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Explicit formulae for Kerov polynomials

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Abstract We prove two formulae expressing the Kerov polynomial Σ_k as a weighted sum over the set of noncrossing partitions of the set $\{1, \dots, k+1\}$. We also give a combinatorial description of a family of symmetric functions specializing in the coefficients of Σ_k.

Keywords Kerov polynomials · Noncrossing partitions · Symmetric group · Normalized characters · Symmetric functions

1 Introduction

The nth free cumulant R_n can be thought of as a function $R_n : \lambda \in \mathcal{Y} \mapsto R_n(\lambda) \in \mathbb{Z}$, defined on the set of all Young diagrams $\mathcal{Y}$, which we identify with the corresponding integer partitions, and taking integer values. Indeed, after a suitable representation of a Young diagram λ as a function in the plane $\mathbb{R}^2$ [4], it is possible to determine the sequences of integers $x_0, \dots, x_m$ and $y_1, \dots, y_m$, consisting of the x-coordinates of the local minima and maxima of λ, respectively. In this way, if we set

$$\mathcal{H}_\lambda(z) = \frac{\prod_{i=0}^{m}(z - x_i)}{\prod_{i=1}^{m}(z - y_i)},$$

then $R_n(\lambda)$ is the coefficient of z^{n-1} in the formal Laurent series expansion of $\mathcal{K}_\lambda(z)$ such that $\mathcal{K}_\lambda(\mathcal{H}_\lambda(z)) = \mathcal{H}_\lambda(\mathcal{K}_\lambda(z)) = z$. It can be shown that $R_1(\lambda) = 0$ for all λ.

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So, the kth Kerov polynomial is a polynomial $\Sigma_k(R_2,\ldots,R_{k+1})$ which satisfies the following identity,

$$\Sigma_k\big(R_2(\lambda),\ldots,R_{k+1}(\lambda)\big)=(n)_k\frac{\chi^\lambda(k,1^{n-k})}{\chi^\lambda(1^n)},$$

where $\chi^\lambda(k,1^{n-k})$ denotes the value of the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by the partition λ on k-cycles. Two remarkable properties of Σ_k have to be stressed. First, it is a "universal polynomial", that is, it depends neither on λ nor on n. Second, its coefficients are nonnegative integers. A combinatorial proof of the positivity of Σ_k is quite recent and is due to Féray [8]. An explicit combinatorial description of such coefficients is due to Dołęga, Féray and Śniady [6]. Until now, several results on Kerov polynomials have been proved and conjectured; see, for instance, [5, 10, 12, 17] and [9] for a more detailed treatment.

Originally, free cumulants arise in the noncommutative context of free probability theory [15]. To the best of our knowledge, their earliest applications in the asymptotic character theory of the symmetric group are due to Biane; see, for instance, [3]. In 1994, Speicher [16] showed that the formulae connecting moments and free cumulants of a noncommutative random variable X obey the Möbius inversion on the lattice of noncrossing partitions of a finite set. This result highlights the strong analogy between free cumulants and classical cumulants, which are related to the moments of a random variable X, defined on a classical probability space, via the Möbius inversion on the lattice of all partitions of a finite set. More recently, Di Nardo, Petrullo and Senato [7] have shown how the classical umbral calculus provides an alternative setting for the cumulant families which passes through a generalization of the Abel polynomials.

In 1997, it was again Biane [2] who showed that the lattice NC_n of noncrossing partitions of $\{1,\ldots,n\}$ can be embedded into the Cayley graph of the symmetric group $\mathfrak{S}_n$. Thus it seems reasonable that a not too complicated expression of the Kerov polynomials involving noncrossing partitions, or the Cayley graph of $\mathfrak{S}_n$, should exist. In particular, such a formula, conjectured in [4], appeared with a rather implicit description in [6, 8].

In this paper, we state two explicit formulae relating Σ_k and the set NC_{k+1} of noncrossing partitions of $\{1,\ldots,k+1\}$. More precisely, if NC^{irr}_{k+1} denotes the subset of NC_{k+1} of partitions having 1 and $k+1$ in the same block, then a new partial order $\le^{\mathrm{irr}}$ on NC^{irr}_{k+1} is considered, thanks to which we have

$$\Sigma_k=\sum_{\tau\in NC^{\mathrm{irr}}_{k+1}}\Bigg[\sum_{\tau\le^{\mathrm{irr}}\pi}(-1)^{\ell_\pi-1}W_\tau(\pi)\Bigg]R_{\dot\tau}.$$

Here, ℓ_π is the number of blocks of π, $W_\tau(\pi)$ is a suitable weight depending on τ and π, and $R_{\dot\tau}=\prod_B R_{|B|}$, with B ranging over the blocks of τ having at least 2 elements. The special structure of the weight $W_\tau(\pi)$ allows us to give a combinatorial description of the symmetric functions $\boldsymbol{g}_\mu(x_1,\ldots,x_k)$'s, that evaluated at $x_i=i$ return the coefficient of $\prod_{i\ge 2}R_i^{m_i}$ in Σ_k, for every integer partition μ of size $k+1$ having m_i parts equal to i.

A second formula expressing Σ_k as a weighted sum over the whole NC_{k+1} is proved by means of the notion of irreducible components of a noncrossing partition. In particular, if d_τ is the number of irreducible components of τ, then we have

$$\Sigma_k = \sum_{\tau \in NC_{k+1}} \big[(-1)^{d_\tau - 1} V_\tau\big] R_{\dot{\tau}},$$

where V_τ is a suitable weight depending on τ.

2 Kerov polynomials

Let n be a positive integer and let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be an integer partition of size n, that is, $1 \le \lambda_1 \le \cdots \le \lambda_l$ and $\sum_i \lambda_i = n$. Denote by $\mathcal{Y}_n$ the set of all Young diagrams of size n, and set $\mathcal{Y} = \bigcup_n \mathcal{Y}_n$. From now on, an integer partition and its Young diagram will be denoted by the same symbol λ. Moreover, as is usual we write $\lambda \vdash n$ if λ is an integer partition of size n.

After a suitable representation of a Young diagram λ as a function in the plane $\mathbb{R}^2$ [4], it is possible to determine the sequences of integers $x_0, \ldots, x_m$ and $y_1, \ldots, y_m$, consisting of the x-coordinates of its local minima and maxima, respectively. Then, by expanding the rational function

$$\mathcal{H}_\lambda(z) = \frac{\prod_{i=0}^m (z - x_i)}{\prod_{i=1}^m (z - y_i)}$$

as a formal power series in z^{-1} one has $\mathcal{H}_\lambda(z) = z^{-1} + \sum_{n\ge 1} M_n(\lambda)\, z^{-(n+1)}$. The integer $M_n(\lambda)$ is said to be the nth *moment* of λ. Now, define $\mathcal{K}_\lambda(z) = \mathcal{H}_\lambda^{\langle -1\rangle}(z)$, that is $\mathcal{K}_\lambda(\mathcal{H}_\lambda(z)) = \mathcal{H}_\lambda(\mathcal{K}_\lambda(z)) = z$, and consider its expansion as a formal Laurent series, $\mathcal{K}_\lambda(z) = z^{-1} + \sum_{n\ge 1} R_n(\lambda)\, z^{n-1}$. Then, the integer $R_n(\lambda)$ is named the nth *free cumulant* of λ. It is not difficult to see that $M_1(\lambda) = R_1(\lambda) = 0$ for all λ. By setting $\mathcal{M}_\lambda(z) = z^{-1}\mathcal{H}_\lambda(z^{-1})$ and $\mathcal{R}_\lambda(z) = z\mathcal{K}_\lambda(z)$, we obtain two formal power series in z, $\mathcal{M}_\lambda(z) = 1 + \sum_{n\ge 1} M_n(\lambda)\, z^n$ and $\mathcal{R}_\lambda(z) = 1 + \sum_{n\ge 1} R_n(\lambda)\, z^n$, such that

$$\mathcal{M}_\lambda(z) = \mathcal{R}_\lambda\big(z\,\mathcal{M}_\lambda(z)\big). \tag{2.1}$$

Let $\mu \vdash n$ and denote by $\chi^\lambda(\mu)$ the value of the irreducible character of $\mathfrak{S}_n$ indexed by λ on the permutations of type μ. So that, if $\mu = (k, 1^{n-k})$, that is, $\mu_1 = k$ and $\mu_2 = \cdots = \mu_{n-k+1} = 1$, then the value $\widehat{\chi}^\lambda(k, 1^{n-k})$ of the normalized character indexed by λ on the k-cycles of $\mathfrak{S}_n$ is given by

$$\widehat{\chi}^\lambda\big(k, 1^{n-k}\big) = (n)_k \frac{\chi^\lambda(k, 1^{n-k})}{\chi^\lambda(1^n)},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. The kth Kerov polynomial is a polynomial Σ_k in k commuting variables which satisfies the following interesting identity,

$$\Sigma_k\big(R_2(\lambda), \ldots, R_{k+1}(\lambda)\big) = \widehat{\chi}^\lambda\big(k, 1^{n-k}\big).$$

If we think of $R_n(\lambda)$ as the image of a map $R_n : \lambda \in \mathcal{Y} \mapsto R_n(\lambda) \in \mathbb{Z}$, then also Kerov polynomials become maps $\Sigma_k = \Sigma_k(R_2, \dots, R_{k+1})$, which are polynomials in the R_n's, such that $\Sigma_k(\lambda) = \widehat{\chi}^\lambda(k, 1^{n-k})$. Since the coefficients of Σ_k depend neither on λ nor on n, but only on k, such polynomials are said to be "universal". A second remarkable property of Kerov polynomials is that all their coefficients are positive integers. This fact is known as the "Kerov conjecture" [11]. The first proof of the Kerov conjecture was given with combinatorial methods by Féray [8]. By using rather different techniques, the same author with Dołęga and Śniady [6] have then stated an explicit combinatorial interpretation of such coefficients. The following formula for Σ_k can be found in Stanley [17]. It is also stated in Biane [4], where the author quotes it as a private communication with A. Okounkov.

Theorem 2.1 *Let $\mathcal{R}(z) = 1 + \sum_{n\geq 2} R_n z^n$. If $\mathcal{F}(z) = \frac{z}{\mathcal{R}(z)}$ and $\mathcal{G}(z) = \frac{z}{\mathcal{F}^{\langle -1\rangle}(z^{-1})}$, then we have*

$$\Sigma_k = -\frac{1}{k}[z^{-1}]_\infty \prod_{j=0}^{k-1} \mathcal{G}(z-j). \tag{2.2}$$

More precisely, if $[z^n]f(z)$ denotes the coefficient of z^n in the formal power series $f(z)$, then $[z^{-1}]_\infty f(z) = [z]f(z^{-1})$. This way, Identity (2.2) states that Σ_k is obtained by expressing the right-hand side in terms of the free cumulants R_n's. Moreover, thanks to the invariance of the residue under translation of the variable, if $\mathcal{M}(z) = 1 + \sum_{n\geq 1} M_n z^n$, then by virtue of (2.1) we have $z\,\mathcal{G}(z)^{-1} = \mathcal{M}(z^{-1})$, and (2.2) can be rewritten in the following equivalent form,

$$\Sigma_k = -\frac{1}{k}[z^{k+1}] \prod_{j=1}^{k} \frac{1-jz}{\mathcal{M}(\frac{z}{1-jz})}. \tag{2.3}$$

For all $j = 1, \dots, k$, we denote by $\lambda \boxplus j$ the image of the diagram λ under the translation of the plane given by $x \mapsto x + j$. The ith local minimum and maximum of $\lambda \boxplus j$ are $x_i + j$ and $y_i + j$, respectively, so that

$$\mathcal{H}_{\lambda\boxplus j}(z) = \frac{\prod_{i=0}^m z-(x_i+j)}{\prod_{i=1}^m z-(y_i+j)} \quad \text{and} \quad \mathcal{M}_{\lambda\boxplus j}(z) = \frac{1}{1-jz}\mathcal{M}_\lambda\left(\frac{z}{1-jz}\right).$$

In this way, we may rewrite (2.3) as follows:

$$\Sigma_k = -\frac{1}{k}[z^{k+1}] \prod_{j=1}^{k} \frac{1}{\mathcal{M}_{\lambda\boxplus j}(z)}. \tag{2.4}$$

Denote by $\mathcal{R}_{\lambda\boxplus j}(z)$ the formal power series such that $\mathcal{M}_{\lambda\boxplus j}(z) = \mathcal{R}_{\lambda\boxplus j}(z\mathcal{M}_{\lambda\boxplus j}(z))$. It is immediate to verify that

$$\mathcal{R}_{\lambda\boxplus j}(z) = jz + \mathcal{R}_\lambda(z). \tag{2.5}$$

3 Irreducible noncrossing partitions

A partition of a finite set S is an unordered sequence $\tau = \{A_1, \dots, A_l\}$ of pairwise disjoint nonempty subsets of S, called the blocks of τ, such that $\bigcup_i A_i = S$. We say that τ *refines* π, written $\tau \le \pi$, if and only if each block of π is a union of blocks of τ. If $T \subset S$, then the restriction of τ to T is the partition $\tau_{|T}$ obtained by removing from τ all the elements in $S \setminus T$.

Denote by $[n]$ the set $\{1, \dots, n\}$. A partition $\tau = \{A_1, \dots, A_l\}$ of $[n]$ is said to be *noncrossing* if and only if $a, c \in A_i$ and $b, d \in A_j$ implies $i = j$, whenever $1 \le a < b < c < d \le n$. The set of all the noncrossing partitions of $[n]$ is usually denoted by NC_n. Its cardinality equals the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The reader interested in this subject may refer to [1] and references therein for further details. Now, if we set $R_\tau = R_{|A_1|} \cdots R_{|A_l|}$ then we can state the following beautiful formula, due to Speicher [16], expressing the moments M_n's in terms of their respective free cumulants R_n's:

$$M_n = \sum_{\tau \in NC_n} R_\tau.$$

Following Lehner [13], if 1 and n lie in the same block of a partition τ of $[n]$, then we say that τ is *irreducible*. Moreover, we denote by NC_n^{irr} the set of all the irreducible noncrossing partitions of $[n]$. Note that a partition of NC_{n+1}^{irr} is obtained from a partition of NC_n simply by inserting $n+1$ in the block containing 1. By taking the sum of the monomials R_τ's, τ ranging in NC_n^{irr} instead of NC_n, one defines a quantity B_n known as a boolean cumulant (see [13])

$$B_n = \sum_{\tau \in NC_n^{\mathrm{irr}}} R_\tau. \tag{3.1}$$

In particular, if $\mathcal{B}(z) = \sum_{n \ge 1} B_n z^n$, then we have

$$\mathcal{M}(z) = \frac{1}{1 - \mathcal{B}(z)}. \tag{3.2}$$

If $\mu = (\mu_1, \dots, \mu_l)$ is an integer partition of size n, set $R_\mu = R_{\mu_1} \cdots R_{\mu_l}$ and define NC_μ^{irr} to be the subset of NC_n^{irr} consisting of all the irreducible noncrossing partitions of type μ. From (3.1) we have

$$B_n = \sum_{\mu \vdash n} \left|NC_\mu^{\mathrm{irr}}\right| R_\mu. \tag{3.3}$$

Moreover, thanks to the Lagrange inversion formula, we recover

$$B_n = \frac{1}{n-1}\left[z^n\right]\mathcal{R}(z)^{n-1} = \sum_{\mu \vdash n} \frac{(n-2)_{\ell_\mu - 1}}{m(\mu)!} R_\mu, \tag{3.4}$$

where $m(\mu)! = m_1(\mu)! \cdots m_n(\mu)!$, and $m_i(\mu)$ is the number of occurrences of i as a part of μ. By comparing (3.3) and (3.4), we deduce

$$\left|NC_\mu^{\mathrm{irr}}\right| = \frac{(n-2)_{\ell_\mu - 1}}{m(\mu)!}. \tag{3.5}$$

The notion of noncrossing partition can be given for any totally ordered set S. In particular, NC_S^{irr} will denote the set of all the noncrossing partitions of S, such that the minimum and the maximum of S lie in the same block. Let us introduce a partial order on NC_S^{irr}.

Definition 3.1 (Irreducible refinement) Let $\tau, \pi \in NC_S^{\mathrm{irr}}$. We say that τ refines π in an *irreducible way*, and write $\tau \le^{\mathrm{irr}} \pi$, if and only if $\tau \le \pi$ and the restriction $\pi_{|A}$, of π to each block A of τ, is in NC_A^{irr}. In particular, we say that π *covers* τ if and only if $\tau \le^{\mathrm{irr}} \pi$ and π is obtained by joining two blocks of τ.

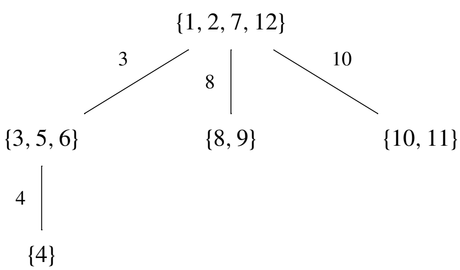
For instance, let $\tau = \{\{1,5\},\{2,3\},\{4\}\}$, $\pi = \{\{1,2,3,5\},\{4\}\}$ and $\sigma = \{\{1,5\},\{2,3,4\}\}$. Then $\tau, \pi, \sigma \in NC_5^{\mathrm{irr}}$ and τ refines both π and σ. However, $\tau \le^{\mathrm{irr}} \pi$ and in particular π covers τ, while it is not true that $\tau \le^{\mathrm{irr}} \sigma$, since $\tau_{|\{2,3,4\}} = \{\{2,3\},\{4\}\}$ is not irreducible.

The singletons of the noncrossing partitions will play a special role. For all $\tau \in NC_n$, we denote by $U(\tau)$ the subset of $[n]$ consisting of all the integers i such that $\{i\}$ is a block of τ, while $\dot{\tau}$ will be the partition obtained from τ by removing its singletons. When $\tau, \pi \in NC_n^{\mathrm{irr}}$ and $\tau \le^{\mathrm{irr}} \pi$, then π_τ is the restriction of π to $U(\tau)$. Note that $\pi_\tau \in NC_{U(\tau)}$.

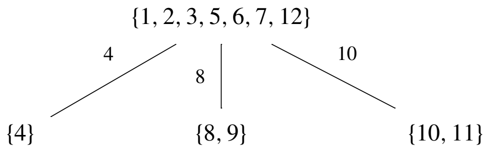
We define a tree-representation for the partitions of NC_n^{irr} in the following way. Assume $\tau = \{A_1, \dots, A_l\} \in NC_n^{\mathrm{irr}}$ and $\min A_i < \min A_{i+1}$. Construct a labeled rooted tree t_τ by the following steps:

- Choose A_1 as the root of t_τ;
- If $2 \le i < j \le l$ then draw an edge between A_i and A_j if and only if i is the biggest integer such that $\min A_i < \min A_j \le \max A_j < \max A_i$;
- Label each edge $\{A_i, A_j\}$ with $\min A_j$.

For example, if $\tau = \{\{1,2,7,12\}, \{3,5,6\}, \{4\}, \{8,9\}, \{10,11\}\}$ then t_τ is the following tree,



Now, let $E(\tau)$ be the set of labels of t_τ, and choose $j \in E(\tau)$. We denote by $t_{\tau,j}$ the tree obtained from t_τ by deleting the edge labeled by j and joining its nodes (i.e., joining the blocks). In the following, we will say that $t_{\tau,j}$ is the tree obtained from t_τ by removing j. Hence, $t_{\tau,3}$ is given by



Now, $t_{\tau,j}$ is the tree-representation of an irreducible noncrossing partition, here denoted by $\tau_{\{j\}}$, whose blocks are the nodes of $t_{\tau,j}$. By construction, we have $\tau \leq^{\mathrm{irr}} \tau_{\{j\}}$ and $E(\tau_{\{j\}}) = E(\tau) \setminus \{j\}$. More generally, given a subset $S \subseteq E(\tau)$, we denote by τ_S the only partition whose tree t_{τ_S} is obtained from t_τ by successively removing all labels in S. We note that $\tau_\emptyset = \tau$ and that τ_S depends only on the set S and not on the order in which labels are chosen. The following proposition is easy to prove.

Proposition 3.1 *Let $\tau, \pi \in NC_n^{\mathrm{irr}}$. Then, we have $\tau \leq^{\mathrm{irr}} \pi$ if and only if $\pi = \tau_S$ for some $S \subseteq E(\tau)$. In particular, if $\ell(\tau)$ is the number of blocks of τ, then we have*

$$\left|\left\{\sigma \mid \tau \leq^{\mathrm{irr}} \sigma\right\}\right| = \left|2^{E(\tau)}\right| = 2^{\ell(\tau)-1},$$

where $2^{E(\tau)}$ is the powerset of $E(\tau)$, and

$$\left|\{\sigma \mid \sigma \text{ covers } \tau\}\right| = \left|E(\tau)\right| = \ell(\tau) - 1.$$

4 Kerov polynomial formulae

By means of the results of previous sections, we are able to give new formulae for the Kerov polynomials Σ_k. In particular, the first formula is related to the partial order $\leq^{\mathrm{irr}}$ on the irreducible noncrossing partitions of the set $[k+1]$, instead the second formula expresses Σ_k as a weighted sum over NC_{k+1}.

4.1 Kerov polynomials and irreducible refinement

Let $\tau, \pi \in NC_{k+1}^{\mathrm{irr}}$ with $\tau \leq^{\mathrm{irr}} \pi$, and let $\pi_\tau = \{A_1, \dots, A_l\}$. Moreover, set $A_i = \emptyset$ for $i > l$ and define $W_\tau(\pi)$ to be such that

$$W_\tau(\pi) = \frac{1}{k!} \sum_{w \in \mathfrak{S}_k} w(1)^{|A_1|} \cdots w(k)^{|A_k|}.$$

Theorem 4.1 *We have*

$$\Sigma_k = \sum_{\tau \in NC_{k+1}^{\mathrm{irr}}} \left[\sum_{\tau \leq^{\mathrm{irr}} \pi} (-1)^{\ell_\pi - 1} W_\tau(\pi) \right] R_{\dot{\tau}}. \tag{4.1}$$

Proof Let $B_n(j) = -[z^n](\mathcal{M}_{\lambda\boxplus j}(z))^{-1}$ for all $n \geq 1$, and set $B_0(j) = 1$. Since $R_1(\lambda) = 0$, then from (2.5), (3.1) and (3.2) we deduce

$$B_n(j) = \sum_{\pi \in NC_n^{\mathrm{irr}}} j^{u(\pi)} R_{\dot{\pi}}(\lambda), \tag{4.2}$$

where $u(\pi) = |U(\pi)|$. The right-hand side in (2.4) is equal to

$$-\frac{1}{k} \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+\dots+a_k=k+1}} \prod_{j=1}^{k} [z^{a_j}] \frac{1}{M_{\lambda\boxplus j}(z)} = \frac{1}{k} \sum_{\substack{\mu\vdash k+1 \\ \ell_\mu \leq k}} \frac{(-1)^{\ell_\mu - 1}}{m(\mu)!(k-\ell_\mu)!} \sum_{w\in\mathfrak{S}_k} \prod_{i=1}^{k} B_{\mu_i}\big(w(i)\big),$$

where $\mu_1, \dots, \mu_{\ell_\mu}$ are the parts of μ and $\mu_i = 0$ when $i > \ell_\mu$. However, by taking into account (3.5), we may rewrite it in the following form:

$$\frac{1}{k!} \sum_{\pi \in NC_{k+1}^{\mathrm{irr}}} (-1)^{\ell_\pi - 1} \sum_{w\in\mathfrak{S}_k} \prod_{i=1}^{k} B_{|A_i|}\big(w(i)\big),$$

where $A_1, \dots, A_{\ell_\mu}$ are the blocks of π and $A_i = \emptyset$ if $i > \ell_\pi$. Moreover, by means of identity (4.2), we obtain

$$\sum_{w\in\mathfrak{S}_k} \prod_{i=1}^{k} B_{|A_i|}\big(w(i)\big) = \sum_{\tau_1,\dots,\tau_k} \sum_{w\in\mathfrak{S}_k} w(1)^{u(\tau_1)} \cdots w(k)^{u(\tau_k)} R_{\dot{\tau}_1}(\lambda) \cdots R_{\dot{\tau}_k}(\lambda),$$

where τ_i ranges over all $NC_{A_i}^{\mathrm{irr}}$, with $NC_{\emptyset}^{\mathrm{irr}} = \{\emptyset\}$.

Now, if we set $\tau = \tau_1 \cup \dots \cup \tau_k$, then $\tau \in NC_{k+1}^{\mathrm{irr}}$, $\tau \leq^{\mathrm{irr}} \pi$ and $R_{\dot{\tau}}(\lambda) = R_{\dot{\tau}_1}(\lambda) \cdots R_{\dot{\tau}_k}(\lambda)$. Finally, $u(\tau_i)$ is the number of singletons in $\tau_i = \tau_{|A_i}$, that is, the cardinality of the set $A_i \cap U(\tau)$, which if nonempty is a block of π_τ. This completes the proof. □

Remark 4.1 Consider the polynomial $\Omega_k(x_1, \dots, x_k)$ defined by

$$\Omega_k(x_1, \dots, x_k) = -\frac{1}{k} [z^{k+1}] \prod_{j=1}^{k} \frac{1 - x_j z}{\mathcal{M}(\frac{z}{1-x_j z})}.$$

Of course, Ω_k is symmetric with respect to the x_i's. Moreover, by virtue of (2.3), we obtain $\Omega_k(1, 2, \dots, k) = \Sigma_k$. A formula for $\Omega_k(x_1, \dots, x_k)$ is obtained from (4.1) simply by replacing $w(j)$ with $x_{w(j)}$ in $W_\tau(\pi)$. More precisely, via Proposition 3.1, if μ is an integer partition of size $k+1$ then the coefficient of $R_{\dot{\mu}}$ in $\Omega_k(x_1 \dots, x_k)$ is

$$\boldsymbol{g}_\mu(x_1, \dots, x_k) = \sum_{\tau\in NC_\mu^{\mathrm{irr}}} \sum_{S\subseteq E(\tau)} (-1)^{|E(\tau)|-|S|}\, W_\tau(S; x_1, \dots, x_k), \tag{4.3}$$

where $R_{\dot{\mu}} = \prod_{i\geq 2} R_i^{m_i(\mu)}$, and where $W_\tau(S; x_1, \dots, x_k)$ is obtained by replacing $w(j)$ with $x_{w(j)}$ in $W_\tau(\tau_S)$. Now, let $\lambda_\tau(S)$ denote the integer partition corresponding to the type of π_τ, with $\pi = \tau_S$. Then, it is not difficult to see that

$$k!W_\tau(S; x_1, \ldots, x_k) = m\big(\lambda_\tau(S)\big)!\big(k - \ell\big(\lambda_\tau(S)\big)\big)!\boldsymbol{m}_{\lambda_\tau(S)}(x_1, \ldots, x_k),$$

with $\boldsymbol{m}_{\lambda_\tau(S)}(x_1, \ldots, x_k)$ being the monomial symmetric function indexed by the partition $\lambda_\tau(S)$ [14]. This way, the coefficient of $R_{\dot{\mu}}$ in $\Omega_k(x_1, \ldots, x_k)$ is a symmetric function $\boldsymbol{g}_\mu(x_1, \ldots, x_k)$ of degree $m_1(\mu)$. Assume that

$$\boldsymbol{g}_\mu(x_1, \ldots, x_k) = \sum_\lambda g_{\mu,\lambda}\boldsymbol{m}_\lambda(x_1, \ldots, x_k).$$

The left-hand side of (4.3) says that, for every λ of size $m_1(\mu)$, we have

$$g_{\mu,\lambda} = \frac{1}{k!}\sum_{\tau \in NC_\mu^{\mathrm{irr}}}\ \sum_{\substack{S \subseteq E(\tau)\\ \lambda_\tau(S)=\lambda}} (-1)^{|E(\tau)|-|S|} m(\lambda)!(k - \ell_\lambda)!,$$

thus we have provided a combinatorial formula for the $g_{\mu,\lambda}$'s.

Finally, we stress that the coefficients in the expressions of $\boldsymbol{g}_\mu$ in terms of the classical basis, and Schur basis, are not positive integers. Indeed, we have

$$\boldsymbol{g}_{(3,1,1,1)} = \frac{4}{5}\boldsymbol{m}_{(1,1,1)} - \frac{3}{5}\boldsymbol{m}_{(1,2)} + \frac{4}{5}\boldsymbol{m}_{(3)}.$$

4.2 Kerov polynomials via irreducible components of noncrossing partitions

We will state a second formula expressing Σ_k as a weighted sum over the whole NC_{k+1}. To this end, we introduce the notion of an irreducible component of a noncrossing partition.

Given $\tau \in NC_n$, let j_1 be the greatest integer lying in the same block as 1. Set $\tau_1 = \tau_{|[j_1]}$ so that τ_1 is an irreducible noncrossing partition of $[j_1]$. Now, let j_2 be the greatest integer lying in the same block of $j_1 + 1$ and set $\tau_2 = \tau_{|[j_1+1,j_2]}$. By iterating this process, we determine the sequence of irreducible noncrossing partitions $\tau_1, \ldots, \tau_d$, which we name the *irreducible components* of τ, such that $\tau = \tau_1 \cup \cdots \cup \tau_d$. For all $\tau \in NC_n$, we denote by d_τ the number of its irreducible components. Note that $d_\tau = 1$ if and only if τ is an irreducible noncrossing partition.

Theorem 4.2 *We have*

$$\Sigma_k = \sum_{\tau \in NC_{k+1}} \big[(-1)^{d_\tau - 1} V_\tau\big] R_{\dot{\tau}}, \tag{4.4}$$

where

$$V_\tau = \frac{1}{k} \sum_{1 \le i_1 < \cdots < i_d \le k} i_1{}^{u(\tau_1)} \cdots i_d{}^{u(\tau_d)},$$

if $d = d_\tau$.

Proof Let $\mathcal{B}_{\lambda\boxplus j}(z) = 1(\mathcal{M}_{\lambda\boxplus j}(z))^{-1}$. From (2.4) we obtain

$$\begin{aligned}\Sigma_k &= -\frac{1}{k}\big[z^{k+1}\big]\prod_{j=1}^{k}\big(1 - B_{\lambda\boxplus j}(z)\big)\\ &= \sum_{d=1}^{k}\frac{(-1)^{d-1}}{k}\sum_{1\le i_1<\cdots<i_d\le k}\big[z^{k+1}\big]\mathcal{B}_{\lambda\boxplus i_1}(z)\cdots\mathcal{B}_{\lambda\boxplus i_d}(z).\end{aligned}$$

Of course, the complex $B_n(j) = [z^n]\mathcal{B}_{\lambda\boxplus j}(z)$ is the n-boolean cumulant of $\lambda\boxplus j$ and satisfies (4.2). This way we deduce

$$\big[z^{k+1}\big]\mathcal{B}_{\lambda\boxplus i_1}(z)\cdots\mathcal{B}_{\lambda\boxplus i_d}(z) = \sum_{\substack{a_1,\dots,a_d\ge 1\\ a_1+\cdots+a_d=k+1}}\sum_{\pi_1,\dots,\pi_d} {i_1}^{u(\pi_1)}\cdots {i_d}^{u(\pi_d)} R_{\dot{\pi}_1}\cdots R_{\dot{\pi}_d},$$

where π_i ranges over $NC^{\mathrm{irr}}_{a_i}$. Let $a_0 = 0$ and consider the intervals $A_i = [a_0 + \cdots + a_{i-1} + 1, a_0 + \cdots + a_i]$ for $i = 1, \dots, k+1$. Each translation $h \in [1, a_i] \mapsto h + a_0 + \cdots + a_{i-1} \in A_i$ induces a bijection $\pi \in NC^{\mathrm{irr}}_{a_i} \mapsto \tau \in NC^{\mathrm{irr}}_{A_i}$. Hence, in the identity above we may replace each π_i with the corresponding τ_i obtaining

$$\big[z^{k+1}\big]\mathcal{B}_{\lambda\boxplus i_1}(z)\cdots\mathcal{B}_{\lambda\boxplus i_d}(z) = \sum_{\substack{a_1,\dots,a_d\ge 1\\ a_1+\cdots+a_d=k+1}}\sum_{\tau_1,\dots,\tau_d} {i_1}^{u(\tau_1)}\cdots {i_d}^{u(\tau_d)} R_{\dot{\tau}_1}\cdots R_{\dot{\tau}_d}.$$

Now, set $\tau = \tau_1\cup\cdots\cup\tau_d$ so that $\tau \in NC_{k+1}$, τ_i is the ith irreducible component of τ, and (4.4) follows. □

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