

Non-abelian representations of the slim dense near hexagons on 81 and 243 points

B. De Bruyn · B.K. Sahoo · N.S.N. Sastry

Received: 13 November 2009 / Accepted: 17 May 2010 / Published online: 3 June 2010
© Springer Science+Business Media, LLC 2010

Abstract We prove that the near hexagon $Q(5, 2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group 2_+^{1+12} and that the near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation in the extra-special 2-group 2_-^{1+18} . The description of the non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$ makes use of a new combinatorial construction of this near hexagon.

Keywords Near hexagon · Non-abelian representation · Extra-special 2-group

1 Introduction

Let $\mathcal{S} = (P, L)$ be a partial linear space with point set P and line set L . We suppose that \mathcal{S} is *slim*, i.e., that every line of \mathcal{S} is incident with precisely three points. For distinct points $x, y \in P$, we write $x \sim y$ if they are collinear. In that case, we denote by xy the unique line containing x and y and define $x * y$ by $xy = \{x, y, x * y\}$. For $x \in P$, we define $x^\perp := \{x\} \cup \{y \in P : y \sim x\}$. If $x, y \in P$, then $d(x, y)$ denotes the distance between x and y in the collinearity graph of \mathcal{S} .

B. De Bruyn (✉)
Department of Mathematics, Ghent University, Krijglaan 281 (S22), 9000 Gent, Belgium
e-mail: bdb@cage.ugent.be

B.K. Sahoo
School of Mathematical Sciences, National Institute of Science Education and Research,
Sainik School Post, Bhubaneswar 751005, Orissa, India
e-mail: bksahoo@niser.ac.in

N.S.N. Sastry
Statistics and Mathematics Unit, Indian Statistical Institute, Mysore Road, R. V. College Post,
Bangalore 560059, India
e-mail: nsastry@isibang.ac.in

A *representation* [9, p. 525] of \mathcal{S} is a pair (R, ψ) , where R is a group and ψ is a mapping from P to the set of involutions of R , satisfying:

- (R1) R is generated by the image of ψ .
- (R2) ψ is one–one on each line $\{x, y, x * y\}$ of \mathcal{S} and $\psi(x)\psi(y) = \psi(x * y)$.

Notice that if $x \sim y$, then $\psi(x)$ and $\psi(y)$ necessarily commute by Condition (R2). The group R is called a *representation group* of \mathcal{S} . A representation (R, ψ) of \mathcal{S} is *faithful* if ψ is injective and is *abelian* or *non-abelian* according as R is abelian or not. Note that, in [9], ‘non-abelian representation’ means that ‘the representation group is not necessarily abelian’. Abelian representations are called embeddings in the literature. For an abelian representation, the representation group is an elementary abelian 2-group and hence can be considered as a vector space over the field \mathbb{F}_2 with two elements. We refer to [8] and [12, Sects. 1 and 2] for more on representations of partial linear spaces with $p + 1$ points per line, where p is a prime.

A finite 2-group G is called *extra-special* if its Frattini subgroup $\Phi(G)$, its commutator subgroup $G' = [G, G]$ and its center $Z(G)$ coincide and have order 2. We refer to [5, Sect. 20, pp. 78–79] or [6, Chap. 5, Sect. 5] for the properties of extra-special 2-groups which we will mention now. An extra-special 2-group is of order 2^{1+2m} for some integer $m \geq 1$. Let D_8 and Q_8 , respectively, denote the dihedral and the quaternion groups of order 8. A non-abelian 2-group of order 8 is extra-special and is isomorphic to either D_8 or Q_8 . If G is an extra-special 2-group of order 2^{1+2m} , $m \geq 1$, then the exponent of G is 4 and either G is a central product of m copies of D_8 , or G is a central product of $m - 1$ copies of D_8 and one copy of Q_8 . If the former (respectively, latter) case occurs, then the extra-special 2-group is denoted by 2_+^{1+2m} (respectively, 2_-^{1+2m}).

A partial linear space $\mathcal{S} = (P, L)$ is called a *near polygon* if for every point p and every line L , there exists a unique point on L nearest to p . If d is the maximal distance between two points of \mathcal{S} , then the near polygon is also called a *near $2d$ -gon*. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. By [1], there are (up to isomorphism) 11 slim dense near hexagons. The paper [13] initiated the study of the non-abelian representations of these dense near hexagons.

Suppose (R, ψ) is a non-abelian representation of a slim dense near hexagon. Then by [13, Proposition 4.1, p. 205], (R, ψ) is necessarily faithful and for $x, y \in P$, $[\psi(x), \psi(y)] \neq 1$ if and only if x and y are at maximal distance 3 from each other. If \mathcal{S} is the (up to isomorphism) unique slim dense near hexagon on 81 points, which will be denoted by $Q(5, 2) \times \mathbb{L}_3$ in the sequel, then it was shown in [13, Theorem 1.6, p. 199] that R is necessarily isomorphic to the extra-special 2-group 2_+^{1+12} . If \mathcal{S} is the (up to isomorphism) unique slim dense near hexagon on 243 points, which will be denoted by $Q(5, 2) \otimes Q(5, 2)$ in the sequel, then it was shown in [13, Theorem 1.6, p. 199], that R is necessarily isomorphic to the extra-special 2-group 2_-^{1+18} . The question whether such non-abelian representations exist remained, however, unanswered in [13]. The following theorem, which is the main result of this paper, deals with these existence problems.

Theorem 1.1

- (1) The slim dense near hexagon $Q(5, 2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group 2_+^{1+12} .
- (2) The slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation in the extra-special 2-group 2_-^{1+18} .

The slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has many substructures isomorphic to $Q(5, 2) \times \mathbb{L}_3$. We will describe a non-abelian representation of $Q(5, 2) \times \mathbb{L}_3$ in Sect. 4. In Sect. 5, we will use this to construct a non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$. To describe the non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$, we make use of a model of $Q(5, 2) \otimes Q(5, 2)$ which we discuss in Sects. 2 and 3.

Remark Two other constructions of non-abelian representations of slim dense near polygons, in particular, of the slim dense near hexagons on 105 and 135 points, can be found in [10].

2 The point–line geometry \mathcal{S}_θ

Near quadrangles are usually called *generalized quadrangles* (GQs). A GQ is said to be of *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines. Up to isomorphism, there exist unique GQs of order $(2, 2)$ and $(2, 4)$, see, e.g., [11]. These GQs are denoted by $W(2)$ and $Q(5, 2)$, respectively. A *spread* of a point–line geometry is a set of lines partitioning its point set. A spread S of $Q(5, 2)$ is called a *spread of symmetry* if for every line $L \in S$ and every two points $x_1, x_2 \in L$, there exists an automorphism of $Q(5, 2)$ fixing each line of S and mapping x_1 to x_2 . By [2, Sect. 7.1], $Q(5, 2)$ has (up to isomorphism) a unique spread of symmetry.

Now, suppose S is a given spread of symmetry of $Q(5, 2)$. If L_1 and L_2 are two distinct lines of S and if G denotes the unique (3×3) -subgrid of $Q(5, 2)$ containing L_1 and L_2 , then the unique line L_3 of G disjoint from L_1 and L_2 is also contained in S .

Suppose θ is a map from $S \times S$ to \mathbb{Z}_3 (the additive group of order three) satisfying the following property:

- (*) If L_1, L_2, L_3 are three lines of S contained in a grid of $Q(5, 2)$, then $\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3)$.

Notice that $\theta(L, L) = 0$ and $\theta(M, L) = -\theta(L, M)$ for all $L, M \in S$. With θ , there is associated a point-line geometry \mathcal{S}_θ . The points of \mathcal{S}_θ are of four types:

- (P1) The points x of $Q(5, 2)$.
- (P2) The symbols \bar{x} , where x is a point of $Q(5, 2)$.
- (P3) The symbols $\bar{\bar{x}}$, where x is a point of $Q(5, 2)$.
- (P4) The triples (x, y, i) , where $i \in \mathbb{Z}_3$ and x, y are distinct collinear points of $Q(5, 2)$ satisfying $xy \in S$.

The lines of \mathcal{S}_θ are of nine types:

- (L1) The lines $\{x, y, z\}$ of $Q(5, 2)$.
- (L2) The sets $\{\bar{x}, \bar{y}, \bar{z}\}$, where $\{x, y, z\}$ is a line of $Q(5, 2)$.
- (L3) The sets $\{\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}\}$, where $\{x, y, z\}$ is a line of $Q(5, 2)$.
- (L4) The sets $\{x, \bar{x}, \bar{\bar{x}}\}$, where x is a point of $Q(5, 2)$.
- (L5) The sets $\{a, (a, b, i), (a, c, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
- (L6) The sets $\{\bar{a}, (b, a, i), (c, a, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
- (L7) The sets $\{\bar{\bar{a}}, (b, c, i), (c, b, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
- (L8) The sets $\{(a, b, i), (b, c, j), (c, a, k)\}$, where $\{i, j, k\} = \mathbb{Z}_3$ and $\{a, b, c\}$ is a line belonging to S .
- (L9) The sets $\{(a, u, i), (b, v, j), (c, w, k)\}$, where (i) $\{a, b, c\}$ and $\{u, v, w\}$ are two disjoint lines of $Q(5, 2)$; (ii) $d(a, u) = d(b, v) = d(c, w) = 1$; (iii) $au, bv, cw \in S$; (iv) $j = i + \theta(au, bv), k = i + \theta(au, cw)$.

Incidence is containment. One can easily show that \mathcal{S}_θ is a partial linear space. In order to show that two distinct points of \mathcal{S}_θ are contained in at most one line of Type (L9), one has to make use of Property (*).

3 An isomorphism $Q(5, 2) \otimes Q(5, 2) \cong \mathcal{S}_\theta$

The aim of this section is to show that the slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ is isomorphic to a point–line geometry \mathcal{S}_θ for a suitable spread of symmetry S of $Q(5, 2)$ and a suitable map $\theta : S \times S \rightarrow \mathbb{Z}_3$ satisfying Property (*). We start with recalling some known properties of the near hexagon $Q(5, 2) \otimes Q(5, 2)$.

- (1) Every two points x and y of $Q(5, 2) \otimes Q(5, 2)$ are contained in a unique convex subspace of diameter 2, called a *quad*. The points and lines which are contained in a given quad define a GQ which is isomorphic to either the (3×3) -grid or $Q(5, 2)$.
- (2) If Q is a $Q(5, 2)$ -quad and $x \notin Q$, then x is collinear with a unique point $\pi_Q(x) \in Q$ and we denote by $\mathcal{R}_Q(x)$ the unique point of $x\pi_Q(x)$ distinct from x and $\pi_Q(x)$. If $x \in Q$, then we define $\pi_Q(x) = \mathcal{R}_Q(x) := x$. The map $x \mapsto \mathcal{R}_Q(x)$ defines an automorphism of $Q(5, 2) \otimes Q(5, 2)$. If Q_1 and Q_2 are two disjoint $Q(5, 2)$ -quads, then the map $Q_1 \rightarrow Q_2; x \mapsto \pi_{Q_2}(x)$ defines an isomorphism between Q_1 and Q_2 .
- (3) There exist two partitions T_1 and T_2 of the point set of $Q(5, 2) \otimes Q(5, 2)$ into $Q(5, 2)$ -quads.
- (4) Every element of T_1 intersects every element of T_2 in a line. As a consequence, $S^\otimes := \{Q_1 \cap Q_2 : Q_1 \in T_1 \text{ and } Q_2 \in T_2\}$ is a spread of $Q(5, 2) \otimes Q(5, 2)$.
- (5) For every $Q \in T_i, i \in \{1, 2\}$, the set $\{Q \cap R : R \in T_{3-i}\}$ is a spread of symmetry of Q .
- (6) Every line L of $Q(5, 2) \otimes Q(5, 2)$ not belonging to S^\otimes is contained in a unique quad of $T_1 \cup T_2$.

Now, let Q and \bar{Q} be two disjoint $Q(5, 2)$ -quads belonging to T_1 and put $\bar{\bar{Q}} := \mathcal{R}_Q(\bar{Q}) = \mathcal{R}_{\bar{Q}}(Q)$. For every point x of Q , put $\bar{x} := \pi_{\bar{Q}}(x)$ and $\bar{\bar{x}} := \pi_{\bar{\bar{Q}}}(x)$.

Put $S = \{Q \cap Q_2 : Q_2 \in T_2\}$. Then S is a spread of symmetry of Q . For every $L \in S$, let R_L denote the unique element of T_2 containing L . Let L^* denote a specific

line of S and put $R^* = R_L^*$. For every $L \in S$, $R_L \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ is a (3×3) -subgrid σ_L of R_L . This (3×3) -subgrid σ_L is contained in precisely three $W(2)$ -subquadrangles of R_L . We denote by W^0, W^1, W^2 the three $W(2)$ -subquadrangles of R^* containing $R^* \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$. For every $L \in S$ and $i \in \mathbb{Z}_3$, put $W_L^i := \pi_{R_L}(W^i)$.

For every $i \in \mathbb{Z}_3$, for every $L \in S$ and for all $x, y \in L$ with $x \neq y$, we denote by (x, y, i) the unique point μ of $R_L \setminus (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ such that $\pi_Q(\mu) = x, \pi_{\overline{Q}}(\mu) = \bar{y}$ and $\mu \in W_L^i$. The point (x, y, i) is the unique point of W_L^i collinear with x and \bar{y} , but not contained in σ_L .

Lemma 3.1 *Every point of $Q(5, 2) \otimes Q(5, 2)$ not contained in $Q \cup \overline{Q} \cup \overline{\overline{Q}}$ has received a unique label.*

Proof Let μ be a point of $Q(5, 2) \otimes Q(5, 2)$ not contained in $Q \cup \overline{Q} \cup \overline{\overline{Q}}$, let R denote the unique element of T_2 containing μ and put $L := R \cap Q$. Then $R = R_L$. There exists a unique $W(2)$ -subquadrangle of R containing μ and σ_L . Let $i \in \mathbb{Z}_3$ such that $\mu \in W_L^i$. Let x and y be the points of L such that $x = \pi_Q(\mu)$ and $\bar{y} = \pi_{\overline{Q}}(\mu)$. If $x = y$, then $\{x, \bar{y}, \mu\}$ is a set of mutually collinear points, implying that $\mu = \bar{x}$, contradicting $\mu \notin \overline{\overline{Q}}$. Hence $x \neq y$ and the point μ has label (x, y, i) . It is also clear that μ cannot be labeled in different ways. □

We will now define a map $\theta : S \times S \rightarrow \mathbb{Z}_3$. For each ordered pair (L_1, L_2) of lines of S , the map $R^* \rightarrow R^*; x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ determines an automorphism of R^* fixing each line of the spread $\{R^* \cap Q_1 : Q_1 \in T_1\}$ of R^* . By [2, Theorem 4.1], such an automorphism either is trivial or acts on any line of the form $R^* \cap Q_1, Q_1 \in T_1$, as a cycle. Since every line $R^* \cap Q_1, Q_1 \in T_1 \setminus \{Q, \overline{Q}, \overline{\overline{Q}}\}$, intersects each $W(2)$ -subquadrangle $W^i, i \in \mathbb{Z}_3$, in a unique point, the map $R^* \rightarrow R^*; x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ is either trivial or permutes the elements of $\{W^0, W^1, W^2\}$ in one of the following ways:

$$W^0 \rightarrow W^1 \rightarrow W^2 \rightarrow W^0, \quad W^0 \rightarrow W^2 \rightarrow W^1 \rightarrow W^0.$$

Hence, there exists a unique $\theta(L_1, L_2) \in \mathbb{Z}_3$ such that

$$\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = W^{i+\theta(L_1, L_2)}$$

for every $i \in \mathbb{Z}_3$.

Lemma 3.2 *The following holds:*

- (i) For every $L \in S, \theta(L, L) = 0$.
- (ii) For any two lines L_1 and L_2 of $S, \theta(L_2, L_1) = -\theta(L_1, L_2)$.
- (iii) If L_1, L_2, L_3 are three lines of S which are contained in a grid, then $\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3)$.
- (iv) If L_1, L_2, L_3 are three lines of S which are not contained in a grid, then $\theta(L_1, L_2) + \theta(L_2, L_3) \neq \theta(L_1, L_3)$.

Proof

- (i) For every $i \in \mathbb{Z}_3$, we have $\pi_{R^*} \circ \pi_{R_L} \circ \pi_{R_L}(W^i) = \pi_{R^*} \circ \pi_{R_L}(W^i) = W^i$. Hence, $\theta(L, L) = 0$.
- (ii) If $\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = W^{i+\theta(L_1, L_2)}$ for every $i \in \mathbb{Z}_3$, then $W^i = \pi_{R^*} \circ \pi_{R_{L_1}} \circ \pi_{R_{L_2}}(W^{i+\theta(L_1, L_2)})$ for every $i \in \mathbb{Z}_3$. It follows that $\theta(L_2, L_1) = -\theta(L_1, L_2)$.
- (iii) Let L_1, L_2, L_3 be three lines of S which are contained in a grid. Then $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}}(W^{i+\theta(L_1, L_2)}) = W^{i+\theta(L_1, L_2)+\theta(L_2, L_3)}$. Hence, $\theta(L_1, L_3) = \theta(L_1, L_2) + \theta(L_2, L_3)$.
- (iv) Let L_1, L_2, L_3 be three lines of S which are not contained in a grid. Suppose that $\theta(L_1, L_3) = \theta(L_1, L_2) + \theta(L_2, L_3)$. Then for every $y \in R^*$, $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(y) = (\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}}) \circ (\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}})(y)$, that is, $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(y) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(y)$. Hence, the map $R_{L_3} \rightarrow R_{L_3}$ defined by $x \mapsto \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ is the identity map on R_{L_3} . This implies that the points $x, \pi_{R_{L_1}}(x), \pi_{R_{L_2}}(x)$ are mutually collinear for every $x \in R_{L_3}$, that is, $\{x, \pi_{R_{L_1}}(x), \pi_{R_{L_2}}(x)\}$ is a line for every $x \in R_{L_3}$. This contradicts the fact that L_1, L_2, L_3 are not contained in a grid. Hence, $\theta(L_1, L_3) \neq \theta(L_1, L_2) + \theta(L_2, L_3)$. □

Proposition 3.3 $Q(5, 2) \otimes Q(5, 2) \cong S_\theta$, where θ is as defined above.

Proof We must show that the set of lines of $Q(5, 2) \otimes Q(5, 2)$ are in bijective correspondence with the sets of Type $(L1), (L2), \dots, (L9)$ defined in Sect. 2. Obviously:

- The set of lines of $Q(5, 2) \otimes Q(5, 2)$ contained in Q correspond to the sets of Type $(L1)$.
- The set of lines of $Q(5, 2) \otimes Q(5, 2)$ contained in \overline{Q} correspond to the sets of Type $(L2)$.
- The set of lines of $Q(5, 2) \otimes Q(5, 2)$ contained in $\overline{\overline{Q}}$ correspond to the sets of Type $(L3)$.
- The set of lines of $Q(5, 2) \otimes Q(5, 2)$ meeting Q, \overline{Q} and $\overline{\overline{Q}}$ correspond to the sets of Type $(L4)$.

Consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point $a \in L$ of Q . Put $L = \{a, b, c\}$. There exists a unique $W(2)$ -subquadrangle W_L^i containing σ_L and M . One readily sees that the points of M have labels $a, (a, b, i)$ and (a, c, i) . So, M corresponds to a set of Type $(L5)$. Conversely, every set of Type $(L5)$ corresponds to a (necessarily unique) line of $Q(5, 2) \otimes Q(5, 2)$.

Next, consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point \bar{a} of \overline{Q} . Put $L = \{a, b, c\}$. Then, there exists a unique $W(2)$ -subquadrangle W_L^i containing σ_L and M . One readily sees that the points of M have labels $\bar{a}, (b, a, i)$ and (c, a, i) . So, M corresponds to a set of Type $(L6)$. Conversely, every set of Type $(L6)$ corresponds to a (necessarily unique) line of $Q(5, 2) \otimes Q(5, 2)$.

Now, consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point \bar{a} of \bar{Q} . Put $L = \{a, b, c\}$. Then, there exists a unique $W(2)$ -subquadrangle W_L^i containing σ_L and M . One readily sees that the points of M have labels \bar{a} , (b, c, i) and (c, b, i) . So, M corresponds to a set of Type (L7). Conversely, every set of Type (L7) corresponds to a (necessarily unique) line of $Q(5, 2) \otimes Q(5, 2)$.

Consider next a line M of R_L which is disjoint from σ_L . Then M intersects each W_L^i , $i \in \mathbb{Z}_3$, in a unique point. Put $L = \{a, b, c\}$. The labels of the points of M are (u, u', i) , (v, v', j) , (w, w', k) , where $\{i, j, k\} = \{0, 1, 2\}$, $\{u, v, w\} = \pi_Q(M) = \{a, b, c\}$, $\{u', v', w'\} = \pi_Q \circ \pi_{\bar{Q}}(M) = \{a, b, c\}$, $u \neq u'$, $v \neq v'$, $w \neq w'$. It readily follows that $\{(u, u', i), (v, v', j), (w, w', k)\}$ is a set of Type (L8). Conversely, one can readily verify that every set of Type (L8) corresponds to a line of $Q(5, 2) \otimes Q(5, 2)$.

Finally, let M be a line of $Q(5, 2) \otimes Q(5, 2)$ not belonging to S^\otimes and contained in a quad of $T_1 \setminus \{Q, \bar{Q}, \overline{\bar{Q}}\}$. With M , there corresponds a set of the form $\{(a, u, i), (b, v, j), (c, w, k)\}$. We have that $\{a, b, c\} = \pi_Q(M)$ is a line of Q not belonging to S . Similarly, $\{u, v, w\} = \pi_Q \circ \pi_{\bar{Q}}(M)$ is a line of Q not belonging to S . Moreover, we have that $au, bv, cw \in S$ and $j = i + \theta(au, bv)$, $k = i + \theta(au, cw)$ by the definition of the map θ . So, M corresponds to a set of Type (L9). Conversely, we show that every set $\{(a, u, i), (b, v, j), (c, w, k)\}$ of Type (L9) corresponds to a line of $Q(5, 2) \otimes Q(5, 2)$ not belonging to S^\otimes and contained in a quad of $T_1 \setminus \{Q, \bar{Q}, \overline{\bar{Q}}\}$. Let x denote the point of $Q(5, 2) \otimes Q(5, 2)$ corresponding to (a, u, i) , let Q_1 denote the unique element of T_1 containing x and let $M = \pi_{Q_1}(\{a, b, c\})$. Then M corresponds to a set of the form $\{(a, u, i), (b, *, *), (c, *, *)\}$. Since v, w, j, k are uniquely determined by a, u, i, b, c , this set is equal to $\{(a, u, i), (b, v, j), (c, w, k)\}$.

By the above discussion, we indeed know that $Q(5, 2) \otimes Q(5, 2) \cong S_\theta$. □

Definitions.

(1) An *admissible triple* is a triple $\Sigma = (\mathcal{L}, G, \Delta)$, where:

- G is a nontrivial additive group whose order $s + 1$ is finite.
- \mathcal{L} is a linear space, different from a point, in which each line is incident with exactly $s + 1$ points. We denote the point set of \mathcal{L} by P .
- Δ is a map from $P \times P$ to G such that the following holds for any three points x, y and z of \mathcal{L} : x, y and z are collinear $\Leftrightarrow \Delta(x, y) + \Delta(y, z) = \Delta(x, z)$.

(2) Suppose $\Sigma_1 = (\mathcal{L}_1, G_1, \Delta_1)$ and $\Sigma_2 = (\mathcal{L}_2, G_2, \Delta_2)$ are two admissible triples, where \mathcal{L}_1 and \mathcal{L}_2 are not lines. Then Σ_1 and Σ_2 are called *equivalent* if there exist an isomorphism α from \mathcal{L}_1 to \mathcal{L}_2 , an isomorphism β from G_1 to G_2 and a map f from the point set of \mathcal{L}_1 to G_1 satisfying $\Delta_2(\alpha(x), \alpha(y)) = (\beta(f(x) + \Delta_1(x, y) - f(y)))^\beta$ for all points x and y of \mathcal{L}_1 .

Let \mathcal{L}_S denote the linear space whose points are the elements of S and whose lines are the unordered triples of lines of S which are contained in a grid, with incidence being containment. Then \mathcal{L}_S is isomorphic to the affine plane $AG(2, 3)$ of order three. By Lemma 3.2, we know that $(\mathcal{L}_S, \mathbb{Z}_3, \theta)$ is an admissible triple.

Proposition 3.4 *Let θ_1 and θ_2 be two maps from $S \times S$ to \mathbb{Z}_3 such that $\Sigma_1 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_1)$ and $\Sigma_2 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_2)$ are admissible triples. If Σ_1 and Σ_2 are equivalent, then $\mathcal{S}_{\theta_1} \cong \mathcal{S}_{\theta_2}$.*

Proof Since Σ_1 and Σ_2 are equivalent, there exists an automorphism α of \mathcal{L}_S , an automorphism β of \mathbb{Z}_3 and a map f from S to \mathbb{Z}_3 satisfying $\theta_2(\alpha(x), \alpha(y)) = (f(x) + \theta_1(x, y) - f(y))^\beta$ for all points x and y of \mathcal{L}_S . There exists an automorphism ϕ of Q such that $\alpha(L) = \phi(L)$ for every line L of S ; see, e.g., [3, Sect. 3, Example 1]. One readily verifies that the map $x \mapsto x^\phi; \bar{x} \mapsto \overline{x^\phi}; \bar{\bar{x}} \mapsto \overline{\overline{x^\phi}}; (a, b, i) \mapsto (a^\phi, b^\phi, (i - f(ab))^\beta)$ defines an isomorphism between \mathcal{S}_{θ_1} and \mathcal{S}_{θ_2} . \square

It is known that the affine plane $AG(2, 3)$ admits, up to equivalence, a unique admissible triple. (This follows, for instance, from [4, Theorem 2.1] and the fact that there exists a unique generalized quadrangle of order $(2, 4)$, namely $Q(5, 2)$, and a unique spread of symmetry in $Q(5, 2)$.) If we coordinatize $AG(2, 3)$ in the standard way, then an admissible triple can be obtained by putting $\Delta[(x_1, y_1), (x_2, y_2)] := x_1y_2 - x_2y_1 \in \mathbb{Z}_3$.

4 A non-abelian representation of the near hexagon $Q(5, 2) \times \mathbb{L}_3$

The slim dense near hexagon $Q(5, 2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of $Q(5, 2)$ and joining the corresponding points to form lines of size 3. In this section, we prove that there exists a non-abelian representation of $Q(5, 2) \times \mathbb{L}_3$.

Let Q and B , respectively, be the point and line set of $Q(5, 2)$. Set $\overline{Q} = \{\bar{x} : x \in Q\}$, $\overline{\overline{Q}} = \{\bar{\bar{x}} : x \in Q\}$, $\overline{B} = \{\{\bar{x}, \bar{y}, \bar{z}\} : \{x, y, z\} \in B\}$ and $\overline{\overline{B}} = \{\{\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}\} : \{x, y, z\} \in B\}$. Then $(\overline{Q}, \overline{B})$ and $(\overline{\overline{Q}}, \overline{\overline{B}})$ are isomorphic to $Q(5, 2)$. The near hexagon $Q(5, 2) \times \mathbb{L}_3$ is isomorphic to the geometry whose point set P is $Q \cup \overline{Q} \cup \overline{\overline{Q}}$ and whose line set L is $B \cup \overline{B} \cup \overline{\overline{B}} \cup \{\{x, \bar{x}, \bar{\bar{x}}\} : x \in Q\}$.

It is known that if $Q(5, 2) \times \mathbb{L}_3$ admits a non-abelian representation, then the representation group must be the extra-special 2-group 2_+^{1+12} [13, Theorem 1.6, p. 199]. Let $R = 2_+^{1+12}$ with $R' = \{1, \lambda\}$. Set $V = R/R'$. Consider V as a vector space over \mathbb{F}_2 . The map $f : V \times V \rightarrow \mathbb{F}_2$ defined by

$$f(xR', yR') = \begin{cases} 0 & \text{if } [x, y] = 1, \\ 1 & \text{if } [x, y] = \lambda \end{cases}$$

for $x, y \in R$, is a non-degenerate symplectic bilinear form on V [5, Theorem 20.4, p. 78]. Write V as an orthogonal direct sum of six hyperbolic planes K_i ($1 \leq i \leq 6$) in V and let H_i be the inverse image of K_i in R (under the canonical homomorphism $R \rightarrow R/R'$). Then each H_i is generated by two involutions x_i and y_i such that $[x_i, y_i] = \lambda$. Let $M = \langle x_i : 1 \leq i \leq 6 \rangle$ and $\overline{M} = \langle y_i : 1 \leq i \leq 6 \rangle$. Then M and \overline{M} are elementary abelian 2-subgroups of R each of order 2^6 . Further, M, \overline{M} and $Z(R)$ pairwise intersect trivially and $R = M\overline{M}Z(R)$. Also, $C_M(\overline{M})$ and $C_{\overline{M}}(M)$ are trivial.

We regard the points and lines of Q as the points and lines of a nonsingular elliptic quadric of the projective space $PG(M)$, where M is regarded as a 6-dimensional vector space over \mathbb{F}_2 . Let (M, τ) be the natural abelian representation of (Q, B) associated with this embedding of Q in $PG(M)$. For every point x of Q , put $m_x = \tau(x)$. There exists a unique non-degenerate symplectic bilinear form g on M such that $m_x^\perp = \langle m_y : y \in x^\perp \rangle$ for every point x of Q ; see, e.g., [7, Sect. 22.3]. Here, the following notational convention has been used: for every $m \in M$, m^\perp denotes the set of all $m' \in M$ for which $g(m, m') = 0$.

Now, let m be an arbitrary element of M . If $m = 1$, then we define $\overline{m} := 1$. Suppose now that $m \neq 1$. Then m^\perp is maximal in M , that is, of index 2 in M . So, the centralizer of m^\perp in \overline{M} is a subgroup $\langle \overline{m} \rangle$ of order 2. Since m^\perp is maximal in M , $\langle m^\perp, m' \rangle = M$ for every $m' \in M \setminus m^\perp$. The triviality of $C_{\overline{M}}(M)$ then implies that $[\overline{m}, m'] = \lambda$ for every $m' \in M \setminus m^\perp$.

We prove that the map $M \rightarrow \overline{M}; m \mapsto \overline{m}$ is an isomorphism. This map is easily seen to be bijective. (Notice that $C_M(\overline{m}) = m^\perp$.) So, it suffices to prove that $\overline{m_1 m_2} = \overline{m_1} \overline{m_2}$ for all $m_1, m_2 \in M$. Clearly, this holds if $1 \in \{m_1, m_2\}$ or $m_1 = m_2$. So, we may suppose that $m_1 \neq 1 \neq m_2 \neq m_1$. The set $\{m_1, m_2, m_1 m_2\}$ corresponds to a line of $PG(M)$. So, for every $m \in (m_1 m_2)^\perp$, $([\overline{m_1}, m], [\overline{m_2}, m])$ is equal to either $(1, 1)$ or (λ, λ) . Then

$$[\overline{m_1} \overline{m_2}, m] = [\overline{m_1}, m][\overline{m_2}, m] = 1.$$

The first equality holds since R has nilpotency class 2. Thus $\overline{m_1} \overline{m_2} \in C_{\overline{M}}((m_1 m_2)^\perp) = \langle \overline{m_1 m_2} \rangle$. Since $\overline{m_1} \overline{m_2} \neq 1$, we have $\overline{m_1} \overline{m_2} = \overline{m_1 m_2}$.

We conclude that if we define $\overline{\tau} : \overline{Q} \rightarrow \overline{M}; \overline{x} \mapsto \overline{m_x}$ for every $x \in Q$, then $(\overline{M}, \overline{\tau})$ is a faithful abelian representation of $(\overline{Q}, \overline{B})$.

Now, let m be an arbitrary element of M . If $m = 1$, then we define $\overline{\overline{m}} := 1$. If $m = m_x$ for some $x \in Q$, then we define $\overline{\overline{m}} := m \overline{m}$. If $m \neq 1$ and $m \neq m_x, \forall x \in Q$, then we define $\overline{\overline{m}} := m \overline{m} \lambda$. Since $m^2 = \overline{m}^2 = \lambda^2 = [m, \overline{m}] = 1$, $\overline{\overline{m}}$ is an involution. We prove that the map $m \mapsto \overline{\overline{m}}$ defines an isomorphism between M and an elementary abelian 2-group $\overline{\overline{M}}$ of order 2^6 . Since $R = M \overline{M} Z(R)$, this map is injective and hence it suffices to prove that $\overline{\overline{m_1 m_2}} = \overline{\overline{m_1}} \overline{\overline{m_2}}$ for all $m_1, m_2 \in M$. Obviously, this holds if $1 \in \{m_1, m_2\}$ or $m_1 = m_2$. So, we may suppose that $m_1 \neq 1 \neq m_2 \neq m_1$. The set $\{m_1, m_2, m_1 m_2\}$ corresponds to a line of $PG(M)$. Suppose $3 - N$ elements of $\{m_1, m_2, m_1 m_2\}$ correspond to points of Q . Then $m_1 \in m_2^\perp$ if and only if N is even.¹ So, $[\overline{m_1}, m_2] = \lambda^N$. If N' is the number of elements of $\{m_1, m_2\}$ corresponding to points of Q , then $2 - N' - N \in \{-1, 0\}$ and $2 - N' - N = 0$ if and only if $m_1 m_2$ corresponds to a point of Q . Hence, $\overline{\overline{m_1}} \overline{\overline{m_2}} = m_1 \overline{m_1} m_2 \overline{m_2} \lambda^{2-N'} = m_1 m_2 \overline{m_1} \overline{m_2} \lambda^{2-N'-N} = m_1 m_2 \overline{m_1 m_2} \lambda^{2-N'-N} = \overline{\overline{m_1 m_2}}$.

So, if we define $\overline{\overline{\tau}} : \overline{\overline{Q}} \rightarrow \overline{\overline{M}}$ by putting $\overline{\overline{\tau}}(\overline{\overline{x}}) := \overline{\overline{m_x}} = m_x \overline{m_x}$ for all $x \in Q$, then $(\overline{\overline{M}}, \overline{\overline{\tau}})$ is a faithful abelian representation of $(\overline{\overline{Q}}, \overline{\overline{B}})$.

¹Perhaps the case $N = 3$ needs more explanation. Suppose $N = 3$ and $m_1 \in m_2^\perp$. Then the hyperplane π of $PG(M)$ corresponding to m_2^\perp intersects Q in a nonsingular parabolic quadric $Q(4, 2)$ of π . Since the point of $PG(M)$ corresponding to m_2 is the kernel of $Q(4, 2)$, the line of $PG(M)$ corresponding to $\{m_1, m_2, m_1 m_2\} \subset \pi$ must meet $Q(4, 2)$, in contradiction with $N = 3$.

Now, define a map $\psi : P \rightarrow R$ which coincides with τ on Q , $\bar{\tau}$ on \overline{Q} and $\overline{\bar{\tau}}$ on $\overline{\overline{Q}}$. Since $R = \langle M, \overline{M} \rangle$, $R = \langle \psi(P) \rangle$. By construction, (R, ψ) also satisfies Property (R2) in the definition of representation. Hence, (R, ψ) is a non-abelian representation of $Q(5, 2) \times \mathbb{L}_3$.

5 A non-abelian representation of the near hexagon $Q(5, 2) \otimes Q(5, 2)$

In this section, we prove that the slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation. By Proposition 3.3, this is equivalent with showing that the partial linear space S_θ has a non-abelian representation, where θ is as defined in Sect. 3.

We continue with the notation introduced in Sect. 3. Let M^* be a line of S^\otimes contained in R^* but distinct from $R^* \cap Q$, $R^* \cap \overline{Q}$ and $R^* \cap \overline{\overline{Q}}$. Then M^* intersects each W^i , $i \in \{1, 2, 3\}$, in a unique point. For every point x of L^* , put $\epsilon(x) := i$ if the unique point of M^* collinear with x belongs to W^i . If $y \in Q \setminus L^*$, then we define $\epsilon(y) := \epsilon(x)$, where x is the unique point of L^* collinear with y .

Lemma 5.1 *Let L_1 and L_2 be two distinct lines in S and let $\alpha_i \in L_i$, $i \in \{1, 2\}$. Then $\alpha_1 \sim \alpha_2$ if and only if $\epsilon(\alpha_2) - \epsilon(\alpha_1) = \theta(L_1, L_2)$.*

Proof Let α'_2 be the unique point of L_2 collinear with α_1 , let x_1 and x_2 be the unique points of L^* nearest to α_1 and α'_2 , respectively, and let z_i , $i \in \{1, 2\}$, denote the unique point of M^* collinear with x_i . The automorphism $R^* \rightarrow R^*$; $x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ of R^* maps x_1 to x_2 and hence z_1 to z_2 . This implies that $W^{\epsilon(x_1) + \theta(L_1, L_2)} = W^{\epsilon(x_2)}$, i.e., $\theta(L_1, L_2) = \epsilon(x_2) - \epsilon(x_1) = \epsilon(\alpha'_2) - \epsilon(\alpha_1)$. Hence, $\alpha_1 \sim \alpha_2$ if and only if $\alpha_2 = \alpha'_2$, i.e., if and only if $\epsilon(\alpha_2) - \epsilon(\alpha_1) = \theta(L_1, L_2)$. □

Lemma 5.2 *Let $N = 2^{1+6}_-$ with $N' = \{1, \lambda\}$ and let $I_2(N)$ be the set of involutions in N . Then there exists a map δ from Q to $I_2(N)$ satisfying the following:*

- (i) δ is one-one.
- (ii) For $x, y \in Q$, $[\delta(x), \delta(y)] = 1$ if and only if $y \in x^\perp$.
- (iii) If $x, y \in Q$ with $x \sim y$, then

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in S, \\ \delta(x)\delta(y)\lambda & \text{if } xy \notin S. \end{cases}$$

- (iv) The image of δ generates N .

Proof We use a model for the generalized quadrangle $Q \cong Q(5, 2)$ which is described in [11, Sect. 6.1, pp. 101–102]. Put $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Let \mathcal{E} be the set of all 2-subsets of Ω and let \mathcal{F} be the set of all partitions of Ω in three 2-subsets of Ω . Then the point set of Q can be identified with the set $\mathcal{E} \cup \Omega \cup \Omega'$ and the line set of Q can be identified with the set

$\mathcal{F} \cup \{\{i, \{i, j\}, j'\} : 1 \leq i, j \leq 6, i \neq j\}$. Now, consider the following nine lines of Q :

$$\begin{aligned} L_1 &= \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}; & L_2 &= \{\{1, 4\}, 1, 4'\}; & L_3 &= \{\{2, 6\}, 2, 6'\}; \\ L_4 &= \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}; & L_5 &= \{\{1, 5\}, 1', 5\}; & L_6 &= \{\{2, 3\}, 2', 3\}; \\ L_7 &= \{\{1, 3\}, \{2, 5\}, \{4, 6\}\}; & L_8 &= \{\{3, 6\}, 3', 6\}; & L_9 &= \{\{4, 5\}, 4, 5'\}. \end{aligned}$$

These 9 lines are mutually disjoint and hence determine a spread S' of Q . Any two distinct lines L_i and L_j of S' are contained in a unique (3×3) -subgrid and the unique line of this subgrid disjoint from L_i and L_j also belongs to S' . A spread of $Q(5, 2)$ having this property is called *regular*. Since any regular spread of $Q(5, 2)$ is also a spread of symmetry [2, Sect. 7.1], and there exists up to isomorphism a unique spread of symmetry in $Q(5, 2)$, we may without loss of generality suppose that $S = S'$.

Put $N = \langle a, b \rangle \circ \langle c, d \rangle \circ Q_8$, where a, b, c, d are involutions and $\langle a, b \rangle \cong \langle c, d \rangle \cong D_8$. So, $[a, b] = [c, d] = \lambda$. Take $Q_8 = \{1, \lambda, i, j, k, i\lambda, j\lambda, k\lambda\}$, where $i^2 = j^2 = k^2 = \lambda, ij = k, jk = i, ki = j$ and $[i, j] = [j, k] = [k, i] = \lambda$. We define $\delta : Q \rightarrow I_2(N)$ as follows:

$$\begin{aligned} \delta(\{1, 2\}) &= a, & \delta(\{3, 4\}) &= c, & \delta(\{5, 6\}) &= ac, \\ \delta(\{1, 4\}) &= abdi, & \delta(1) &= cdj, & \delta(4') &= abck\lambda, \\ \delta(\{2, 6\}) &= abi\lambda, & \delta(2) &= acdk, & \delta(6') &= bcdj\lambda, \\ \delta(\{1, 6\}) &= b, & \delta(\{2, 4\}) &= bd, & \delta(\{3, 5\}) &= d, \\ \delta(\{1, 5\}) &= abci, & \delta(1') &= cdk\lambda, & \delta(5) &= abdj, \\ \delta(\{2, 3\}) &= bcdi\lambda, & \delta(2') &= acdj\lambda, & \delta(3) &= abk, \\ \delta(\{1, 3\}) &= abcd\lambda, & \delta(\{2, 5\}) &= bc\lambda, & \delta(\{4, 6\}) &= ad\lambda, \\ \delta(\{3, 6\}) &= acdi\lambda, & \delta(3') &= abj\lambda, & \delta(6) &= bcdk, \\ \delta(\{4, 5\}) &= cdi, & \delta(4) &= abcj, & \delta(5') &= abdk\lambda. \end{aligned}$$

Put $W = N/N'$. Suppose $\{x_1, x_2, \dots, x_6\}$ is a set of 6 points of Q such that the smallest subspace $[x_1, x_2, \dots, x_6]$ of Q containing $\{x_1, x_2, \dots, x_6\}$ coincides with Q . If τ is an abelian representation of Q in W , then by Property (R1) in the definition of representation, $W = \langle \tau(x_1), \dots, \tau(x_6) \rangle$ and hence $\{\tau(x_1), \dots, \tau(x_6)\}$ is a basis of W (regarded as \mathbb{F}_2 -vector space). Conversely, if $\{w_1, \dots, w_6\}$ is a basis of W , then the map $x_i \mapsto w_i, i \in \{1, \dots, 6\}$, can be extended to a unique abelian representation τ of Q in W . (Since there exists an abelian representation of Q in W , there must exist an abelian representation τ for which $\tau(x_i) = w_i, i \in \{1, \dots, 6\}$. The uniqueness of τ follows from the fact that $\tau(y_1 * y_2) = \tau(y_1)\tau(y_2)$ for any two distinct collinear points y_1 and y_2 of Q .) Consider now the special case where $x_1 = \{1, 2\}, x_2 = \{3, 4\}, x_3 = \{3, 5\}, x_4 = \{1, 6\}, x_5 = \{4, 5\}, x_6 = 1, w_1 = aN', w_2 = cN', w_3 = dN', w_4 = bN', w_5 = cdiN'$ and $w_6 = cdjN'$. One indeed readily verifies that $[x_1, \dots, x_6] = Q$ and that $\{w_1, \dots, w_6\}$ is a basis of the \mathbb{F}_2 -vector space W . Let δ' denote the unique abelian representation of Q in W for which $\delta'(x_i) = w_i, i \in \{1, \dots, 6\}$. Then, using

the fact that $\delta'(y_1 * y_2) = \delta'(y_1)\delta'(y_2)$ for any two distinct collinear points y_1 and y_2 of Q , one can verify that $\delta'(y) = \delta(y)N'$ for every $y \in Q$. This implies that $\delta(y_1 * y_2)$ is equal to either $\delta(y_1)\delta(y_2)$ or $\delta(y_1)\delta(y_2)\lambda$ for any two distinct collinear points y_1 and y_2 of Q .

Clearly, the map $\delta : Q \rightarrow I_2(N)$ satisfies the properties (i) and (iv) of the lemma. We will now prove that also property (ii) of the lemma is satisfied. So, if $\{y_1, y_2\}$ is one of the 351 unordered pairs of distinct points of Q , then we need to prove that $[\delta(y_1), \delta(y_2)] = 1$ if and only if $y_1 \in y_2^\perp$. Since $Q = [x_1, \dots, x_6]$, it suffices to prove the following three statements: (I) the above claim holds if $\{y_1, y_2\} \subseteq \{x_1, \dots, x_6\}$; (II) if $[\delta(y_1), \delta(y_2)] = 1$ for some distinct collinear points y_1 and y_2 , then also $[\delta(y_1), \delta(y_1 * y_2)] = 1$; (III) if the above claim holds for unordered pairs $\{y_1, y_2\}$ and $\{y_1, y_3\}$ of points where $y_2 \sim y_3$ and $y_1 \notin y_2y_3$, then it also holds for the unordered pair $\{y_1, y_2 * y_3\}$. Statement (I) is easily verified by considering all 15 pairs $\{x_i, x_j\}$ where $i, j \in \{1, \dots, 6\}$ with $i \neq j$. As to Statement (II), notice that $[\delta(y_1), \delta(y_1 * y_2)]$ is equal to either $[\delta(y_1), \delta(y_1)\delta(y_2)]$ or $[\delta(y_1), \delta(y_1)\delta(y_2)\lambda]$ which is in any case equal to 1. We now prove Statement (III). Since $\delta(y_2 * y_3)$ is equal to either $\delta(y_2)\delta(y_3)$ or $\delta(y_2)\delta(y_3)\lambda$, we have $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)\delta(y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)]$. If y_1 is collinear with precisely one of y_2, y_3 , then y_1 is not collinear with $y_2 * y_3$ and $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = 1 \cdot \lambda = \lambda$. If y_1 is collinear with $y_2 * y_3$, then $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = \lambda \cdot \lambda = 1$. So, this proves Statement (III) and finishes the proof of property (ii) of the lemma.

Property (iii) of the lemma is verified by considering all 45 lines L of Q and an ordered pair (x, y) of distinct points of L . Notice that by property (ii) of the lemma, we only need to consider one ordered pair (x, y) for each line L of Q . □

It is known that if the near hexagon $Q(5, 2) \otimes Q(5, 2)$ admits a non-abelian representation, then the representation group must be the extra-special 2-group 2_-^{1+18} [13, Theorem 1.6, p. 199]. We next construct a non-abelian representation of $S_\theta \cong Q(5, 2) \otimes Q(5, 2)$ in the group 2_-^{1+18} .

Let $R = 2_-^{1+18}$ with $R' = \{1, \lambda\}$. Write R as a central product $R = M \circ N$, where $M = 2_+^{1+12}$ and $N = 2_-^{1+6}$. Let $Y = Q \cup \overline{Q} \cup \overline{\overline{Q}}$. Then the subgeometry of S_θ whose point set is Y together with the lines of types (L1)–(L4) is isomorphic to $Q(5, 2) \times \mathbb{L}_3$. Let P be the point set of S_θ and let δ be a map from Q to $I_2(N)$ satisfying the conditions of Lemma 5.2. We extend δ to the set $P \setminus Y$ using the map $\epsilon : Q \rightarrow \mathbb{Z}_3$ which we defined in the beginning of this section:

For $L_1 \in S$, distinct points $a, b \in L_1$ and $j \in \mathbb{Z}_3$, we define $\delta(a, b, j) := \delta(u)$, where u is the unique point of L_1 with $\epsilon(u) = j$.

Now, fix a non-abelian representation (M, ϕ) of Y . Such a representation exists by Sect. 4. Let ψ be the following map from P to R :

- If $q \in Y$, then $\psi(q) := \phi(q)$.
- If $q = (a, b, i) \in P \setminus Y$, then $\psi(q) = \psi(a, b, i) := \phi(b)\phi(\bar{a})\delta(a, b, i)$.

We prove the following.

Theorem 5.3 (R, ψ) is a non-abelian representation of S_θ .

Proof Since the image of ϕ generates M and the image of δ generates N , we have $R = \langle \psi(P) \rangle$. For every line $L_1 \in S$ and distinct $a, b \in L_1$, we have $[\phi(a), \phi(\bar{b})] = 1$, since a and \bar{b} are at distance two from each other. This implies that $\psi(q)$ is an involution for every $q \in P$. We need to verify Condition (R2) in the definition of representation. This is true for all lines of types (L1)–(L4), since they are also lines of Y and ψ coincides with ϕ on Y .

Let $\{a, (a, b, i), (a, c, i)\}$ be a line of type (L5). Since $\delta(a, b, i) = \delta(a, c, i)$, we have $\psi(a, b, i)\psi(a, c, i) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{a}) = \phi(b)\phi(c) = \phi(a) = \psi(a)$. Similar argument holds for lines of types (L6) and (L7).

Next, consider a line $\{(a, b, i), (b, c, j), (c, a, k)\}$ of type (L8). We have $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{b})\delta(a, b, i)\delta(b, c, j)$. Since $\{i, j, k\} = \mathbb{Z}_3$, $\{\delta(a, b, i), \delta(b, c, j), \delta(c, a, k)\} = \{\delta(a), \delta(b), \delta(c)\}$. Since $\{a, b, c\} \in S$, Lemma 5.2(iii) implies that $\delta(a, b, i)\delta(b, c, j) = \delta(c, a, k)$. So, $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(c)\phi(\bar{a})\phi(\bar{b})\delta(c, a, k) = \phi(a)\phi(\bar{c})\delta(c, a, k) = \psi(c, a, k)$. Notice that the second equality holds since $\{a, b, c\}$ and $\{\bar{a}, \bar{b}, \bar{c}\}$ are lines of Y .

Finally, consider a line $\{(a, u, i), (b, v, j), (c, w, k)\}$ of type (L9). Here the lines au, bv, cw are in S , $j = i + \theta(au, bv)$ and $k = i + \theta(au, cw)$. Let $\delta(a, u, i) = \delta(\alpha)$, $\delta(b, v, j) = \delta(\beta)$ and $\delta(c, w, k) = \delta(\gamma)$, where $\alpha \in au, \beta \in bv$ and $\gamma \in cw$. So $\epsilon(\alpha) = i, \epsilon(\beta) = j$ and $\epsilon(\gamma) = k$. Since $\epsilon(\beta) - \epsilon(\alpha) = j - i = \theta(au, bv)$, Lemma 5.1 implies that $\alpha \sim \beta$. Similarly, $\alpha \sim \gamma$. Thus $\{\alpha, \beta, \gamma\}$ is a line of Q not contained in S . Then by Lemma 5.2(iii), $\delta(a, u, i)\delta(b, v, j) = \delta(\alpha)\delta(\beta) = \delta(\gamma)\lambda = \delta(c, w, k)\lambda$. So $\psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(\bar{a})\phi(v)\phi(\bar{b})\delta(c, w, k)\lambda$. Since v and \bar{a} are at distance three from each other, $[\phi(\bar{a}), \phi(v)] = \lambda$. So $\phi(\bar{a})\phi(v) = \phi(v)\phi(\bar{a})[\phi(\bar{a}), \phi(v)] = \phi(v)\phi(\bar{a})\lambda$. Then $\psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(v)\phi(\bar{a})\phi(\bar{b})\delta(c, w, k) = \phi(w)\phi(\bar{c})\delta(c, w, k) = \psi(c, w, k)$. The second equality holds since $\{\bar{a}, \bar{b}, \bar{c}\}$ and $\{u, v, w\}$ are lines of Y . This completes the proof. □

References

1. Brouwer, A.E., Cohen, A.M., Hall, J.I., Wilbrink, H.A.: Near polygons and Fischer spaces. *Geom. Dedic.* **49**, 349–368 (1994)
2. De Bruyn, B.: Generalized quadrangles with a spread of symmetry. *Eur. J. Comb.* **20**, 759–771 (1999)
3. De Bruyn, B.: On the number of nonisomorphic glued near hexagons. *Bull. Belg. Math. Soc. Simon Stevin* **7**, 493–510 (2000)
4. De Bruyn, B., Thas, K.: Generalized quadrangles with a spread of symmetry and near polygons. *III. J. Math.* **46**, 797–818 (2002)
5. Doerk, K., Hawkes, T.: *Finite Soluble Groups*. de Gruyter Expositions in Mathematics, vol. 4. Walter de Gruyter, Berlin (1992)
6. Gorenstein, D.: *Finite Groups*. Chelsea, New York (1980)
7. Hirschfeld, J.W.P., Thas, J.A.: *General Galois Geometries*. Oxford Mathematical Monographs. Oxford Science Publications. Clarendon, New York (1991)
8. Ivanov, A.A.: Non-abelian representations of geometries. In: “Groups and Combinatorics”—In Memory of Michio Suzuki. *Adv. Stud. Pure Math.*, vol. 32, pp. 301–314. Math. Soc. Japan, Tokyo (2001)
9. Ivanov, A.A., Pasechnik, D.V., Shpectorov, S.V.: Non-abelian representations of some sporadic geometries. *J. Algebra* **181**, 523–557 (1996)
10. Patra, K.L., Sahoo, B.K.: A non-abelian representation of the dual polar space $DQ(2n, 2)$. *Innov. Incidence Geom.* **9**, 177–188 (2009)

11. Payne, S.E., Thas, J.A.: *Finite Generalized Quadrangles*, 2nd edn. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich (2009)
12. Sahoo, B.K., Sastry, N.S.N.: A characterization of finite symplectic polar spaces of odd prime order. *J. Comb. Theory, Ser. A* **114**, 52–64 (2007)
13. Sahoo, B.K., Sastry, N.S.N.: On the order of a non-abelian representation group of a slim dense near hexagon. *J. Algebraic Comb.* **29**, 195–213 (2009)