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On balanced colorings of the n-cube

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Abstract A 2-coloring of the n-cube in the n-dimensional Euclidean space can be considered as an assignment of weights of 1 or 0 to the vertices. Such a colored n-cube is said to be balanced if its center of mass coincides with its geometric center. Let $B_{n,2k}$ be the number of balanced 2-colorings of the n-cube with $2k$ vertices having weight 1. Palmer, Read, and Robinson conjectured that for $n \geq 1$, the sequence $\{B_{n,2k}\}_{k=0,1,\ldots,2^{n-1}}$ is symmetric and unimodal. We give a proof of this conjecture. We also propose a conjecture on the log-concavity of $B_{n,2k}$ for fixed k, and by probabilistic method we show that it holds when n is sufficiently large.

Keywords Unimodality · n-Cube · Balanced coloring

1 Introduction

This paper is concerned with a conjecture of Palmer, Read, and Robinson [5] on 2-colorings of the n-cube in the n-dimensional Euclidean space. A 2-coloring of the n-cube is considered as an assignment of weights of 1 or 0 to the vertices. The black vertices are considered as having weight 1, whereas the white vertices are considered as having weight 0. We say that a 2-coloring of the n-cube is balanced if the colored n-cube is balanced, namely, the center of mass is located at its geometric center.

Let $\mathcal{B}_{n,2k}$ denote the set of balanced 2-colorings of the n-cube with exactly $2k$ black vertices and $B_{n,2k} = |\mathcal{B}_{n,2k}|$. Palmer, Read, and Robinson proposed the conjecture that the sequence $\{B_{n,2k}\}_{0\leq k\leq 2^{n-1}}$ is unimodal with the maximum at $k = 2^{n-2}$

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for any $n \geq 1$. For example, when $n = 4$, the sequence $\{B_{n,2k}\}$ reads

$$1, 8, 52, 152, 222, 152, 52, 8, 1.$$

A sequence $\{a_i\}_{0\leq i\leq m}$ is called unimodal if there exists k such that

$$a_0 \leq \cdots \leq a_k \geq \cdots \geq a_m$$

and is called strictly unimodal if

$$a_0 < \cdots < a_k > \cdots > a_m.$$

A sequence $\{a_i\}_{0\leq i\leq m}$ of real numbers is said to be log-concave if

$$a_i^2 \geq a_{i+1}a_{i-1}$$

for all $1 \leq i \leq m-1$.

Palmer, Read, and Robinson [3–5] used Pólya's theorem to derive a formula for $B_{n,2k}$, which is a sum over integer partitions of $2k$. However, the unimodality of the sequence $\{B_{n,2k}\}$ does not seem to be an easy consequence since the summation involves negative terms. In Sect. 2, we shall establish a relation on a refinement of the numbers $B_{n,2k}$ from which the unimodality easily follows. In Sect. 3, we conjecture that the sequence $\{B_{n,2k}\}$ is log-concave for fixed k and show that it holds when n is sufficiently large.

2 The unimodality

In this section, we give a proof of the unimodality conjecture of Palmer, Read, and Robinson. Let Q_n be the n-dimensional cube represented by a graph whose vertices are sequences of 1's and -1's of length n, where two vertices are adjacent if they differ only at one position. Let V_n denote the set of vertices of Q_n, namely,

$$V_n = \big\{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \,|\, \epsilon_i = -1 \text{ or } 1,\ 1 \leq i \leq n\big\}.$$

By a 2-coloring of the Q_n we mean an assignment of weights 1 or 0 to the vertices of Q_n. The weight of a 2-coloring is the sum of weights or the numbers of vertices with weight 1. The center of mass of a coloring f with $w(f) \neq 0$ is the point whose coordinates are given by

$$\frac{1}{w(f)} \sum (\epsilon_1, \epsilon_2, \ldots, \epsilon_n),$$

where the sum ranges over all black vertices. If $w(f) = 0$, we take the center of mass to be the origin. A 2-coloring is balanced if its center of mass coincides with the origin. A pair of vertices of the n-cube is called an antipodal pair if it is of the form $(v, -v)$. A 2-coloring is said to be antipodal if any vertex v and its antipodal have the same color.

The key idea of our proof relies on the following classification of the set $\mathcal{B}_{n,2k}$ of balanced 2-colorings.

Theorem 2.1 *Let $\mathcal{B}_{n,2k,i}$ denote the set of balanced 2-colorings in $\mathcal{B}_{n,2k}$ containing exactly i antipodal pairs of black vertices. Then we have*

$$\left(2^{n-1} - 2k + i\right)|\mathcal{B}_{n,2k,i}| = (i+1)|\mathcal{B}_{n,2k+2,i+1}| \tag{2.1}$$

for $0 \le i \le k$ and $0 \le k \le 2^{n-2} - 1$.

Proof We aim to show that both sides of (2.1) count the number of ordered pairs (F, G), where $F \in \mathcal{B}_{n,2k,i}$ and $G \in \mathcal{B}_{n,2k+2,i+1}$ are such that G can be obtained by changing a pair of antipodal white vertices of F to black vertices. Equivalently, F can be obtained from G by changing a pair of antipodal black vertices to white vertices.

First, for each $F \in \mathcal{B}_{n,2k,i}$, we wish to obtain G in $\mathcal{B}_{n,2k+2,i+1}$ by changing a pair of antipodal white vertices to black vertices. By the definition of $\mathcal{B}_{n,2k,i}$, for each F, there are i antipodal pairs of black vertices and $2k-2i$ black vertices whose antipodal vertices are colored by white. Since $k \le 2^{n-2} - 1$, that is, $2^{n-1} - 2(k-i) - i > 0$, there are exactly $2^{n-1} - 2(k-i) - i$ antipodal pairs of white vertices in F. Thus, from each $F \in \mathcal{B}_{n,2k,i}$ we can obtain $2^{n-2} - 2k + i$ different 2-colorings in $\mathcal{B}_{n,2k+2,i+1}$ by changing a pair of antipodal white vertices of F to black vertices. Hence, the number of ordered pairs (F, G) equals $(2^{n-1} - 2k + i)|\mathcal{B}_{n,2k,i}|$.

On the other hand, for each $G \in \mathcal{B}_{n,2k+2,i+1}$, since there are $i+1$ antipodal pairs of black vertices in G, we see that from G we can obtain $i+1$ different 2-colorings in $\mathcal{B}_{n,2k,i}$ by changing a pair of antipodal black vertices to white vertices. So the number of ordered pairs (F, G) equals $(i+1)|\mathcal{B}_{n,2k+2,i+1}|$. This completes the proof. □

Theorem 2.2 *For $n \ge 1$, the sequence $\{B_{n,2k}\}_{0 \le k \le 2^{n-1}}$ is strictly unimodal with the maximum attained at $k = 2^{n-2}$.*

Proof It is easily seen that $\{B_{n,2k}\}_{0 \le k \le 2^{n-1}}$ is symmetric for any $n \ge 1$. Given a balanced coloring of the n-cube, if we exchange the colors on all vertices, the complementary coloring is still balanced. Thus, it is sufficient to prove that $B_{n,2k} < B_{n,2k+2}$ for $0 \le k \le 2^{n-2} - 1$.

Clearly,

$$B_{n,2k} = \sum_{i=0}^{k} |\mathcal{B}_{n,2k,i}|.$$

We wish to establish the inequality

$$|\mathcal{B}_{n,2k,i}| < |\mathcal{B}_{n,2k+2,i+1}|. \tag{2.2}$$

If it is true, then

$$B_{n,2k} = \sum_{i=0}^{k} |\mathcal{B}_{n,2k,i}| < \sum_{i=1}^{k+1} |\mathcal{B}_{n,2k+2,i}| \le \sum_{i=0}^{k+1} |\mathcal{B}_{n,2k+2,i}| = B_{n,2k+2}$$

for $0 \le k \le 2^{n-2} - 1$, as claimed in the theorem. Thus, it remains to prove (2.2). Since $0 \le k \le 2^{n-2} - 1$, it is clear that

$$\left(2^{n-1} - 2k + i\right) - (i+1) = 2^{n-1} - 2k - 1 \ge 1.$$

Applying Theorem 2.1, we find that

$$|\mathcal{B}_{n,2k,i}| < |\mathcal{B}_{n,2k+2,i+1}|$$

for $0 \le i \le k$ and $1 \le k \le 2^{n-2} - 1$, and hence (2.2) holds. This completes the proof. □

3 The log-concavity for fixed *k*

Log-concave sequences and polynomials often arise in combinatorics, algebra, and geometry, see, for example, Brenti [1] and Stanley [6]. While $\{B_{n,2k}\}_k$ is not log-concave in general, we shall show that the sequence $\{B_{n,2k}\}_n$ is log-concave for fixed k and sufficiently large n, and we conjecture that the log-concavity holds for any given k.

Conjecture 3.1 *When* $0 \le k \le 2^{n-1}$, *we have*

$$B_{n,2k}^2 \ge B_{n-1,2k} B_{n+1,2k}.$$

Palmer, Read, and Robinson [5] have shown that

$$B_{n,2} = 2^{n-1}$$

and

$$B_{n,4} = \frac{1}{4^n}\left((4!)^{n-1} - 2^{3n-3}\right).$$

It is easy to verify that the sequences $\{B_{n,2}\}_{n\ge 1}$ and $\{B_{n,4}\}_{n\ge 2}$ are both log-concave. In the remaining of this paper, we shall be concerned with the case $k \ge 3$. To be more specific, we shall show that Conjecture 3.1 is true for $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$. Our proof utilizes the well-known Bonferroni inequality, which can be stated as follows. Let $P(E_i)$ be the probability of the event E_i, and let $P(\bigcup_{i=1}^n E_i)$ be the probability that at least one of the events $E_1, E_2, \ldots, E_n$ will occur. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} P(E_i).$$

Before we present the proof of the asymptotic log-concavity of the sequence $\{B_{n,2k}\}$ for fixed k, let us introduce the (0, 1)-matrices associated with a balanced 2-coloring of the n-cube with $2k$ vertices having weight 1. Since such a 2-coloring is uniquely determined by the set of vertices having weight 1, we may represent a 2-coloring by these vertices with weight 1. This leads us to consider the set $\mathcal{M}_{n,2k}$ of

$n \times 2k$ matrices such that each row contains k $+1$'s and k -1's without two identical columns. Let $M_{n,2k} = |\mathcal{M}_{n,2k}|$. It is obvious that

$$M_{n,2k} = (2k)! B_{n,2k}.$$

Hence the log-concavity of the sequence $\{M_{n,2k}\}_{n \geq \log_2 k+1}$ is equivalent to the log-concavity of the sequence $\{B_{n,2k}\}_{n \geq \log_2 k+1}$.

Canfield, Gao, Greenhill, McKay, and Robinson [2] obtained the following estimate.

Theorem 3.2 *If* $0 \leq k \leq o(2^{n/2})$, *then*

$$M_{n,2k} = \binom{2k}{k}^n \left(1 - O\left(\frac{k^2}{2^n}\right)\right).$$

To prove the asymptotic log-concavity of $M_{n,2k}$ for fixed k, we need the following monotone property which implies Theorem 3.2.

Theorem 3.3 *Let* $c_{n,k}$ *be the real number such that*

$$M_{n,2k} = \binom{2k}{k}^n \left(1 - c_{n,k}\left(\frac{k^2}{2^n}\right)\right). \tag{3.3}$$

Then we have

$$c_{n,k} > c_{n+1,k}$$

for $k \geq 3$ *and* $n \geq 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$.

Proof Let $\mathcal{L}_{n,2k}$ be the set of matrices with every row consisting of k -1's and k $+1$'s that do not belong to $\mathcal{M}_{n,2k}$ and $L_{n,2k} = |\mathcal{L}_{n,2k}|$. In other words, any matrix in $\mathcal{L}_{n,2k}$ has two identical columns. Since the number of $n \times 2k$ matrices with each row consisting of k $+1$'s and k -1's equals $\binom{2k}{k}^n$, from (3.3) it is easily checked that

$$L_{n,2k} = c_{n,k} \frac{k^2}{2^n} \binom{2k}{k}^n. \tag{3.4}$$

We now proceed to give an upper bound on the cardinality of $\mathcal{L}_{n+1,2k}$. For each $M \in \mathcal{L}_{n+1,2k}$, it is easy to see that the matrix M' obtained from M by deleting the $(n+1)$st row contains two identical columns as well. Therefore, every matrix in $\mathcal{L}_{n+1,2k}$ can be obtained from a matrix in $\mathcal{L}_{n,2k}$ by adding a suitable row to a matrix in $\mathcal{L}_{n,2k}$ as the $(n+1)$st row. This observation enables us to construct three classes of matrices M from $\mathcal{L}_{n+1,2k}$ by the properties of M'. It is obvious that any matrix in $\mathcal{L}_{n+1,2k}$ belongs to one of these three classes. Note that the classes are not necessarily exclusive.

Class 1: There exist at least three identical columns in M'. For each row of M', the probability that the three prescribed positions of this row are identical equals

$$2\binom{2k-3}{k} \Big/ \binom{2k}{k}.$$

Here the factor 2 indicates that there are two choices for the values at the prescribed positions. Consequently, the probability that the three prescribed columns in M' are identical equals

$$\left(2\binom{2k-3}{k}\Big/\binom{2k}{k}\right)^n = \left(\frac{k-2}{2(2k-1)}\right)^n < \frac{1}{4^n}.$$

By the Bonferroni inequality, the probability that there are at least three identical columns in M' is bounded by $\frac{8k^3}{4^n}$. Because the number of $(n+1)\times 2k$ matrices with each row consisting of k $+1$'s and k -1's is $\binom{2k}{k}^{n+1}$, the number of matrices M in $\mathcal{L}_{n+1,2k}$ with M' containing at least three identical columns is bounded by

$$\frac{8k^3}{4^n}\binom{2k}{k}^{n+1}.$$

Class 2: There exist at least two pairs of identical columns in M'. For any two prescribed pairs (i_1, i_2) and (j_1, j_2) of columns, let us estimate the probability that in M' the i_1th column is identical to the i_2th column and that the j_1th column is identical to the j_2th column. We have two cases for each row of M'. The first case is that the values at the positions i_1, i_2, j_1, and j_2 are all identical. The probability for any given row to be in this case equals

$$2\binom{2k-4}{k-4}\Big/\binom{2k}{k}.$$

Again, the factor 2 comes from the two choices for the values at the prescribed positions.

The second case is that the value of the i_1th position is different from the value of the j_1th position. In this case, we have either that the values at the i_1th and i_2th positions are $+1$ and the values at the j_1th and j_2th positions are -1 or that the values at i_1th and i_2th position are -1 and the values at the j_1th and j_2th positions are $+1$. Thus, the probability for any given row to be in this case equals

$$2\binom{2k-4}{k-2}\Big/\binom{2k}{k}.$$

Combining the above two cases, we see that for $k \geq 3$, the probability that M' has two prescribed pairs of identical columns equals

$$\left(2\binom{2k-4}{k-4}\Big/\binom{2k}{k} + 2\binom{2k-4}{k-2}\Big/\binom{2k}{k}\right)^n < \frac{1}{4^n}.$$

Again, by the Bonferroni inequality, the probability that there exist at least two pairs of identical columns of M' is bounded by $\frac{16k^4}{4^n}$. It follows that the number of matrices M in $\mathcal{L}_{n+1,2k}$ with M' containing at least two pairs of identical columns is bounded by

$$\frac{16k^4}{4^n}\binom{2k}{k}^{n+1}.$$

Class 3: There exists exactly one pair of identical columns in M'. By the definition, the number of matrices M' containing exactly one pair of identical columns is bounded by $L_{n,2k}$. On the other hand, it is easy to see that for each M' containing exactly one pair of identical columns, there are

$$2\binom{2k-2}{k} = \frac{k-1}{2k-1}\binom{2k}{k} \tag{3.5}$$

matrices of $\mathcal{L}_{n+1,2k}$ which can be obtained by adding a suitable row as the $(n+1)$st row. Combining (3.4) and (3.5), we find that the number of matrices M of $\mathcal{L}_{n+1,2k}$ such that M' contains exactly one pair of identical columns is bounded by

$$\frac{k-1}{2k-1} c_{n,k} \frac{k^2}{2^n} \binom{2k}{k}^{n+1}.$$

Clearly, $L_{n+1,2k}$ is bounded by the sum of the cardinalities of the above three classes. This yields the upper bound

$$L_{n+1,2k} < \frac{8k^3}{4^n}\binom{2k}{k}^{n+1} + \frac{16k^4}{4^n}\binom{2k}{k}^{n+1} + \frac{k-1}{2k-1} c_{n,k} \frac{k^2}{2^n}\binom{2k}{k}^{n+1}$$

for $k \geq 3$.

We claim that

$$\frac{8k^3}{4^n} + \frac{16k^4}{4^n} < \frac{1}{4k-2} c_{n,k} \frac{k^2}{2^n} \tag{3.6}$$

when

$$n \geq 5 \log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96. \tag{3.7}$$

Notice that the probability that two specified columns in M' are identical is

$$\left(2\binom{2k-2}{k}\Big/\binom{2k}{k}\right)^n = \left(\frac{k-1}{2k-1}\right)^n.$$

Since $c_{n,k}\frac{k^2}{2^n}$ is the probability that there exists at least two identical columns in M', for $k \geq 2$, we deduce that

$$c_{n,k}\frac{k^2}{2^n} > \left(2\binom{2k-2}{k}\Big/\binom{2k}{k}\right)^n = \left(\frac{k-1}{2k-1}\right)^n > \frac{1}{3^n}.$$

But under condition (3.7), we have

$$\frac{8k^3}{4^n} + \frac{16k^4}{4^n} < \frac{1}{3^n(4k-2)},$$

which implies (3.6). Since $\frac{k-1}{2k-1} + \frac{1}{4k-2} = \frac{1}{2}$, it follows from (3.6) that

$$L_{n+1,2k} < c_{n,k}\frac{k^2}{2^{n+1}}\binom{2k}{k}^{n+1}, \tag{3.8}$$

subject to condition (3.7). Restating formula (3.4) for $n+1$, we have

$$L_{n+1,2k} = c_{n+1,k}\frac{k^2}{2^{n+1}}\binom{2k}{k}^{n+1}. \tag{3.9}$$

Combining (3.8) and (3.9) gives

$$c_{n,k} > c_{n+1,k},$$

given condition (3.7). This completes the proof. □

Applying Theorem 3.3, we arrive at the following inequality.

Theorem 3.4 *When* $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, *we have*

$$M_{n,2k}^2 > M_{n-1,2k}M_{n+1,2k}.$$

Proof We only consider the case $k \geq 3$. Let

$$M_{n,2k} = \binom{2k}{k}^n\left(1 - c_{n,k}\frac{k^2}{2^n}\right).$$

Then

$$\begin{aligned} &M_{n,2k}^2 - M_{n-1,2k}M_{n+1,2k} \\ &= \binom{2k}{k}^{2n}\left[\left(1-c_{n,k}\frac{k^2}{2^n}\right)^2 - \left(1-c_{n+1,k}\frac{k^2}{2^{n+1}}\right)\left(1-c_{n-1,k}\frac{k^2}{2^{n-1}}\right)\right] \\ &= \binom{2k}{k}^{2n}\left[-c_{n,k}\frac{k^2}{2^{n-1}} + c_{n,k}^2\frac{k^4}{4^n} + c_{n+1,k}\frac{k^2}{2^{n+1}} + c_{n-1,k}\frac{k^2}{2^{n-1}} - c_{n-1,k}c_{n+1,k}\frac{k^4}{4^n}\right]. \end{aligned}$$

By Theorem 3.3, we have $c_{n-1,k} > c_{n,k}$ for $k \geq 3$ and $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$. This implies that

$$c_{n,k}\frac{k^2}{2^{n-1}} < c_{n-1,k}\frac{k^2}{2^{n-1}}$$

for $k \geq 3$ and $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$.

Now we claim $c_{n,k} < 4$ for any n. The probability that a specified pair of columns are equal is given by

$$\left(2\binom{2k-2}{k}\Big/\binom{2k}{k}\right)^n = \left(\frac{k-1}{2k-1}\right)^n < \frac{1}{2^n}.$$

Since there are $2k$ columns in every M, by the Bonferroni inequality, the probability that there exist at least two identical columns in M is bounded by $\frac{4k^2}{2^n}$. This implies that $c_{n,k} < 4$ for any n.

Since

$$5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96 > 2\log_2 k + 3,$$

using condition (3.7), we have

$$c_{n-1,k}c_{n+1,k}\frac{k^4}{4^n} < c_{n+1,k}\frac{k^4}{4^{n-1}} \le c_{n+1,k}\frac{k^2}{2^{n+1}}.$$

Hence,

$$M_{n,2k}^2 > M_{n-1,2k}M_{n+1,2k}.$$

This completes the proof. □

Since $M_{n,2k} = (2k)!B_{n,2k}$, Theorem 3.4 implies the asymptotic log-concavity of $B_{n,2k}$ for fixed k.

Corollary 3.5 *When* $n > 5\log_{\frac{4}{3}} k + \log_{\frac{4}{3}} 96$, *we have*

$$B_{n,2k}^2 > B_{n-1,2k}B_{n+1,2k}.$$

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References

1. Brenti, F.: Unimodal, log-concave, and Pólya frequency sequences in combinatorics. Mem. Am. Math. Soc. **413**, 1–106 (1989)
2. Canfield, E.R., Gao, Z., Greenhill, C., McKay, B.D., Robinson, R.W.: Asymptotic enumeration of correlation-immune Boolean functions. arXiv:0909.3321
3. Palmer, E.M., Robinson, R.W.: Enumeration under two representations of the wreath product. Acta Math. **131**, 123–143 (1973)
4. Palmer, E.M., Robinson, R.W.: Enumeration of self-dual configurations. Pac. J. Math. **110**, 203–221 (1984)
5. Palmer, E.M., Read, R.C., Robinson, R.W.: Balancing the n-cube: A census of colorings. J. Algebr. Comb. **1**, 257–273 (1992)
6. Stanley, R.P.: Log-concave and unimodal sequences in algebra, combinatorics and geometry. Ann. N.Y. Acad. Sci. **576**, 500–535 (1989)