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Geometric combinatorics of Weyl groupoids

István Heckenberger · Volkmar Welker

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Abstract We extend properties of the weak order on finite Coxeter groups to Weyl groupoids admitting a finite root system. In particular, we determine the topological structure of intervals with respect to weak order, and show that the set of morphisms with fixed target object forms an ortho-complemented meet semilattice. We define the Coxeter complex of a Weyl groupoid with finite root system and show that it coincides with the triangulation of a sphere cut out by a simplicial hyperplane arrangement. As a consequence, one obtains an algebraic interpretation of many hyperplane arrangements that are not reflection arrangements.

Keywords Coxeter complex · Simplicial arrangements · Weak order · Weyl groupoid

1 Introduction

Finite crystallographic Coxeter groups, also known as finite Weyl groups, play a prominent role in many branches of mathematics like combinatorics, Lie theory, number theory, and geometry. In the late 1960s, V. Kac and R.V. Moody (see [16]) discovered independently a class of infinite dimensional Lie algebras. In their approach, the Weyl group is defined in terms of a generalized Cartan matrix. Later in the 1970s, V. Kac also introduced Lie superalgebras using even more general Cartan matrices [15], and observed that different Cartan matrices may give rise to isomorphic Lie superalgebras. S. Khoroshkin and V. Tolstoy [17, p. 77] observed that the Weyl group

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I. Heckenberger · V. Welker (✉)
Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, 35032 Marburg, Germany
e-mail: welker@mathematik.uni-marburg.de

I. Heckenberger
e-mail: heckenberger@mathematik.uni-marburg.de

symmetry of simple Lie algebras can be generalized to a Weyl groupoid symmetry of contragredient Lie superalgebras, without working out the details. Independently, Weyl groupoids turned out to be the main tool for the study of finiteness properties of Nichols algebras [1] over groups. We give the definition and examples of Weyl groupoids in Sect. 2.

Motivated by these developments, an axiomatic study of Weyl groupoids was initiated by H. Yamane and the first author [13]. The theory was further extended by a series of papers of M. Cuntz and the first author, and a satisfactory classification result of finite Weyl groupoids of rank two and three was achieved [7, 8]. Interestingly, not all finite Weyl groupoids obtained via the classification are related to known Nichols algebras. A possible explanation could be the existence of an additional axiom which holds for the Weyl groupoid of any Nichols algebra. However, no such axiom has been found yet, and a more systematic study is needed to find some clue.

In this paper, we analyze two structures associated to a Weyl groupoid—the weak order and the Coxeter complex. Both are generalizations from the classical case. Most of our results are known for Coxeter groups from the work of A. Björner (see [2, 3, 5]). In our work, we find the appropriate definition of the weak order and the Coxeter complex for Weyl groupoids. From definitions we deduce the generalizations of classical results. For the proofs, either a careful adaption of the classical proofs is required or the lack of group structure forces new proofs which in some cases seem to be simpler than the usual ones.

A first structure prominently associated to a Weyl group is the weak order. The weak order for Weyl groupoids is defined using the length function. It proved its relevance for Coxeter groups, and it also has an interpretation for Nichols algebras in terms of right coideal subalgebras [12]. We work out an example (Example 3.1) which shows that the weak order on a Weyl groupoid may have significantly different properties than the one on a Coxeter group. As a consequence, our results cover a much wider class of partially ordered sets than the classical ones. We verify the existence of longest elements of parabolic subgroupoids and investigate their properties. We show in Proposition 3.7 that the subposet of the weak order consisting of the longest elements is isomorphic to the poset of subsets of the set of simple reflections ordered by inclusion. In Theorem 3.10, we prove that the set of morphisms with fixed target object is a meet semilattice. It is worthwhile to mention that this result is usually proved using the exchange condition, which is not available for Weyl groupoids [13]. For our proof, we take advantage of our knowledge on longest elements. In addition, with Theorem 3.21 we find a formula involving the letters of the meet of two words in the weak order. With Theorem 3.13 we clarify the topological structure of intervals in weak order, and in Theorem 3.18 it is shown that the set of morphisms with fixed target object is ortho-complemented.

Coxeter groups, in particular Weyl groups, are a source of important classes of examples for simplicial hyperplane arrangements (see, for example, the seminal work of P. Deligne [10]). Roughly speaking, a simplicial hyperplane arrangement is a family of hyperplanes in a Euclidean space that cuts space into simplicial cones. However, most simplicial arrangements have no interpretation in terms of Coxeter groups. Therefore, there is no canonical algebraic structure which hints toward a description of the fundamental group of the complement of the complexification as described in [10]. Also, in general, simplicial arrangements lack a relation to Lie algebras. To each Weyl groupoid there is an associated arrangement of hyperplanes—the set of hyperplanes defined by the root system. A priori the geometric structure of this arrangement of hyperplanes is not clear. It was observed in [7] that for Weyl groupoids of rank three this arrangement is simplicial and therefore can be seen as an arrangement of lines in the projective plane cutting out triangles. Interestingly, the classification of such arrangements is not yet completed [11]. It was noted in [7] that most known exceptional arrangements, in particular the largest one, can be explained via Weyl groupoids. In Sect. 4, we clarify the geometric structure of the arrangement of a Weyl groupoid of arbitrary rank. This effort is motivated by the second prominent structure we analyze in this paper—the Coxeter complex. We give two different definitions of the Coxeter complex associated to a fixed object of a Weyl groupoid. From the definition in terms of cosets of parabolic subgroupoids, it is immediate that the Coxeter complex is a (abstract) simplicial complex. The other definition of the Coxeter complex is geometric and is given by the cell decomposition of the unit sphere cut out by the arrangement of the Weyl groupoid. We prove in Corollary 4.6 that the two definitions yield isomorphic complexes, and hence the Coxeter complex is a simplicial complex which can be seen as the complex induced by a simplicial hyperplane arrangement on the unit sphere. Note that the mathematical reasoning of Sect. 4 is independent of Sect. 3. Nevertheless, the combinatorics of the weak order and the Coxeter complex is linked in the same way as in the classical case. Indeed, at the end of Sect. 4 in Theorem 4.9 we note that any linear extension of any weak order associated to a Weyl groupoid induces a shelling order on its Coxeter complex.

In classical Coxeter group theory, the Bruhat order on the elements of the group also plays an important role. It is a refinement of the weak order. But classically the definition of the Bruhat order uses the conjugation in the group. In general, for groupoids the notion of conjugation is not well defined. Therefore, we leave the definition and study of a Bruhat order for Weyl groupoids as an open problem.

2 Basic concepts

2.1 Weyl groupoids

We mainly follow the notation in [8, 9]. The foundations of the general theory have been developed in [13]. Let us start by recalling the main definitions.

Let I and A be finite sets with $A \neq \emptyset$. Let $\{\alpha_i \,|\, i \in I\}$ be the standard basis of $\mathbb{Z}^I$. For all $i \in I$, let $\rho_i : A \to A$ be a map, and for all $a \in A$ let $C^a = (c^a_{jk})_{j,k\in I}$ be a generalized Cartan matrix in the sense of [16, Sect. 1.1], where $c^a_{jk} \in \mathbb{Z}$ for all $j, k \in I$. More precisely,

- $c^a_{jj} = 2$ for all $j \in I$,
- $c^a_{jk} < 0$ for all $j, k \in I$ and $j \neq k$ and
- $c^a_{jk} = 0$ implies $c^a_{kj} = 0$ for all $j, k \in I$.

The quadruple

$$\mathcal{C} = \mathcal{C}\big(I, A, (\rho_i)_{i\in I}, \big(C^a\big)_{a\in A}\big)$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c_{ij}^a = c_{ij}^{\rho_i(a)}$ for all $a \in A$ and $i, j \in I$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$, define $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \quad \text{for all } j \in I. \tag{2.1}$$

Then σ_i^a is a reflection in the sense of [6, Chap. V, Sect. 2]. The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ_i^a with $i \in I$ and $a \in A$, where σ_i^a is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The category $\mathcal{W}(\mathcal{C})$ is a groupoid. The set of morphisms of $\mathcal{W}(\mathcal{C})$ is also denoted by $\mathcal{W}(\mathcal{C})$, and we use the notation

$$\mathrm{Hom}\big(\mathcal{W}(\mathcal{C}), a\big) = \bigcup_{b\in A} \mathrm{Hom}(b, a) \quad \text{(disjoint union)}.$$

Example 2.1 Let (W, S) be a Coxeter system for a crystallographic Coxeter group W. Then (W, S) can be seen as a Weyl groupoid $\mathcal{W}(\mathcal{C})$ with a single object a and $\mathrm{Hom}(a, a) = \langle S\rangle = W$ with Cartan scheme $\mathcal{C} = \mathcal{C}(\{1, \ldots, |S|\}, \{a\}, (\rho_i = \mathrm{id})_{i=1,\ldots,|S|}, (C^a))$ where C^a is the usual Cartan matrix of W. Note that the classical Cartan matrices are positive definite, which is not required for the generalized Cartan matrices of Weyl groupoids. Conversely, if $\mathcal{C} = \mathcal{C}(I, \{a\}, (\rho_i = \mathrm{id})_{i\in I}, (C^a))$ is a Cartan scheme with one object a, then $(\mathcal{W}(\mathcal{C}), S)$ with $S = \{\rho_i \mid i \in I\}$ is the Coxeter system for the crystallographic Coxeter group $\mathcal{W}(\mathcal{C})$. In particular, for any Cartan scheme on one object a the Cartan matrix C^a has to be positive definite.

For notational convenience we will often neglect upper indices referring to elements of A if they are uniquely determined by the context. For example, the morphism

$$\sigma_{i_1}^{\rho_{i_2}\cdots\rho_{i_k}(a)} \cdots \sigma_{i_{k-1}}^{\rho_{i_k}(a)} \sigma_{i_k}^a \in \mathrm{Hom}(a, b),$$
$$\text{where } k \in \mathbb{N}_0,\ i_1, \ldots, i_k \in I, \text{ and } b = \rho_{i_1} \cdots \rho_{i_k}(a),$$

will be denoted by $\sigma_{i_1} \cdots \sigma_{i_k}^a$ or by $\mathrm{id}^b \sigma_{i_1} \cdots \sigma_{i_k}$. The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan scheme is called *simply connected*, if for all $a, b \in A$ the set $\mathrm{Hom}(a, b)$ consists of at most one element.

Let $\mathcal{C}$ be a Cartan scheme. For all $a \in A$, let

$$\big(R^{\mathrm{re}}\big)^a = \big\{\mathrm{id}^a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0,\ i_1, \ldots, i_k, j \in I\big\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a)—this notion is adopted from [16, Sect. 5.1]. The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a\in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called *positive* (resp., *negative*) if $\alpha \in \mathbb{N}_0^I$ (resp., $\alpha \in$

$-\mathbb{N}_0^I$). In contrast to real roots associated to a single generalized Cartan matrix (e.g., Example 2.1), $(R^{\mathrm{re}})^a$ may contain elements which are neither positive nor negative. A good general theory can be obtained if $(R^{\mathrm{re}})^a$ satisfies additional properties.

Let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme. For all $a\in A$ let $R^a\subseteq\mathbb{Z}^I$, and define $m_{i,j}^a=|R^a\cap(\mathbb{N}_0\alpha_i+\mathbb{N}_0\alpha_j)|$ for all $i,j\in I$ and $a\in A$. One says that

$$\mathcal{R}=\mathcal{R}\big(\mathcal{C},\big(R^a\big)_{a\in A}\big)$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms:

(R1) $R^a=R_+^a\cup -R_+^a$, where $R_+^a=R^a\cap\mathbb{N}_0^I$, for all $a\in A$.
(R2) $R^a\cap\mathbb{Z}\alpha_i=\{\alpha_i,-\alpha_i\}$ for all $i\in I$, $a\in A$.
(R3) $\sigma_i^a(R^a)=R^{\rho_i(a)}$ for all $i\in I$, $a\in A$.
(R4) If $i,j\in I$ and $a\in A$ such that $i\neq j$ and $m_{i,j}^a$ is finite, then $(\rho_i\rho_j)^{m_{i,j}^a}(a)=a$.

Example 2.2 Let (W,S) be a Coxeter system for a finite crystallographic Coxeter group W acting on some real vector space V seen as a Weyl groupoid as in Example 2.1. Then by [14, p. 6] a root system of W is a set of vectors R from V such that:

(R1′) $R\cap\mathbb{R}\alpha=\{\alpha,-\alpha\}$ for all $\alpha\in R$.
(R2′) $\sigma R=R$ for all reflections σ from W.

Clearly, (R1′) implies (R2), and from the finiteness and the crystallographic condition we infer that (R2) implies (R1′). It is obvious that (R2′) implies (R3). Since any reflection is a product of simple reflections, it follows that (R3) implies (R2′). Since our groupoid has only one object, Axiom (R4) is vacuous. As a consequence [14, p. 8] of (R1′) and (R2′) every set of positive roots contains a unique simple system. Then the definition of a simple system and the crystallographic condition imply (R1). Thus we have shown that for finite crystallographic Coxeter groups conditions (R1′)–(R2′) and (R1)–(R3) are equivalent.

Axioms (R2) and (R3) are always fulfilled for $\mathcal{R}^{\mathrm{re}}$. A root system $\mathcal{R}$ is called *finite* if for all $a\in A$ the set R^a is finite. By [9, Proposition 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}=\mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case.

In [9, Definition 4.3], the concept of an *irreducible* root system of type $\mathcal{C}$ was defined. By [9, Proposition 4.6], if $\mathcal{C}$ is a connected Cartan scheme and $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}$ is irreducible if and only if for all $a\in A$ (or, equivalently, for some $a\in A$) the generalized Cartan matrix C^a is indecomposable.

Let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme. Let Γ be an undirected graph, such that the vertices of Γ correspond to the elements of A. Assume that for all $i\in I$ and $a\in A$ with $\rho_i(a)\neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i, and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of $\mathcal{C}$.

Now we introduce parabolic subgroupoids which will play a crucial role in the sequel.

Definition 2.3 Let $\mathcal{C}=\mathcal{C}(I,A,(\rho_i)_{i\in I},(C^a)_{a\in A})$ be a Cartan scheme and let $J\subseteq I$. The *parabolic subgroupoid* $\mathcal{W}_J(\mathcal{C})$ is the smallest subgroupoid of $\mathcal{W}(\mathcal{C})$ which contains all objects of $\mathcal{W}(\mathcal{C})$ and all morphisms σ_j^a with $j\in J$ and $a\in A$.

Example 2.4 1. Assume that $\mathcal{C}$ is a Cartan scheme such that A consists of a single element. Then the parabolic subgroupoids of $\mathcal{W}(\mathcal{C})$ are just the standard parabolic subgroups of the Coxeter group $\mathcal{W}(\mathcal{C})$.

2. Let (W,S) be a crystallographic Coxeter system. Let $\mathcal{C}$ be a connected and simply connected Cartan scheme such that all Cartan matrices C^a with $a\in A$ coincide with the Cartan matrix of W. Then the connected components of $\mathcal{W}_J(\mathcal{C})$, where $J\subseteq I$, can be interpreted as the parabolic subgroups of W conjugate to W_J.

In general, parabolic subgroupoids are not connected, even if $\mathcal{C}$ is connected.

In what follows, we will consider only Cartan schemes $\mathcal{C}$ which admit a root system $\mathcal{R}(\mathcal{C},(R^a)_{a\in A})$.

The most important tools for the study of the weak order in the next section will be the length functions of the parabolic subgroupoids $\mathcal{W}_J(\mathcal{C})$ of $\mathcal{W}(\mathcal{C})$, where $J\subseteq I$. For all $J\subseteq I$ let $\ell_J:\mathcal{W}_J(\mathcal{C})\to\mathbb{N}_0$ be such that

$$\ell_J(w)=\min\bigl\{k\in\mathbb{N}_0\mid w=\sigma_{i_1}\cdots\sigma_{i_k}^a, i_1,\dots,i_k\in J\bigr\} \tag{2.2}$$

for all $a,b\in A$ and $w\in\mathrm{Hom}(a,b)$. For $J=I$ this is the adaption of the usual length function from classical Coxeter groups to Weyl groupoids defined in [13]. We write $\ell(w)$ instead of $\ell_I(w)$. For $w\in\mathcal{W}(\mathcal{C})$ we say that $w=\sigma_{i_1}\cdots\sigma_{i_k}$ is a *reduced decomposition* of w if $k=\ell(w)$.

The length function on Weyl groupoids has similar properties as the usual length function on Coxeter groups, see [13]. In particular, the following holds.

Lemma 2.5 (Lemma 8(iii) [13]) *Let $a,b\in A$ and $w\in\mathrm{Hom}(a,b)$. Then*

$$\ell(w)=\bigl|\bigl\{\alpha\in R_+^a\mid w(\alpha)\in -R_+^b\bigr\}\bigr|.$$

Lemma 2.6 (Corollary 3 [13]) *Let $a,b\in A$, $w\in\mathrm{Hom}(a,b)$, and $i\in I$. Then $\ell(w\sigma_i)=\ell(w)-1$ if and only if $w(\alpha_i)\in -R_+^b$. Equivalently, $\ell(w\sigma_i)=\ell(w)+1$ if and only if $w(\alpha_i)\in R_+^b$.*

Before we proceed with studying the length function itself, we clarify the structure of the set of subsets $J\subseteq I$ for which $w\in\mathrm{Hom}(a,b)$ is also a morphism in $\mathcal{W}_J(\mathcal{C})$.

Proposition 2.7 *Let $w\in\mathrm{Hom}(a,b)$. If $w=\sigma_{i_1}\cdots\sigma_{i_k}^a$ is a reduced decomposition of w and $w=\sigma_{j_1}\cdots\sigma_{j_l}^a$ is another decomposition, where $k,l\in\mathbb{N}_0$ and $i_1,\dots,i_k,j_1,\dots,j_l\in I$, then as sets*

$$\{i_1,\dots,i_k\}\subseteq\{j_1,\dots,j_l\}.$$

In particular, if $k=l$ then $\{i_1,\dots,i_k\}=\{j_1,\dots,j_k\}$.

Proof Set $J := \{i_1, \dots, i_k\}$ and $J' = \{j_1, \dots, j_l\}$. Assume that $J \not\subseteq J'$. Let $m \in \{1, \dots, k\}$ be such that $i_m \notin J'$ and $i_{m'} \in J'$ for all $m' < m$. Let $\alpha = \mathrm{id}^a \sigma_{i_k} \sigma_{i_{k-1}} \cdots \sigma_{i_{m+1}}(\alpha_{i_m})$. Then $\alpha \in R^a_+$ by the fact that $w = \sigma_{i_1} \cdots \sigma^a_{i_k}$ is a reduced decomposition and by Lemma 2.6. Moreover,

$$w(\alpha) = \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m}(\alpha_{i_m}) = -\sigma_{i_1} \cdots \sigma_{i_{m-1}}(\alpha_{i_m}) \in -\alpha_{i_m} + \mathrm{span}_{\mathbb{Z}}\{\alpha_j \mid j \in J'\}. \tag{2.3}$$

Let $\alpha = \alpha' + \alpha''$ with $\alpha' \in \mathrm{span}_{\mathbb{N}_0}\{\alpha_j \mid j \notin J'\}$ and $\alpha'' \in \mathrm{span}_{\mathbb{N}_0}\{\alpha_j \mid j \in J'\}$. Since $w \in \mathcal{W}_{J'}(\mathcal{C})$, we conclude that $w(\alpha) \in \alpha' + \mathrm{span}_{\mathbb{Z}}\{\alpha_j \mid j \in J'\}$. This is a contradiction to (2.3) since $i_m \notin J'$. Hence $J \subseteq J'$. □

For all $a, b \in A$, $w \in \mathrm{Hom}(a, b)$ and reduced decompositions $w = \sigma_{i_1} \cdots \sigma^a_{i_k}$ we set $J(w) := \{i_1, \dots, i_k\}$. By Proposition 2.7, this definition is independent of the chosen reduced decomposition. Moreover, for any subset $J \subseteq I$ and any $w \in \mathcal{W}_J(\mathcal{C})$ the reduced decompositions of w are also contained in $\mathcal{W}_J(\mathcal{C})$. Observe also that $J(w) = J(w^{-1})$ for all $w \in \mathcal{W}(\mathcal{C})$ and that $J(uv) = J(u) \cup J(v)$ for all $u, v \in \mathcal{W}(\mathcal{C})$ with $\ell(uv) = \ell(u) + \ell(v)$.

Corollary 2.8 *Let $J \subseteq I$. Then $\ell_J(w) = \ell(w)$ for all $w \in \mathcal{W}_J(\mathcal{C})$.*

Proof If there is a decomposition of w having only factors σ_i with $i \in J$ then by Proposition 2.7 all reduced decompositions have this property. The assertion follows. □

One can characterize $J(w)$ for any $w \in \mathcal{W}(\mathcal{C})$ in terms of roots.

Lemma 2.9 *Let $a, b \in A$, $J \subseteq I$, and let $w \in \mathrm{Hom}(b, a)$. Then $J(w) \subseteq J$ if and only if $w(R^b_+) \subseteq R^a_+ \cup \sum_{j \in J} \mathbb{Z}\alpha_j$.*

Proof The implication $\Rightarrow$ follows from the definition of simple reflections and from Axioms (R1), (R3). Assume now that $w(R^b_+) \subseteq R^a_+ \cup \sum_{j \in J} \mathbb{Z}\alpha_j$ and that $J(w) \not\subseteq J$. Then $J(\sigma_i w) \not\subseteq J$ and $\sigma_i w(R^b_+) \subseteq R^{\rho_i(a)}_+ \cup \sum_{j \in J} \mathbb{Z}\alpha_j$ for all $i \in J$, and hence by multiplying w from the left by an appropriate element of $\mathcal{W}_J(\mathcal{C})$ we may assume that $\ell(\sigma_j w) = \ell(w) + 1$ for all $j \in J$. It follows that $w^{-1}(\alpha_j) \in R^b_+$ for all $j \in J$ by Lemma 2.6. Hence $w(R^b_+) \subseteq R^a_+$, and therefore $w = \mathrm{id}^a$ by Lemma 2.5. This is a contradiction to $J(w) \not\subseteq J$. □

Let $J \subseteq I$ and for all $a \in A$ let $C'^a = (c'^a_{jk})_{j,k \in J}$. Then $\mathcal{C}' = \mathcal{C}'(J, A, (\rho_j)_{j \in J}, (C'^a)_{a \in A})$ is a Cartan scheme. It is denoted by $\mathcal{C}|_J$ and is called the *restriction of $\mathcal{C}$ to J*. As noted in [9, Sect. 4], if $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a root system of type $\mathcal{C}$, then $\mathcal{R}^{\mathrm{re}}(\mathcal{C}|_J)$ is a root system of type $\mathcal{C}|_J$, and finiteness of $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ implies finiteness of $\mathcal{R}^{\mathrm{re}}(\mathcal{C}|_J)$. We compare restrictions with parabolic subgroupoids.

Lemma 2.10 *Let $J \subseteq I$, $a \in A$, $k \in \mathbb{N}_0$, and $i_1, \dots, i_k \in J$ such that $\sigma_{i_1} \cdots \sigma^a_{i_k}|_{\mathbb{Z}^J} = \mathrm{id}^a|_{\mathbb{Z}^J}$. Then $\sigma_{i_1} \cdots \sigma^a_{i_k} = \mathrm{id}^a$.*

Proof By assumption $\sigma_{i_1}\cdots\sigma_{i_k}^a(\alpha_j)=\alpha_j$ for all $j\in J$. Since $i_1,\dots,i_k\in J$, the definition of σ_j^b for $j\in J$, $b\in A$ implies that $\sigma_{i_1}\cdots\sigma_{i_k}^a(\alpha_i)\in\alpha_i+\mathbb{Z}^J$ for all $i\in I\setminus J$. Hence $\sigma_{i_1}\cdots\sigma_{i_k}^a(\alpha_i)\in\mathbb{N}_0^I$ for all $i\in I\setminus J$ by Axioms (R1) and (R3). Then $\ell(\sigma_{i_1}\cdots\sigma_{i_k}^a)=0$ by Lemma 2.5 and hence $\sigma_{i_1}\cdots\sigma_{i_k}^a=\mathrm{id}^a$. □

Proposition 2.11 *For all $J\subseteq I$ there is a unique functor $E_J:\mathcal{W}(\mathcal{C}|_J)\to\mathcal{W}(\mathcal{C})$ with $E_J(a)=a$ and $E_J(\sigma_j^a)=\sigma_j^a$ for all $a\in A$ and $j\in J$. This functor induces an isomorphism of groupoids between $\mathcal{W}(\mathcal{C}|_J)$ and $\mathcal{W}_J(\mathcal{C})$.*

Proof The uniqueness of E_J follows from the definition of $\mathcal{W}(\mathcal{C}|_J)$, and $E_J(w)\in\mathcal{W}_J(\mathcal{C})$ for all $w\in\mathcal{W}(\mathcal{C}|_J)$. The functor E_J is well-defined by Lemma 2.10. It is clear that $E_J(w)=\mathrm{id}^a$ for some $a\in A$ and $w\in\mathcal{W}(\mathcal{C}|_J)$ implies that $w=\mathrm{id}^a$, and hence E_J is an isomorphism. □

Finally, we state an analogue of a well-known decomposition theorem for Coxeter groups. Following [5, Definition 2.4.2], let

$$\mathcal{W}^J(\mathcal{C})=\bigl\{w\in\mathcal{W}(\mathcal{C})\mid\ell(w\sigma_j)=\ell(w)+1\text{ for all }j\in J\bigr\}.\tag{2.4}$$

Proposition 2.12 *Let $J\subseteq I$ and $w\in\mathcal{W}(\mathcal{C})$. Then the following hold*:

(i) *There exist unique elements $u\in\mathcal{W}^J(\mathcal{C})$ and $v\in\mathcal{W}_J(\mathcal{C})$ such that $w=uv$.*
(ii) *Let u,v be as in* (i). *Then $\ell(w)=\ell(u)+\ell(v)$.*

Proof The existence in (i) and the claim in (ii) can be shown inductively on the length of w; see, for example, [5, Proposition 2.4.4]. If $w\in\mathcal{W}^J(\mathcal{C})$, then $w=w\mathrm{id}$ is a desired decomposition. Otherwise, let $j\in J$ be such that $\ell(w\sigma_j)=\ell(w)-1$. By induction hypothesis, there exist $u\in\mathcal{W}^J(\mathcal{C})$ and $v_1\in\mathcal{W}_J(\mathcal{C})$ such that $w\sigma_j=uv_1$ and $\ell(w\sigma_j)=\ell(u)+\ell(v_1)$. We obtain that $w=uv$, where $v=v_1\sigma_j\in\mathcal{W}_J(\mathcal{C})$. Moreover,

$$\begin{aligned}\ell(u)+\ell(v)&\le\ell(u)+\ell(v_1)+1=\ell(uv_1)+1\\&=\ell(w\sigma_j)+1=\ell(w)=\ell(uv)\le\ell(u)+\ell(v),\end{aligned}$$

and hence (ii) holds.

Let now $u_1,u_2\in\mathcal{W}^J(\mathcal{C})$ and $v_1,v_2\in\mathcal{W}_J(\mathcal{C})$ be such that $w=u_1v_1=u_2v_2$. Then

$$u_1=u_2v_2(v_1)^{-1}.\tag{2.5}$$

Assume that $v_2\ne v_1$. Then there exists $j\in J$ such that $\ell(v_2v_1^{-1}\sigma_j)=\ell(v_2v_1^{-1})-1$, and hence $v_2v_1^{-1}(\alpha_j)\in-\sum_{k\in J}\mathbb{N}_0\alpha_k$ by Lemma 2.6. Since $u_2\in\mathcal{W}^J(\mathcal{C})$, it follows again by Lemma 2.6 that $u_2v_2v_1^{-1}(\alpha_j)\in-\mathbb{N}_0^I$. On the other hand, $u_1(\alpha_j)\in\mathbb{N}_0^I$ by Lemma 2.6 since $u_1\in\mathcal{W}^J(\mathcal{C})$. This is a contradiction to (2.5), and hence $v_1=v_2$ and $u_1=u_2$. □

An immediate consequence of Proposition 2.12 is the following.

Corollary 2.13 *Let $J \subseteq I$. Then every left coset $w\mathcal{W}_J(\mathcal{C})$, where $w \in \mathcal{W}(\mathcal{C})$, has a unique representative of minimal length. The system of such representatives is $\mathcal{W}^J(\mathcal{C})$.*

2.2 Geometric combinatorics

Let P be a partially ordered set with order relation $\preceq$. A *chain* of length i in P is a linearly ordered subset $p_0 \prec \cdots \prec p_i$ of $i+1$ elements of P. A chain is called *maximal* if it is an inclusionwise maximal linearly ordered subset of P. The order complex $\Delta(P)$ of P is the abstract simplicial complex on ground set P whose i-simplices are the chains of length i. If $p \preceq q$ are two elements of P then we denote by $[p,q]$ the *closed interval* $\{r \in P \mid p \preceq r \preceq q\}$. Analogously, one defines the *open interval* $(p,q) := [p,q] \setminus \{p,q\}$. We write $\Delta(p,q)$ to denote the *order complex* of (p,q). For $p \in P$ we write $P_{\prec p}$ for the subposet of all $q \in P$ with $q \prec p$.

Via the geometric realization $|\Delta(P)|$ of P, one can speak of topological properties of partially ordered sets P. In particular, we can speak of P being homotopy equivalent or homeomorphic to another partially ordered set or topological space. If P is a partially ordered set with unique maximal element $\hat{1}$ or unique minimal element $\hat{0}$ then $\Delta(P)$ is a cone over $\hat{1}$ (resp., $\hat{0}$) and therefore contractible. Hence in order to be able to capture non-trivial topology, one considers for partially ordered sets P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ the proper part $\hat{P} := P \setminus \{\hat{0}, \hat{1}\}$ of P. For example, $\widehat{[p,q]} = (p,q)$. The following simple example will be useful in the subsequent sections.

Example 2.14 Let Ω be a finite set and 2^Ω be the Boolean lattice of all subsets of Ω ordered by inclusion. Then 2^Ω has the unique minimal element $\hat{0} = \emptyset$ and the unique maximal element $\hat{1} = \Omega$. Then $\Delta(\widehat{2^\Omega})$ is the barycentric subdivision (see, for example, [18, Sect. 15]) of the boundary of the $(|\Omega|-1)$-simplex and hence homeomorphic to an $(|\Omega|-2)$-sphere.

For our purposes, the following well known result on the topology of partially ordered sets will be crucial.

Theorem 2.15 (Corollary 10.12 [4]) *Let P be a partially ordered set and let $f : P \to P$ be a map such that*:

(i) $p \preceq q$ *implies* $f(p) \preceq f(q)$;
(ii) $f(p) \preceq p$.

Then P and $f(P)$ are homotopy equivalent.

In order to set up the next tool, it is most convenient to work in the context of (abstract) simplicial complexes. For a simplicial complex Δ, we call $A \in \Delta$ a *face* of Δ and denote by $\dim A = \#A - 1$ its *dimension*. We call Δ *pure* if all inclusionwise maximal faces have the same dimension. The order complex $\Delta(P)$ of a partially ordered set P is pure if and only if all maximal chains in P have the same length. A pure simplicial complex Δ is called *shellable* if there is a numbering $F_1, \ldots, F_r$ of the set of its maximal faces such that for all $1 \le i < j \le r$ there is an $\ell < j$ and an $\omega \in F_j$ such that $F_i \cap F_j \subseteq F_\ell \cap F_j = F_j \setminus \{\omega\}$.

It is well known (see, e.g., [4]) that if Δ is a shellable simplicial complex of dimension d then the geometric realization is homotopy equivalent to a wedge of spheres of dimension d. For the subsequent applications, we are interested in situations when Δ is homeomorphic to a sphere. This can also be verified using shellability when Δ is a pseudomanifold. A pure d-dimensional simplicial complex Δ is called a *pseudomanifold* if for all faces $F \in \Delta$ of dimension $d-1$ there are at most 2 faces of dimension d containing F.

Theorem 2.16 (Theorem 11.4 [4]) *Let Δ be a shellable d-dimensional pseudomanifold. If every face of dimension $d-1$ is contained in exactly 2 faces of dimension d then Δ is homeomorphic to a d-sphere, otherwise Δ is homeomorphic to a d-ball.*

3 Weak order

In this section, we define and study the weak order on a Weyl groupoid. We are interested in combinatorial and geometric properties of this partial order. We show in Theorems 3.10 and 3.18 that this order is indeed an ortho-complemented lattice. In Theorem 3.13, we identify the homotopy types of order complexes of intervals in the weak order as spheres or points.

Throughout this section, let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system.

The (right) *weak order* or *Duflo order* $\le_R$ on Weyl groupoids is the natural generalization of the (right) weak order on Coxeter groups, see [5, Chap. 3]: for any $a, b, c \in A$ and $u \in \mathrm{Hom}(b, a)$, $v \in \mathrm{Hom}(c, b)$ we define

$$u \le_R uv \quad :\Leftrightarrow \quad \ell(u) + \ell(v) = \ell(uv).$$

For all $a \in A$, the weak order is a partial ordering on $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. As shown in [12], the weak order has an algebraic interpretation in terms of right coideal subalgebras of Nichols algebras.

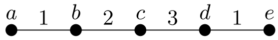
Example 3.1 Let $I = \{1, 2, 3\}$ and $A = \{a, b, c, d, e\}$. There is a unique Cartan scheme $\mathcal{C}$ with

$$C^a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad C^b = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

$$C^c = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad C^d = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

$$C^e = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix},$$

Fig. 1 The object change diagram for Example 3.1



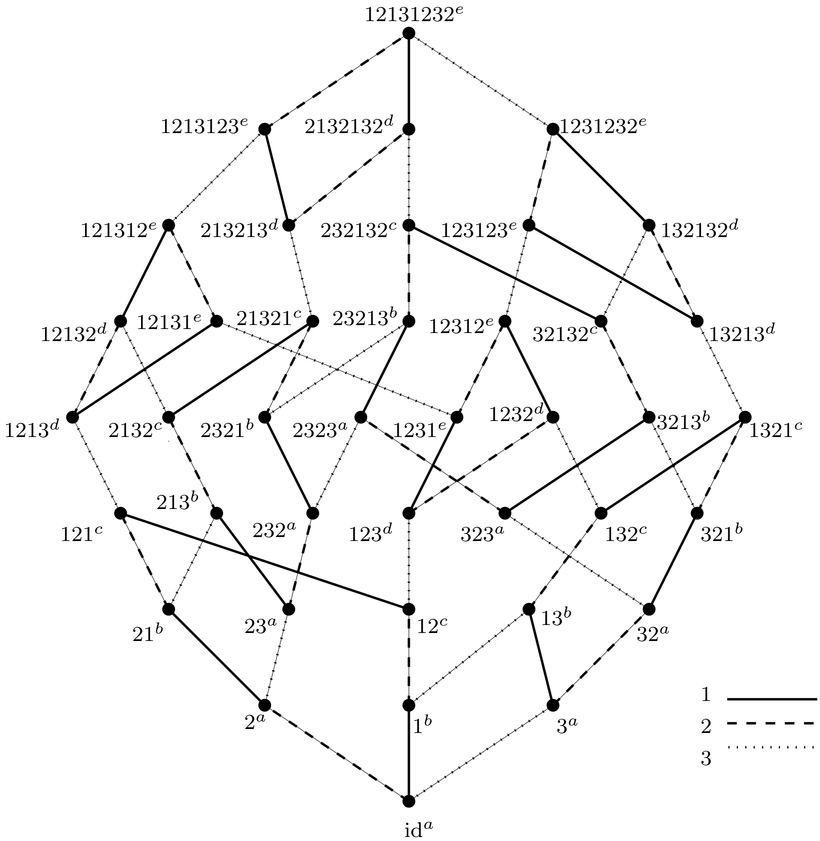


Fig. 2 The weak order for Example 3.1 in object a

where the object change diagram is as in Fig. 1.

The rank of the Cartan scheme is 3 and the length of the longest element in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ (see below) is 8, and hence none of the posets $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$, $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), b)$ and $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), c)$ with the weak order depicted in Figs. 2, 3 and 4 can be obtained from a Coxeter group. In this respect, a particularly interesting case is Fig. 4. Note that for Coxeter groups W the polynomial $\sum_{w \in W} t^{\ell(w)}$ is a product of factors of the form $1 + t + \cdots + t^e$. In particular, it follows that the coefficient sequence of $\sum_{w \in W} t^{\ell(w)}$ is unimodal, i.e., weakly increases and weakly decreases along increasing t powers. Now despite the fact that they cannot arise from Coxeter groups for Figs. 2 and 3, the analogously defined polynomial still has the nice factorization. But in the example Fig. 4 this fails, and moreover the coefficient sequence $1, 3, 6, 7, 6, 7, 6, 3, 1$ is not unimodal.

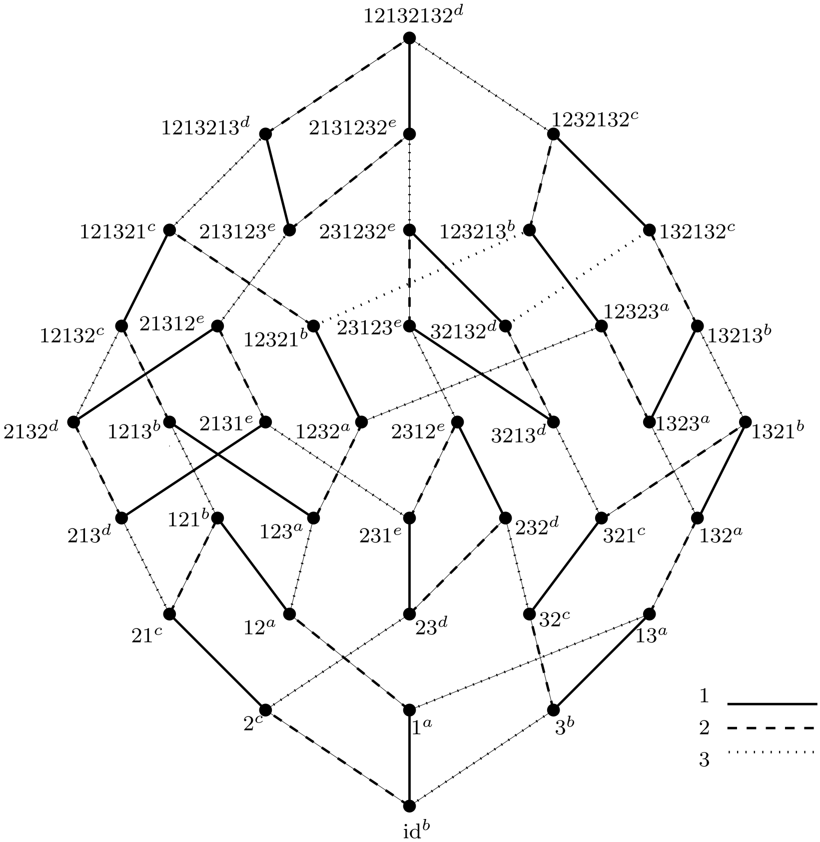


Fig. 3 The weak order for Example 3.1 in object b

In what follows, for all $a \in A$ we consider $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ as a poset with respect to the weak order.

Lemma 3.2 *Let $a \in A$. Then all maximal chains in* $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ *have the same length. This number is independent of a in the connected component of $\mathcal{C}$ containing a. Hence, $\Delta(\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a))$ is a pure simplicial complex.*

Proof A chain $u_0 <_R u_1 <_R \cdots <_R u_k$ in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$, where $k \in \mathbb{N}_0$, is maximal if and only if $\ell(u_j) = j$ for all $j \in \{0, 1, \dots, k\}$ and

$$\ell(u_k\sigma_i) \le \ell(u_k) \quad \text{for all } i \in I. \tag{3.1}$$

Lemma 2.6 and (3.1) imply that $u_k(\alpha_i) \in -R^a_+$ for all $i \in I$. Hence $u_k(\alpha) \in -R^a_+$ for all $\alpha \in R^b_+$, where $b \in A$ such that $u_k \in \mathrm{Hom}(b, a)$. Then $k = \ell(u_k) = |R^b_+| = |R^a_+| = |R^a|/2$ by Lemma 2.5. In the connected component of $\mathcal{C}$ containing a, the number of roots per object is constant by Axiom (R3). $\square$

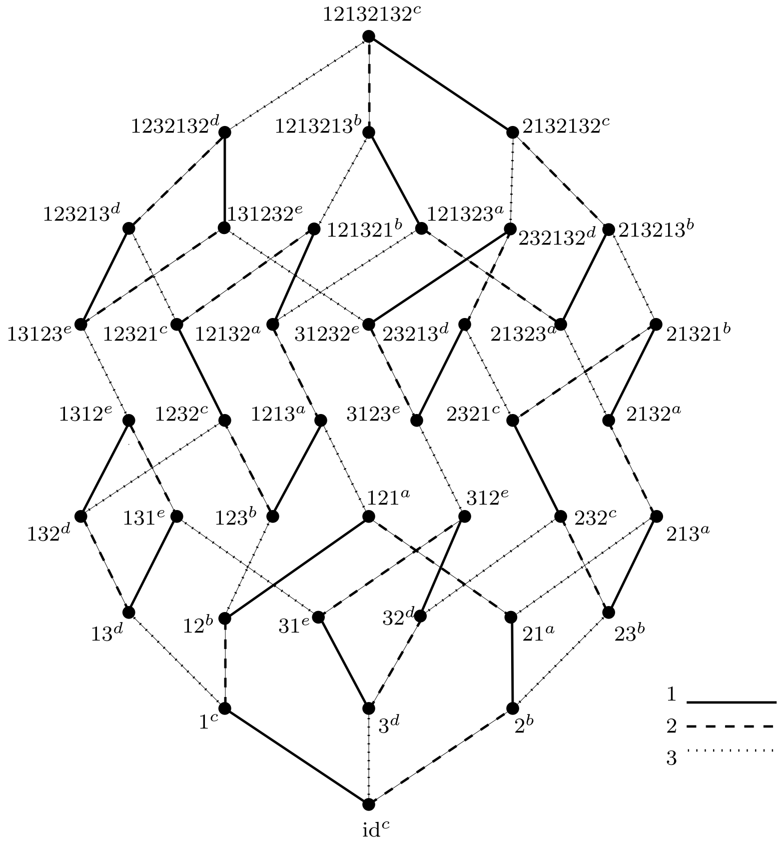


Fig. 4 The weak order for Example 3.1 in object c

Corollary 3.3 *Let $a \in A$ and $J \subseteq I$. There is a unique minimal and a unique maximal element in* $\mathrm{Hom}(\mathcal{W}_J(\mathcal{C}), a)$.

Proof By Proposition 2.11, the groupoid $\mathcal{W}_J(\mathcal{C})$ is isomorphic to the Weyl groupoid of a Cartan scheme. The length function on $\mathcal{W}_J(\mathcal{C})$ is ℓ_J, which itself coincides with the restriction of the length function of $\mathcal{W}(\mathcal{C})$ by Proposition 2.8. Thus we may assume that $J = I$.

The unique minimal element in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ is id^a. In view of the proof of Lemma 3.2, maximal elements have length $|R^a_+|$. By [13, Corollary 5], there is a unique element in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ of maximal length, which implies the claim. □

Definition 3.4 For all $a \in A$ and $J \subseteq I$ we write w_J for the unique maximal element of $\mathrm{Hom}(\mathcal{W}_J(\mathcal{C}), a)$ with respect to weak order. We say that w_J is the *longest word* over J.

The element w_J in Definition 3.4 depends on the object a. Nevertheless, for brevity we omit a in the notation, since usually it is clear from the context what it is.

Lemma 3.5 *Let $a \in A$, $J \subseteq I$ and let w_J be the unique maximal element of* $\mathrm{Hom}(\mathcal{W}_J(\mathcal{C}), a)$ *with respect to weak order. Then $J(w_J) = J$.*

Proof This follows from Lemma 2.6. □

In [5, p. 17], left descent sets and left descents of elements of Coxeter groups have been defined. We generalize the definition to our setting, and introduce a related notion.

For all $a \in A$ and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$, let

$$D_L(w) = \big\{s \in \mathrm{Hom}\big(\mathcal{W}(\mathcal{C}), a\big) \mid \ell(s) = 1,\ s \le_R w\big\}, \tag{3.2}$$

$$I_L(w) = \big\{i \in I \mid \mathrm{id}^a \sigma_i \in D_L(w)\big\}. \tag{3.3}$$

The set $D_L(w)$ is called the *left descent set* of w and its elements are called the left descents of w. Clearly, every element $w \ne \mathrm{id}^a$ has left descents. Similarly, let

$$\bar{D}_L(w) = \big\{w_J \in \mathrm{Hom}\big(\mathcal{W}(\mathcal{C}), a\big) \mid J \subseteq I,\ w_J \le_R w\big\}. \tag{3.4}$$

Since $w_{\{j\}} = \mathrm{id}^a \sigma_j$ for all $j \in I$, we have a natural inclusion $D_L(w) \subseteq \bar{D}_L(w)$. In the sequel, we will consider $\bar{D}_L(w)$ as a subposet of $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ ordered by the weak order.

Lemma 3.6 *Let $a \in A$, $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $J = I_L(w)$. Then $w_J \le_R w$.*

Proof Set $x := w^{-1} w_J$. Then $w = w_J x^{-1}$. To prove that $w_J \in \bar{D}_L(w)$, we have to show that $\ell(x) = \ell(w) - \ell(w_J)$. By definition of $I_L(w)$ and Lemma 2.6 we conclude that $w^{-1}(\alpha_j) \in -\mathbb{N}_0^I$ and $w_J(\alpha_j) \in -\mathrm{span}_{\mathbb{N}_0}\{\alpha_m \mid m \in J\}$ for all $j \in J$. Hence $x(\alpha_j) \in \mathbb{N}_0^I$ for all $j \in J$. Therefore, $x \in \mathcal{W}^J(\mathcal{C})$ by Lemma 2.6, and hence $\ell(x w_J^{-1}) = \ell(x) + \ell(w_J^{-1})$ by Proposition 2.12(ii). This yields the claim. □

Proposition 3.7 *Let $a \in A$ and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. The map $2^{I_L(w)} \to \bar{D}_L(w)$, $J \mapsto w_J$, is an isomorphism of posets.*

Proof Well-defined: By Lemma 3.6, the map $2^{I_L(w)} \to \bar{D}_L(w)$ is well defined.

Injectivity: This follows immediately from Lemma 3.5.

Surjectivity: Let $J \subseteq I$ be such that $w_J \le_R w$. The definition of w_J implies that $\mathrm{id}^a \sigma_j \le_R w_J$ for all $j \in J$, and hence $J \subseteq I_L(w_J) \subseteq I_L(w)$. Thus the map $2^{I_L(w)} \to \bar{D}_L(w)$ is surjective.

Poset-Isomorphism: Definition 3.4 implies that $w_J \le_R w_{J'}$ whenever $J \subseteq J' \subseteq I$. Conversely, let $J, J' \subseteq I$ with $w_J \le_R w_{J'}$. By Corollary 3.5, it follows that $J = J(w_J)$ and $J' = J(w_{J'})$. Hence from $w_J \le_R w_{J'}$ we infer $J \subseteq J'$. □

Proposition 3.8 *Let $a, b \in A$, $u \in \mathrm{Hom}(b, a)$ and $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be such that $u <_R v$.*

(i) *The map $w \mapsto u^{-1}w$ is an isomorphism of posets from the interval $[u, v]$ to the interval $[\mathrm{id}^b, u^{-1}v]$.*
(ii) *The map $w \mapsto u^{-1}w$ is an isomorphism of posets from the interval (u, v) to the interval $(\mathrm{id}^b, u^{-1}v)$.*

Proof Follow the proof of [5, Proposition 3.1.6]. This uses only basic properties of the length function which hold also for the length function of $\mathcal{W}(\mathcal{C})$. The arguments are the same for both (i) and (ii), and work also if one considers intervals which are open on one side and closed on the other. □

Let $(P, \le)$ be a poset and $U \subseteq P$ a subset. An element $z \in P$ is called the *meet* of U if

- $z \le u$ for all $u \in U$, and
- $y \le z$ for all $y \in P$ with $y \le u$ for all $u \in U$.

If it exists, the meet of U is unique and is denoted by $\bigwedge U$. The meet of two elements $x, y \in P$ is denoted by $x \wedge y$. Similarly, an element $z \in P$ is called the *join* of U if

- $u \le z$ for all $u \in U$, and
- $z \le y$ for all $y \in P$ with $u \le y$ for all $u \in U$.

If it exists, the join of U is unique and is denoted by $\bigvee U$. The join of two elements $x, y \in P$ is denoted by $x \vee y$. In the sequel, we write $\vee$ for the join and $\wedge$ for the meet in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ with respect to the weak order.

A poset is called a *meet semilattice*, if every finite non-empty subset has a meet. Finite Coxeter groups with weak order form a meet semilattice by [5, Theorem 3.2.1], but the proof uses the exchange condition which is not available in our setting (see Remark 3.11 for the case of infinite Coxeter groups and Weyl groupoids). We present for Weyl groupoids of Cartan schemes a proof which is based on Proposition 3.7. The following lemma is one step in our proof.

Lemma 3.9 *Let $a \in A$ and $u, v, w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be such that $w \le_R u$ and $w \le_R v$. If $I_L(w) \subsetneq I_L(u) \cap I_L(v)$ then there exists $w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ such that $w <_R w'$ and $w' \le_R u$, $w' \le_R v$.*

Proof We proceed by induction on the length of w. If $\ell(w) = 0$ then $w = \mathrm{id}^a$ and the claim holds with $w' = w_{I_L(u) \cap I_L(v)}$ by Lemma 3.6.

Assume now that $\ell(w) > 0$. Let $J = I_L(u) \cap I_L(v)$, and let $w_0 \in \mathrm{Hom}(\mathcal{W}_J(\mathcal{C}), a)$ be maximal with respect to weak order such that $w_0 \le_R w$. Then $\ell(w_0) > 0$ since $\ell(w) > 0$ and $I_L(w) \subseteq J$. Further, $w_0 \ne \mathrm{id}^a w_J$ since $I_L(w) \ne J$. Let $b \in A$ and $u_1, v_1, w_1 \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), b)$ be such that $w = w_0 w_1$, $u = w_0 u_1$, and $v = w_0 v_1$. Then $w_0 \le_R u$ and $w_0 \le_R v$ by transitivity of $\le_R$, and hence $w_1 \le_R u_1$, $w_1 \le_R v_1$ by Proposition 3.8. Moreover, $I_L(w_1) \cap J = \emptyset$ by the maximality of w_0, and $I_L(u_1) \cap I_L(v_1) \cap J \ne \emptyset$ since $w_0 \ne \mathrm{id}^a w_J$. Since $\ell(w_1) < \ell(w)$, induction hypothesis provides us with $w'' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), b)$ such that $w_1 <_R w''$ and $w'' \le_R u_1$, $w'' \le_R v_1$. Then the lemma holds with $w' = w_0 w''$ by Proposition 3.8. □

Theorem 3.10 *Let $a \in A$. Then* $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ *is a meet semilattice.*

Proof For all $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$, the set $\{w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a) \mid w \le_R v\}$ is finite. Hence it suffices to show that any pair of elements of $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ has a meet.

Let $u, v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. We prove by induction on the length of u that the set $\{u, v\}$ has a meet.

For all $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ with $w \le_R u$ and $w \le_R v$, it follows that $I_L(w) \subseteq I_L(u) \cap I_L(v)$. Thus if $I_L(u) \cap I_L(v) = \emptyset$, then $w = \mathrm{id}^a$, and hence $u \wedge v = \mathrm{id}^a$. This happens in particular if $\ell(u) = 0$.

Assume now that $J := I_L(u) \cap I_L(v) \ne \emptyset$, and let $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be maximal with respect to weak order such that $w_i \le_R u$ and $w_i \le_R v$ for all $i \in \{1, 2\}$. We show that $w_1 = w_2$. The maximality assumption and Lemma 3.9 imply that $I_L(w_1) = I_L(w_2) = J$. Hence $\mathrm{id}^a w_J \le_R w_i$ for all $i \in \{1, 2\}$ by Lemma 3.6. Therefore, there exist unique $b \in A$, $u', v', w'_1, w'_2 \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), b)$ such that $\mathrm{id}^a w_J \in \mathrm{Hom}(b, a)$, $w_i = \mathrm{id}^a w_J w'_i$, $u = \mathrm{id}^a w_J u'$, $v = \mathrm{id}^a w_J v'$. Proposition 3.8 implies that w'_1, w'_2 are maximal. Since $\ell(u') < \ell(u)$, induction hypothesis implies that $w'_1 = w'_2$, and hence $w_1 = w_2$. Thus the theorem is proven. □

Remark 3.11 The proof of Theorem 3.10 does not use the assumption that $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ is finite. Thus analogously to the case of Coxeter groups (see [5, Theorem 3.2.1]) in the weak order of Weyl groupoids the meet of an arbitrary subset exists, and therefore the weak order forms a complete meet semilattice.

Recall that a poset P is called a lattice if every (finite) subset of P has a join and a meet. Since $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ is a finite meet-semilattice by Theorem 3.10 and has a unique maximal element by Corollary 3.3, the following corollary holds by standard arguments from lattice theory.

Corollary 3.12 *Let $a \in A$. Then* $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ *is a lattice.*

The following result is the extension to Weyl groupoids of Theorem 3.2.7 from [5].

Theorem 3.13 *Let $a \in A$ and $u, v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be such that $u \le_R v$. Let $J = I_L(u^{-1}v)$. If $u^{-1}v \ne w_J$ then (u, v) is contractible. If $u^{-1}v = w_J$ then (u, v) is homotopy equivalent to a sphere of dimension $|J| - 2$.*

Proof By Proposition 3.8, it follows that we only need to consider the case $u = \mathrm{id}^a$. Consider the map $f : (\mathrm{id}^a, v) \to (\mathrm{id}^a, v)$ sending $w \in (\mathrm{id}^a, v)$ to $w_{I_L(w)}$.

Let $w, w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ with $w \le_R w'$. Then $I_L(w) \subseteq I_L(w')$ and hence $f(w) \le_R f(w') \le_R w'$. Hence, by Theorem 2.15, it follows that (id^a, v) and its image under f are homotopy equivalent. From Proposition 3.7, we infer that the image of $[\mathrm{id}^a, v]$ under f is as a poset isomorphic to $2^{I_L(v)}$ ordered by inclusion.

If $v = w_{I_L(v)}$ then Proposition 3.7 implies that the image of the open interval (id^a, v) under f is isomorphic to the open interval $(\emptyset, I_L(v))$, and hence by Example 2.14 homeomorphic to a $|I_L(v)| - 2$ sphere. If $v \ne w_{I_L(v)}$ then $w_{I_L(v)}$ is the unique maximal element of the image of (id^a, v) under f. In particular, the image is isomorphic to the half open interval $(\emptyset, I_L(v)]$. Since a poset with unique maximal element is contractible the rest of the assertion follows. □

Remark 3.14 For all $a\in A$ let $\tau(a)\in A$ be such that $w_I\in \mathrm{Hom}(\tau(a),a)$. Since w_I maps positive roots to negative roots, Lemma 2.6 implies that w_I^{-1} is a maximal element in $\mathrm{Hom}(a,\tau(a))$. Hence $\tau^2(a)=a$ by Corollary 3.3 and the definition of τ. Thus $\tau: A\to A$, $a\mapsto \tau(a)$, is an involution of A.

The longest element of a Weyl group induces an automorphism of the corresponding Dynkin diagram. This automorphism can be generalized to Weyl groupoids as follows. Let $a\in A$. Since $w_I\in \mathrm{Hom}(a,\tau(a))$ maps positive roots to negative roots, Axiom (R1) implies that there exists a permutation $\tau_I^a\in \mathfrak{S}_I$ such that $w_I\mathrm{id}^a(\alpha_j)=-\alpha_{\tau_I^a(j)}$.

Lemma 3.15 (i) *For all $a\in A$, the permutation τ_I^a is an involution and $\tau_I^b=\tau_I^a$ for all $b\in A$ in the connected component of a in $\mathcal{C}$.*

(ii) *For all $a\in A$ and $i\in I$, we have $w_I\sigma_i^a w_I=\sigma_{\tau_I^a(i)}^{\tau(a)}$.*

Proof The definition of τ_I^a and the formula $w_Iw_I\mathrm{id}^a=\mathrm{id}^a$ imply that $\tau_I^{\tau(a)}\tau_I^a=\mathrm{id}$ for all $a\in A$.

(ii) Let $a\in A$ and $i,j\in I$. Then $w_I\sigma_i w_I\sigma_j^{\rho_j(\tau(a))}\in \mathrm{Hom}(\rho_j(\tau(a)),\tau(\rho_i(a)))$. Assume that $\tau_I^{\tau(a)}(j)=i$, that is, $j=\tau_I^a(i)$. Then

$$
\begin{aligned}
w_I\sigma_i w_I\sigma_j^{\rho_j(\tau(a))}(\alpha_j) &= -w_I\sigma_i w_I\mathrm{id}^{\tau(a)}(\alpha_j)=w_I\sigma_i^a(\alpha_i)=-w_I\mathrm{id}^{\rho_i(a)}(\alpha_i)\\
&=\alpha_{\tau_I^{\rho_i(a)}(i)}.
\end{aligned}
\tag{3.5}
$$

Moreover, $w_I\sigma_i w_I\sigma_j^{\rho_j(\tau(a))}$ maps any positive root different from α_j to a positive root since w_I maps positive roots to negative roots and for all $l\in I$, $b\in A$ the map σ_l^b sends positive roots different from α_l to positive roots, see [13, Lemma 1]. Thus $\ell(w_I\sigma_i w_I\sigma_j^{\rho_j(\tau(a))})=0$ by Lemma 2.5, and hence $w_I\sigma_i^a w_I=\sigma_j^{\tau(a)}$.

(i) Since for all $a\in A$ the object $\tau(a)$ is in the same connected component as a, it suffices to show that for all $a\in A$ and $i\in I$ the permutations τ_I^a and $\tau_I^{\rho_i(a)}$ are equal. Let $a\in A$ and $i\in I$. By (ii), we obtain that

$$
\sigma_{\tau_I^a(i)}w_I\sigma_i^a=w_I\mathrm{id}^a,
\tag{3.6}
$$

and (3.5) gives that $\tau_I^{\rho_i(a)}(i)=j=\tau_I^a(i)$. Applying (3.6) to all α_k with $k\in I$ implies that $\tau_I^a=\tau_I^{\rho_i(a)}$. □

For all $a\in A$ define the map $t^a:\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)\to \mathrm{Hom}(\mathcal{W}(\mathcal{C}),\tau(a))$ by

$$
t^a\big(\mathrm{id}^a\sigma_{i_1}\cdots\sigma_{i_k}\big)=\mathrm{id}^{\tau(a)}\sigma_{\tau_I^a(i_1)}\cdots\sigma_{\tau_I^a(i_k)}\quad \text{for all } k\in\mathbb{N}_0,\ i_1,\dots,i_k\in I.
$$

By the following proposition, for different objects a the sets $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ may be isomorphic as posets with the weak order.

Proposition 3.16 *Let $a \in A$. Then $t^a(w) = w_I w w_I$ and $\ell(t^a(w)) = \ell(w)$ for all $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. The map t^a is an isomorphism of posets with respect to weak order.*

Proof Lemma 3.15(i) and (ii) imply that

$$\begin{aligned}\mathrm{id}^{\tau(a)} w_I \sigma_{i_1} \cdots \sigma_{i_k} w_I &= \mathrm{id}^{\tau(a)} (w_I \sigma_{i_1} w_I)(w_I \sigma_{i_2} w_I) \cdots (w_I \sigma_{i_k} w_I) \\ &= \mathrm{id}^{\tau(a)} \sigma_{\tau_I^a(i_1)} \sigma_{\tau_I^a(i_2)} \cdots \sigma_{\tau_I^a(i_k)}\end{aligned}$$

for all $a \in A$, $k \in \mathbb{N}_0$, and $i_1, \ldots, i_k \in I$. Hence t^a is well-defined and the first claim holds. Since $w_I w_I \mathrm{id}^a = \mathrm{id}^a$, we conclude that $t^{\tau(a)} t^a(w) = w$ for all $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $t^a t^{\tau(a)}(w) = w$ for all $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), \tau(a))$, and hence t^a is bijective. It is clear from the definition and bijectivity of t^a that t^a preserves length, and therefore it preserves and reflects weak order. □

A lattice P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ is called *ortho-complemented* if there is a map $\perp: P \to P$ such that (O1) $p \wedge p^\perp = \hat{0}$, (O2) $p \vee p^\perp = \hat{1}$, (O3) For all $p \in P$ we have $(p^\perp)^\perp = p$, and (O4) for all $p \preceq q$ in P we have $q^\perp \preceq p^\perp$.

Lemma 3.17 *Let $a \in A$ and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. Then the following hold*:

(i) $\ell(w) + \ell(w w_I) = \ell(w_I)$.
(ii) $I_L(w) \cap I_L(w w_I) = \emptyset$.
(iii) *For $i \in I$ we have $i \in I_L(w)$ if and only if $i \notin I_L(w w_I)$.*

Proof (i) For any $b \in A$ and $v \in \mathrm{Hom}(b, a)$ we have $\ell(v) = \#\{\alpha \in R_+^b \mid v(\alpha) \in -R_+^a\}$. Now $w_I(\alpha) \in -R_+^b$ for all $\alpha \in R_+^{\tau(b)}$. Thus for $\alpha \in R_+^b$ we have

$$w(\alpha) \in -R_+^a \quad \Leftrightarrow \quad w w_I\big(-w_I(\alpha)\big) \in R_+^a.$$

This implies that $\ell(w) + \ell(w w_I) = \ell(w_I)$.

(ii) Let $i \in I_L(w) \cap I_L(w w_I)$. Then $\ell(\sigma_i w) = \ell(w) - 1$ and $\ell(\sigma_i w w_I) = \ell(w w_I) - 1$. Hence

$$\ell(\sigma_i w) + \ell(\sigma_i w w_I) = \ell(w) - 1 + \ell(w w_I) - 1 = \ell(w_I) - 2.$$

This contradicts (i), and hence $I_L(w) \cap I_L(w w_I) = \emptyset$.

(iii) By (ii), it suffices to show that $I_L(w) \cup I_L(w w_I) = I$. Assume there is an $i \in I \setminus (I_L(w) \cup I_L(w w_I))$. Then $\ell(\sigma_i w) = \ell(w) + 1$ and $\ell(\sigma_i w w_I) = \ell(w w_I) + 1$. Analogously to (ii), we obtain that

$$\ell(\sigma_i w) + \ell(\sigma_i w w_I) = \ell(w) + 1 + \ell(w w_I) + 1 = \ell(w_I) + 2,$$

which is a contradiction to (i), and we are done. □

Theorem 3.18 *Let $a \in A$. Then the map $\perp$: $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a) \to \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ defined by $w^\perp := w w_I$ satisfies* (O1)–(O4). *Thus* $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ *with the weak order is an ortho-complemented lattice.*

Proof (O1) This follows immediately from Lemma 3.17(ii).

(O2) By Lemma 3.17(iii), we know that $I_L(w) \cup I_L(w w_I) = I$. Thus any $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ with $w \le_R v$, $w w_I \le_R v$ satisfies $w_I \le_R v$ by Lemma 3.6. Hence $w \vee w w_I = w_I$.

(O3) This follows from the definition of $\perp$ and Remark 3.14.

(O4) Let $u, v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ with $u \le_R v$. If $\ell(u) = 0$ then clearly $v^\perp \le_R u^\perp = w_I$. Now proceed by induction on $\ell(u)$. Assume that $\ell(u) \ge 1$ and let $i \in I_L(u)$. Then $i \in I_L(v)$ and we find $\bar{u}$ and $\bar{v}$ in $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), \rho_i(a))$ such that $u = \sigma_i \bar{u}$ and $v = \sigma_i \bar{v}$. Then $\bar{u} \le_R \bar{v}$. By the induction hypothesis, we obtain that $\bar{v}^\perp \le_R \bar{u}^\perp$. Since $i \notin I_L(\bar{v})$ and $i \notin I_L(\bar{u})$, it follows from Lemma 3.17(iii) and the definition of $\perp$ that $i \in I_L(\bar{v}^\perp)$ and $i \in I_L(\bar{u}^\perp)$. Hence $\sigma_i \bar{v}^\perp \le_R \sigma_i \bar{u}^\perp$. By the definition of $\perp$, this implies that $v w_I = \sigma_i \bar{v} w_I \le_R \sigma_i \bar{u} w_I = u w_I$. Hence $v^\perp \le_R u^\perp$. □

The following proposition strengthens Proposition 3.7 by showing that the embedding is indeed an embedding of lattices.

Proposition 3.19 *Let $a \in A$ and $J, J' \subseteq I$. Then $w_J \wedge w_{J'} = w_{J \cap J'}$ and $w_J \vee w_{J'} = w_{J \cup J'}$. In particular, the map $2^I \to \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$, $J \mapsto w_J$ is an embedding of lattices.*

Proof ($\wedge$) By Proposition 3.7, it follows that $w_{J \cap J'} \le_R w_J, w_{J'}$. By Theorem 3.10, there is a meet $w := w_J \wedge w_{J'}$ and hence $w_{J \cap J'} \le_R w$. Let $b \in A$ be such that $w \in \mathrm{Hom}(b, a)$. From $w \le_R w_J$ and $w \le_R w_{J'}$, we deduce that there are $u, u' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), b)$ such that $w_J = wu$, $w_{J'} = wu'$ and $\ell(w_J) = \ell(w) + \ell(u)$, $\ell(w_{J'}) = \ell(w) + \ell(u')$. From $w_{J \cap J'} \le_R w$, we deduce that there is $v \in \mathcal{W}(\mathcal{C})$ such that $w = w_{J \cap J'} v$ and $\ell(w) = \ell(w_{J \cap J'}) + \ell(v)$. Since $w_{J \cap J'} v u = w_J$ and $w_{J \cap J'} v u' = w_{J'}$, it follows that $I_L(v) \subseteq J \cap J'$. However, by the fact that $w_{J \cap J'}$ is the longest word in $J \cap J'$ and $\ell(w_{J \cap J'}) + \ell(v) = \ell(w_{J \cap J'} v)$, it follows that $v = \mathrm{id}^a$ and hence $w = w_{J \cap J'}$.

($\vee$) By Proposition 3.7, it follows that $w_J, w_{J'} \le_R w_{J \cup J'}$. Let now $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be such that $w_J, w_{J'} \le_R w$. We have to show that $w_{J \cup J'} \le_R w$. By Proposition 3.7, with $w = w_J$ we conclude that $I_L(w_J) = J$, and similarly $I_L(w_{J'}) = J'$. Thus $J \cup J' = I_L(w_J) \cup I_L(w_{J'}) \subseteq I_L(w)$. Lemma 3.6 and Proposition 3.7 imply that $w_{J \cup J'} \le_R w_{I_L(w)} \le_R w$, and we are done. □

The following is an immediate consequence of Proposition 3.19.

Corollary 3.20 *Let $a \in A$. Then for all $J \subseteq I$ we have*

$$\bigvee_{i \in J} \mathrm{id}^a \sigma_i = \mathrm{id}^a w_J.$$

In particular, for all $w \in \mathcal{W}(\mathcal{C})$ we have

$$\bigvee_{i \in I_L(w)} \mathrm{id}^a \sigma_i = \mathrm{id}^a w_{I_L(w)}.$$

Next we present a formula about the factors appearing in a reduced decomposition of the meet of two morphisms.

Theorem 3.21 *Let $a \in A$ and $u, v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. Then*

$$J(u) \cup J(v) = J(u \wedge v) \cup J(u^{-1}v).$$

Proof Since $u \wedge v \le_R u$ and $u \wedge v \le_R v$, it follows that $J(u \wedge v) \subseteq J(u) \cap J(v)$. Moreover, $J(u^{-1}v) \subseteq J(u) \cup J(v)$, and hence the inclusion $\supseteq$ in the theorem holds.

Now we prove the inclusion $\subseteq$ by induction on $\ell(u) + \ell(v)$. If $\ell(u) = \ell(v) = 0$ then the claim clearly holds. Assume now that $\ell(u) + \ell(v) > 0$.

Case 1. $u \wedge v \ne \mathrm{id}^a$. Then there exists $i \in I_L(u) \cap I_L(v)$. Let $u_0, v_0 \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), \rho_i(a))$ be such that $u = \sigma_i u_0$, $v = \sigma_i v_0$. Then

$$J(u) = J(u_0) \cup \{i\}, \qquad J(v) = J(v_0) \cup \{i\},$$
$$J(u \wedge v) = J\big(\sigma_i(u_0 \wedge v_0)\big) = J(u_0 \wedge v_0) \cup \{i\},$$

and $u^{-1}v = u_0^{-1}v_0$. Thus the claim follows from the induction hypothesis.

Case 2. $u \wedge v = \mathrm{id}^a$*,* $J(u) \not\subseteq J(v)$. By Proposition 2.12, there exist unique elements $u^J \in \mathcal{W}^{J(v)}(\mathcal{C})$, $u_J \in \mathcal{W}_{J(v)}(\mathcal{C})$ such that $u^{-1} = u^J u_J$. Then $u = u_J^{-1}(u^J)^{-1}$ and $\ell(u^J) + \ell(u_J) = \ell(u)$ and hence $J(u^J) \cup J(u_J) = J(u)$. We have $\ell(u_J) < \ell(u)$ since $u \notin \mathcal{W}_{J(v)}(\mathcal{C})$. Further,

$$u_J^{-1} \wedge v = \mathrm{id}^a \tag{3.7}$$

since $u \wedge v = \mathrm{id}^a$. Thus

$$\begin{aligned} J(u) \cup J(v) &= J\big(u^J\big) \cup J\big(u_J^{-1}\big) \cup J(v) = J\big(u^J\big) \cup \big(J\big(u_J^{-1} \wedge v\big) \cup J(u_J v)\big) \\ &= J\big(u^J\big) \cup J(u_J v) = J\big(u^J u_J v\big) = J\big(u^{-1}v\big). \end{aligned}$$

Here the second equation holds by induction hypothesis and the third by (3.7). The fourth equation follows from $u_J v \in \mathcal{W}_{J(v)}(\mathcal{C})$, $u^J \in \mathcal{W}^{J(v)}(\mathcal{C})$ and Proposition 2.12(ii).

Case 3. $u \wedge v = \mathrm{id}^a$*,* $J(v) \not\subseteq J(u)$. Replace u and v and apply Case 2.

Case 4. $u \wedge v = \mathrm{id}^a$*,* $J(u) = J(v)$. Let $J = J(u^{-1}v)$. We have to show that $J(u) \subseteq J$. By Corollary 2.13, there exists a unique minimal element $w \in u\mathcal{W}_J(\mathcal{C})$.

Since $v = u(u^{-1}v)$ and $u^{-1}v \in \mathcal{W}_J(\mathcal{C})$, there exist $u_1, v_1 \in \mathcal{W}_J(\mathcal{C})$ such that

$$u = wu_1, \qquad v = wv_1, \qquad \ell(u) = \ell(w) + \ell(u_1), \qquad \ell(v) = \ell(w) + \ell(v_1).$$

Therefore, $w \le u \wedge v = \mathrm{id}^a$, and hence $u \in w\mathcal{W}_J(\mathcal{C}) = \mathcal{W}_J(\mathcal{C})$. Thus $J(u) \subseteq J$. $\square$

4 Coxeter complex

In this section, we study the cell decomposition of the unit sphere induced by the set of hyperplanes associated to a Weyl groupoid. It is shown that this decomposition is indeed a triangulation and that the underlying abstract simplicial complex can be defined purely algebraically in terms of cosets of parabolic subgroupoids. This complex is called Coxeter complex. Finally, we note in Theorem 4.9 that any linear extension of any of the weak orders of the Weyl groupoid induces a shelling order on the simplicial complex.

Throughout this section, let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and let $a \in A$. Assume that $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$ is a finite root system of type $\mathcal{C}$.

Definition 4.1 Let

$$\Omega^a_{\mathcal{C}} := \bigl\{ w\mathcal{W}_J(\mathcal{C}) \mid w \in \mathrm{Hom}\bigl(\mathcal{W}(\mathcal{C}), a\bigr), J \subseteq I, |J| = |I| - 1 \bigr\}.$$

We call the subset $\Delta^a_{\mathcal{C}}$ of the powerset $2^{\Omega^a_{\mathcal{C}}}$ whose elements are the non-empty subsets $F \subseteq \Omega^a_{\mathcal{C}}$ such that

$$\bigcap_{w\mathcal{W}_J(\mathcal{C}) \in F} w\mathcal{W}_J(\mathcal{C}) \neq \emptyset$$

the *Coxeter complex* of $\mathcal{C}$ at a.

By definition, the Coxeter complex $\Delta^a_{\mathcal{C}}$ is a simplicial complex. If $\mathcal{C}$ is a Cartan scheme with only one object a, then $\Delta^a_{\mathcal{C}}$ is just the Coxeter complex of the crystallographic Coxeter group $\mathcal{W}(\mathcal{C})$ as defined in [14, Sect. 1.15]. Note that for technical reasons our simplicial complexes do not contain the empty set. Our goal in this section will be to give a second construction of the Coxeter complex. This way we obtain additional information on the structure of faces.

Lemma 4.2 *Let $J, K \subseteq I$ and $u, v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ be such that $u\mathcal{W}_J(\mathcal{C}) \cap v\mathcal{W}_K(\mathcal{C}) \neq \emptyset$. Then*

$$u\mathcal{W}_J(\mathcal{C}) \cap v\mathcal{W}_K(\mathcal{C}) = w\mathcal{W}_{J\cap K}(\mathcal{C}) \tag{4.1}$$

for some $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. In particular, if $J \subseteq K$ then $w\mathcal{W}_J(\mathcal{C}) = u\mathcal{W}_J(\mathcal{C})$ and if $J = K$ then $w\mathcal{W}_J(\mathcal{C}) = u\mathcal{W}_J(\mathcal{C}) = v\mathcal{W}_J(\mathcal{C})$.

Proof Assume first that $v=\mathrm{id}^a$. By Proposition 2.12, there exist $u_0\in\mathcal{W}^J(\mathcal{C})$ and $u_1\in\mathcal{W}_J(\mathcal{C})$ such that $u=u_0u_1$. Then

$$u\mathcal{W}_J(\mathcal{C})=u_0\mathcal{W}_J(\mathcal{C})$$

and $J(u_0)\subseteq J(w)$ for all $w\in u\mathcal{W}_J(\mathcal{C})$ by Corollary 2.13. Hence $J(u_0)\subseteq J(w)\subseteq K$ for all $w\in u\mathcal{W}_J(\mathcal{C})\cap v\mathcal{W}_K(\mathcal{C})$ which is non-empty by assumption. Thus

$$\begin{aligned}u\mathcal{W}_J(\mathcal{C})\cap v\mathcal{W}_K(\mathcal{C})&=u_0\bigl(\mathcal{W}_J(\mathcal{C})\cap u_0^{-1}\mathcal{W}_K(\mathcal{C})\bigr)=u_0\bigl(\mathcal{W}_J(\mathcal{C})\cap\mathcal{W}_K(\mathcal{C})\bigr)\\&=u_0\mathcal{W}_{J\cap K}(\mathcal{C}).\end{aligned}$$

Let now $v\in\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)$ be an arbitrary element. Then

$$u\mathcal{W}_J(\mathcal{C})\cap v\mathcal{W}_K(\mathcal{C})=v\bigl(v^{-1}u\mathcal{W}_J(\mathcal{C})\cap\mathcal{W}_K(\mathcal{C})\bigr)=vw_0\mathcal{W}_{J\cap K}(\mathcal{C})$$

for some $w_0\in\mathrm{Hom}(\mathcal{W}(\mathcal{C}),b)$, where $b\in A$ with $v\in\mathrm{Hom}(b,a)$, by the first part of the proof. This implies the claim. □

In [14, Sect. 1.15], the Coxeter complex of a reflection group was defined by means of hyperplanes in a Euclidean space. We introduce an analogous complex for the pair $(\mathcal{C},a)$. We show that the complex defined this way is isomorphic to the Coxeter complex $\Delta^a_{\mathcal{C}}$.

Let $(\cdot,\cdot)$ be a scalar product on $\mathbb{R}^I$. For any subset $J\subseteq I$ and any $w\in\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)$, let

$$F^w_J=\bigl\{\lambda\in\mathbb{R}^I\mid\bigl(\lambda,w(\alpha_j)\bigr)=0\text{ for all }j\in J,\bigl(\lambda,w(\alpha_i)\bigr)>0\text{ for all }i\in I\setminus J\bigr\}.$$

The subsets F^w_J are intersections of hyperplanes and of open half-spaces, and are called *faces*. For brevity, we will omit their dependence on the scalar product. By construction, the faces do not depend on connected components of $\mathcal{C}$ not containing a. Also, up to the choice of a scalar product the set of faces F^w_J does not change when passing from an object a to an object a' from a covering Cartan scheme once a' lies in the connected component covering the connected component of a.

The next lemma is the analog of [14, Lemma 1.12].

Lemma 4.3 *Let $(\cdot,\cdot)$ be a scalar product on $\mathbb{R}^I$.*

(i) *For all $\lambda\in\mathbb{R}^I$ there exist $w\in\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)$ and $J\subseteq I$ such that $\lambda\in F^w_J$.*
(ii) *Let $w_1,w_2\in\mathrm{Hom}(\mathcal{W}(\mathcal{C}),a)$ and let $J_1,J_2\subseteq I$. If $w_1\mathcal{W}_{J_1}(\mathcal{C})=w_2\mathcal{W}_{J_2}(\mathcal{C})$ then $F^{w_1}_{J_1}=F^{w_2}_{J_2}$. If $w_1\mathcal{W}_{J_1}(\mathcal{C})\neq w_2\mathcal{W}_{J_2}(\mathcal{C})$ then $F^{w_1}_{J_1}\cap F^{w_2}_{J_2}=\emptyset$.*

Proof (i) Let $k=|\{\beta\in R^a_+\mid(\lambda,\beta)<0\}|$. We proceed by induction on k. If $k=0$ then the claim holds with $w=\mathrm{id}^a$.

Assume that $k>0$. Then there exists $i\in I$ such that $(\lambda,\alpha_i)<0$. Let $\lambda'=\sigma^a_i(\lambda)$ and define a scalar product $(\cdot,\cdot)'$ on $\mathbb{R}^I$ by $(\mu,\nu)'=(\sigma_i^{\rho_i(a)}(\mu),\sigma_i^{\rho_i(a)}(\nu))$

for all $\mu, \nu \in \mathbb{R}^I$. Then for all $\beta \in R_+^{\rho_i(a)}$ we have $(\lambda', \beta)' < 0$ if and only if $(\lambda, \sigma_i^{\rho_i(a)}(\beta)) < 0$. Moreover,

$$(\lambda', \alpha_i)' = \left(\sigma_i^{\rho_i(a)}\sigma_i^a(\lambda), \sigma_i^{\rho_i(a)}(\alpha_i)\right) = -(\lambda, \alpha_i) > 0,$$

and $\sigma_i^{\rho_i(a)}$ is a bijection between $R_+^{\rho_i(a)} \setminus \{\alpha_i\}$ and $R_+^a \setminus \{\alpha_i\}$ by (R1)–(R3). Hence

$$\left|\left\{\beta \in R_+^{\rho_i(a)} \mid (\lambda', \beta)' < 0\right\}\right| = k - 1.$$

By induction hypothesis, there exist $J \subseteq I$ and $w' \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), \rho_i(a))$ such that $\lambda' \in F_J^{w'}$. Then $\lambda \in \sigma_i^{\rho_i(a)} F_J^{w'} = F_J^w$, where $w = \sigma_i^{\rho_i(a)} w'$.

(ii) Suppose that $w_1\mathcal{W}_{J_1}(\mathcal{C}) = w_2\mathcal{W}_{J_2}(\mathcal{C})$. Then $J_1 = J_2$ and $w_2 = w_1 x$ for some $x \in \mathcal{W}_{J_1}(\mathcal{C})$. Therefore,

$$\left(\lambda, w_2(\alpha_i)\right) = \left(\lambda, w_1 x(\alpha_i)\right) = \left(\lambda, w_1\left(\alpha_i + \sum_{j\in J_1} a_{ji}\alpha_j\right)\right) = \left(\lambda, w_1(\alpha_i)\right)$$

for all $\lambda \in F_{J_1}^{w_1}$ and all $i \in I$, where $x(\alpha_i) = \alpha_i + \sum_{j\in J_1} a_{ji}\alpha_j$ for some $a_{ji} \in \mathbb{Z}$ for all $j \in J_1$. We conclude that $F_{J_1}^{w_1} \subseteq F_{J_2}^{w_2}$, and similarly $F_{J_2}^{w_2} \subseteq F_{J_1}^{w_1}$. This proves the first claim.

The converse will be proven indirectly. Assume that $w_1\mathcal{W}_{J_1}(\mathcal{C}) \neq w_2\mathcal{W}_{J_2}(\mathcal{C})$ and that there exists $\lambda \in F_{J_1}^{w_1} \cap F_{J_2}^{w_2}$. Let $b_1, b_2 \in A$ be such that $w_1 \in \mathrm{Hom}(b_1, a)$ and $w_2 \in \mathrm{Hom}(b_2, a)$. Let $x = w_1^{-1}w_2 \in \mathrm{Hom}(b_2, b_1)$. By the choice of λ and the definition of x, we have $(\lambda, w_1(\alpha_j)) \geq 0$ and $(\lambda, w_1 x(\alpha_j)) \geq 0$ for all $j \in I$. Moreover, equality holds if and only if $j \in J_1$, respectively $j \in J_2$. Since $x(\alpha_j) \in R_+^{b_1} \cup -R_+^{b_1}$ for all $j \in I$, we conclude that $x(\alpha_j) \in \sum_{k\in J_1} \mathbb{Z}\alpha_k$ for all $j \in J_2$ and that $x(\alpha_j) \in R_+^{b_1} \setminus \sum_{k\in J_1} \mathbb{Z}\alpha_k$ for all $j \in I \setminus J_2$. Hence $J(x) \subseteq J_1$ by Lemma 2.9. It follows that

$$w_2 \in w_1\mathcal{W}_{J_1}(\mathcal{C}). \tag{4.2}$$

By the first part of the proof, we obtain that $F_{J_2}^{w_2} = F_{J_2}^{w_2'}$ for all $w_2' \in w_2\mathcal{W}_{J_2}(\mathcal{C})$. Hence $J(xx') \subseteq J_1$ for all $x' \in \mathcal{W}_{J_2}(\mathcal{C})$, and therefore $J_2 \subseteq J_1$. Symmetry yields that $J_1 = J_2$. Thus $w_1\mathcal{W}_{J_1}(\mathcal{C}) = w_2\mathcal{W}_{J_2}(\mathcal{C})$ by (4.2), a contradiction. Hence $F_{J_1}^{w_1} \cap F_{J_2}^{w_2} = \emptyset$. □

By definition, for any $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $J \subseteq I$ the face F_J^w is a relative open polyhedral cone in $\mathbb{R}^I$. In particular, it is a relative open cell. By Lemma 4.3, the set of all F_J^w stratifies $\mathbb{R}^I$. Clearly, this stratification depends on the choice of $\mathcal{C}$, a, and the scalar product on $\mathbb{R}^I$. In order to show that the stratification indeed gives a regular CW-decomposition of $\mathbb{R}^I$, we have to clarify the structure of the closures of the cells.

Theorem 4.4 *Let $K \subseteq I$ and let $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. Then $\overline{F_K^w}$ is the disjoint union of the faces F_J^w for $J \supseteq K$. Moreover, for $v \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $J \subseteq I$ we have $F_J^v \subseteq \overline{F_K^w}$ if and only if $v\mathcal{W}_J(\mathcal{C}) \supseteq w\mathcal{W}_K(\mathcal{C})$.*

Proof The first assertion follows from the definition of F_K^w.

By Lemma 4.3, the space $\mathbb{R}^I$ is the disjoint union of faces, and hence $F_J^v \subseteq \overline{F_K^w}$ if and only if $F_J^v = F_L^w$ for some $L \supseteq K$. Lemma 4.3(ii) implies that the latter is equivalent to $v\mathcal{W}_J(\mathcal{C}) = w\mathcal{W}_L(\mathcal{C})$. Clearly, if $v\mathcal{W}_J(\mathcal{C}) = w\mathcal{W}_L(\mathcal{C})$ for some $L \supseteq K$ then $v\mathcal{W}_J(\mathcal{C}) \supseteq w\mathcal{W}_K(\mathcal{C})$. Conversely, if $v\mathcal{W}_J(\mathcal{C}) \supseteq w\mathcal{W}_K(\mathcal{C})$ then $v^{-1}w\mathcal{W}_K(\mathcal{C}) \subseteq \mathcal{W}_J(\mathcal{C})$, and hence $v^{-1}w \in \mathcal{W}_J(\mathcal{C})$ and $K \subseteq J$. Thus $v\mathcal{W}_J(\mathcal{C}) = w\mathcal{W}_J(\mathcal{C})$ and the theorem is proven. □

Corollary 4.5 *The cells F_K^w for $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $K \subseteq I$ define a regular CW-decomposition of $\mathbb{R}^I$.*

Proof From the fact that any root system contains a basis, it follows that $F_I^w = \{0\}$. Hence it follows from Theorem 4.4 and the fact that all F_J^w are relative open polyhedral cones in $\mathbb{R}^I$ that $\dim F_J^w = \#I - \#J$. Since by Lemma 4.3 the cells F_J^w are a stratification of $\mathbb{R}^I$, they actually define a regular CW-decomposition of $\mathbb{R}^I$. □

Now we define the regular CW-complex $\mathcal{K}_\mathcal{C}^a$ as the regular CW-complex whose cells are the intersections $F_J^w \cap S^{\#I-1}$ of the relative open cones F_J^w with the unit sphere in $\mathbb{R}^I$ for $J \subseteq I$, $J \neq I$. From Corollary 4.5 and the fact that all F_J^w are relative open cones with apex in the origin, it follows that $\mathcal{K}_\mathcal{C}^a$ is a regular CW-decomposition of $S^{\#I-1}$.

Corollary 4.6 *The Coxeter complex $\Delta_\mathcal{C}^a$ at $a \in A$ is isomorphic to the complex $\mathcal{K}_\mathcal{C}^a$.*

Proof Since by Corollary 4.6 the complex $\mathcal{K}_\mathcal{C}^a$ is a regular CW-complex and $\Delta_\mathcal{C}^a$ is by definition a regular CW-complex, it suffices to show that there is an inclusion preserving bijection between the faces of $\mathcal{K}_\mathcal{C}^a$ and $\Delta_\mathcal{C}^a$.

Definition 4.1 and Lemma 4.2 imply that the faces of the Coxeter complex are in bijection with the left cosets $w\mathcal{W}_J(\mathcal{C})$, where $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ and $J \subsetneq I$. By Lemma 4.3(ii), the faces of $\mathcal{K}_\mathcal{C}^a$ are also in bijection with these left cosets. Hence it remains to show that in both complexes the inclusion of closures of faces corresponds to the inclusion of left cosets. For the Coxeter complex, this holds by definition. For the complex $\mathcal{K}_\mathcal{C}^a$, the claim follows from Theorem 4.4. □

Let $\mathcal{A}_\mathcal{C}^a$ be the set of hyperplanes $H_\alpha = \{\lambda \in \mathbb{R}^I \mid (\lambda, \alpha) = 0\}$ for $\alpha \in (R^{\mathrm{re}})_+^a$. Then the complement $\mathbb{R}^I \setminus \bigcup_{H \in \mathcal{A}_\mathcal{C}^a} H$ of the arrangement of hyperplanes $\mathcal{A}_\mathcal{C}^a$ is the disjoint union of connected components which are in bijection with the maximal faces of $\mathcal{K}_\mathcal{C}^a$. It follows by Corollary 4.6 that $\mathcal{K}_\mathcal{C}^a$ and $\Delta_\mathcal{C}^a$ are isomorphic. Since $\Delta_\mathcal{C}^a$ is a simplicial complex, it follows that all connected components of $\mathbb{R}^I \setminus \bigcup_{H \in \mathcal{A}_\mathcal{C}^a} H$ are open simplicial cones. In general, an arrangement satisfying this property is called *simplicial arrangement*.

Corollary 4.7 *The arrangement of hyperplanes $\mathcal{A}_\mathcal{C}^a$ is a simplicial arrangement.*

From the fact that by Corollary 4.6 the Coxeter complex $\Delta_\mathcal{C}^a$ is a triangulation of a sphere, the next corollary follows immediately.

Corollary 4.8 *The simplicial complex $\Delta^a_{\mathcal{C}}$ is pure of dimension $|I|-1$ and each codimension 1 face of $\Delta^a_{\mathcal{C}}$ is contained in exactly two faces of maximal dimension. In particular, $\Delta^a_{\mathcal{C}}$ is a pseudomanifold.*

Using Theorem 4.4, one can identify the maximal simplices of $\Delta^a_{\mathcal{C}}$ with the elements of $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. Hence any linear extension of the weak order on $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$ defines a linear order on the maximal simplices of $\Delta^a_{\mathcal{C}}$. Indeed, it can be shown by the same proof as for the analogous statement for Coxeter groups [3, Theorem 2.1] that any linear extension of the weak order defines a shelling order for $\Delta^a_{\mathcal{C}}$. The crucial facts about Coxeter groups used by Björner are verified for Weyl groupoids in Lemma 4.2 and Theorem 4.4.

Theorem 4.9 *Let $\preceq$ be any linear extension of the weak order $\le_R$ on $\mathrm{Hom}(\mathcal{W}(\mathcal{C}), a)$. Then $\preceq$ is a shelling order for $\Delta^a_{\mathcal{C}}$.*

We omit the detailed verification of Theorem 4.9 here since the main topological consequence Corollary 4.6 is already known. Indeed, Corollary 4.8 together with Theorems 4.9 and 2.16 imply that $\Delta^a_{\mathcal{C}}$ is a triangulation of a PL-sphere.

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