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Group theoretic characterizations of Buekenhout–Metz unitals in PG$(2, q^2)$

Giorgio Donati · Nicola Durante

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Abstract Let G be the group of projectivities stabilizing a unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ and let A, B be two distinct points of $\mathcal{U}$. In this paper we prove that, if G has an elation group of order q with center A and a group of projectivities stabilizing both A and B of order a divisor of $q-1$ greater than $2(\sqrt{q}-1)$, then $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital. From this result two group theoretic characterizations of orthogonal Buekenhout–Metz unitals are given.

Keywords Unitals · Projectivities · Elations

1 Introduction

There are many group theoretic characterizations of orthogonal Buekenhout–Metz unitals embedded in $\mathrm{PG}(2, q^2)$. An important question is how much information concerning the group of projectivities stabilizing a unital is needed to prove that the unital is orthogonal Buekenhout–Metz. Abatangelo and Larato in [2] determine the group of projectivities stabilizing the orthogonal Buekenhout–Metz unital $\mathcal{U}$ expressed as a partial pencil of conics, q odd, independently constructed by Baker and Ebert [4] and by Hirschfeld and Szőnyi [18] and prove that some properties of this group characterize $\mathcal{U}$. Abatangelo in [1] proves that, for q even, $\mathcal{U}$ is an orthogonal Buekenhout–Metz unital with respect to a point P if and only if the collineation group G stabilizing $\mathcal{U}$ fixes P, has a normal subgroup acting on the points of $\mathcal{U} \setminus \{P\}$ as a sharply transitive group and the one point stabilizer of G has order $q-1$ for any point of $\mathcal{U} \setminus \{P\}$.

G. Donati · N. Durante (✉)
Dipartimento di Matematica e Applicazioni, Università di Napoli "Federico II", Complesso di Monte S. Angelo, Edificio T, via Cintia, 80126, Napoli, Italy
e-mail: ndurante@unina.it

G. Donati
e-mail: giorgio.donati@unina.it

Baker and Ebert [5] and Ebert [14] determine the group of projectivities G stabilizing a non-classical orthogonal Buekenhout–Metz unital $\mathcal{U}$ in $\mathrm{PG}(2,q^2)$ respectively for q odd or q even. They prove that G is a semidirect product $S \rtimes R$ where S has order q^3 and R is a cyclic group of order either $2(q-1)$ or $q-1$. The first case occurs if and only if $\mathcal{U}$ can be expressed as a partial pencil of conics and q is odd. Ebert and Wantz [15] prove that $\mathcal{U}$ is an orthogonal Buekenhout–Metz unital if and only if the group of projectivities stabilizing $\mathcal{U}$ contains a semidirect product $S \rtimes R$ where S has order q^3 and R has order $q-1$. Moreover S is abelian if and only if $\mathcal{U}$ is a union of conics, in which case q is necessarily odd and S is elementary abelian.

2 Preliminary results

Let $\mathrm{PG}(2,q^2)$, $q=p^h$, p a prime number, be the projective plane over the Galois field $\mathrm{GF}(q^2)$. A *unital* in $\mathrm{PG}(2,q^2)$ is a set $\mathcal{U}$ of q^3+1 points meeting every line of $\mathrm{PG}(2,q^2)$ in either 1 or $q+1$ points. Lines meeting a unital $\mathcal{U}$ in 1 or $q+1$ points are called respectively *tangent* or *secant* lines to $\mathcal{U}$. Through each point of $\mathcal{U}$ there pass q^2 secant lines and one tangent line. Through each point P not on $\mathcal{U}$ there pass q^2-q secant lines and $q+1$ tangent lines.

An example is the *classical* unital, that is the set of the absolute points of a non-degenerate unitary polarity of $\mathrm{PG}(2,q^2)$. We now recall the André/Bruck–Bose representation of $\mathrm{PG}(2,q^{2n})$ in $\mathrm{PG}(4n,q)$. Let ℓ_∞ be a line of $\mathrm{PG}(2,q^{2n})$ that we consider as the line at infinity. Let Σ_∞ be a hyperplane of $\mathrm{PG}(4n,q)$ and let $\mathcal{S}$ be a regular spread of $(2n-1)$-dimensional subspaces of Σ_∞. The points of the affine plane $\mathrm{PG}(2,q^{2n}) \setminus \ell_\infty$ are represented by the points of $\mathrm{PG}(4n,q) \setminus \Sigma_\infty$; the lines of $\mathrm{PG}(2,q^{2n}) \setminus \ell_\infty$ are represented by the $2n$-dimensional subspaces of $\mathrm{PG}(4n,q)$ which meet Σ_∞ exactly in an element of $\mathcal{S}$. The incidence relation of $\mathrm{PG}(2,q^{2n}) \setminus \ell_\infty$ is represented by set theoretic inclusion in $\mathrm{PG}(4n,q)$. We complete this representation by the addition of the points of ℓ_∞ that are represented in $\mathrm{PG}(4n,q)$ by the elements of the spread $\mathcal{S}$. From now on if $\mathcal{A}$ is a set of points of $\mathrm{PG}(2,q^{2n})$, the corresponding set of points of $\mathrm{PG}(4n,q)$ will be denoted by $\mathcal{A}^*$. The spread $\mathcal{S}$ can be obtained as the unique short orbit on $(2n-1)$-dimensional subspaces of a Singer group S of Σ_∞. For any element $T \in \mathcal{S}$ the stabilizer of T under the action of S is a Singer group of T and its unique short orbit on $(n-1)$-dimensional subspaces of T is the so called *induced spread* $\mathcal{S}_T$ (see [13]). In this representation a Baer subline ℓ_0 with a unique point P in ℓ_∞ becomes an n-dimensional subspace meeting Σ_∞ exactly in an element of the induced spread $\mathcal{S}_{P^*}$. For more details of the André/Bruck–Bose representation see [3, 7, 9, 10, 23].

For $n=1$, let V and P be two points on a line T^* of the spread $\mathcal{S}$ and let $\mathcal{O}$ be an ovoid in some three-dimensional subspace of $\mathrm{PG}(4,q)$ not containing V and such that $\mathcal{O}$ meets Σ_∞ exactly in the point P. Let $\mathcal{U}^*$ be the cone with vertex V projecting $\mathcal{O}$. The set $\mathcal{U}$ of points of $\mathrm{PG}(2,q^2)$ corresponding to $\mathcal{U}^*$ is called an *ovoidal Buekenhout–Metz* unital with respect to the point T (see [11]). Recall that all known ovoids in $\mathrm{PG}(3,q)$ are either elliptic quadrics or Tits ovoids (see [16]). Moreover every ovoid in $\mathrm{PG}(3,q)$, q odd or $q \in \{2,4\}$ is an elliptic quadric (see [6, 20, 21]).

If $\mathcal{O}$ is an elliptic quadric, then $\mathcal{U}$ is called an *orthogonal Buekenhout–Metz* unital; if $q=2^{2h+1}$, $h \geq 1$ and $\mathcal{O}$ is a Tits ovoid, then $\mathcal{U}$ is called a *Buekenhout–Tits* unital.

Recall that from [11] a classical unital, tangent to ℓ_∞, is an orthogonal Buekenhout–Metz unital but the converse need not be true, since there also exist non-classical orthogonal Buekenhout–Metz unitals (see [19]).

Every unital in $\mathrm{PG}(2,2^2)$ is classical (see [17, Corollary 1.12]) and thus it is an orthogonal Buekenhout–Metz unital. Hence from now on we can assume $q>2$. If $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital with respect to a point T, then every secant line through T intersects $\mathcal{U}$ in a Baer subline. Casse, O'Keefe and Penttila [12] for $q>2$ even or $q=3$ and Quinn and Casse [22] for $q>3$, obtain a characterization of ovoidal Buekenhout–Metz unitals proving the following:

Theorem 2.1 *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q^2), q>2$. Then $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital if, and only if, there exists a point T of $\mathcal{U}$ such that the points of $\mathcal{U}$ on each of the q^2 secant lines to $\mathcal{U}$ through T form a Baer subline.*

For more information on unitals in projective planes see [8].

Next we recall that a *central* collineation of $\mathrm{PG}(2,q^2)$ is a collineation α fixing every point of a line ℓ (the *axis* of α) and fixing every line through a point C (the *center* of α). If $C\in\ell$, then α is an *elation*; otherwise α is a *homology*. It is well known that given a line ℓ and three distinct collinear points C, P, P' of $\mathrm{PG}(2,q^2)$, with $P, P'\notin\ell$, there is a unique central collineation with axis ℓ and center C mapping P onto P'.

If (x_1,x_2,x_3) denote the homogeneous projective coordinates of a point of $\mathrm{PG}(2,q^2)$, then every elation with axis the line $x_2=0$ and center the point $(1,0,0)$ is given by

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where a is an element of $\mathrm{GF}(q^2)$.

Every elation of $\mathrm{PG}(2,q^2)$ with axis ℓ and center C induces on every line r through C a projectivity that will be called an *elation*, with center C, of the line r. Finally note that a non-identity elation g of $\mathrm{PG}(2,q^2)$ stabilizing a unital $\mathcal{U}$ has center a point C of $\mathcal{U}$ and axis the tangent line ℓ to $\mathcal{U}$ at C. Indeed suppose by way of contradiction that C is not on $\mathcal{U}$ and let s be a secant line to $\mathcal{U}$ on C, different from ℓ. The set $s\cap\mathcal{U}$ is partitioned into orbits of size p under the action of the group $\langle g\rangle$. It follows that $p|(q+1)$, a contradiction. Hence C is on $\mathcal{U}$. Next suppose by way of contradiction that ℓ is a secant line to $\mathcal{U}$. Let R be a point of $\mathcal{U}$ on the line ℓ, distinct from C. Then g permutes the q^2 lines on R distinct from ℓ. Hence g fixes the tangent line to $\mathcal{U}$ at R, contradicting the fact that g is an elation with center C.

Finally we mention the following result whose proof can be easily obtained by induction on n.

Lemma 2.2 *The minimum number of proper subspaces pairwise disjoint covering the full point-set of $\mathrm{PG}(n,q)$ is $q^{n/2+1}+1$ if n is even and $q^{(n+1)/2}+1$ if n is odd.*

Note that this minimum number can be obtained for n odd by choosing a spread $\mathcal{S}$ of $\frac{n-1}{2}$-dimensional subspaces and for n even by taking a hyperplane section of the spread $\mathcal{S}$.

3 Characterizations

Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2,q^2)$, $q=p^h$, p a prime number, and let A and B be two distinct points of $\mathcal{U}$. Throughout the paper we will denote by G the group of projectivities of $\mathrm{PG}(2,q^2)$ stabilizing $\mathcal{U}$, by t_A the tangent line to $\mathcal{U}$ at A, by ℓ the line AB, by ℓ_0 the set $\mathcal{U}\cap\ell$ and by C the common point to the tangent lines to $\mathcal{U}$ at A and at B.

Proposition 3.1 *Let $\mathcal{U}$ be a unital in* $\mathrm{PG}(2,q^2)$, $q=p^h$, *p a prime number, and let s be a secant line to $\mathcal{U}$ through A. If $\mathcal{U}$ is stabilized by a group of elations G_A of order q with center A then, in the André/Bruck–Bose representation of* $\mathrm{PG}(2,q^2)$ *in* $\mathrm{PG}(4h,p)$ *with t_A as line at infinity, $(\mathcal{U}\cap s)^*$ is an h-dimensional subspace of* $\mathrm{PG}(4h,p)$ *not contained in Σ_∞.*

Proof Let f be a non-identity elation of G_A and let $\mathcal{O}_P$ be the orbit of a point P of $\mathcal{U}\cap s\setminus\{A\}$ under the group $\langle f\rangle$. To every non-identity elation of $\langle f\rangle$ corresponds in $\mathrm{PG}(4h,p)$ an elation with axis Σ_∞ and center a point V of A^* stabilizing $\mathcal{U}^*$. Hence the orbit $\mathcal{O}_P$ is represented by an affine line of s^* with V as point at infinity. It follows that $(\mathcal{U}\cap s\setminus\{A\})^*$ is the union of p^{h-1} distinct affine lines, contained in s^*, all with V as point at infinity. If $h=1$, then $(\mathcal{U}\cap s\setminus\{A\})^*$ is a line in $\mathrm{PG}(4,p)$. Hence we may assume, for the rest of the proof, that $h>1$. Let f' be an elation of G_A such that $\langle f'\rangle\neq\langle f\rangle$. To every non-identity elation of $\langle f'\rangle$ also corresponds an elation of $\mathrm{PG}(4h,p)$ with axis Σ_∞ and center a point V' of A^* stabilizing $\mathcal{U}^*$. Suppose, by way of contradiction, that $V=V'$. In such a case every point X of $\mathcal{U}\cap s\setminus\{A\}$ has the same orbit under the groups $\langle f\rangle$ and $\langle f'\rangle$. Therefore $f'(X)=f^k(X)$ for some $k\in\{0,\dots,p-1\}$ and so $\langle f\rangle=\langle f'\rangle$. The set $\mathcal{V}$ of the centers of the elations of $\mathrm{PG}(4h,p)$ which corresponds to the elations of G_A is an $(h-1)$-dimensional subspace of A^*. Indeed let V and V' be two distinct centers of $\mathcal{V}$ corresponding to distinct subgroups $\langle f\rangle$ and $\langle f'\rangle$ of G_A. Let P^* be a point of $(\mathcal{U}\cap s\setminus\{A\})^*$. The orbit of P under $\langle f\rangle$ is represented, in $\mathrm{PG}(4h,p)$, by the affine line $P^*V\setminus\{V\}\subseteq(\mathcal{U}\cap s\setminus\{A\})^*$. Let now Q^* be an arbitrary point of $P^*V\setminus\{V\}$. The orbit of Q under $\langle f'\rangle$ is represented, in $\mathrm{PG}(4h,p)$, by the affine line $Q^*V'\setminus\{V'\}\subseteq(\mathcal{U}\cap s\setminus\{A\})^*$. It follows that the plane π_{P^*} spanned by the line VV' and the point P^* is contained in $(\mathcal{U}\cap s)^*$. Let V'' be a point of the line VV' different from V and from V'. Every non-identity elation f''^* of $\mathrm{PG}(4h,p)$ with axis Σ_∞ and center V'' stabilizes π_{P^*} for any point P^* of $(\mathcal{U}\cap s\setminus\{A\})^*$ and hence stabilizes $(\mathcal{U}\cap s)^*$. Since this property holds for any secant line to $\mathcal{U}$ through A, it follows that f''^* stabilizes $\mathcal{U}^*$, and so f'' is an element of G_A. This gives that V'' is a point of $\mathcal{V}$, so the set $\mathcal{V}$ is a projective subspace of A^*. As $|\mathcal{V}|=\frac{p^h-1}{p-1}$, we have that $\mathcal{V}$ is an $(h-1)$-dimensional subspace of A^*. Finally, let R^* be a point of $(\mathcal{U}\cap s\setminus\{A\})^*$ and let W be an arbitrary point of $\mathcal{V}$. The set $R^*W\setminus\{W\}$ represents in $\mathrm{PG}(4h,p)$ the orbit of R under a subgroup $\langle g\rangle$ of G_A, where g^* is an elation of $\mathrm{PG}(4h, p)$ with axis Σ_∞ and center W. Hence the h-dimensional subspace spanned by $\mathcal{V}$ and by R^* represents the orbit of R under the group G_A together with the point A, that is the set $\mathcal{U} \cap s$. □

If q is a prime number and f is a non-identity elation in G with center a point A, then the group $G_A = \langle f \rangle$ has order q and hence from the previous proposition we have that every secant line to $\mathcal{U}$ through A meets $\mathcal{U}$ in a Baer subline. From Theorem 2.1 we obtain the following characterization of orthogonal Buekenhout–Metz unitals.

Proposition 3.2 *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, p^2)$, p a prime number. Then $\mathcal{U}$ is an orthogonal Buekenhout–Metz unital if, and only if, G contains a non-identity elation.*

Proposition 3.3 *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$ and let A be a point of $\mathcal{U}$. If G has a subgroup G_A of elations with center A of order q and there exists a line m through A which intersects $\mathcal{U}$ in a Baer subline, then $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital with respect to A.*

Proof Denote by m_0 the Baer subline $\mathcal{U} \cap m$ and put $m_0 = \{A, P_1, \dots, P_q\}$. There exists a unique elation $f_i \in G_A$ mapping P_1 onto P_i, $i = 1, \dots, q$. Let s be a line through A different from m and different from the tangent line t_A to $\mathcal{U}$ at A. Let R_1 be a point of $\mathcal{U} \cap s$ different from A. As f_i stabilizes $\mathcal{U} \cap s$, the points A and $R_i = f_i(R_1)$, $i = 1, \dots, q$, are all in $\mathcal{U} \cap s$, hence $\mathcal{U} \cap s = \{A, R_1, \dots, R_q\}$. Since f_i is an elation with center A and axis t_A, it follows that P_i, R_i, and $P_1R_1 \cap t_A$ are collinear points. Therefore the sets $\mathcal{U} \cap s$ and m_0 are in perspective from the point $P_1R_1 \cap t_A$, so $\mathcal{U} \cap s$ is a Baer subline of s. Hence every secant line through A to $\mathcal{U}$ meets $\mathcal{U}$ in a Baer subline. From Theorem 2.1 it follows that $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital with respect to A. □

Theorem 3.4 *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$. If G has a subgroup G_A of elations with center A of order q and a subgroup G_{AB} fixing both A and B of order a divisor of $q - 1$ greater than $2(\sqrt{q} - 1)$, then $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital with respect to A.*

Proof Let H_{AB} be the group of homologies of G_{AB} with axis AB and center C. As H_{AB} is a subgroup of G_{AB}, $|H_{AB}|$ divides $q - 1$. Let s be a secant line to $\mathcal{U}$ through C such that $s \cap \ell_0 = \emptyset$ and let $s_0 = s \cap \mathcal{U}$. The unique homology of H_{AB} fixing a point P of s_0 is the identity, hence $|Orb_{H_{AB}}(P)| = |H_{AB}|$, and $|H_{AB}|$ divides also $q + 1$. It follows that $|H_{AB}|$ is either 1 or 2. Let Q be a point of ℓ_0 different from A and from B. Since the stabilizer of Q under the action of G_{AB} is H_{AB}, it follows that $|G_{AB}| = |Orb_{G_{AB}}(Q)||H_{AB}|$, and hence the number of orbits of G_{AB} on $\ell_0 \setminus \{A, B\}$ is either $(q - 1)/|G_{AB}|$ or $2(q - 1)/|G_{AB}|$ according to $|H_{AB}|$ is either 1 or 2. In both cases the number of orbits is less than $\sqrt{q} + 1$. We will show that this fact leads us to a contradiction. The group G_{AB} induces on the line ℓ a group of projectivities L_{AB} fixing both A and B and stabilizing ℓ_0 which is isomorphic to the quotient group $\frac{G_{AB}}{H_{AB}}$. Let U be a point of ℓ_0 different from A and from B. Without loss of generality we may assume that, with respect to the induced coordinates on the line ℓ, $A = (1, 0)$,

$B=(0,1)$ and $U=(1,1)$. The map

$$\Phi:\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}\in L_{AB}\to x\in \mathrm{GF}\big(q^2\big),$$

is a monomorphism between L_{AB} and the multiplicative group of GF(q^2). The orbit of U under L_{AB} is the set $\{(a,1)\in\ell : a\in\Phi(L_{AB})\}$ and since $|\Phi(L_{AB})|=|L_{AB}|$, it follows that $a^{|L_{AB}|}=1$ for any $a\in\Phi(L_{AB})$. Let $\mathcal{O}_U$ be the orbit of U under G_{AB}. Since the stabilizer of U under the action of G_{AB} is H_{AB}, it follows that $|L_{AB}|=|G_{AB}|/|H_{AB}|=|\mathcal{O}_U|$, therefore $|L_{AB}|$ divides $q-1$ and so $a^{q-1}=1$ for any $a\in\Phi(L_{AB})$. This proves that the orbit $\mathcal{O}_U$ is contained in a Baer subline b of ℓ containing A and B. In PG($4h,p$) the set b^* is an h-dimensional subspace meeting Σ_∞ in an $(h-1)$-dimensional subspace of the induced spread. Since the unital $\mathcal{U}$ is stabilized by the group G_A, from Proposition 3.1 it follows that ℓ_0 is represented in PG($4h,p$) by an h-dimensional subspace.

Suppose, by way of contradiction, that ℓ_0 is not a Baer subline. Then, from Lemma 2.2, we have that ℓ_0^* meets Σ_∞ in at least $\sqrt{q}+1$ elements of the induced spread of A^*. Hence $\ell_0\setminus\{A,B\}$ meets nontrivially at least $\sqrt{q}+1$ different Baer sublines on A and B and so it is the union of at least $\sqrt{q}+1$ different orbits, a contradiction. Hence ℓ_0 is a Baer subline and from Proposition 3.3 we have that $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital. □

Now we are able to prove the following characterization of orthogonal Buekenhout–Metz unitals.

Theorem 3.5 *Suppose that q is either odd or $q\in\{2,4\}$. A unital $\mathcal{U}$ in* PG($2,q^2$) *is an orthogonal Buekenhout–Metz unital if, and only if, there exist two distinct points A and B on $\mathcal{U}$ such that G has a subgroup G_A of elations with center A of order q and a subgroup G_{AB} fixing both A and B of order a divisor of $q-1$ greater than* $2(\sqrt{q}-1)$.

Proof Suppose that $\mathcal{U}$ is an orthogonal Buekenhout–Metz unital with respect to a point A. The group G, as shown in [5] for q odd and in [14] for q even, has a subgroup G_A of elations with center A of order q and there exists a point B on $\mathcal{U}$ such that G has a subgroup G_{AB} of order $q-1$ fixing both A and B. Vice versa, if the group G satisfies the hypothesis, then from Theorem 3.4 we have that $\mathcal{U}$ is an ovoidal Buekenhout–Metz unital. Since q is either odd or $q\in\{2,4\}$, it follows that every ovoidal cone in PG($4,q$) is an elliptic cone (see [6, 20, 21]), thus $\mathcal{U}$ is an orthogonal Buekenhout–Metz unital. □

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