [26] computed its basic enumerative invariants listed below:

- The cardinality of $NC^B(n)$ is equal to $\binom{2n}{n}$.
- The number of elements of rank k is equal to $\binom{n}{k}^2$.
- The zeta polynomial satisfies $Z(NC^B(n), m) = \binom{mn}{n}$.
- The number of maximal chains is equal to n^n.
- The Möbius function satisfies $\mu_n(\hat{0}, \hat{1}) = (-1)^n \binom{2n-1}{n}$.

In this section we focus on the enumerative properties of another interesting special case of $L(k, r)$, namely the lattice $\mathcal{L}_n := L(0, n)$. It is the interval $[e, v]$ of $\mathrm{Abs}(B_n)$, where v is the element $[1][2]\cdots[n]$ of B_n. First we describe this poset explicitly. Each element of $\mathcal{L}_n$ can be obtained from $[1][2]\cdots[n]$ by applying repeatedly the following steps:

- Delete some $[i]$.
- Replace a product $[i][j]$ with $((i, j))$ or $((i, -j))$.

Thus $w \in \mathcal{L}_n$ if and only if every nontrivial cycle of w is a reflection. In that case there is a poset isomorphism $[e, w] \cong \mathcal{L}_k \times \mathcal{B}_l$, where k and l are the numbers of balanced and paired cycles of w, respectively and $\mathcal{B}_l$ denotes the lattice of subsets of the set $\{1, 2, \ldots, l\}$, ordered by inclusion. It is worth pointing out that $\mathcal{L}_n$ coincides with the subposet of $\mathrm{Abs}(B_n)$ induced on the set of involutions. Figure 5 illustrates the Hasse diagram of $\mathcal{L}_3$.

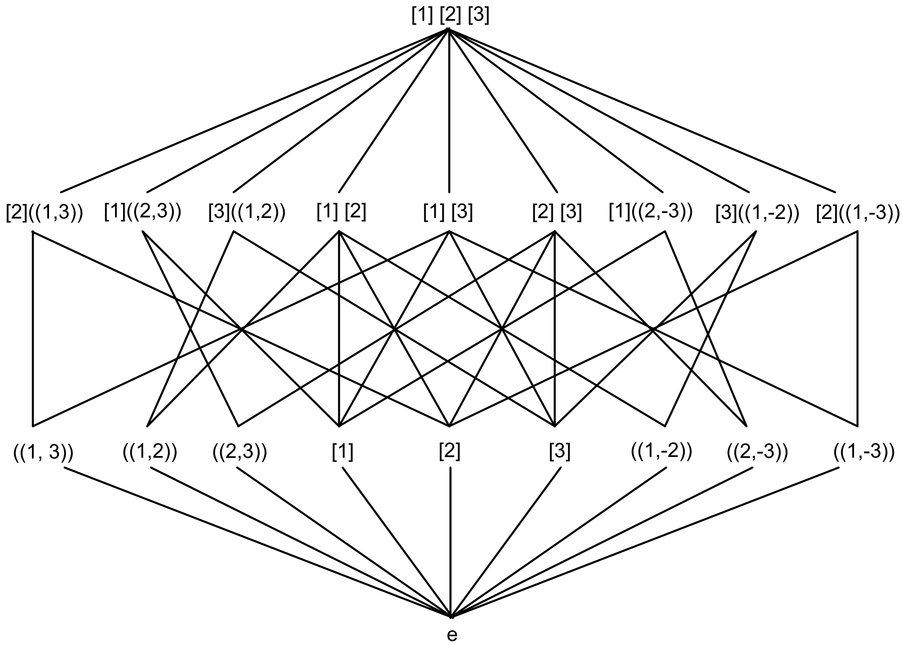


Fig. 5 The interval $[e, [1][2][3]]$ in $\mathrm{Abs}(B_3)$

In Proposition 3 we give the analogue of the previous list for the lattice $\mathcal{L}_n$. We recall that the zeta polynomial $Z(P,m)$ of a finite poset P counts the number of multichains $x_1 \leq x_2 \leq \cdots \leq x_{m-1}$ of P. It is known (see [17], [27, Proposition 3.11.1]) that $Z(P,m)$ is a polynomial function of m of degree n, where n is the length of P and that $Z(P,2) = \#P$. Moreover, the leading coefficient of $Z(P,m)$ is equal to the number of maximal chains divided by $n!$ and if P is bounded, then $Z(P,-1) = \mu(\hat{0},\hat{1})$. Finally, we recall that $n!! = 1 \cdot 3 \cdot \ldots \cdot (n-2) \cdot n$, where n is a positive odd integer.

Proposition 3 *For the lattice $\mathcal{L}_n$ the following hold*:

(i) *The number of elements of $\mathcal{L}_n$ is equal to*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-k} (2k-1)!!.$$

(ii) *The number of elements of $\mathcal{L}_n$ of rank r is equal to*

$$\sum_{k=0}^{\min\{r,n-r\}} \frac{n!}{k!(r-k)!(n-r-k)!}.$$

(iii) *The zeta polynomial Z_n of $\mathcal{L}_n$ is given by the formula*

$$Z_n(m)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}m^{n-k}(m-1)^k(2k-1)!!.$$

(iv) *The number of maximal chains of $\mathcal{L}_n$ is equal to*

$$n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(2k-1)!!.$$

(v) *For the Möbius function μ_n of $\mathcal{L}_n$ we have*

$$\mu_n(\hat{0},\hat{1})=(-1)^n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}2^k(2k-1)!!,$$

where $\hat{0}$ and $\hat{1}$ denotes the minimum and the maximum element of $\mathcal{L}_n$, respectively.

Proof Suppose that x has k paired reflections. These can be chosen in

$$2^k\binom{n}{2k}(2k-1)!!$$

ways. On the other hand, the balanced reflections of w can be chosen in 2^{n-2k} ways. Therefore the cardinality of $\mathcal{L}_n$ is equal to

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}2^{n-k}(2k-1)!!.$$

The same argument shows that the number of elements of $\mathcal{L}_n$ of rank r, where $r\le\lfloor\frac{n}{2}\rfloor$ is equal to

$$\begin{aligned}\sum_{k=0}^{r}2^k\binom{n}{2k}(2k-1)!!\binom{n-2k}{r-k}&=\sum_{k=0}^{r}2^k\binom{n}{2k}\frac{(2k)!}{2^kk!}\binom{n-2k}{r-k}\\&=\sum_{k=0}^{r}\frac{n!}{k!(r-k)!(n-r-k)!}.\end{aligned}$$

Since $\mathcal{L}_n$ is self-dual, the number of elements in $\mathcal{L}_n$ of rank r is equal to the number of those that have rank $n-r$. The number of multichains in $\mathcal{L}_n$ in which k distinct paired reflections appear, is equal to $\binom{n}{2k}(2k-1)!!(m(m-1))^km^{n-2k}$. Therefore, the zeta polynomial of $\mathcal{L}_n$ is given by

$$Z_n(m)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(2k-1)!!m^{n-k}(m-1)^k.$$

Finally, computing the coefficient of m^n in this expression for $Z_n(m)$ and multiplying by $n!$ we conclude that the number of maximal chains of $\mathcal{L}_n$ is equal to

$$n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(2k-1)!!$$

and setting $m=-1$ we get

$$\mu_n(\hat{0},\hat{1}) = Z_n(-1) = (-1)^n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(2k-1)!!2^k. \qquad \square$$

Remark 3 By Proposition 1, the lattice $\mathcal{L}_n$ is EL-shellable. We describe two more EL-labelings for $\mathcal{L}_n$.

(i) Let $\Lambda = \{[i] : i = 1,2,\dots,n\} \cup \{((i,j)) : i,j = 1,2,\dots,n, i<j\}$. We linearly order the elements of Λ in the following way. We first order the balanced reflections so that $[i] <_\Lambda [j]$ if and only if $i<j$. Then we order the paired reflections lexicographically. Finally, we define $[n] <_\Lambda ((1,2))$. The map $\lambda_1 : C(B_n) \to \Lambda$ defined as

$$\lambda_1(a,b) = \begin{cases} [i] & \text{if } a^{-1}b = [i], \\ ((i,j)) & \text{if } a^{-1}b = ((i,j)) \text{ or } ((i,-j)) \end{cases}$$

is an EL-labeling for $\mathcal{L}_n$.

(ii) Let $\mathcal{T}$ be the set of reflections of B_n. We define a total order $<_\mathcal{T}$ on $\mathcal{T}$, which extends the order $<_\Lambda$, by ordering the reflections $((i,-j))$, for $1\le i<j\le n$, lexicographically and letting $((n-1,n)) <_\mathcal{T} ((1,-2))$. For example, if $n=3$ we have the order $[1]_\mathcal{T} <_\mathcal{T} [2] <_\mathcal{T} [3] <_\mathcal{T} ((1,2)) <_\mathcal{T} ((1,3)) <_\mathcal{T} ((2,3)) <_\mathcal{T} ((1,-2)) <_\mathcal{T} ((1,-3)) <_\mathcal{T} ((2,-3))$. Let t_i be the ith reflection in the order above. We define a map $\lambda_2 : C(B_n) \to \{1,2,\dots,n^2\}$ as

$$\lambda_2(a,b) = \min_{1\le i\le n^2}\{i : t_i \vee a = b\}.$$

The map λ_2 is an EL-labeling for $\mathcal{L}_n$.

See Fig. 6 for an example of these two EL-labelings when $n=2$.

7 Enumerative combinatorics of $L(k,r)$

In this section we compute the cardinality, zeta polynomial and Möbius function of the lattice $L(k,r)$, where k,r are nonnegative integers with $k+r=n$. The case $k=n-1$ was treated by Goulden, Nica and Oancea in their work [18] on the posets of annular noncrossing partitions; see also [23, 24] for related work. We will use their results, as well as the formulas for cardinality and zeta polynomial for $NC^B(n)$ and Proposition 3, to find the corresponding formulas for $L(k,r)$.

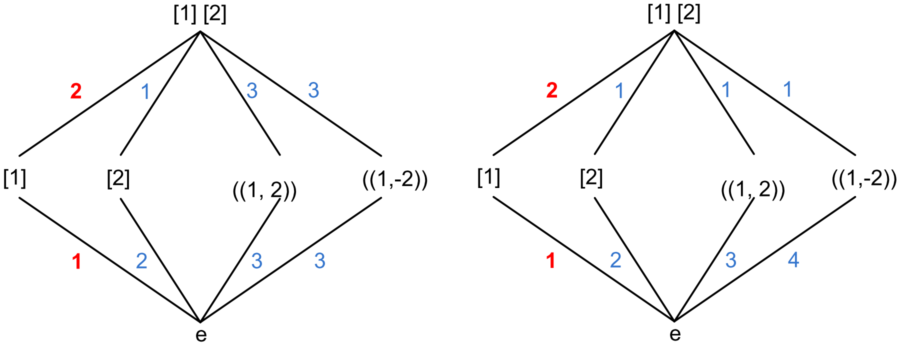


Fig. 6 EL-labelings for the interval $[e, [1][2]]$ in $\mathrm{Abs}(B_2)$

Proposition 4 *Let* $\alpha_r = |\mathcal{L}_r|$, $\beta_r(m) = Z(\mathcal{L}_r, m)$ *and* $\mu_r = \mu_r(\mathcal{L}_r)$, *where* $\alpha_r = \beta_r(m) = \mu_r = 1$ *for* $r = 0, 1$. *For fixed nonnegative integers* k, r *such that* $k + r = n$, *the cardinality, zeta polynomial and Möbius function of* $L(k, r)$ *are given by*

- $\#L(k,r) = \binom{2k}{k}\left(\frac{2rk}{k+1}\alpha_{r-1} + a_r\right)$.
- $Z(L(k,r), m) = \binom{mk}{k}\left(\frac{2rk}{k+1}(m-1)\beta_{r-1}(m) + \beta_r(m)\right)$.
- $\mu(L(k,r)) = (-1)^n\binom{2k-1}{k}\left(\frac{4rk}{k+1}|\mu_{r-1}| + |\mu_r|\right)$.

Proof We denote by A the subset of $L(k,r)$ which consists of the elements x with the following property: every cycle of x that contains at least one of $\pm 1, \pm 2, \dots, \pm k$ is less than or equal to the element $[1, 2, \dots, k]$ in $\mathrm{Abs}(B_n)$. Equivalently, $x \in A$ if and only if for every cycle x_i of x we have either $x_i \preceq [1, 2, \dots, k]$ or $x_i \preceq [k+1] \cdots [k+r]$. Let $x = x_1 x_2 \cdots x_\nu \in A$, written as a product of disjoint cycles. Without loss of generality, we may assume that there is a $t \in \{0, 1, \dots, \nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1, 2, \dots, k]$ and $x_{t+1} x_{t+2} \cdots x_\nu \preceq [k+1][k+2] \cdots [k+r]$. Observe that if $t = 0$ then $x \preceq [k+1][k+2] \cdots [k+r]$ in $\mathrm{Abs}(B_n)$, while if $t = \nu$ then $x \preceq [1, 2, \dots, k]$. Clearly, there exists a poset isomorphism

$$\begin{aligned} f : A &\to NC^B(k) \times \langle [k+1] \cdots [k+r] \rangle \\ x &\mapsto (x_1 \cdots x_t,\ x_{t+1} \cdots x_\nu), \end{aligned}$$

so that

$$A \cong NC^B(k) \times \mathcal{L}_r. \tag{7}$$

Let $C = L(k,r) \setminus A$ and $x = x_1 x_2 \cdots x_\nu \in C$, written as a product of disjoint cycles. Then there is exactly one paired cycle x_1 of x and one reflection $((i, l))$ with $i \in \{\pm 1, \pm 2, \dots, \pm k\}$, $l \in \{k+1, k+2, \dots, k+r\}$, such that $((i, l)) \preceq x_1$. For every $l \in \{k+1, k+2, \dots, k+r\}$ denote by C_l the set of permutations $x \in L(k,r)$ which have a cycle, say x_1, such that $((i, l)) \preceq x_1$ for some $i \in \{\pm 1, \pm 2, \dots, \pm k\}$. It follows that $C_l \cap C_{l'} = \varnothing$ for $l \neq l'$. Clearly, $C_l \cong C_{l'}$ for $l \neq l'$ and $C = \bigcup_{l=k+1}^{k+r} C_l$.

Summarizing, for every $x \in C$ there exists an ordering $x_1, x_2, \dots, x_\nu$ of the cycles of x and a unique index $t \in \{1, 2, \dots, \nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1, 2, \dots, k][l]$ and

$x_{t+1}x_{t+2}\cdots x_\nu \preceq [k+1][k+2]\cdots[l-1][l+1]\cdots[k+r]$. Let

$$E_l = \big\{x \in C : x \preceq [1,2,\dots,k][l]\big\}.$$

We remark that no permutation of E_l has a balanced cycle in its cycle decomposition. Clearly, there exists a poset isomorphism

$$\begin{aligned} g_l : C_l &\to E_l \times \big\langle [k+1]\cdots[l-1][l+1]\cdots[k+r]\big\rangle \\ x &\mapsto (x_1\cdots x_t,\ x_{t+1}\cdots x_\nu) \end{aligned}$$

so that

$$C_l \cong E_l \times \mathcal{L}_{r-1} \tag{8}$$

for every $l \in \{k+1, k+2, \dots, k+r\}$. Using (7) and (8), we proceed to the proof of Proposition 4 as follows. From our previous discussion we have $L(k,r) = \#A + r(\#C_{k+1})$. From (7) we have

$$\#A = \binom{2k}{k}\alpha_r$$

and (8) implies that $\#C_{k+1} = (\#E_{k+1})(\#\mathcal{L}_{r-1}) = (\#E_{k+1})\alpha_{r-1}$. Since E_{k+1} consists of the permutations in $\langle [1,2,\dots,k][k+1]\rangle \cap C$, it follows from [18, Sect. 5] that $\#E_{k+1} = 2\binom{2k}{k-1}$. Therefore,

$$\#L(k,r) = 2r\binom{2k}{k-1}\alpha_{r-1} + \binom{2k}{k}\alpha_r = \binom{2k}{k}\left(\frac{2rk}{k+1}\alpha_{r-1} + a_r\right).$$

Recall that the zeta polynomial $Z(L(k,r),m)$ counts the number of multichains $\pi_1 \preceq \pi_2 \preceq \cdots \preceq \pi_{m-1}$ in $L(k,r)$. We distinguish two cases. If $\pi_{m-1} \in C$, then $\pi_{m-1} \in C_l$ for some $l \in \{k+1,\dots,k+r\}$. Isomorphism (8) then implies that there are $Z(E_l,m)Z(\mathcal{L}_{r-1},m)$ such multichains. From [18, Sect. 5] we have $Z(E_l,m) = 2\binom{mk}{k+1}$, therefore $Z(E_l,m)Z(\mathcal{L}_{r-1},m) = 2\binom{mk}{k+1}\beta_{r-1}$. Since there are r choices for the set C_l, we conclude that the number of multichains $\pi_1 \preceq \pi_2 \preceq \cdots \preceq \pi_{m-1}$ in $L(k,r)$ for which $\pi_{m-1} \in C$ is equal to

$$2r\binom{mk}{k+1}\beta_{r-1}(m). \tag{9}$$

If $\pi_{m-1} \in A$, then $\pi_{m-1} \in NC^B(k) \times \mathcal{L}_r$ and therefore number of such multichains is equal to

$$\binom{mk}{k}\beta_r(m). \tag{10}$$

The proposed expression for the zeta polynomial of $L(k,r)$ follows by summing the expressions (9) and (10) and straightforward calculation.

The expression for the Möbius function follows once again from that of the zeta polynomial by setting $m = -1$. □

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Appendix

In this section we prove the following lemmas.

Lemma 11 *The order ideal of* $\mathrm{Abs}(S_n)$ *generated by all cycles* $u \in S_n$ *for which* $\pi_n(u) = (1\,2\,\cdots\,n-1)$ *is homotopy Cohen–Macaulay of rank* $n-1$.

Lemma 12

(i) *The order ideal of* $\mathrm{Abs}(B_n)$ *generated by all cycles* $u \in B_n$ *for which* $\pi_n(u) = ((1, 2, \ldots, n-1))$ *is homotopy Cohen–Macaulay of rank* $n-1$.
(ii) *The order ideal of* $\mathrm{Abs}(B_n)$ *generated by all cycles* u *of* B_n *for which* $\pi_n(u) = [1, 2, \ldots, n-1]$ *is homotopy Cohen–Macaulay of rank* n.

A.1 Proof of Lemma 11

We will show that the order ideal considered in Lemma 11 is in fact strongly constructible. The following remark will be used in the proof.

Remark 4 Let $u_1, u_2, \ldots, u_m \in S_n$ be elements of absolute length k and let $v \in S_n$ be a cycle of absolute length r which is disjoint from u_i for each $i \in \{1, 2, \ldots, m\}$. Suppose that the union $\bigcup_{i=1}^m [e, u_i]$ is strongly constructible of rank k. Then

$$\bigcup_{i=1}^{m} [e, vu_i] \cong \bigcup_{i=1}^{m} \big([e, v] \times [e, u_i]\big) = [e, v] \times \bigcup_{i=1}^{m} [e, u_i],$$

is strongly constructible of rank $k + r$, by Lemma 1(i).

Lemma 13 *For* $i \in \{1, 2, \ldots, n-1\}$, *consider the element*

$$u_i = (1\,i+1\,\cdots\,n-1)(2\,3\,\cdots\,i\,n) \in S_n.$$

The union $\bigcup_{i=1}^m [e, u_i]$ *is strongly constructible of rank* $n-2$ *for all* $1 \le m \le n-1$.

Proof We denote by $I(n, m)$ the union in the statement of the lemma and proceed by induction on n and m, in this order. We may assume that $n \ge 3$ and $m \ge 2$, since otherwise the result is trivial. Suppose that the result holds for positive integers smaller than n. We will show that it holds for n as well. By induction on m, it suffices to show that $[e, u_m] \cap I(n, m-1)$ is strongly constructible of rank $n-3$. Indeed, we have $[e, u_m] \cap I(n, m-1) = \bigcup_{i=1}^{m-1} [e, u_m] \cap [e, u_i]$ and

$$[e, u_m] \cap [e, u_i] = \big[e, (1\, m+1\, m+2 \cdots n-1)(2\, 3 \cdots i\, n)(i+1\, i+2 \cdots m)\big].$$

Since the cycle $(1\, m+1\, m+2 \cdots n-1)$ is present in the disjoint cycle decomposition of each maximal element of $[e, u_m] \cap I(n, m-1)$, the desired statement follows easily from Remark 4 by induction on n. □

Example 1 If $n = 6$ and $m = 3$, then $I(n, m)$ is the order ideal of $\mathrm{Abs}(S_n)$ generated by the elements $u_1 = (1\,2\,3\,4\,5)(6)$, $u_2 = (1\,3\,4\,5)(2\,6)$ and $u_3 = (1\,4\,5)(2\,3\,6)$. The intersection

$$[e, u_3] \cap ([e, u_1] \cup [e, u_2]) = \big[e, (1\,4\,5)(2\,3)(6)\big] \cup [e, (1\,4\,5)(3)(2\,6)]$$

is strongly constructible of rank 3 and $I(n, m)$ is strongly constructible of rank 4.

Lemma 14 *For $i \in \{1, 2, \ldots, n-2\}$, consider the element*

$$v_i = (1\, n\, i+2 \cdots n-1)(2\, 3 \cdots i+1) \in S_n.$$

The union $\bigcup_{i=1}^{m} [e, v_i]$ is strongly constructible of rank $n-2$ for all $1 \le m \le n-2$.

Proof The proof is similar to that of Lemma 13 and is omitted. □

Lemma 15 *Let $u_1, u_2, \ldots, u_{n-1} \in S_n$ and $v_1, v_2, \ldots, v_{n-2} \in S_n$ be defined as in Lemmas 13 and 14, respectively. If $I_n = \bigcup_{i=1}^{n-1} [e, u_i]$ and $I'_n = \bigcup_{i=1}^{n-2} [e, v_i]$, then $I_n \cap I'_n$ is strongly constructible of rank $n-3$.*

Proof We proceed by induction on n. For $n = 3$ the result is trivial, so assume that $n \ge 4$. For $i, j \in \{2, 3, \ldots, n-1\}$ we set

$$z_{ij} = (1\, j+1 \cdots n-1)(2\, 3 \cdots i)(n)(i+1 \cdots j)$$

and

$$w_{ij} = (1\, i+1 \cdots n-1)(2\, 3 \cdots j)(j+1 \cdots i\, n).$$

We observe that

$$[e, u_i] \cap [e, v_j] = \begin{cases} z_{ij}, & \text{if } i < j, \\ w_{ij}, & \text{if } i \ge j. \end{cases}$$

Let M_i be the order ideal of $\mathrm{Abs}(S_n)$ generated by the elements w_{ij} for $2 \le j \le i-1$. Since $z_{ij} \preceq w_{ij}$ for all $i, j \in \{2, 3, \ldots, n-1\}$ with $i \ne j$, we have $I_n \cap I'_n = \bigcup_{i=2}^{n-1} M_i$. Each of the ideals M_i is strongly constructible of rank $n-3$, by Remark 4 and Lemma 14. We prove by induction on k that $\bigcup_{i=2}^{k} M_i$ is strongly constructible of rank $n-3$ for every $k \le n-1$. Suppose that this holds for positive integers smaller than k. We need to show that $M_k \cap (\bigcup_{i=2}^{k-1} M_i)$ is strongly constructible of rank $n-4$. For $i \leq k-1$ we have

$$M_k \cap M_i = \langle v\,(2\,3 \cdots j)(j+1 \cdots i\,n)(i+1 \cdots k) : j = 2, 3, \dots, i-1 \rangle,$$

where $v = (1\,k+1 \cdots n-1)$. Remark 4 and Lemma 14 imply that $M_k \cap M_i$ is a strongly constructible poset of rank $n-3$. Since v is present in the disjoint cycle decomposition of each maximal element of $M_k \cap (\bigcup_{i=2}^{k-1} M_i)$, it follows by Remark 4 and induction on n that $M_k \cap (\bigcup_{i=2}^{k-1} M_i)$ is strongly constructible of rank $n-3$ as well. This concludes the proof of the lemma. □

Proof of Lemma 11 We denote by C_n the order ideal in the statement of the lemma. We will show that C_n is strongly constructible of rank $n-1$ by induction on n. The result is easy to check for $n \leq 3$, so suppose that $n \geq 4$. We have $C_n = \bigcup_{i=1}^{n-1} [e, w_i]$, where $w_1 = (1\,2 \cdots n-1\,n)$, $w_2 = (1\,2 \cdots n\,n-1), \dots, w_{n-1} = (1\,n\,2 \cdots n-1)$. By induction and Remark 4, it suffices to show that $[e, w_{n-1}] \cap (\bigcup_{i=1}^{n-2} [e, w_i])$ is strongly constructible of rank $n-2$. We observe that for $1 \leq i \leq n-2$ the intersection $[e, w_{n-1}] \cap [e, w_i]$ is equal to the ideal generated by $(1\,2 \cdots n-1)$ and the elements

$$u_{n-i} = (1\,n-i+1 \cdots n-1)(2 \cdots n-i\,n),$$
$$v_{n-i-1} = (1\,n\,n-i+1 \cdots n-1)(2 \cdots n-i),$$

considered in Lemmas 13 and 14, respectively. Hence

$$[e, w_{n-1}] \cap \left(\bigcup_{i=1}^{n-2} [e, w_i] \right) = I_n \cup I'_n$$

and the result follows from Lemmas 13, 14 and 15. □

A.2 Proof of Lemma 12

Part (i) of Lemma 12 is equivalent to Lemma 11. The proof of part (ii) is analogous to that of Lemma 11, with the following minor modifications in the statements of the various lemmas involved and the proofs.

Remark 5 Let $u_1, u_2, \dots, u_m \in B_n$ be elements of absolute length k which are products of disjoint paired cycles and let $v \in B_n$ be a cycle of absolute length r which is disjoint from u_i for each $i \in \{1, 2, \dots, m\}$. Suppose that the union $\bigcup_{i=1}^{m} [e, u_i]$ is strongly constructible of rank k. Then

$$\bigcup_{i=1}^{m} [e, vu_i] \cong \bigcup_{i=1}^{m} \big([e, v] \times [e, u_i]\big) = [e, v] \times \bigcup_{i=1}^{m} [e, u_i]$$

is strongly constructible of rank $k+r$, by Lemma 3(i).

Lemma 16 *For $i \in \{1, 2, \dots, n-1\}$ consider the element*

$$u_i = [1, i+1, \dots, n-1]((2, 3, \dots, i, n)) \in B_n.$$

The union $\bigcup_{i=1}^{m} [e, u_i]$ is strongly constructible of rank $n-1$ for all $1 \le m \le n-1$.

Proof The proof is similar to that of Lemma 13. □

Example 2 Let $I(n, m)$ be the union in the statement of the Lemma 16. If $n = 6$ and $m = 3$, then $I(n, m)$ is the order ideal of $\mathrm{Abs}(B_n)$ generated by the elements $u_1 = [1, 2, 3, 4, 5]((6))$, $u_2 = [1, 3, 4, 5]((2, 6))$ and $u_3 = [1, 4, 5]((2, 3, 6))$. We have

$$[e, u_3] \cap \big([e, u_1] \cup [e, u_2]\big) = \big[e, [1, 4, 5]((2, 3))((6))\big] \cup \big[e, [1, 4, 5]((3))((2, 6))\big].$$

This intersection is strongly constructible of rank 4 and $I(n, m)$ is strongly constructible of rank 5.

Lemma 17 *For $i \in \{1, 2, \dots, n-2\}$ consider the element*

$$v_i = [1, n, i+2, \dots, n-1]((2, 3, \dots, i+1)) \in B_n.$$

The union $\bigcup_{i=1}^{m} [e, v_i]$ is strongly constructible of rank $n-1$ for all $1 \le m \le n-2$.

Proof The proof is similar to that of Lemma 14. □

Lemma 18 *Let $u_1, u_2, \dots, u_{n-1} \in B_n$ and $v_1, v_2, \dots, v_{n-1} \in B_n$ be defined as in Lemmas 16 and 17, respectively. If $I_n = \bigcup_{i=1}^{n-1} [e, u_i]$ and $I'_n = \bigcup_{i=1}^{n-2} [e, v_i]$, then $I_n \cap I'_n$ is strongly constructible of rank $n-2$.*

Proof We proceed by induction on n. For $n = 3$ the result is trivial, so assume that $n \ge 4$. Let M_i be the order ideal of $\mathrm{Abs}(B_n)$ generated by the elements w_{ij} for $j \in \{2, 3, \dots, i-1\}$, where

$$w_{ij} = [1, i+1, \dots, n-1]((2, 3, \dots, j))((j+1, \dots, i, n)).$$

We observe that $I_n \cap I'_n = \bigcup_{i=2}^{n-1} M_i$. Each of the ideals M_i is strongly constructible of rank $n-2$, by Remark 5 and Lemma 17. As in the proof of Lemma 15, it can be shown by induction on k that $\bigcup_{i=2}^{k} M_i$ is strongly constructible for every $k \le n-1$. □

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