J Algebr Comb (2011) 33: 325–348
DOI 10.1007/s10801-010-0246-4

Minimal half-spaces and external representation of tropical polyhedra

Stéphane Gaubert · Ricardo D. Katz

Received: 11 August 2009 / Accepted: 1 July 2010 / Published online: 30 July 2010

Abstract We give a characterization of the minimal tropical half-spaces containing a given tropical polyhedron, from which we derive a counter-example showing that the number of such minimal half-spaces can be infinite, contradicting some statements which appeared in the tropical literature, and disproving a conjecture of F. Block and J. Yu. We also establish an analogue of the Minkowski–Weyl theorem, showing that a tropical polyhedron can be equivalently represented internally (in terms of extreme points and rays) or externally (in terms of half-spaces containing it). A canonical external representation of a polyhedron turns out to be provided by the extreme elements of its tropical polar. We characterize these extreme elements, showing in particular that they are determined by support vectors.

Keywords Max-plus semiring · Max-plus convexity · Tropical convexity · Polyhedra · Polytopes · Minkowski–Weyl theorem · Supporting half-spaces

1 Introduction

Max-plus or tropical convexity has been developed by several researchers under different names, with various motivations. It goes back at least to the work of Zim-

The first author was partially supported by the Arpege programme of the French National Agency of Research (ANR), project "ASOPT", number ANR-08-SEGI-005 and by the Digiteo project DIM08 "PASO" number 3389.

S. Gaubert (✉)
INRIA and Centre de Mathématiques Appliquées (CMAP), École Polytechnique, 91128 Palaiseau Cedex, France
e-mail: Stephane.Gaubert@inria.fr

R.D. Katz (✉)
CONICET, Instituto de Matemática "Beppo Levi", Universidad Nacional de Rosario, Avenida Pellegrini 250, 2000 Rosario, Argentina
e-mail: rkatz@fceia.unr.edu.ar

mermann [35]. It was studied by Litvinov, Maslov, and Shpiz [29], in relation to problems of calculus of variations, and by Cohen, Gaubert, and Quadrat [12, 13], motivated by discrete event system problems (max-plus polyhedra represent invariant spaces of max-plus linear dynamical systems [11]). Some of this work was pursued with Singer (see [14]), with motivations from generalized convexity [32]. The work of Briec and Horvath [7] is also in the setting of generalized convexity. Develin and Sturmfels [16] pointed out some remarkable relations with tropical geometry, and developed a new approach, thinking of tropical polyhedra as polyhedral complexes in the usual sense. This was the starting point of several works of the same authors, of Joswig [26] and of Block and Yu [6]. Some of the previously mentioned researchers, and some other ones, including Allamigeon, Butkovič, Goubault, Katz, Nitica, Meunier, Sergeev, Schneider, have recently made a number of works in the field, we refer the reader to [3, 5, 10, 19–22, 27, 28, 30] for a representative set of contributions.

A closed convex set can be represented classically in two different ways, either internally, in terms of extreme elements (extreme points and rays) and lineality space, or externally, as the intersection of (closed) half-spaces.

The max-plus or tropical analogue of the external representation, esspecially in the case of polyhedra, is the main object of this paper.

The existence of an external representation relies on separation arguments. In the max-plus setting, several separations theorems have been obtained, with various degrees of generality and precision, by Zimmermann [35], by Samborskiĭ and Shpiz [31], by Cohen, Gaubert, and Quadrat [12, 13] with a further refinement in a work with Singer [14], and by Develin and Sturmfels [16]. In particular, the results of [12–14] yield a simple geometric construction of the separating half-space, showing the analogy with the Hilbert space case. This geometric approach was extended to the case of the separation of several convex sets by Gaubert and Sergeev [24], using a cyclic projection method. Briec and Horvath derived a separation theorem for two convex sets using a different approach [8].

The existence of an internal representation relies on Krein–Milman type theorems. Results of this kind were established by Butkovič, Schneider, and Sergeev [10] and by the authors [20], who also studied in [21] the analogue of the polar of a convex set, which consists of the set of inequalities satisfied by its elements.

Polyhedra are usually defined by the condition that they have a finite external or internal representation, the equivalence of both conditions being the classical Minkowski–Weyl theorem.

In the max-plus setting, a first result of this nature was established by Gaubert in [18, Chap. III, Theorem 1.2.2], who showed that a finitely generated max-plus cone can be characterized by finitely many max-plus linear inequalities. One element of the proof is an argument showing that the set of solutions of a system of max-plus linear equations is finitely generated, an observation which was already made by Butkovič and Hegedus [9]. Some accounts in English of the result of [18] appeared in [2, 19, 23].

To address the same issue, Joswig [26] introduced the very interesting notion of minimal half-spaces (tropical half-spaces that are minimal for inclusion among the ones containing a given tropical polytope), he stated that the apex of such a tropical half-space is a vertex of the classical polyhedral complex arising from the polytope, and deduced that there are only finitely many such minimal half-spaces. This statement was refined by a conjecture of Block and Yu, characterizing the minimal half-spaces, in the generic case [6, Conj. 14].

The finiteness of the number of minimal half-spaces is an appealing property, which is geometrically quite obvious in dimension 2. It came to us as a surprise that it does not hold in higher dimensions. We give here a counter-example (Example 2 below), contradicting the finiteness of the number of minimal half-spaces containing a tropical polyhedron (Corollary 3.4 of [26]) and disproving Conjecture 14 of [6]. The analysis of the present counter-example is based on a general characterization of the minimal half-spaces, Theorem 4 below, the main result of this paper, which gives some answer to the question at the origin of the conjecture of Block and Yu.

In a preliminary section, we establish an analogue of the Minkowski–Weyl theorem (Theorem 2), showing that a tropical polyhedron can be equivalently described either as the sum of the convex hull of finitely many points and of the cone generated by finitely many vectors, or as the intersection of finitely many half-spaces (there is no tropical analogue of the lineality space). The proof is based on the idea of [18] (Theorem 1 below), which is combined with the results of [20].

In the final section, we characterize the extreme elements of the polar of a tropical polyhedral cone. The set of extreme elements of this polar has the property that any inequality satisfied by all the elements of the cone is a max-plus linear combination of the inequalities represented by these extreme elements, and it is the unique minimal set with this property (up to a scaling). In particular, these extreme elements provide a finite representation of the original polyhedron as the intersection of half-spaces. Theorem 5 below characterizes these extreme elements, showing in particular that each of them is determined by support vectors.

2 The tropical Minkowski–Weyl theorem

Let us first recall some basic definitions. The max-plus semiring, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $(a, b) \mapsto \max(a, b)$ and the multiplication $(a, b) \mapsto a + b$. To emphasize the semiring structure, we write $a \oplus b := \max(a, b)$, $ab := a + b$, $\mathbb{0} := -\infty$ and $\mathbb{1} := 0$. The term "tropical" is now used essentially as a synonym of max-plus. The semiring operations are extended in the natural way to matrices over the max-plus semiring: $(A \oplus B)_{ij} := A_{ij} \oplus B_{ij}$, $(AB)_{ij} := \bigoplus_k A_{ik}B_{kj}$ and $(\lambda A)_{ij} := \lambda A_{ij}$ for all i, j, where A, B are matrices of compatible sizes and $\lambda \in \mathbb{R}_{\max}$. We denote by $\mathrm{e}^k \in \mathbb{R}^n_{\max}$ the k-th unit vector, i.e. the vector defined by: $(\mathrm{e}^k)_k := \mathbb{1}$ and $(\mathrm{e}^k)_h := \mathbb{0}$ if $h \neq k$.

We consider $\mathbb{R}_{\max}$ equipped with the usual topology (resp. order), which can be defined by the metric: $d(a, b) := |\exp(a) - \exp(b)|$. The set $\mathbb{R}^n_{\max}$ is equipped with the product topology (resp. order). Note that the semiring operations are continuous with respect to this topology.

A subset $\mathscr{V}$ of $\mathbb{R}^n_{\max}$ is said to be a *max-plus or tropical cone* if it is stable by max-plus linear combinations, meaning that

$$\lambda u \oplus \mu v \in \mathscr{V} \tag{1}$$

for all $u, v \in \mathscr{V}$ and $\lambda, \mu \in \mathbb{R}_{\max}$. Note that, in the max-plus setting, positivity constraints are implicit because any scalar $\lambda \in \mathbb{R}_{\max}$ satisfies $\lambda \geq \mathbb{0}$. As a consequence, max-plus cones turn out to share many properties with classical convex cones. This analogy leads us to define *max-plus convex subsets* $\mathscr{C}$ of $\mathbb{R}^n_{\max}$ by requiring them to be stable by max-plus convex combinations, meaning that $\lambda u \oplus \mu v \in \mathscr{C}$ holds for all $u, v \in \mathscr{C}$ and $\lambda, \mu \in \mathbb{R}_{\max}$ such that $\lambda \oplus \mu = \mathbb{1}$. We denote by $\operatorname{cone}(\mathscr{X})$ the smallest cone containing a subset $\mathscr{X}$ of $\mathbb{R}^n_{\max}$, and by $\operatorname{co}(\mathscr{X})$ the smallest convex set containing it. Therefore, $\operatorname{cone}(\mathscr{X})$ (resp. $\operatorname{co}(\mathscr{X})$) is the set of all max-plus linear (resp. convex) combinations of finitely many elements of $\mathscr{X}$. A cone $\mathscr{V}$ is said to be finitely generated if there exists a finite set $\mathscr{X}$ such that $\mathscr{V} = \operatorname{cone}(\mathscr{X})$, which equivalently means that $\mathscr{V} = \{Cw \mid w \in \mathbb{R}^t_{\max}\}$ for some matrix $C \in \mathbb{R}^{n\times t}_{\max}$.

A *half-space* of $\mathbb{R}^n_{\max}$ is a set of the form

$$\mathscr{H} = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{1\le i\le n} a_i x_i \le \bigoplus_{1\le j\le n} b_j x_j \right\},$$

where $a, b \in \mathbb{R}^n_{\max}$, and an *affine half-space* of $\mathbb{R}^n_{\max}$ is a set of the form

$$\mathscr{H} = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \left(\bigoplus_{1\le i\le n} a_i x_i \right) \oplus c \le \left(\bigoplus_{1\le j\le n} b_j x_j \right) \oplus d \right\},$$

where $a, b \in \mathbb{R}^n_{\max}$ and $c, d \in \mathbb{R}_{\max}$. With the classical notation, the latter set can be written as

$$\mathscr{H} = \left\{ x \in \mathbb{R}^n_{\max} \mid \max\Bigl(\max_{1\le i\le n} a_i + x_i, c \Bigr) \le \max\Bigl(\max_{1\le j\le n} b_j + x_j, d \Bigr) \right\}.$$

Note that half-spaces are max-plus cones.

Classical polyhedra can be defined either as a finite intersection of affine half-spaces, or in terms of finite sets of vertices and rays, i.e. as the Minkowski sum of a polytope and a finitely generated cone. Here we adopt the first approach and define a *max-plus or tropical polyhedron* as the intersection of finitely many affine half-spaces. We warn the reader that our notion of polyhedra is more general than the one used in [16] (the latter reference deals with max-plus cones having a finite generating family consisting of vectors with finite entries).

The following "conic" form of the Minkowski–Weyl theorem is equivalent to a result established in [18], showing that a finitely generated max-plus cone is characterized by finitely many max-plus linear equalities. This result was reproduced (but without its proof) in the surveys [2, 19, 23]. For the convenience of the reader, we include the proof here. The "if" part is equivalent to the existence of a finite set of generators of a system of max-plus linear equations, which was first shown in [9]. There has recently been progress on these issues, leading to a faster algorithm, see [3, 4].

Theorem 1 (Compare with [18, Chap. III, Theorem 1.2.2] and [23, Theorem 9]) *A max-plus cone is finitely generated if, and only if, it is the intersection of finitely many half-spaces.*

Proof Let $\mathscr{V} \subset \mathbb{R}^n_{\max}$ be an intersection of p half-spaces. We next prove that $\mathscr{V}$ is finitely generated by induction on p.

When $p=1$, as $\mathscr{V}=\{x\in\mathbb{R}^n_{\max}\mid\bigoplus_{1\le i\le n}a_ix_i\le\bigoplus_{1\le j\le n}b_jx_j\}=\bigcup_{1\le j\le n}\mathscr{V}_j$, where

$$\mathscr{V}_j:=\left\{x\in\mathbb{R}^n_{\max}\;\middle|\;a_ix_i\le b_jx_j,\ \forall i=1,\dots,n\right\},$$

to prove that $\mathscr{V}$ is finitely generated it suffices to show that the cones $\mathscr{V}_j$ are all finitely generated. If $b_j\neq\mathbb{0}$ and $a_j\le b_j$, then it can be checked that $\mathscr{V}_j=\operatorname{cone}(\mathscr{X}_j)$, where $\mathscr{X}_j:=\{b_j\mathrm{e}^i\oplus a_i\mathrm{e}^j\mid i=1,\dots,n\}$. If $b_j=\mathbb{0}$ or $a_j>b_j$, then $\mathscr{V}_j=\operatorname{cone}(\mathscr{X}_j)$, where $\mathscr{X}_j:=\{\mathrm{e}^i\mid a_i=\mathbb{0}\}$.

Assume now that the intersection of p half-spaces is finitely generated and let

$$\mathscr{V}:=\left\{x\in\mathbb{R}^n_{\max}\;\middle|\;Ax\le Bx\right\}\cap\left\{x\in\mathbb{R}^n_{\max}\;\middle|\;ax\le bx\right\},$$

where $A,B\in\mathbb{R}^{p\times n}_{\max}$ and $a,b\in\mathbb{R}^{1\times n}_{\max}$, be an intersection of $p+1$ half-spaces. Then, we know that there exists a matrix $C\in\mathbb{R}^{n\times t}_{\max}$, for some $t\in\mathbb{N}$, such that $\{x\in\mathbb{R}^n_{\max}\mid Ax\le Bx\}=\{Cw\mid w\in\mathbb{R}^t_{\max}\}$. As $\mathscr{H}:=\{w\in\mathbb{R}^t_{\max}\mid aCw\le bCw\}$ is a half-space, there exists another matrix $D\in\mathbb{R}^{t\times r}_{\max}$, for some $r\in\mathbb{N}$, such that $\mathscr{H}=\{Du\mid u\in\mathbb{R}^r_{\max}\}$. Therefore, $\mathscr{V}=\{Cw\mid aCw\le bCw\}=\{CDu\mid u\in\mathbb{R}^r_{\max}\}$ is finitely generated.

Conversely, let $\mathscr{V}=\{Cw\mid w\in\mathbb{R}^t_{\max}\}$, where $C\in\mathbb{R}^{n\times t}_{\max}$, be a finitely generated cone. Then, as finitely generated cones are closed (see [10, Cor. 27] or [20, Lemma 2.20]), it follows from the separation theorem for closed cones of [14, 31, 35] that $\mathscr{V}$ is the intersection of the half-spaces of $\mathbb{R}^n_{\max}$ in which it is contained. Note that a half-space $\{x\in\mathbb{R}^n_{\max}\mid ax\le bx\}$ contains $\mathscr{V}$ if, and only if, the row vectors a and b satisfy $aC\le bC$. Since $\{(a,b)\in\mathbb{R}^{1\times 2n}_{\max}\mid aC\le bC\}$ is a finite intersection of half-spaces, we know by the first part of the proof that there exist matrices A and B such that (a,b) satisfies $aC\le bC$ if, and only if, (a,b) is a max-plus linear combination of the rows of the matrix (A,B). Therefore, $\mathscr{V}=\{x\in\mathbb{R}^n_{\max}\mid Ax\le Bx\}$, i.e. $\mathscr{V}$ is an intersection of finitely many half-spaces. □

Recall that the *recession cone* [20] of a max-plus convex set $\mathscr{C}$ consists of the vectors u for which there exists a vector $x\in\mathscr{C}$ such that $x\oplus\lambda u\in\mathscr{C}$ for all $\lambda\in\mathbb{R}_{\max}$. This property is known to be independent of the choice of $x\in\mathscr{C}$ as soon as $\mathscr{C}$ is closed.

Given a max-plus cone $\mathscr{V}\subset\mathbb{R}^n_{\max}$, a non-zero vector $v\in\mathscr{V}$ is said to be an *extreme vector* of $\mathscr{V}$ if the following property is satisfied

$$v=u\oplus w,\ u,w\in\mathscr{V}\Rightarrow v=u\text{ or }v=w.$$

The set of scalar multiples of v is an *extreme ray* of $\mathscr{V}$. Given a max-plus convex set $\mathscr{C}\subset\mathbb{R}^n_{\max}$, a vector $v\in\mathscr{C}$ is said to be an *extreme point* of $\mathscr{C}$ if

$$v=\lambda u\oplus\mu w,\ u,w\in\mathscr{C},\ \lambda,\mu\in\mathbb{R}_{\max},\ \lambda\oplus\mu=\mathbb{1}\Rightarrow v=u\text{ or }v=w.$$

As a corollary of Theorem 1 we obtain a max-plus analogue of the Minkowski–Weyl theorem, the first part of which was announced in [19]. A picture illustrating the decomposition can be found in [19, 20].

Theorem 2 (Tropical Minkowski–Weyl Theorem) *The max-plus polyhedra are precisely the sets of the form*

$$\mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y}),$$

where $\mathscr{Z}, \mathscr{Y}$ are finite sets. The set $\mathrm{cone}(\mathscr{Y})$ *in such a representation is unique, it coincides with the recession cone of the polyhedron. Any minimal set $\mathscr{Y}$ in such a representation can be obtained by selecting precisely one non-zero vector in each extreme ray of the recession cone of the polyhedron. The minimal set $\mathscr{Z}$ in such a representation consists of the extreme points of the polyhedron.*

Here, $\oplus$ denotes the max-plus Minkowski sum of two subsets, which is defined as the set of max-plus sums of a vector of the first set and a vector of the second one.

Proof Let $\mathscr{C} \subset \mathbb{R}^n_{\max}$ be a max-plus polyhedron. Then, there exist matrices A, B and column vectors c, d such that $\mathscr{C} = \{x \in \mathbb{R}^n_{\max} \mid Ax \oplus c \leq Bx \oplus d\}$. Consider the max-plus cone

$$\mathscr{V} := \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+1}_{\max} \;\middle|\; Ax \oplus c\lambda \leq Bx \oplus d\lambda \right\}.$$

Since $\mathscr{V}$ is an intersection of finitely many half-spaces, by Theorem 1 it follows that $\mathscr{V} = \mathrm{cone}(\mathscr{X})$, for some finite subset $\mathscr{X}$ of $\mathbb{R}^{n+1}_{\max}$. Note that we can assume, without loss of generality, that

$$\mathscr{X} = \left\{ \begin{pmatrix} z \\ \mathbb{1} \end{pmatrix} \in \mathbb{R}^{n+1}_{\max} \;\middle|\; z \in \mathscr{Z} \right\} \cup \left\{ \begin{pmatrix} y \\ \mathbb{0} \end{pmatrix} \in \mathbb{R}^{n+1}_{\max} \;\middle|\; y \in \mathscr{Y} \right\} \tag{2}$$

for some finite subsets $\mathscr{Z}, \mathscr{Y}$ of $\mathbb{R}^n_{\max}$. Therefore, we have

$$\begin{aligned} x \in \mathscr{C} &\iff \begin{pmatrix} x \\ \mathbb{1} \end{pmatrix} \in \mathscr{V} \iff \begin{pmatrix} x \\ \mathbb{1} \end{pmatrix} = \left(\bigoplus_{z \in \mathscr{Z}} \lambda_z \begin{pmatrix} z \\ \mathbb{1} \end{pmatrix} \right) \oplus \left(\bigoplus_{y \in \mathscr{Y}} \lambda_y \begin{pmatrix} y \\ \mathbb{0} \end{pmatrix} \right) \\ &\iff x = \left(\bigoplus_{z \in \mathscr{Z}} \lambda_z z \right) \oplus \left(\bigoplus_{y \in \mathscr{Y}} \lambda_y y \right), \quad \bigoplus_{z \in \mathscr{Z}} \lambda_z = \mathbb{1} \\ &\iff x \in \mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y}), \end{aligned}$$

which shows that $\mathscr{C} = \mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y})$.

Conversely, let $\mathscr{C} = \mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y})$, where $\mathscr{Z}, \mathscr{Y}$ are finite subsets of $\mathbb{R}^n_{\max}$. Note that x belongs to $\mathscr{C}$ if, and only if, $\binom{x}{\mathbb{1}}$ belongs to $\mathscr{V} := \mathrm{cone}(\mathscr{X})$, where $\mathscr{X}$ is the finite subset of $\mathbb{R}^{n+1}_{\max}$ defined in (2). Since $\mathscr{V}$ is a finitely generated cone, we know by Theorem 1 that there exist matrices A, B and column vectors c, d such that $\mathscr{V} = \{\binom{x}{\lambda} \in \mathbb{R}^{n+1}_{\max} \mid Ax \oplus c\lambda \leq Bx \oplus d\lambda\}$. Therefore, $\mathscr{C} = \{x \in \mathbb{R}^n_{\max} \mid Ax \oplus c \leq Bx \oplus d\}$, i.e. $\mathscr{C}$ is a max-plus polyhedron.

Now let $\mathscr{C} = \mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y})$ be a max-plus polyhedron. From the definition of recession cones, it readily follows that $\mathrm{cone}(\mathscr{Y})$ is contained in the recession cone of $\mathscr{C}$. Assume that u is a vector in the recession cone of $\mathscr{C}$. By the first part of the proof, if we define $\mathscr{V} := \mathrm{cone}(\mathscr{X})$, where $\mathscr{X}$ is the finite subset of $\mathbb{R}_{\max}^{n+1}$ defined in (2), then there exist matrices A, B and column vectors c, d such that $\mathscr{V} = \{\binom{x}{\lambda} \in \mathbb{R}_{\max}^{n+1} \mid Ax \oplus c\lambda \leq Bx \oplus d\lambda\}$ and $\mathscr{C} = \{x \in \mathbb{R}_{\max}^{n} \mid Ax \oplus c \leq Bx \oplus d\}$. Since u is in the recession cone of $\mathscr{C}$, there exists $x \in \mathscr{C}$ such that $x \oplus \lambda u \in \mathscr{C}$ for all $\lambda \in \mathbb{R}_{\max}$. This means that $A(x \oplus \lambda u) \oplus c \leq B(x \oplus \lambda u) \oplus d$ for all $\lambda \in \mathbb{R}_{\max}$, so we conclude that $Au \leq Bu$. Therefore, $\binom{u}{\mathbb{0}} \in \mathscr{V} = \mathrm{cone}(\mathscr{X})$, which implies that $u \in \mathrm{cone}(\mathscr{Y})$ by the definition of $\mathscr{X}$. In consequence, $\mathrm{cone}(\mathscr{Y})$ is equal to the recession cone of $\mathscr{C}$.

Assume that z is an extreme point of $\mathscr{C}$. We next show that necessarily $z \in \mathscr{Z}$. To this end, by the definition of extreme points, it suffices to show that $z \in \mathrm{co}(\mathscr{Z})$. To the contrary, assume that $z = x \oplus u$, where $x \in \mathrm{co}(\mathscr{Z})$, $u \in \mathrm{cone}(\mathscr{Y})$ and $x_i < u_i$ for some $i \in \{1, \dots, n\}$. Then, given any non-zero scalar $\lambda < \mathbb{1}$, we have $z = (x \oplus \lambda u) \oplus \lambda(x \oplus (-\lambda)u)$, which contradicts the fact that z is an extreme point of $\mathscr{C}$ because $x \oplus \lambda u$ and $x \oplus (-\lambda)u$ are two elements of $\mathscr{C}$ different from z. Therefore, any extreme point of $\mathscr{C}$ must belong to $\mathscr{Z}$.

Now let y be an extreme vector of the recession cone of $\mathscr{C}$. Since the recession cone of $\mathscr{C}$ is equal to $\mathrm{cone}(\mathscr{Y})$, by the definition of extreme vectors, it follows that a non-zero scalar multiple of y must belong to $\mathscr{Y}$.

Finally, since $\mathscr{C}$ is closed because it is a finite intersection of closed sets, from Theorem 3.3 of [20] it follows that any minimal set $\mathscr{Y}$ in the representation $\mathscr{C} = \mathrm{co}(\mathscr{Z}) \oplus \mathrm{cone}(\mathscr{Y})$ can be obtained by selecting precisely one non-zero vector in each extreme ray of the recession cone of $\mathscr{C}$, and a minimal set $\mathscr{Z}$ in this representation is given by the extreme points of $\mathscr{C}$. □

3 The partially ordered set of half-spaces

In this section we prove the existence of minimal half-spaces with respect to a max-plus cone $\mathscr{V}$. With this aim, it is convenient to start with the following lemma which shows that any half-space $\mathscr{H}$ of $\mathbb{R}_{\max}^n$ can be written as

$$\mathscr{H} = \left\{x \in \mathbb{R}_{\max}^n \;\middle|\; \bigoplus_{i \in I} a_i x_i \leq \bigoplus_{j \in J} a_j x_j \right\},$$

where I and J are disjoint subsets of $\{1, \dots, n\}$ and $a_k \in \mathbb{R}$ for all $k \in I \cup J$. Henceforth, all the half-spaces we consider will be written in this way.

Lemma 1 *Let $a, b, c, d \in \mathbb{R}_{\max}$. Then,*

$$\{x \in \mathbb{R}_{\max} \mid ax \oplus c \leq bx \oplus d\} = \{x \in \mathbb{R}_{\max} \mid ax \oplus c \leq d\}$$

if $a > b$, and

$$\{x \in \mathbb{R}_{\max} \mid ax \oplus c \leq bx \oplus d\} = \{x \in \mathbb{R}_{\max} \mid c \leq bx \oplus d\}$$

if $a \leq b$.

Proof We only prove the case $a > b$ because the other one is straightforward.

Assume that $ax \oplus c \leq bx \oplus d$. If $x = \mathbb{0}$, necessarily $c \leq d$ and thus $ax \oplus c = c \leq d$. If $x \neq \mathbb{0}$, then, as $ax \oplus c \geq ax > bx$ and $ax \oplus c \leq bx \oplus d$, it follows that $ax \oplus c \leq d$.

Conversely, assume that $ax \oplus c \leq d$. Then, we have $ax \oplus c \leq d \leq bx \oplus d$. □

Given $v \in \mathbb{R}^n_{\max}$ and $\mathscr{V} \subset \mathbb{R}^n_{\max}$, the *supports* of v and $\mathscr{V}$ are respectively defined by

$$\operatorname{supp} v := \{k \mid v_k \neq \mathbb{0}\} \quad \text{and} \quad \operatorname{supp} \mathscr{V} := \bigcup_{v \in \mathscr{V}} \operatorname{supp} v.$$

We shall say that a max-plus cone $\mathscr{V} \subset \mathbb{R}^n_{\max}$ has *full support* if $\operatorname{supp} \mathscr{V} = \{1, \dots, n\}$. For any non-zero scalar $\lambda \in \mathbb{R}_{\max}$, we define $\lambda^- := -\lambda$, and we extend this notation to vectors of $\mathbb{R}^n_{\max}$ with only finite entries, so that x^- represents the vector with entries $-x_i$ for $i \in \{1, \dots, n\}$.

Lemma 2 *Let*

$$\mathscr{H} := \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i \in I} a_i x_i \leq \bigoplus_{j \in J} a_j x_j \right\}$$

and

$$\mathscr{H}' := \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i \in I'} b_i x_i \leq \bigoplus_{j \in J'} b_j x_j \right\}$$

be two half-spaces. Then, when $I \neq \emptyset$, $\mathscr{H}' \subset \mathscr{H}$ if, and only if, $I \subset I'$, $J' \subset J$ and $b_j (b_i)^- \leq a_j (a_i)^-$ for all $i \in I$ and $j \in J'$.

Proof (⇒) Assume that $I \not\subset I'$. Pick any $i \in I \setminus I'$. Then, $\mathrm{e}^i \in \mathscr{H}'$ and $\mathrm{e}^i \notin \mathscr{H}$, which contradicts the fact that $\mathscr{H}' \subset \mathscr{H}$. Therefore, $I \subset I'$.

Now assume that $J' \not\subset J$. Pick any $j \in J' \setminus J$ and $i \in I \subset I'$, and define the vector $x \in \mathbb{R}^n_{\max}$ by

$$x_k := \begin{cases} {b_k}^- & \text{if } k \in \{i, j\}, \\ \mathbb{0} & \text{otherwise.} \end{cases} \tag{3}$$

Then, $x \in \mathscr{H}'$ and $x \notin \mathscr{H}$, which is a contradiction. Therefore, $J' \subset J$.

Finally, since the vector x defined in (3) belongs to $\mathscr{H}'$ for any $i \in I \subset I'$ and $j \in J'$, it follows that it also belongs to $\mathscr{H}$ and thus $a_i (b_i)^- \leq a_j (b_j)^-$. Therefore, $b_j (b_i)^- \leq a_j (a_i)^-$ for all $i \in I$ and $j \in J'$.

(⇐) Since

$$\begin{aligned} x \in \mathscr{H}' &\Rightarrow b_i x_i \leq \bigoplus_{j \in J'} b_j x_j, \ \forall i \in I' \Rightarrow b_i x_i \leq \bigoplus_{j \in J'} b_j x_j, \ \forall i \in I \\ &\Rightarrow x_i \leq \bigoplus_{j \in J'} b_j (b_i)^- x_j, \ \forall i \in I \Rightarrow x_i \leq \bigoplus_{j \in J'} a_j (a_i)^- x_j, \ \forall i \in I \\ &\Rightarrow a_i x_i \leq \bigoplus_{j \in J'} a_j x_j, \ \forall i \in I \Rightarrow a_i x_i \leq \bigoplus_{j \in J} a_j x_j, \ \forall i \in I \Rightarrow x \in \mathscr{H}, \end{aligned}$$

it follows that $\mathscr{H}' \subset \mathscr{H}$. □

Lemma 3 *Let $\mathscr{V} \subset \mathbb{R}^n_{\max}$ be a max-plus cone with full support and $\{\mathscr{H}_r\}_{r\in\mathbb{N}}$ be a decreasing sequence of half-spaces such that $\mathscr{V} \subset \mathscr{H}_r$ for all $r \in \mathbb{N}$. Then, there exists a half-space $\mathscr{H}$ such that $\mathscr{H} = \bigcap_{r\in\mathbb{N}} \mathscr{H}_r$.*

Proof Assume that

$$\mathscr{H}_r = \left\{x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I_r} a^r_i x_i \leq \bigoplus_{j\in J_r} a^r_j x_j\right\},$$

where for all $r \in \mathbb{N}$, I_r and J_r are disjoint subsets of $\{1,\dots,n\}$ and $a^r_k \in \mathbb{R}$ for all $k \in I_r \cup J_r$. By Lemma 2 we may assume, without loss of generality, that there exist $I, J \subset \{1,\dots,n\}$ such that $I_r = I$ and $J_r = J$ for all $r \in \mathbb{N}$. If $I = \emptyset$, we have $\mathscr{H}_r = \mathbb{R}^n_{\max}$ for all $r \in \mathbb{N}$, so in this case the result is obvious. We next consider the case $I \neq \emptyset$. Note that in this case we also have $J \neq \emptyset$, because $\mathscr{V} \subset \mathscr{H}_r$ for all $r \in \mathbb{N}$ and $\operatorname{supp} \mathscr{V} = \{1,\dots,n\}$.

We may also assume, without loss of generality, that $\bigoplus_{j\in J} a^r_j = \mathbb{1}$ for all $r \in \mathbb{N}$. Then, since $\operatorname{supp} \mathscr{V} = \{1,\dots,n\}$ and $\bigoplus_{i\in I} a^r_i x_i \leq \bigoplus_{j\in J} a^r_j x_j$ for $r \in \mathbb{N}$ and $x \in \mathscr{V}$, it follows that the sequence $\{a^r_i\}_{r\in\mathbb{N}}$ is bounded from above for all $i \in I$. Therefore, we may assume, taking sub-sequences if necessary, that there exists $a_i \in \mathbb{R}_{\max}$ such that $\lim_{r\to\infty} a^r_i = a_i$ for $i \in I$.

We claim that $a_i \neq \mathbb{0}$ for all $i \in I$. To the contrary, assume that $a_h = \mathbb{0}$ for some $h \in I$. Since by Lemma 2 we have $a^r_j (a^r_h)^- \leq a^1_j (a^1_h)^-$ for all $j \in J$ and $r \in \mathbb{N}$, this implies that $\lim_{r\to\infty} a^r_j = \mathbb{0}$ for all $j \in J$, which contradicts the fact that $J \neq \emptyset$ and $\bigoplus_{j\in J} a^r_j = \mathbb{1}$ for all $r \in \mathbb{N}$. This proves our claim.

Since $\bigoplus_{j\in J} a^r_j = \mathbb{1}$ for all $r \in \mathbb{N}$, the sequence $\{a^r_j\}_{r\in\mathbb{N}}$ is also bounded from above for all $j \in J$. Then, taking sub-sequences if necessary, we may assume that there exists $a_j \in \mathbb{R}_{\max}$ such that $\lim_{r\to\infty} a^r_j = a_j$ for all $j \in J$. If we define $J' := \{j \in J \mid a_j \neq \mathbb{0}\}$ and

$$\mathscr{H} := \left\{x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I} a_i x_i \leq \bigoplus_{j\in J'} a_j x_j\right\},$$

then it follows that $\mathscr{H} = \bigcap_{r\in\mathbb{N}} \mathscr{H}_r$. Indeed, if $x \in \bigcap_{r\in\mathbb{N}} \mathscr{H}_r$, we have

$$\bigoplus_{i\in I} a_i x_i = \lim_{r\to\infty} \left(\bigoplus_{i\in I} a^r_i x_i\right) \leq \lim_{r\to\infty} \left(\bigoplus_{j\in J} a^r_j x_j\right) = \bigoplus_{j\in J'} a_j x_j,$$

and thus $x \in \mathscr{H}$. Therefore, $\bigcap_{r\in\mathbb{N}} \mathscr{H}_r \subset \mathscr{H}$. Conversely, since $J' \subset J$ and $a_j (a_i)^- \leq a^r_j (a^r_i)^-$ for all $i \in I$, $j \in J'$ and $r \in \mathbb{N}$, by Lemma 2 it follows that $\mathscr{H} \subset \mathscr{H}_r$ for all $r \in \mathbb{N}$. Therefore, we also have $\mathscr{H} \subset \bigcap_{r\in\mathbb{N}} \mathscr{H}_r$. □

Remark 1 Lemma 3 does not hold if $\mathscr{V}$ does not have full support. For instance, consider

$$\mathscr{V} = \left\{x \in \mathbb{R}^3_{\max} \;\middle|\; x_2 = \mathbb{0}, x_1 \leq x_3\right\}$$

and the decreasing sequence of half-spaces

$$\mathscr{H}_r = \{x \in \mathbb{R}^3_{\max} \mid x_1 \oplus r x_2 \leq x_3\},$$

where $r \in \mathbb{N}$. Then, $\bigcap_{r\in\mathbb{N}} \mathscr{H}_r = \mathscr{V}$, but $\mathscr{V}$ is not a half-space of $\mathbb{R}^3_{\max}$.

Theorem 3 *Let $\mathscr{V} \subset \mathbb{R}^n_{\max}$ be a max-plus cone with full support. If $\mathscr{V}$ is contained in the half-space $\mathscr{H}$, then there exists a half-space $\mathscr{H}'$ such that $\mathscr{V} \subset \mathscr{H}' \subset \mathscr{H}$ and $\mathscr{H}'$ is minimal for inclusion with respect to this property.*

Proof By Zorn's Lemma it suffices to show that for any chain $\{\mathscr{H}_\alpha\}_{\alpha\in\Delta}$ of half-spaces which satisfies $\mathscr{V} \subset \mathscr{H}_\alpha \subset \mathscr{H}$ for all $\alpha \in \Delta$, there exists a half-space $\mathscr{H}'$ such that $\mathscr{H}' = \bigcap_{\alpha\in\Delta} \mathscr{H}_\alpha$.

According to Lemma 2, we may assume that

$$\mathscr{H}_\alpha = \left\{x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I} a_i^\alpha x_i \leq \bigoplus_{j\in J} a_j^\alpha x_j\right\}$$

for all $\alpha \in \Delta$, where I and J are disjoint subsets of $\{1,\dots,n\}$ and $a_k^\alpha \in \mathbb{R}$ for all $k \in I \cup J$. Again, if $I = \emptyset$ the previous assertion is trivial, so assume $I \neq \emptyset$. Consider any sequence $\{\alpha_r\}_{r\in\mathbb{N}} \subset \Delta$ such that the sequence $\{a_j^{\alpha_r}(a_i^{\alpha_r})^-\}_{r\in\mathbb{N}}$ is decreasing and

$$\lim_{r\to\infty} a_j^{\alpha_r}\left(a_i^{\alpha_r}\right)^- = \inf_{\alpha\in\Delta} a_j^\alpha\left(a_i^\alpha\right)^-, \tag{4}$$

for all $i \in I$ and $j \in J$. Then, by Lemma 3 there exists a half-space $\mathscr{H}'$ such that $\mathscr{H}' = \bigcap_{r\in\mathbb{N}} \mathscr{H}_{\alpha_r}$. Since (4) is satisfied, by Lemma 2, for any $\alpha \in \Delta$ there exists $r \in \mathbb{N}$ such that $\mathscr{H}_{\alpha_r} \subset \mathscr{H}_\alpha$. Therefore, we have $\mathscr{H}' = \bigcap_{\alpha\in\Delta} \mathscr{H}_\alpha$. □

As a consequence of Theorem 3 and the separation theorem for closed cones of [14, 31, 35], it follows that any closed cone $\mathscr{V}$ with full support can be expressed as the intersection of a family of minimal half-spaces with respect to $\mathscr{V}$. When $\mathscr{V}$ is finitely generated, by Theorem 1 we conclude that it is possible to select a finite number of minimal half-spaces with respect to $\mathscr{V}$ such that their intersection is equal to $\mathscr{V}$. However, like in the classical case, even in the finitely generated case, the number of minimal half-spaces with respect to $\mathscr{V}$ need not be finite, as it is shown in the next section.

4 Characterization of minimal half-spaces

Throughout this section, $\mathscr{V} \subset \mathbb{R}^n_{\max}$ will represent a fixed max-plus cone generated by the vectors $v^r \in \mathbb{R}^n_{\max}$, where $r = 1,\dots,p$. For the sake of simplicity, in this section we shall assume that all the vectors we consider have only finite entries. We next recall basic definitions and properties related to the natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$. We refer the reader to [16] for a complete presentation, but we warn that the results of [16] are in the setting of the min-plus semiring $\mathbb{R}_{\min} := (\mathbb{R} \cup \{+\infty\}, \min, +)$, which is however equivalent to the setting considered here.

We define the type of a vector $x \in \mathbb{R}^n_{\max}$ relative to the generators v^r as the n-tuple $\operatorname{type}(x) = (S_1(x), \ldots, S_n(x))$ of subsets $S_j(x) \subset \{1, \ldots, p\}$ defined as follows:

$$S_j(x) := \left\{ r \mid v^r_j(x_j)^- = \bigoplus_{1\le k\le n} v^r_k(x_k)^- \right\}. \tag{5}$$

Note that $v^r_j(x_j)^- < \bigoplus_{1\le k\le n} v^r_k(x_k)^-$ if $r \notin S_j(x)$ and that any $r \in \{1, \ldots, p\}$ belongs to some $S_j(x)$.

Given an n-tuple $S = (S_1, \ldots, S_n)$ of subsets of $\{1, \ldots, p\}$, consider like in [16] the set X_S of all the vectors whose type contains S, i.e.

$$X_S := \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; S_j \subset S_j(x),\ \forall j = 1, \ldots, n \right\}. \tag{6}$$

Lemma 10 of [16] shows that the sets X_S are closed convex polyhedra (both in the max-plus and usual sense) which are given by

$$X_S = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; x_j v^r_i \le x_i v^r_j,\ \forall i, j \in \{1, \ldots, n\} \text{ with } r \in S_j \right\}.$$

The natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$ is the collection of convex polyhedra X_S, where S ranges over all the possible types. This cell decomposition has in particular the property that $\mathscr{V}$ is the union of its bounded cells, where a cell is said to be bounded if it is bounded in the $(n-1)$-dimensional *max-plus or tropical projective space* $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ (see [16] for details).

Given a cell X_S, if we define the undirected graph G_S with set of nodes $\{1, \ldots, n\}$ and an edge connecting the nodes i and j if and only if $S_i \cap S_j \neq \emptyset$, then by Proposition 17 of [16] the dimension of X_S is given by the number of connected components of G_S (in [16] the dimension of X_S is one less the one considered here because it refers to the projective space). Any non-zero vector in a cell of dimension one, which is therefore of the form $\{\lambda x \mid \lambda \in \mathbb{R}_{\max}\}$ for some $x \in \mathbb{R}^n$, is called a *vertex* of the natural cell decomposition.

Example 1 Consider the max-plus cone $\mathscr{V} \subset \mathbb{R}^3_{\max}$ generated by the vectors: $v^1 = (1, 2, 3)^T$, $v^2 = (2, 4, 6)^T$ and $v^3 = (3, 6, 9)^T$. This cone is represented on the left-hand side of Fig. 1 by the bounded dark gray region together with the two line segments joining the points v^1 and v^3 to it. On the same figure we show the type of a vector, for each cell X_S contained in $\mathscr{V}$. For instance, the type of the vector $a = (0, 1, 3)^T$ is $S = \operatorname{type}(a) = (\{1\}, \{1, 2\}, \{2, 3\})$. Then, since the graph G_S has only one connected component, a is a vertex. If we take $b = (0, 1, 2.5)^T$, then $S = \operatorname{type}(b) = (\{1\}, \{1\}, \{2, 3\})$ so that in this case the cell X_S is two-dimensional. In Fig. 1 this cell is represented by the line segment which connects the points v^1 and a. The natural cell decomposition of $\mathbb{R}^3_{\max}$ induced by the generators of $\mathscr{V}$ has six vertices, fifteen two-dimensional cells (six of them bounded) and ten three-dimensional cells (only one of them bounded, which is represented by the bounded dark gray region labeled by the type $S = (\{1\}, \{2\}, \{3\})$ on the left-hand side of Fig. 1).

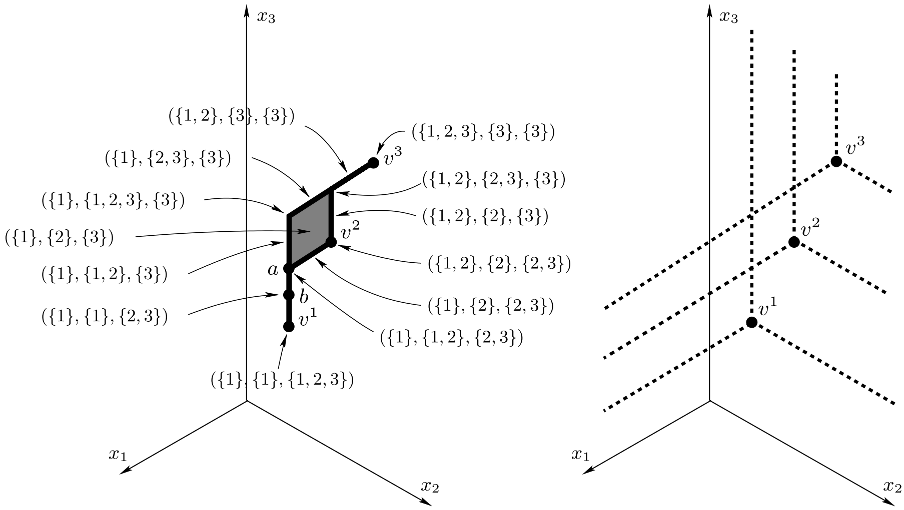

Fig. 1 Illustration of the combinatorial types (*left*) and of the natural cell decomposition of $\mathbb{R}^3_{\max}$ induced by the generators of a max-plus cone (*right*), as defined by Develin and Sturmfels [16]

A simple geometric construction of the natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$ can be obtained if we consider the min-plus hyperplanes whose apices are the generators of $\mathscr{V}$. Given $a \in \mathbb{R}^n_{\max}$, the *min-plus hyperplane* with apex a^- is the set of vectors $x \in \mathbb{R}^n_{\max}$ such that the minimum $\min_{1\leq i\leq n} a_i + x_i$ is attained at least twice (we refer the reader to [26] for details on hyperplanes and their relation with half-spaces). By Proposition 16 of [16], the cell decomposition induced by the generators of $\mathscr{V}$ is the common refinement of the fans defined by the p min-plus hyperplanes whose apices are the vectors v^r, for $r = 1, \dots, p$. In the case of our example, these min-plus hyperplanes are represented on the right-hand side of Fig. 1, where it can be seen that $\mathscr{V}$ is the union of the bounded cells.

Assume that

$$\mathscr{H} = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I} a_i x_i \leq \bigoplus_{j\in J} a_j x_j \right\}$$

is a minimal half-space with respect to $\mathscr{V}$. Then, we necessarily have $I \cup J = \{1, \dots, n\}$. Indeed, if $h \notin I \cup J$, defining $a_h = \min_{1\leq r\leq p}\{\bigoplus_{j\in J} a_j v^r_j (v^r_h)^-\}$, it follows that the half-space

$$\mathscr{H}' = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; a_h x_h \oplus \left(\bigoplus_{i\in I} a_i x_i \right) \leq \bigoplus_{j\in J} a_j x_j \right\}$$

contains $\mathscr{V}$ because it contains its generators, which by Lemma 2 contradicts the minimality of $\mathscr{H}$. For this reason, in this section we shall assume that $I \cup J = \{1, \dots, n\}$ since we are interested in studying minimal half-spaces, and like in [26] we shall call the vector $a^- \in \mathscr{H} \subset \mathbb{R}^n_{\max}$ the *apex* of $\mathscr{H}$.

The following lemma gives a necessary and sufficient condition for $\mathscr{V}$ to be contained in a half-space in terms of the type of its apex.

Lemma 4 *The max-plus cone $\mathscr{V}$ is contained in the half-space*

$$\mathscr{H}=\left\{x\in\mathbb{R}^n_{\max}\;\middle|\;\bigoplus_{i\in I}a_ix_i\leq\bigoplus_{j\in J}a_jx_j\right\}$$

with apex a^- if, and only if, $\bigcup_{j\in J}S_j(a^-)=\{1,\dots,p\}$.

Proof Assume that $\bigcup_{j\in J}S_j(a^-)\neq\{1,\dots,p\}$. Then, there exists $r\in\{1,\dots,p\}$ such that $r\notin S_j(a^-)$ for all $j\in J$, and so $a_jv^r_j<\bigoplus_{1\leq k\leq n}a_kv^r_k$ for all $j\in J$. Therefore, we have

$$\bigoplus_{j\in J}a_jv^r_j<\bigoplus_{1\leq k\leq n}a_kv^r_k=\bigoplus_{i\in I}a_iv^r_i,$$

which means that v^r does not belong to $\mathscr{H}$ and so $\mathscr{V}$ is not contained in $\mathscr{H}$. This shows the "only if" part of the lemma.

Now assume that $\bigcup_{j\in J}S_j(a^-)=\{1,\dots,p\}$. Then, for each $r\in\{1,\dots,p\}$ there exists $j\in J$ such that $r\in S_j(a^-)$, which means that $a_jv^r_j=\bigoplus_{1\leq k\leq n}a_kv^r_k$. Therefore, we have

$$\bigoplus_{i\in I}a_iv^r_i\leq\bigoplus_{1\leq k\leq n}a_kv^r_k=\bigoplus_{j\in J}a_jv^r_j.$$

Since this holds for all $r\in\{1,\dots,p\}$, it follows that $\mathscr{H}$ contains the generators of $\mathscr{V}$, and thus $\mathscr{V}$ is contained in $\mathscr{H}$. This proves the "if" part of the lemma. □

Now we can characterize the minimality of a half-space with respect to $\mathscr{V}$ in terms of the type of its apex.

Theorem 4 *The half-space*

$$\mathscr{H}=\left\{x\in\mathbb{R}^n_{\max}\;\middle|\;\bigoplus_{i\in I}a_ix_i\leq\bigoplus_{j\in J}a_jx_j\right\}$$

with apex a^- is minimal with respect to the max-plus cone $\mathscr{V}$ if, and only if, the following conditions are satisfied:

(i) *For each $i\in I$ there exists $j\in J$ such that $S_i(a^-)\cap S_j(a^-)\neq\emptyset$,*
(ii) *For each $j\in J$ there exists $i\in I$ such that $S_i(a^-)\cap S_j(a^-)\not\subset\bigcup_{k\in J\setminus\{j\}}S_k(a^-)$,*
(iii) *$\bigcup_{j\in J}S_j(a^-)=\{1,\dots,p\}$.*

Proof If $\mathscr{H}$ is minimal with respect to $\mathscr{V}$, for each $i\in I$ there exists $r\in\{1,\dots,p\}$ such that $a_iv^r_i=\bigoplus_{j\in J}a_jv^r_j$ because otherwise there would exist $\delta>0$ such that $\delta a_iv^r_i\oplus(\bigoplus_{h\in I\setminus\{i\}}a_hv^r_h)\leq\bigoplus_{j\in J}a_jv^r_j$ for all $r\in\{1,\dots,p\}$, contradicting by Lemma 2 the minimality of $\mathscr{H}$. Therefore, for each $i\in I$ there exist $r\in\{1,\dots,p\}$

and $j \in J$ such that $a_i v_i^r = a_j v_j^r \geq a_k v_k^r$ for all k, which implies that $r \in S_i(a^-) \cap S_j(a^-)$ and so $S_i(a^-) \cap S_j(a^-) \neq \emptyset$.

Analogously, if $\mathscr{H}$ is minimal with respect to $\mathscr{V}$, for each $j \in J$ there exists $r \in \{1, \ldots, p\}$ such that $\bigoplus_{i \in I} a_i v_i^r = a_j v_j^r > \bigoplus_{k \in J \setminus \{j\}} a_k v_k^r$. Otherwise, there would exist $\delta < 0$ such that $\bigoplus_{i \in I} a_i v_i^r \leq \delta a_j v_j^r \oplus (\bigoplus_{k \in J \setminus \{j\}} a_k v_k^r)$ for all $r \in \{1, \ldots, p\}$, which by Lemma 2 contradicts the minimality of $\mathscr{H}$. Therefore, for each $j \in J$ there exist $r \in \{1, \ldots, n\}$ and $i \in I$ such that $a_i v_i^r = a_j v_j^r \geq a_k v_k^r$ for all k, where the inequality is strict for $k \in J \setminus \{j\}$, which implies that $r \in S_i(a^-) \cap S_j(a^-)$ but $r \notin \bigcup_{k \in J \setminus \{j\}} S_k(a^-)$.

Finally, since any minimal half-space with respect to $\mathscr{V}$ contains in particular $\mathscr{V}$, it follows that $\bigcup_{j \in J} S_j(a^-) = \{1, \ldots, p\}$ by Lemma 4. This completes the proof of the "only if" part of the theorem.

Now assume that the three conditions of the theorem are satisfied. By Lemma 4, Condition (iii) implies that $\mathscr{V}$ is contained in $\mathscr{H}$. Let

$$\mathscr{H}' = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i \in I'} b_i x_i \leq \bigoplus_{j \in J'} b_j x_j \right\}$$

be a minimal half-space with respect to $\mathscr{V}$ contained in $\mathscr{H}$, which we know exists by Theorem 3. Then, since $\mathscr{H}' \subset \mathscr{H}$, by Lemma 2 we have $I \subset I'$, $J' \subset J$ and $b_j(b_i)^- \leq a_j(a_i)^-$ for all $i \in I$ and $j \in J'$.

We first show that $I = I'$, and thus $J = J'$. To the contrary, assume that $I \neq I'$ and let $h \in I' \setminus I \subset J$. Then, by Condition (ii), there exist $l \in I$ and $r \in \{1, \ldots, p\}$ such that $a_l v_l^r = a_h v_h^r > a_k v_k^r$ for all $k \in J \setminus \{h\}$. Therefore, since $b_j(b_i)^- \leq a_j(a_i)^-$ for all $i \in I$ and $j \in J'$, we have

$$v_l^r > \bigoplus_{k \in J \setminus \{h\}} a_k(a_l)^- v_k^r \geq \bigoplus_{k \in J'} a_k(a_l)^- v_k^r \geq \bigoplus_{k \in J'} b_k(b_l)^- v_k^r$$

and so

$$\bigoplus_{i \in I'} b_i v_i^r \geq \bigoplus_{i \in I} b_i v_i^r \geq b_l v_l^r > \bigoplus_{k \in J'} b_k v_k^r ,$$

contradicting the fact that $v^r \in \mathscr{V} \subset \mathscr{H}'$. Therefore, we conclude that $I = I'$ and $J = J'$.

Note that by Conditions (i) and (ii), it follows that $S_k(a^-) \neq \emptyset$ for all k. This implies, by the covering theorem of Vorobyev [33, Theorem 2.6] and Zimmermann [34, Chap. 3] (see [1] for a complete recent discussion, including generalizations; see also Corollary 12 and Theorem 15 of [16]), that the apex a^- of $\mathscr{H}$ belongs to $\mathscr{V}$. Therefore, since $\mathscr{V} \subset \mathscr{H}'$, we have $\bigoplus_{i \in I} b_i(a_i)^- \leq \bigoplus_{j \in J} b_j(a_j)^-$. Without loss of generality, we may assume that $\bigoplus_{i \in I} b_i(a_i)^- \leq \bigoplus_{j \in J} b_j(a_j)^- = \mathbb{1}$. Then, since $b_j(a_j)^- \leq b_i(a_i)^-$ for all $i \in I$ and $j \in J$, we must have $b_i(a_i)^- = \mathbb{1}$, i.e. $a_i = b_i$, for all $i \in I$.

Now assume that $a \neq b$. Then, there exists $j \in J$ such that $b_j < a_j$ (note that $b_k \leq a_k$ for all $k \in J$ because $\bigoplus_{k \in J} b_k(a_k)^- = \mathbb{1}$), and by Condition (ii) there exist

$i \in I$ and $r \in \{1, \dots, p\}$ such that

$$a_i v_i^r = a_j v_j^r > a_k v_k^r$$

for all $k \in J \setminus \{j\}$. Therefore, it follows that

$$b_i v_i^r = a_i v_i^r = a_j v_j^r > b_j v_j^r$$

and

$$b_i v_i^r = a_i v_i^r > a_k v_k^r \geq b_k v_k^r$$

for all $k \in J \setminus \{j\}$, implying that

$$\bigoplus_{h \in I} b_h v_h^r \geq b_i v_i^r > \bigoplus_{k \in J} b_k v_k^r ,$$

which contradicts the fact that $v^r \in \mathscr{V} \subset \mathscr{H}'$. In consequence, we conclude that $a = b$, and so $\mathscr{H} = \mathscr{H}'$, showing that $\mathscr{H}$ is a minimal half-space with respect to $\mathscr{V}$. □

Note that the theorem above tells us that the property of being minimal with respect to $\mathscr{V}$ depends on the type of the apex of a half-space. More precisely, if $a^- \in \mathbb{R}^n$ is the apex of a minimal half-space with respect to $\mathscr{V}$ and b^- is in the relative interior of X_S, where $S = \text{type}(a^-)$, then b^- is also the apex of a minimal half-space with respect to $\mathscr{V}$. Observe also that, as it was shown in the proof of Theorem 4, the conditions in this theorem imply that the apex of a minimal half-space with respect to $\mathscr{V}$ must belong to $\mathscr{V}$. However, these conditions do not imply that the apex of a minimal half-space with respect to $\mathscr{V}$ should be a vertex of the natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$. In other words, if a^- is the apex of a minimal half-space with respect to $\mathscr{V}$ and $S = \text{type}(a^-)$, then G_S need not have only one connected component. Indeed, this is not the case, except when $n = 3$.

Example 2 Consider the max-plus cone $\mathscr{V} \subset \mathbb{R}^4_{\max}$ generated by the following vectors: $v^r = (1r, 2r, 3r, 4r)^T$ for $r = 1, \dots, 4$, where the product is in the usual algebra. Note that these vectors are in general position, as defined in [16]. Indeed, this kind of cones were already studied in [5, 6] and can be seen as the max-plus analogues of the cyclic polytopes.

If we take $a = (8, 6, 3.5, (-0.5))^T$, then $S = \text{type}(a^-) = (\{1, 2\}, \{2\}, \{3, 4\}, \{4\})$, so a^- is not a vertex. However, since the conditions of Theorem 4 are satisfied for $I = \{2, 4\}$ and $J = \{1, 3\}$, it follows that

$$\mathscr{H} = \left\{x \in \mathbb{R}^4_{\max} \;\middle|\; 6x_2 \oplus (-0.5)x_4 \leq 8x_1 \oplus 3.5x_3\right\}, \tag{7}$$

or

$$\mathscr{H} = \left\{x \in \left(\mathbb{R} \cup \{-\infty\}\right)^4 \;\middle|\; \max(6 + x_2, -0.5 + x_4) \leq \max(8 + x_1, 3.5 + x_3)\right\}$$

with the usual notation, is a minimal half-space with respect to $\mathscr{V}$. Indeed, since

$$X_S = \left\{x \in \mathbb{R}^4_{\max} \;\middle|\; x_2 = 2x_1,\ x_4 = 4x_3,\ 4x_1 \leq x_3 \leq 5x_1\right\},$$

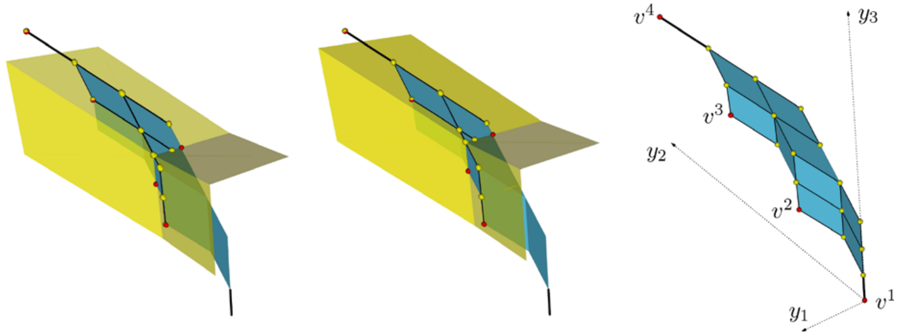

Fig. 2 The counter-example: a max-plus polyhedral cone (*right*) and its intersections with two members of an infinite family of minimal half-spaces containing it (*left* and *middle*). Note that the apex of each of these minimal half-spaces is not a vertex of the natural cell decomposition of $\mathbb{R}^4_{\max}$ induced by the generators of the cone

any half-space of the form

$$\{x \in \mathbb{R}^4_{\max} \mid 6x_2 \oplus \delta x_4 \leq 8x_1 \oplus \delta 4x_3\}, \tag{8}$$

where $-1 < \delta < 0$, is minimal with respect to $\mathscr{V}$ because its apex belongs to the relative interior of X_S. Moreover, this also shows that, even if we assume that the generators of a max-plus cone are in general position, the number of minimal half-spaces need not be finite.

This is illustrated in Fig. 2, which shows the max-plus cone $\mathscr{V}$ (rightmost picture, in blue) together with two minimal half-spaces containing it corresponding to the choice of $\delta = -0.33$ (leftmost picture) and $\delta = -0.67$ (middle picture). The apex of each of these half-spaces belongs to the max-plus segment joining the vectors v^2 and v^4. The existence of an infinite family of minimal half-spaces can be seen on the picture: when the apex of the half-space slides along the middle part of this max-plus segment, the intersection of the boundary of the half-space (in yellow) with the max-plus cone yields an infinite family of sets, two instances of which are represented. The pictures of the max-plus polytopes were generated with POLYMAKE [25], the latest version of which contains an extension dealing with tropical polytopes [27]. We plotted them with JAVAVIEW. The bounded parts of the half-spaces and their intersections with the max-plus cone were computed in SCICOSLAB using the MAX-PLUS TOOLBOX [15]. A vector $x = (x_1, \dots, x_4)^T$ in $\mathbb{R}^4$ is represented by the vector $y = (y_1, y_2, y_3)^T \in \mathbb{R}^3$ with $y_i = x_{i+1} - x_1$. The axes y_1, y_2, y_3 starting from the origin v^1 are represented at the right of the picture.

Remark 2 Like in the classical theory of convex cones, we could define a face of a max-plus cone $\mathscr{V} \subset \mathbb{R}^n_{\max}$ as its intersection with the closure of the complement of a minimal half-space with respect to $\mathscr{V}$, which is also a half-space. However, unlike the classical case, the extreme vectors of a face of a max-plus cone defined in this way need not be extreme vectors of the cone, even in the finitely generated case. To see this, consider the cone $\mathscr{V} \subset \mathbb{R}^4_{\max}$ defined in Example 2 above and the minimal half-space with respect to $\mathscr{V}$ given by (7). Then it can be checked that the face defined by this minimal half-space has extreme vectors which are not extreme vectors of $\mathscr{V}$. Two similar faces are visible in Fig. 2.

When $n=3$, the conditions in Theorem 4 imply that the apex of a minimal half-space with respect to $\mathscr{V}$ must be a vertex of the natural cell decomposition of $\mathbb{R}^3_{\max}$ induced by the generators of $\mathscr{V}$. This means that $\mathscr{V}$ can be expressed as a finite intersection of half-spaces whose apices are vertices. We next show that this property is also valid in higher dimensions. With this aim, we shall need the following immediate consequence of Proposition 19 of [16].

Lemma 5 *Let X_S be a bounded cell of the natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$. Then, $x \in X_S$ if and only if it can be expressed as $x = \min_{1\leq s\leq m} \lambda_s a^s$ for some scalars $\lambda_s \in \mathbb{R}$, where a^s for $s \in \{1,\dots,m\}$ are vertices of the natural cell decomposition which belong to X_S (in other words, X_S is the min-plus cone generated by its vertices).*

As a consequence, we have the following separation theorem in the special case of finitely generated max-plus cones whose generators have only finite entries.

Proposition 1 *Assume that $y \in \mathbb{R}^n_{\max}$ does not belong to the max-plus cone $\mathscr{V}$. Then, there exists a half-space containing $\mathscr{V}$ but not y whose apex is a vertex of the natural cell decomposition of $\mathbb{R}^n_{\max}$ induced by the generators of $\mathscr{V}$.*

Proof By the separation theorem for closed cones of [14], there exists a half-space

$$\mathscr{H} = \left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I} b_i x_i \leq \bigoplus_{j\in J} b_j x_j \right\}$$

containing $\mathscr{V}$ but not y whose apex b^- belongs to $\mathscr{V}$. To be more precise, in [14] it is shown that we can take $b^- = \max\{x \in \mathscr{V} \mid x \leq y\}$. Let $S = \operatorname{type}(b^-)$ be the type of b^-. According to Lemma 4, we have $\bigcup_{j\in J} S_j(b^-) = \{1,\dots,p\}$, and so the half-space

$$\left\{ x \in \mathbb{R}^n_{\max} \;\middle|\; \bigoplus_{i\in I} c_i x_i \leq \bigoplus_{j\in J} c_j x_j \right\}$$

contains $\mathscr{V}$ if c^- belongs to the (bounded) cell X_S.

Let $(a^s)^- \in \mathbb{R}^n_{\max}$, where $s \in \{1,\dots,m\}$ for some $m \in \mathbb{N}$, be the vertices which belong to X_S. Then, by Lemma 5 we know that there exist scalars λ_s such that $b^- = \min_{1\leq s\leq m} \lambda_s (a^s)^-$, and thus

$$b = \bigoplus_{1\leq s\leq m} (\lambda_s)^- a^s.$$

Since y does not belong to $\mathscr{H}$ we have

$$\bigoplus_{i\in I}\left(\bigoplus_{1\leq s\leq m} (\lambda_s)^- a^s_i \right) y_i = \bigoplus_{i\in I} b_i y_i > \bigoplus_{j\in J} b_j y_j = \bigoplus_{j\in J}\left(\bigoplus_{1\leq s\leq m} (\lambda_s)^- a^s_j \right) y_j,$$

and so there exists $r \in \{1, \dots, m\}$ such that

$$\bigoplus_{i \in I} (\lambda_r)^- a_i^r y_i > \bigoplus_{j \in J} (\lambda_r)^- a_j^r y_j .$$

This means that the half-space

$$\left\{ x \in \mathbb{R}_{\max}^n \;\middle|\; \bigoplus_{i \in I} a_i^r x_i \leq \bigoplus_{j \in J} a_j^r x_j \right\},$$

whose apex is the vertex $(a^r)^-$, separates $\mathscr{V}$ from y. □

The previous proposition leads us to study minimal half-spaces with a fixed apex.

Lemma 6 *The maximal number of incomparable half-spaces of $\mathbb{R}_{\max}^n$ with a given apex is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof By Lemma 2 two half-spaces with the same apex $a^- \in \mathbb{R}_{\max}^n$

$$\mathscr{H}' = \left\{ x \in \mathbb{R}_{\max}^n \;\middle|\; \bigoplus_{i \in I'} a_i x_i \leq \bigoplus_{j \in J'} a_j x_j \right\}$$

and

$$\mathscr{H} = \left\{ x \in \mathbb{R}_{\max}^n \;\middle|\; \bigoplus_{i \in I} a_i x_i \leq \bigoplus_{j \in J} a_j x_j \right\}$$

satisfy $\mathscr{H}' \subset \mathscr{H}$ if, and only if, $I \subset I'$. Therefore, the maximal number of incomparable half-spaces with a given apex is equal to the maximal number of incomparable subsets of $\{1, \dots, n\}$, which is equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ according to Sperner's Theorem (see [17]). □

Remark 3 There exist cones $\mathscr{V} \subset \mathbb{R}_{\max}^n$ which have $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ minimal half-spaces with a given apex. For example, consider n odd and define $\mathscr{V}$ as the cone generated by the following vectors:

$$v_i^I := \begin{cases} \mathbb{0} & \text{if } i \in I, \\ \mathbb{1} & \text{otherwise,} \end{cases}$$

where I is any subset of $\{1, \dots, n\}$ with exactly $\lfloor \frac{n}{2} \rfloor$ elements. Then, applying Theorem 4, it can be checked that any half-space of the form

$$\left\{ x \in \mathbb{R}_{\max}^n \;\middle|\; \bigoplus_{i \in I} x_i \leq \bigoplus_{j \notin I} x_j \right\},$$

where again I is any subset of $\{1, \dots, n\}$ with exactly $\lfloor \frac{n}{2} \rfloor$ elements, is minimal with respect to $\mathscr{V}$.

When the generators of a cone $\mathscr{V} \subset \mathbb{R}^n_{\max}$ are in general position, it is possible to have (at least) $P(n)$ minimal half-spaces with respect to $\mathscr{V}$ with the same apex, where $\{P(n)\}_{n\in\mathbb{N}}$ is the Padovan sequence, which is defined by the recurrence

$$P(n) = P(n-2) + P(n-3) \tag{9}$$

with $P(1) = P(2) = P(3) = 1$. More precisely, the max-plus cone $\mathscr{V} \subset \mathbb{R}^n_{\max}$ generated by the vectors $v^r = (1r, 2r, \dots, nr)^T$ for $r = 1, \dots, n$, where the product is in the usual algebra, has $P(n)$ minimal half-spaces with apex a, where $a_i := \sum_{k=1}^{i} k$ for $i = 1, \dots, n$.

To see this, in the first place note that due to the definition of a, the type of a is given by $S_n(a) = \{n\}$ and $S_k(a) = \{k, k+1\}$ for $1 \leq k < n$. Since by Lemmas 2 and 4 minimal half-spaces with a fixed apex a correspond to subsets $J \subset \{1, \dots, n\}$ such that $\{S_j(a)\}_{j\in J}$ is a minimal covering of $\{1, \dots, n\}$, it follows that each time $S_{r-1}(a)$ and $S_r(a)$ belong to such a covering, then $S_{r+1}(a)$ cannot belong to it because $S_r(a) \subset S_{r-1}(a) \cup S_{r+1}(a)$. Observe also that if $S_{r-1}(a)$ belongs to a minimal covering but $S_r(a)$ does not, then $S_{r+1}(a)$ must belong to it. Finally, since $S_n(a) \subset S_{n-1}(a)$, precisely one of these two sets must belong to a minimal covering.

Let $\{S_j(a)\}_{j\in J}$ be a minimal covering of $\{1, \dots, n\}$, and assume that $S_n(a)$ belongs to it. Then, $S_{n-2}(a)$ must also belong to the covering. If we define the sets $S'_j(a)$ for $j = 1, \dots, n-3$ by $S'_{n-3}(a) := \{n-3\}$ and $S'_k(a) := S_k(a)$ for $1 \leq k < n-3$, then it can be checked that there is a bijection between minimal coverings of $\{1, \dots, n-3\}$ by the sets $S'_j(a)$ and minimal coverings of $\{1, \dots, n\}$ by the sets $S_j(a)$ which contain $S_n(a)$.

Analogously, if we now assume that $S_{n-1}(a)$ belongs to a minimal covering of $\{1, \dots, n\}$, and if we define the sets $S''_j(a)$ for $j = 1, \dots, n-2$ by $S''_{n-2}(a) := \{n-2\}$ and $S''_k(a) := S_k(a)$ for $1 \leq k < n-2$, it can be checked that there is a bijective correspondence between minimal coverings of $\{1, \dots, n-2\}$ by the sets $S''_j(a)$ and minimal coverings of $\{1, \dots, n\}$ by the sets $S_j(a)$ which now contain $S_{n-1}(a)$.

In consequence, the number of minimal coverings of $\{1, \dots, n\}$ by the sets $S_j(a)$ is given by the Padovan sequence because these numbers satisfy the recurrence relation that defines this sequence.

Example 3 Taking $n = 4$, the argument used to establish the recurrence (9) defining the Padovan sequence shows that at $a = (1, 3, 6, 10)^T$, there are two minimal coverings of $\{1, \dots, 4\}$ by the sets $S_j(a)$. One consists of $S_1(a) = \{1, 2\}$ and $S_3(a) = \{3, 4\}$, and corresponds to the half-space

$$(-3)x_2 \oplus (-10)x_4 \leq (-1)x_1 \oplus (-6)x_3$$

which coincides with the one in (8) when $\delta = -1$ and has the same shape as the ones in Fig. 2. The second minimal covering consists of $S_1(a) = \{1, 2\}$, $S_2(a) = \{2, 3\}$, $S_4(a) = \{4\}$, it corresponds to the half-space

$$(-6)x_3 \leq (-1)x_1 \oplus (-3)x_2 \oplus (-10)x_4$$

which is represented in Fig. 3.

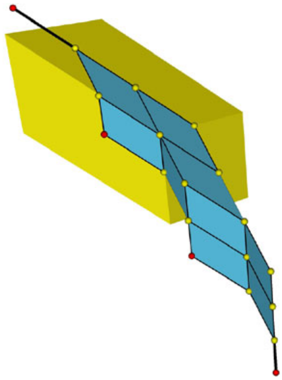

Fig. 3 One of the two minimal half-spaces with apex $(1, 3, 6, 10)^T$

5 Relation between the extreme rays of the polar and minimal half-spaces

In the classical theory of convex cones, it is known that the extreme rays of the polar of a convex cone correspond to its supporting half-spaces. Since the notion of extreme ray carries over to the max-plus setting [10, 20] as well as the notion of polar [21], it is natural to investigate the relation between the minimal half-spaces with respect to a max-plus cone and the extreme rays of its polar.

Following [21], we define the *polar* of a max-plus cone $\mathscr{V} \subset \mathbb{R}^n_{\max}$ as

$$\mathscr{V}^\circ := \left\{ (a, b) \in \left(\mathbb{R}^n_{\max}\right)^2 \,\middle|\, \bigoplus_{1 \le i \le n} a_i x_i \le \bigoplus_{1 \le j \le n} b_j x_j,\ \forall x \in \mathscr{V} \right\},$$

i.e. $\mathscr{V}^\circ$ represents the set of all the half-spaces which contain $\mathscr{V}$. Conversely, we may consider the max-plus cone defined by the intersection of a set of half-spaces. This leads to define (see [21]), for all $\mathscr{W} \subset (\mathbb{R}^n_{\max})^2$, a "dual" polar cone

$$\mathscr{W}^\diamond := \left\{ x \in \mathbb{R}^n_{\max} \,\middle|\, \bigoplus_{1 \le i \le n} a_i x_i \le \bigoplus_{1 \le j \le n} b_j x_j,\ \forall (a, b) \in \mathscr{W} \right\}.$$

Then, by the separation theorem for closed cones ([14, 31, 35]), it follows that a closed cone $\mathscr{V}$ is characterized by its polar cone:

$$\mathscr{V} = \left(\mathscr{V}^\circ\right)^\diamond.$$

In particular, when $\mathscr{V}$ is finitely generated, this means that $\mathscr{V} = \mathscr{W}^\diamond$, where $\mathscr{W} \subset (\mathbb{R}^n_{\max})^2$ is the (finite) set of extreme vectors of $\mathscr{V}^\circ$. Thus, the extreme vectors of the polar of $\mathscr{V}$ determine a finite family of max-plus linear inequalities defining $\mathscr{V}$.

The following theorem characterizes the extreme vectors of $\mathscr{V}^\circ$ in terms of the generators of $\mathscr{V}$.

Theorem 5 *Assume that $\mathscr{V} \subset \mathbb{R}^n_{\max}$ is a max-plus cone with full support generated by the vectors $v^r \in \mathbb{R}^n_{\max}$, where $r = 1, \dots, p$. Then, up to a non-zero scalar multiple, the extreme vectors of $\mathscr{V}^\circ$ are either $(\mathbb{0}, \mathrm{e}^i)$ or $(\mathrm{e}^i, \mathrm{e}^i)$, for $i = 1, \dots, n$, or have the* *form* $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ *for some* $i \in \{1,\dots,n\}$, *where* $J \subset \{1,\dots,n\} \setminus \{i\}$. *Moreover, a vector of* $\mathscr{V}^\circ$ *of the latter form is extreme if, and only if, the following condition is satisfied*:

$$\textit{For each } j \in J \textit{ there exists } r \in \{1,\dots,p\} \textit{ such that } v_i^r = b_j v_j^r > \bigoplus_{k\in J\setminus\{j\}} b_k v_k^r. \quad (10)$$

Proof In the first place, note that by the definition of $\mathscr{V}^\circ$, the vectors $(\mathbb{0}, \mathrm{e}^i)$, for $i = 1,\dots,n$, belong to $\mathscr{V}^\circ$, so these vectors are clearly extreme vectors of $\mathscr{V}^\circ$ and the only ones of the form $(\mathbb{0}, b)$. Moreover, since $\mathscr{V}$ has full support, $\mathscr{V}^\circ$ does not contain vectors of the form $(\mathrm{e}^i, b_i \mathrm{e}^i)$ with $b_i < \mathbb{1}$. This implies that $(\mathrm{e}^i, \mathrm{e}^i)$ is also an extreme vector of $\mathscr{V}^\circ$ for any $i = 1,\dots,n$.

Since $\mathscr{V}^\circ$ satisfies

$$(a' \oplus a'', b) \in \mathscr{V}^\circ \Rightarrow (a', b) \in \mathscr{V}^\circ \text{ and } (a'', b) \in \mathscr{V}^\circ,$$

it follows that (a, b) is an extreme vector of $\mathscr{V}^\circ$ with $a \neq \mathbb{0}$ only if there exists $i \in \{1,\dots,n\}$ such that either $\operatorname{supp}(a) = \{i\} \not\subset \operatorname{supp}(b)$ or (a, b) is a non-zero scalar multiple of $(\mathrm{e}^i, \mathrm{e}^i)$. Therefore, in the former case we may assume that $a = \mathrm{e}^i$ for some $i \in \{1,\dots,n\} \setminus \operatorname{supp}(b)$.

Let $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$, with $i \notin J \subset \{1,\dots,n\}$, be a vector of $\mathscr{V}^\circ$ which satisfies Condition (10). Assume that $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j) = \bigoplus_{1\le s\le m} (a^s, b^s)$, where $m \in \mathbb{N}$ and $(a^s, b^s) \in \mathscr{V}^\circ$ for all $s \in \{1,\dots,m\}$. Then, there exists $l \in \{1,\dots,m\}$ such that $a^l = \mathrm{e}^i$. We claim that $b^l = \bigoplus_{j\in J} b_j \mathrm{e}^j$, which implies that $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ is an extreme vector of $\mathscr{V}^\circ$. To the contrary, assume that $b^l \neq \bigoplus_{j\in J} b_j \mathrm{e}^j$. Then, since $b^l \le \bigoplus_{j\in J} b_j \mathrm{e}^j$, we must have $b_j^l < b_j$ for some $j \in J$. By Condition (10) for this $j \in J$ there exists $r \in \{1,\dots,p\}$ such that $v_i^r = b_j v_j^r > (\bigoplus_{k\in J\setminus\{j\}} b_k v_k^r)$, and thus

$$\bigoplus_{1\le h\le n} a_h^l v_h^r = v_i^r = b_j v_j^r > \left(\bigoplus_{k\in J\setminus\{j\}} b_k v_k^r\right) \oplus b_j^l v_j^r \ge \bigoplus_{1\le k\le n} b_k^l v_k^r,$$

which contradicts the fact that $(a^l, b^l) \in \mathscr{V}^\circ$. This proves the "if" part of the second statement of the theorem.

Now assume that $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ is an extreme vector of $\mathscr{V}^\circ$. If Condition (10) was not satisfied, there would exist $j \in J$ and $\delta < 0$ such that

$$v_i^r \le \left(\bigoplus_{k\in J\setminus\{j\}} b_k v_k^r\right) \oplus \delta b_j v_j^r$$

for all $r \in \{1,\dots,p\}$, implying that $(\mathrm{e}^i, (\bigoplus_{k\in J\setminus\{j\}} b_k \mathrm{e}^k) \oplus \delta b_j \mathrm{e}^j) \in \mathscr{V}^\circ$. Then, we would have

$$\left(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j\right) = \left(\mathrm{e}^i, \left(\bigoplus_{k\in J\setminus\{j\}} b_k \mathrm{e}^k\right) \oplus \delta b_j \mathrm{e}^j\right) \oplus \left(\mathbb{0}, b_j \mathrm{e}^j\right)$$

which contradicts the fact that $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ is extreme because $(\mathbb{0}, b_j\mathrm{e}^j) \in \mathscr{V}^\circ$. This completes the proof of the theorem. □

More generally, there is a hypergraph characterization of the extreme points of a max-plus cone defined by finitely many linear inequalities [5]. In the special case of the polar, Theorem 5 shows that this hypergraph reduces to a star-like graph.

The following proposition shows that the extreme vectors of the polar $\mathscr{V}^\circ$ are special minimal half-spaces, up to a projection of $\mathscr{V}$. Here, $\mathbb{R}_{\max}^{J\cup\{i\}}$ denotes the vectors obtained by keeping only the entries of vectors of $\mathbb{R}_{\max}^n$ whose indices belong to the set $J \cup \{i\}$.

Proposition 2 *A vector* $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ *of the polar* $\mathscr{V}^\circ$ *is extreme if, and only if,*

$$\left\{x \in \mathbb{R}_{\max}^{J\cup\{i\}} \;\middle|\; x_i \leq \bigoplus_{j\in J} b_j x_j\right\}$$

is a minimal half-space with respect to the projection of $\mathscr{V}$ *on* $\mathbb{R}_{\max}^{J\cup\{i\}}$.

Proof This follows readily from Theorem 5 and Lemma 2. □

Remark 4 When the entries of the generators of $\mathscr{V}$ are all finite, if $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ is an extreme generator of $\mathscr{V}^\circ$, Proposition 17 of [16] and Condition (10) imply that the projection of $(\bigoplus_{j\in J} (b_j)^- \mathrm{e}^j) \oplus \mathrm{e}^i$ on $\mathbb{R}_{\max}^{J\cup\{i\}}$ is a vertex of the natural cell decomposition of $\mathbb{R}_{\max}^{J\cup\{i\}}$ induced by the projection of the generators of $\mathscr{V}$ on $\mathbb{R}_{\max}^{J\cup\{i\}}$.

Example 4 Consider again the max-plus cone $\mathscr{V} \subset \mathbb{R}_{\max}^4$ of Example 2. Applying Theorem 5, it can be checked that $(\mathrm{e}^2, 2\mathrm{e}^1 \oplus (-3)\mathrm{e}^3)$ is an extreme vector of $\mathscr{V}^\circ$. The projection of $\mathscr{V}$ on $\mathbb{R}_{\max}^{\{1,2,3\}}$ is represented in Fig. 4 by the bounded dark gray region together with the two line segments joining the points $(0, 1, 2)^T$ and $(0, 4, 8)^T$ to it. The unbounded light gray region represents the projection of the half-space $\{x \in \mathbb{R}_{\max}^4 \mid x_2 \leq 2x_1 \oplus (-3)x_3\}$. The fact that this projection is minimal with respect to the projection of $\mathscr{V}$ is geometrically clear from the figure.

Remark 5 Condition (10) of Theorem 5 shows that when $(\mathrm{e}^i, \bigoplus_{j\in J} b_j \mathrm{e}^j)$ is an extreme vector of the polar $\mathscr{V}^\circ$, the hyperplane

$$\mathscr{H}^= = \left\{x \in \mathbb{R}_{\max}^n \;\middle|\; x_i = \bigoplus_{j\in J} b_j x_j\right\}$$

contains at least $|J|$ generators v^r of $\mathscr{V}$. The latter may be thought of as *support vectors*. It also follows from this theorem that the coefficients b_j of this hyperplane are uniquely determined by these support vectors.

Remark 6 We noted above that the set $\mathscr{W}$ of extreme vectors of the polar $\mathscr{V}^\circ$ satisfies $\mathscr{W}^\diamond = \mathscr{V}$, in other words, it yields a finite family of max-plus linear inequalities

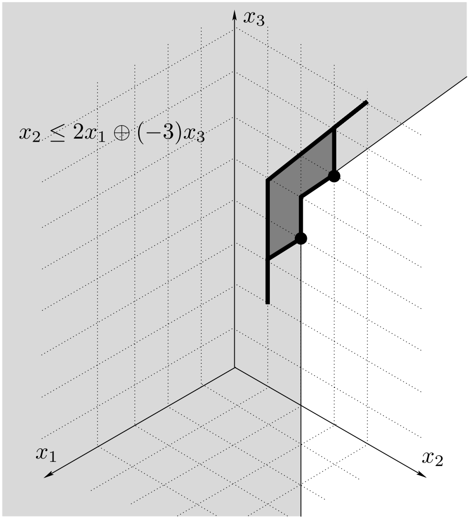

Fig. 4 Illustration of the relation between extreme vectors of the polar and minimal half-spaces. The minimal half-space is in *light gray*. The two support vectors are represented by *bold points*

defining $\mathscr{V}$, the size of which can be bounded by using the results of [5]. However, the bipolar theorem of [21] shows that $\mathscr{W}$ is not always a minimal set with this property.

References

1. Akian, M., Gaubert, S., Kolokoltsov, V.N.: Set coverings and invertibility of functional Galois connections. In: Litvinov, G.L., Maslov, V.P. (eds.) Idempotent Mathematics and Mathematical Physics. Contemporary Mathematics, pp. 19–51. American Mathematical Society, Providence (2005). Also ESI Preprint 1447, arXiv:math.FA/0403441
2. Akian, M., Bapat, R., Gaubert, S.: Max-plus algebras. In: Hogben, L. (ed.) Handbook of Linear Algebra. Discrete Mathematics and Its Applications, vol. 39. Chapman & Hall/CRC, London (2006). Chap. 25
3. Allamigeon, X., Gaubert, S., Goubault, É.: Computing the extreme points of tropical polyhedra. Eprint arXiv:0904.3436 (v2) (2009)
4. Allamigeon, X., Gaubert, S., Goubault, É.: The tropical double description method. In: Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science, STACS'2010, March 4–6, Nancy, France. Leibniz Center in Informatics (2010)
5. Allamigeon, X., Gaubert, S., Katz, R.D.: The number of extreme points of tropical polyhedra. J. Comb. Theory Ser. A (to appear). doi:10.1016/j.jcta.2010.04.003. Also Eprint arXiv:0906.3492
6. Block, F., Yu, J.: Tropical convexity via cellular resolutions. J. Algebr. Comb. **24**(1), 103–114 (2006). Also eprint arXiv:math.MG/0503279
7. Briec, W., Horvath, C.: $\mathbb{B}$-convexity. Optimization **53**, 103–127 (2004)
8. Briec, W., Horvath, C.: Halfspaces and Hahn–Banach like properties in $\mathbb{B}$-convexity and max-plus convexity. Pac. J. Optim. **4**(2), 293–317 (2008)
9. Butkovič, P., Hegedüs, G.: An elimination method for finding all solutions of the system of linear equations over an extremal algebra. Ekon.-Mat. Obz. **20**(2), 203–215 (1984)
10. Butkovič, P., Schneider, H., Sergeev, S.: Generators, extremals and bases of max cones. Linear Algebra Appl. **421**(2–3), 394–406 (2007). Also eprint arXiv:math.RA/0604454
11. Cohen, G., Gaubert, S., Quadrat, J.P.: Max-plus algebra and system theory: where we are and where to go now. Annu. Rev. Control **23**, 207–219 (1999)
12. Cohen, G., Gaubert, S., Quadrat, J.P.: Hahn–Banach separation theorem for max-plus semimodules. In: Menaldi, J.L., Rofman, E., Sulem, A. (eds.) Optimal Control and Partial Differential Equations, pp. 325–334. IOS Press, Amsterdam (2001)

13. Cohen, G., Gaubert, S., Quadrat, J.P.: Duality and separation theorems in idempotent semimodules. Linear Algebra Appl. **379**, 395–422 (2004). doi:10.1016/j.laa.2003.08.010, arXiv:math.FA/0212294
14. Cohen, G., Gaubert, S., Quadrat, J.P., Singer, I.: Max-plus convex sets and functions. In: Litvinov, G.L., Maslov, V.P. (eds.) Idempotent Mathematics and Mathematical Physics. Contemporary Mathematics, pp. 105–129. American Mathematical Society, Providence (2005). Also ESI Preprint 1341, arXiv:math.FA/0308166
15. Cohen, G., Gaubert, S., McGettrick, M., Quadrat, J.P.: Maxplus toolbox of SCILAB. Available at http://minimal.inria.fr/gaubert/maxplustoolbox/; now integrated into SCICOSLAB. http://www.scicoslab.org
16. Develin, M., Sturmfels, B.: Tropical convexity. Doc. Math. **9**, 1–27 (2004) (Erratum pp. 205–206). Also eprint arXiv:math.MG/0308254
17. Engel, K.: Sperner Theory. Encyclopedia of Mathematics and Its Applications, vol. 65. Cambridge University Press, Cambridge (1997)
18. Gaubert, S.: Théorie des systèmes linéaires dans les dioïdes. Thèse, École des Mines de Paris (July 1992)
19. Gaubert, S., Katz, R.D.: Max-plus convex geometry. In: Schmidt, R.A. (ed.) Proceedings of the 9th International Conference on Relational Methods in Computer Science and 4th International Workshop on Applications of Kleene Algebra (RelMiCS/AKA 2006). Lecture Notes in Comput. Sci., vol. 4136, pp. 192–206. Springer, Berlin (2006)
20. Gaubert, S., Katz, R.D.: The Minkowski theorem for max-plus convex sets. Linear Algebra Appl. **421**(2–3), 356–369 (2007). Also eprint arXiv:math.MG/0605078
21. Gaubert, S., Katz, R.D.: The tropical analogue of polar cones. Linear Algebra Appl. **431**(5–7), 608–625 (2009). Also eprint arXiv:0805.3688
22. Gaubert, S., Meunier, F.: Carathéodory, Helly and the others in the max-plus world. Discrete Comput. Geom. **43**(3), 648–662 (2010). Also eprint arXiv:0804.1361
23. Gaubert, S., Plus, M.: Methods and applications of (max,+) linear algebra. In: Reischuk, R., Morvan, M. (eds.) STACS'97, Lübeck, Lecture Notes in Comput. Sci., vol. 1200. Springer, Berlin (March 1997)
24. Gaubert, S., Sergeev, S.N.: Cyclic projectors and separation theorems in idempotent convex geometry. J. Math. Sci. **155**(6), 815–829 (2008). Russian version published in Fundam. Prikl. Mat. **13**(4), 33–52 (2007)
25. Gawrilow, E., Joswig, M.: POLYMAKE: a framework for analyzing convex polytopes. In: Kalai, G., Ziegler, G.M. (eds.) Polytopes—Combinatorics and Computation, pp. 43–74. Birkhäuser, Basel (2000). http://www.math.tu-berlin.de/polymake/
26. Joswig M.: Tropical halfspaces. In: Combinatorial and Computational Geometry. Math. Sci. Res. Inst. Publ., vol. 52, pp. 409–431. Cambridge Univ. Press, Cambridge (2005). Also eprint arXiv:math.CO/0312068
27. Joswig M.: Tropical convex hull computations. In: Litvinov, G.L., Sergeev, S.N. (eds.) Proceedings of the International Conference on Tropical and Idempotent Mathematics. Contemporary Mathematics, vol. 495, pp. 193–212. American Mathematical Society, Providence (2009). Also eprint arXiv:0809.4694
28. Joswig, M., Sturmfels, B., Yu J.: Affine buildings and tropical convexity. Albanian J. Math. **1**(4), 187–211 (2007). Also eprint arXiv:0706.1918
29. Litvinov, G.L., Maslov, V.P., Shpiz G.B.: Idempotent functional analysis: an algebraic approach. Math. Notes **69**(5), 696–729 (2001). Also eprint arXiv:math.FA/0009128
30. Nitica, V., Singer I.: Max-plus convex sets and max-plus semispaces. I. Optimization **56**(1–2), 171–205 (2007)
31. Samborskiĭ, S.N., Shpiz, G.B.: Convex sets in the semimodule of bounded functions. In: Idempotent Analysis, pp. 135–137. American Mathematical Society, Providence (1992)
32. Singer, I.: Abstract Convex Analysis. Wiley, New York (1997)
33. Vorobyev, N.N.: Extremal algebra of positive matrices. Elektron. Inf. Kybern. **3**, 39–71 (1967) (in Russian)
34. Zimmermann, K.: Extremální Algebra Ekonomický ùstav ČSAV, Praha (1976) (in Czech)
35. Zimmermann, K.: A general separation theorem in extremal algebras. Ekon.-Mat. Obz. **13**(2), 179–201 (1977)