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Recurrence formulas for Macdonald polynomials of type A

Michel Lassalle · Michael J. Schlosser

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Abstract We consider products of two Macdonald polynomials of type A, indexed by dominant weights which are respectively a multiple of the first fundamental weight and a weight having zero component on the kth fundamental weight. We give the explicit decomposition of any Macdonald polynomial of type A in terms of this basis.

Keywords Macdonald polynomials · Pieri formula · Multidimensional matrix inverse

1 Introduction

In the 1980s, Macdonald [6–8] introduced a class of orthogonal polynomials which are Laurent polynomials in several variables and generalize the Weyl characters of compact simple Lie groups. In the simplest situation, given a root system R, these polynomials are elements of the group algebra of the weight lattice of R, indexed by the dominant weights and depending on two parameters (q,t).

When R is of type A_n, these Macdonald polynomials are in bijective correspondence with the symmetric functions $\mathcal{P}_\lambda(q,t)$ indexed by partitions, introduced by Macdonald some years before [4, 5]. In fact, they correspond to $\mathcal{P}_\lambda(q,t)(x_1,\dots,x_{n+1})$, for a partition $\lambda=(\lambda_1,\dots,\lambda_n)$ of length n, with the $n+1$ variables $(x_1,\dots,x_{n+1})$ linked by the condition $x_1\cdots x_{n+1}=1$.

M. Lassalle (✉)
Centre National de la Recherche Scientifique, Institut Gaspard-Monge,
Université de Marne-la-Vallée, 77454 Marne-la-Vallée Cedex, France
e-mail: lassalle@univ-mlv.fr
url: http://igm.univ-mlv.fr/~lassalle

M.J. Schlosser
Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, 1090 Vienna, Austria
e-mail: michael.schlosser@univie.ac.at
url: http://www.mat.univie.ac.at/~schlosse

The purpose of this article is to extend the result of [3], given for the symmetric functions $\mathcal{P}_\lambda(q,t)$, to the framework of the root system A_n.

More precisely, in [3, Theorem 4.1] we obtained a recurrence formula giving the symmetric function $\mathcal{P}_{(\lambda_1,\dots,\lambda_n)}(q,t)$ as a sum

$$\mathcal{P}_{(\lambda_1,\dots,\lambda_n)} = \sum_{\theta\in\mathbb{N}^{n-1}} C_{\theta_1,\dots,\theta_{n-1}} \mathcal{P}_{(\lambda_1+\theta_1,\dots,\lambda_{n-1}+\theta_{n-1})} \mathcal{P}_{\lambda_n-|\theta|} \tag{1.1}$$

with $|\theta| = \sum_{i=1}^{n-1}\theta_i$ and $\mathbb{N}$ the set of nonnegative integers. This formula was obtained by inverting the "Pieri formula," which conversely expresses the product $\mathcal{P}_{(\lambda_1,\dots,\lambda_{n-1})}\mathcal{P}_{\lambda_n}$ as a sum

$$\mathcal{P}_{(\lambda_1,\dots,\lambda_{n-1})}\mathcal{P}_{\lambda_n} = \sum_{\theta\in\mathbb{N}^{n-1}} c_{\theta_1,\dots,\theta_{n-1}}\, \mathcal{P}_{(\lambda_1+\theta_1,\dots,\lambda_{n-1}+\theta_{n-1},\lambda_n-|\theta|)}.$$

Both expansions are identities between symmetric functions, valid for any number of variables.

These identities may also be written in terms of Macdonald polynomials of type A_n. For this purpose, let $\{\omega_i, 1\le i\le n\}$ be the n fundamental weights of the root system A_n. Let P_λ denote the Macdonald polynomial associated with the dominant weight $\lambda = \sum_{i=1}^n \lambda_i\omega_i$. The recurrence formula (1.1), written for $n+1$ variables $(x_1,\dots,x_{n+1})$ linked by $x_1\cdots x_{n+1}=1$, yields

$$P_\lambda = \sum_{\theta\in\mathbb{N}^{n-1}} C_{\theta_1,\dots,\theta_{n-1}}\, P_{(\lambda_n-|\theta|)\omega_1}\, P_\mu \tag{1.2}$$

with $\mu = \sum_{i=1}^{n-2}(\lambda_i+\theta_i-\theta_{i+1})\omega_i + (\lambda_{n-1}+\lambda_n+\theta_{n-1})\omega_{n-1}$. This alternative formulation is obvious and does not bring anything new.

However the method of [3], when applied in the A_n root system framework, allows us to get a much stronger result. Indeed, let k be a fixed integer with $1\le k\le n$. In this paper we shall write the Macdonald polynomial P_λ in terms of products $P_{r\omega_1}P_\mu$ with $\mu=\sum_{i=1}^n \mu_i\omega_i$ and $\mu_k=0$. There are n such recurrence formulas, (1.2) being the particular case $k=n$ of the latter.

This paper is organized as follows. In Sect. 2 we introduce our notation for the root system A_n and recall general facts about the corresponding Macdonald polynomials. Their Pieri formula, which involves a specific infinite-multidimensional matrix, is studied in Sect. 3, starting from the one given by Macdonald for the symmetric functions $\mathcal{P}_\lambda(q,t)$ [5, p. 340]. In Sect. 4 we invert the Pieri matrix by applying a particular multidimensional matrix inverse, given separately in the Appendix. This matrix inverse is equivalent to one previously obtained in [3, Sect. 2] by using operator methods. As a result of inverting the Pieri formula, we obtain recurrence formulas for A_n Macdonald polynomials. Finally, in Sect. 5 we detail the examples of the A_2 and A_3 cases and compare them to earlier results.

2 Macdonald polynomials of type A

The standard references for Macdonald polynomials associated with root systems are [6–8].

Let us consider the space $\mathbb{R}^{n+1}$ endowed with the usual scalar product and the quotient space $V = \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$, where $\mathbb{R}(1,\dots,1)$ is the subspace spanned by the vector $(1,\dots,1)$. Let $\varepsilon_1,\dots,\varepsilon_{n+1}$ denote the images in V of the coordinate vectors of $\mathbb{R}^{n+1}$, linked by $\sum_{i=1}^{n+1}\varepsilon_i = 0$.

The root system of type A_n is formed by the vectors $\{\varepsilon_i - \varepsilon_j, i \neq j\}$. The positive roots are $\{\varepsilon_i - \varepsilon_j, i < j\}$, and the simple roots are $\varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le n$. The Weyl group is the symmetric group $W = S_{n+1}$ acting by permutation of the coordinates.

The weight lattice P is formed by integral linear combinations of the fundamental weights $\{\omega_i, 1 \le i \le n\}$ defined by $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Let $\omega_i = 0$ for $i = 0, n+1$. We denote by P^+ the set of dominant weights $\lambda = \sum_{i=1}^n \lambda_i \omega_i$, which are nonnegative integral linear combinations of the fundamental weights.

There is the following correspondence between dominant weights and partitions. Given a dominant weight, if we write it as

$$\lambda = \sum_{i=1}^{n} \lambda_i \omega_i = \sum_{i=1}^{n+1} \mu_i \varepsilon_i,$$

the sequence $\mu = (\mu_1,\dots,\mu_{n+1})$ is a partition with length $\le n+1$. We have

$$\lambda_i = \mu_i - \mu_{i+1} \quad \text{and} \quad \mu_i = \mu_{n+1} + \sum_{j=i}^{n} \lambda_j.$$

Thus μ is defined up to μ_{n+1}, and two partitions μ, ν correspond to the same weight λ if and only if $\mu_1 - \nu_1 = \cdots = \mu_{n+1} - \nu_{n+1}$. We denote by $\mathcal{C}_\lambda$ the family of partitions thus defined.

Let A denote the group algebra over $\mathbb{R}$ of the free Abelian group P. For each $\lambda \in P$, let e^λ denote the corresponding element of A, subject to the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$. The set $\{e^\lambda, \lambda \in P\}$ forms an $\mathbb{R}$-basis of A.

The Weyl group $W = S_{n+1}$ acts on P and on A. Let $W\lambda$ denote the orbit of $\lambda \in P$ and A^W the subspace of W-invariants in A. There are two important bases of A^W, both indexed by dominant weights. The first one is given by the orbit-sums

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu.$$

The second one is provided by the Weyl characters

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \det(w) e^{w(\lambda+\rho)}$$

with $\rho = \sum_{i=1}^n (n-i+1)\varepsilon_i$ and $\delta = \sum_{w\in W} \det(w) e^{w(\rho)}$. The Macdonald polynomials $\{P_\lambda, \lambda \in P^+\}$ form another basis defined as the eigenvectors of a specific self-adjoint operator (which we do not describe here).

For $1 \le i \le n+1$, define $x_i = e^{\varepsilon_i}$, so that the variables x_i are linked by $x_1 \cdots x_{n+1} = 1$. Then δ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$. There is a correspondence between A^W and the symmetric polynomials restricted to $n+1$ variables $x = (x_1, \ldots, x_{n+1})$ linked by the previous condition.

In terms of bases this correspondence may be described as follows. Let λ be any dominant weight, and let $x_1 \cdots x_{n+1} = 1$. All monomial symmetric functions $m_\mu(x_1, \ldots, x_{n+1})$ with $\mu \in \mathcal{C}_\lambda$ are equal, and their common value is the orbit-sum m_λ. Similarly, the Weyl character χ_λ is the common value of the Schur functions $s_\mu(x_1, \ldots, x_{n+1})$, $\mu \in \mathcal{C}_\lambda$, whereas the Macdonald polynomial P_λ is the common value of the symmetric polynomials $\mathcal{P}_\mu(q,t)(x_1, \ldots, x_{n+1})$ with $\mu \in \mathcal{C}_\lambda$ and $\mathcal{P}_\mu(q,t)$ the symmetric function studied in Chap. 6 of [5].

Given a positive integer r and a dominant weight λ, the "Pieri formula" expands the product

$$P_{r\omega_1} P_\lambda = \sum_\rho c_\rho P_{\lambda+\rho}$$

in terms of Macdonald polynomials, where the range of ρ and the values of the coefficients c_ρ are to be determined.

Let Q denote the root lattice spanned by the simple roots. For any vector τ, define

$$\Sigma(\tau) = C(\tau) \cap (\tau + Q)$$

with $C(\tau)$ the convex hull of the Weyl group orbit of τ. Since the orbit of $\omega_1 = \varepsilon_1$ is the set $\{\varepsilon_i = \omega_i - \omega_{i-1}, 1 \le i \le n+1\}$, it is clear that $\Sigma(r\omega_1)$ is formed by vectors

$$\sum_{i=1}^{n+1} \theta_i(\omega_i - \omega_{i-1}) = \sum_{i=1}^{n} (\theta_i - \theta_{i+1})\omega_i$$

with $\theta = (\theta_1, \ldots, \theta_{n+1}) \in \mathbb{N}^{n+1}$ and $|\theta| = \sum_{i=1}^{n+1} \theta_i = r$.

By general results [8, (5.3.8), p. 104], it is known that the sum on the right-hand side of the Pieri formula is restricted to vectors ρ such that $\rho \in \Sigma(r\omega_1)$ and $\lambda + \rho \in P^+$. In the next section we shall give a direct proof of this result and make the value of the coefficient c_ρ explicit.

3 Pieri formula

Let $0 < q < 1$. For any integer r, the classical q-shifted factorial $(u;q)_r$ is defined by

$$(u;q)_\infty = \prod_{j \ge 0} \left(1 - uq^j\right), \qquad (u;q)_r = (u;q)_\infty / \left(uq^r;q\right)_\infty.$$

Let $u = (u_1, \ldots, u_m)$ be m indeterminates, and $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{N}^m$. For clarity of display, throughout this paper, any time such a pair (u, θ) is given, we shall implicitly assume m auxiliary variables $v = (v_1, \ldots, v_m)$ to be defined by $v_i = q^{\theta_i} u_i$.

Macdonald polynomials of type A_n satisfy the following Pieri formula.

Theorem 3.1 *Let $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ be a dominant weight, and $r \in \mathbb{N}$. For any $1 \le i \le n+1$, define*

$$u_i = q^{\sum_{j=i}^n \lambda_j} t^{-i}$$

and for $\theta \in \mathbb{N}^{n+1}$,

$$d_\theta(u_1, \dots, u_{n+1}; r) = \frac{(q;q)_r}{(t;q)_r} \prod_{j=1}^{n+1} \frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}} \prod_{1 \le i < j \le n+1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}.$$

We have

$$P_{r\omega_1} P_\lambda = \sum_{\substack{\theta \in \mathbb{N}^{n+1} \\ |\theta| = r}} d_\theta(u_1, \dots, u_{n+1}; r) P_{\lambda+\rho}$$

with $\rho = \sum_{i=1}^n (\theta_i - \theta_{i+1})\omega_i$.

Proof In a first step, we write the Pieri formula for arbitrary $\mathcal{P}_\mu(q,t)$ with $\mu = (\mu_1, \dots, \mu_n)$ being a partition having length $\le n$. We start from [5, p. 340, (6.24)(i)] and [5, p. 342, Example 2(a)]. Replacing g_r by $(t;q)_r/(q;q)_r \mathcal{P}_{(r)}$, we have

$$\mathcal{P}_{(r)} \mathcal{P}_\mu = \sum_{\kappa \supset \mu} \varphi_{\kappa/\mu} \mathcal{P}_\kappa,$$

where the skew-diagram $\kappa - \mu$ is a horizontal r-strip, i.e., has at most one node in each column. The Pieri coefficient $\varphi_{\kappa/\mu}$ is given by

$$\begin{aligned} \frac{(t;q)_r}{(q;q)_r} \varphi_{\kappa/\mu} &= \prod_{1 \le i \le j \le l(\kappa)} \frac{f(q^{\kappa_i - \kappa_j} t^{j-i})}{f(q^{\kappa_i - \mu_j} t^{j-i})} \frac{f(q^{\mu_i - \mu_{j+1}} t^{j-i})}{f(q^{\mu_i - \kappa_{j+1}} t^{j-i})} \\ &= \prod_{1 \le i \le j \le l(\kappa)} \frac{w_{\kappa_j - \mu_j}(q^{\kappa_i - \kappa_j} t^{j-i})}{w_{\kappa_{j+1} - \mu_{j+1}}(q^{\mu_i - \kappa_{j+1}} t^{j-i})} \end{aligned}$$

with $f(u) = (tu;q)_\infty/(qu;q)_\infty$ and $w_s(u) = (tu;q)_s/(qu;q)_s$.

Since $\kappa - \mu$ is a horizontal strip, the length $l(\kappa)$ of κ is at most equal to $n+1$, so we can write $\kappa = (\mu_1 + \theta_1, \dots, \mu_n + \theta_n, \theta_{n+1})$ with $|\theta| = r$. Then

$$\begin{aligned} \frac{(t;q)_r}{(q;q)_r} \varphi_{\kappa/\mu} &= \prod_{1 \le i \le j \le l(\kappa)} w_{\theta_j}\big(q^{\kappa_i - \kappa_j} t^{j-i}\big) \prod_{1 \le i < j \le l(\kappa)+1} \big(w_{\theta_j}\big(q^{\mu_i - \kappa_j} t^{j-i-1}\big)\big)^{-1} \\ &= \prod_{j=1}^{n+1} \frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}} \prod_{1 \le i < j \le n+1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}, \end{aligned}$$

where for $1 \le i \le n+1$, we set $u_i = q^{\mu_i} t^{-i}$ and $v_i = q^{\kappa_i} t^{-i} = q^{\theta_i} u_i$.

In a second step we translate this result in terms of A_n Macdonald polynomials. Given the dominant weight λ, we choose $\mu = (\mu_1, \dots, \mu_{n+1})$ to be the unique element of $\mathcal{C}_\lambda$ such that $\mu_{n+1}=0$, i.e., with length $\le n$. For $1\le i\le n$, we have $\mu_i=\sum_{j=i}^n \lambda_j$. As for the partition κ (with length $\le n+1$), it belongs to $\mathcal{C}_\sigma$ with $\sigma=\sum_{k=1}^n(\kappa_k-\kappa_{k+1})\omega_k=\sum_{k=1}^n(\lambda_k+\theta_k-\theta_{k+1})\omega_k$. Hence the statement. □

Remark On the right-hand side of the Pieri formula, the condition $\lambda+\rho\in P^+$ is necessarily satisfied as soon as $d_\theta(u_1,\dots,u_{n+1};r)\neq 0$. Using the correspondence between dominant weights and partitions, this may be verified on the Pieri formula

$$\mathcal{P}_{(r)}\mathcal{P}_\mu=\sum_{\kappa=(\mu_1+\theta_1,\dots,\mu_n+\theta_n,\theta_{n+1})}\varphi_{\kappa/\mu}\mathcal{P}_\kappa.$$

We only have to show that $\varphi_{\kappa/\mu}$ necessarily vanishes when the multiinteger κ is not a partition. But then there is an index i such that $\kappa_i<\kappa_{i+1}$, so that the factor $(qu_i/tv_{i+1};q)_{\theta_{i+1}}$ in $\varphi_{\kappa/\mu}$ writes out as

$$\big(1-q^{1+\mu_i-\kappa_{i+1}}\big)\cdots\big(1-q^{\mu_i-\mu_{i+1}}\big).$$

Due to $\kappa_i<\kappa_{i+1}$, this product would be $\neq 0$ only if $\mu_i<\mu_{i+1}$, which is impossible since μ is a partition.

From now on, we fix some integer $1\le k\le n$. Substituting $r-|\theta|$ for θ_k, the Pieri formula may be written in the more explicit form

$$P_{r\omega_1}P_\lambda=\sum_{\substack{\theta=(\theta_1,\dots,\theta_{k-1},0,\theta_{k+1},\dots,\theta_{n+1})\in\mathbb{N}^n\\ |\theta|\le r}}\hat{d}_\theta(u_1,\dots,u_{n+1};r)P_{\lambda+\rho}$$

with

$$\rho=\sum_{\substack{1\le i\le n\\ i\neq k-1,k}}(\theta_i-\theta_{i+1})\omega_i+\theta_{k-1}\omega_{k-1}+\big(r-|\theta|\big)(\omega_k-\omega_{k-1})-\theta_{k+1}\omega_k$$

and

$$\begin{aligned}\hat{d}_\theta(u_1,\dots,u_{n+1};r)=&\frac{(q;q)_r}{(t;q)_r}\frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}}\prod_{\substack{j=1\\ j\neq k}}^{n+1}\frac{(t;q)_{\theta_j}}{(q;q)_{\theta_j}}\\ &\times\prod_{\substack{1\le i<j\le n+1\\ j\neq k}}\frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}}\frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}\\ &\times\prod_{i=1}^{k-1}\frac{(tv_i/v_k;q)_{r-|\theta|}}{(qv_i/v_k;q)_{r-|\theta|}}\frac{(qu_i/tv_k;q)_{r-|\theta|}}{(u_i/v_k;q)_{r-|\theta|}}.\end{aligned}$$

Here u_i,v_i $(1\le i\le n+1)$ are as in Theorem 3.1, except $v_k=q^{r-|\theta|}u_k$. The sum is restricted to $|\theta|\le r$ since $1/(q;q)_s=0$ for $s<0$.

In a second step, we concentrate on the situation $\lambda_k = 0$. Then each term on the right-hand side vanishes unless $\theta_{k+1} = 0$. Indeed, if $\lambda_k = 0$, one has $u_k = tu_{k+1}$ and $v_{k+1} = q^{\theta_{k+1}} u_{k+1}$. Hence, for $i = k$ and $j = k+1$, the factor $(qu_i/tv_j; q)_{\theta_j}$ evaluates as

$$(qu_k/tv_{k+1}; q)_{\theta_{k+1}} = \left(q^{1-\theta_{k+1}}; q\right)_{\theta_{k+1}} = \delta_{\theta_{k+1},0}.$$

Therefore, if $\lambda_k = 0$, the Pieri formula can be written as

$$\begin{aligned} &P_{r\omega_1} P_\lambda \\ &\quad= \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{k-1},0,0,\theta_{k+2},\ldots,\theta_{n+1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} \tilde{d}_\theta(u_1,\ldots,u_{k-1},u_k,u_{k+2},\ldots,u_{n+1};k,r) P_{\lambda+\rho} \end{aligned}$$

with

$$\rho = \sum_{\substack{1\le i\le n\\ i\ne k-1,k,k+1}} (\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\omega_{k-1} + \big(r - |\theta|\big)(\omega_k - \omega_{k-1}) - \theta_{k+2}\omega_{k+1}$$

and

$$\begin{aligned} &\tilde{d}_\theta(u_1,\ldots,u_{k-1},u_k,u_{k+2},\ldots,u_{n+1};k,r) \\ &\quad= \frac{(q;q)_r}{(t;q)_r} \frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}} \prod_{\substack{i=1\\ i\ne k,k+1}}^{n+1} \frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}} \prod_{\substack{1\le i<j\le n+1\\ i\ne k,k+1\\ j\ne k,k+1}} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}} \\ &\qquad\times \prod_{i=1}^{k-1} \frac{(tv_i/v_k;q)_{r-|\theta|}}{(qv_i/v_k;q)_{r-|\theta|}} \frac{(qu_i/tv_k;q)_{r-|\theta|}}{(u_i/v_k;q)_{r-|\theta|}} \prod_{j=k+2}^{n+1} \frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}} \frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}. \end{aligned}$$

Here the notation is the same as before, including $v_k = q^{r-|\theta|}u_k$. For $j \ge k+2$, we have used

$$\begin{aligned} &\frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}} \frac{(qu_k/tv_j;q)_{\theta_j}}{(u_k/v_j;q)_{\theta_j}} \frac{(tv_{k+1}/v_j;q)_{\theta_j}}{(qv_{k+1}/v_j;q)_{\theta_j}} \frac{(qu_{k+1}/tv_j;q)_{\theta_j}}{(u_{k+1}/v_j;q)_{\theta_j}} \\ &\quad= \frac{(tv_k/v_j;q)_{\theta_j}}{(qv_k/v_j;q)_{\theta_j}} \frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}, \end{aligned}$$

which is a direct consequence of $v_{k+1} = u_{k+1} = u_k/t$.

In a third step, we perform some relabeling in order to remove the two 0's appearing in θ. For that purpose, for n indeterminates $(u_0, u_1, \ldots, u_{n-1})$ and $\theta = (\theta_1, \ldots, \theta_{n-1}) \in \mathbb{N}^{n-1}$, we define

$$\begin{aligned} &D_\theta(u_0,u_1,\ldots,u_{n-1};k,r) \\ &\quad= (q/t)^{|\theta|} \frac{(t^2u_0;q)_{|\theta|}}{(qtu_0;q)_{|\theta|}} \prod_{i=1}^{n-1} \frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}} \frac{(q^{|\theta|+1}u_i;q)_{\theta_i}}{(q^{|\theta|}tu_i;q)_{\theta_i}} \end{aligned}$$

$$\times \prod_{1\le i<j\le n-1} \frac{(tv_i/v_j;q)_{\theta_j}}{(qv_i/v_j;q)_{\theta_j}} \frac{(qu_i/tv_j;q)_{\theta_j}}{(u_i/v_j;q)_{\theta_j}}$$

$$\times \prod_{i=1}^{k-1} \frac{(u_i/u_0;q)_{\theta_i}}{(qu_i/tu_0;q)_{\theta_i}} \frac{(qu_i/tu_0;q)_{\theta_i-r+|\theta|}}{(u_i/u_0;q)_{\theta_i-r+|\theta|}} \frac{(u_i/tu_0;q)_{\theta_i-r+|\theta|}}{(qu_i/t^2u_0;q)_{\theta_i-r+|\theta|}}$$

$$\times \prod_{i=k}^{n-1} \frac{(tu_i/u_0;q)_{\theta_i}}{(qu_i/u_0;q)_{\theta_i}}.$$

Lemma *If we write*

$$w_i = \begin{cases} q^{-r}t^{-2}, & i=0, \\ q^{-r}u_i/tu_k, & 1\le i\le k-1, \\ q^{-r}u_{i+2}/tu_k, & k\le i\le n-1, \end{cases}$$

we have

$$D_\theta(w_0,w_1,\dots,w_{n-1};k,r)$$
$$= \tilde{d}_{(\theta_1,\dots,\theta_{k-1},0,0,\theta_k,\dots,\theta_{n-1})}(u_1,\dots,u_{k-1},u_k,u_{k+2},\dots,u_{n+1};k,r).$$

Proof Merely by substitution, and using $v_k = q^{r-|\theta|}u_k$, we only have to prove

$$(q/t)^{|\theta|} \frac{(q^{-r};q)_{|\theta|}}{(q^{1-r}/t;q)_{|\theta|}} \prod_{j=k+2}^{n+1} \frac{(q^{|\theta|-r+1}u_j/tu_k;q)_{\theta_j}}{(q^{|\theta|-r}u_j/u_k;q)_{\theta_j}} \frac{(t^2u_j/u_k;q)_{\theta_j}}{(qtu_j/u_k;q)_{\theta_j}}$$

$$\times \prod_{i=1}^{k-1} \frac{(q^{|\theta|-r+1}u_i/tu_k;q)_{\theta_i}}{(q^{|\theta|-r}u_i/u_k;q)_{\theta_i}} \frac{(tu_i/u_k;q)_{\theta_i}}{(qu_i/u_k;q)_{\theta_i}}$$

$$\times \prod_{i=1}^{k-1} \frac{(qu_i/u_k;q)_{\theta_i-r+|\theta|}}{(tu_i/u_k;q)_{\theta_i-r+|\theta|}} \frac{(u_i/u_k;q)_{\theta_i-r+|\theta|}}{(qu_i/tu_k;q)_{\theta_i-r+|\theta|}}$$

$$= \frac{(q;q)_r}{(t;q)_r} \frac{(t;q)_{r-|\theta|}}{(q;q)_{r-|\theta|}} \prod_{i=1}^{k-1} \frac{(tv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}{(qv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}} \frac{(qu_i/tq^{r-|\theta|}u_k;q)_{r-|\theta|}}{(u_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}$$

$$\times \prod_{j=k+2}^{n+1} \frac{(tq^{r-|\theta|}u_k/v_j;q)_{\theta_j}}{(q^{r-|\theta|+1}u_k/v_j;q)_{\theta_j}} \frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}.$$

We have obviously

$$\frac{(q^{|\theta|-r+1}u_i/tu_k;q)_{\theta_i}}{(q^{|\theta|-r}u_i/u_k;q)_{\theta_i}} \frac{(u_i/u_k;q)_{\theta_i-r+|\theta|}}{(qu_i/tu_k;q)_{\theta_i-r+|\theta|}} = \frac{(qu_i/tq^{r-|\theta|}u_k;q)_{r-|\theta|}}{(u_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}.$$

Using the identities

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}(a/b)^n,$$

$$\frac{(a;q)_n}{(b;q)_n}\frac{(b;q)_{n-k}}{(a;q)_{n-k}}=\frac{(q^{1-n}/a;q)_k}{(q^{1-n}/b;q)_k}(a/b)^k,$$

we get

$$\begin{aligned}\frac{(tu_i/u_k;q)_{\theta_i}}{(qu_i/u_k;q)_{\theta_i}}\frac{(qu_i/u_k;q)_{\theta_i-r+|\theta|}}{(tu_i/u_k;q)_{\theta_i-r+|\theta|}}&=\frac{(q^{1-\theta_i}u_k/tu_i;q)_{r-|\theta|}}{(q^{-\theta_i}u_k/u_i;q)_{r-|\theta|}}(t/q)^{r-|\theta|}\\&=\frac{(tv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}{(qv_i/q^{r-|\theta|}u_k;q)_{r-|\theta|}}.\end{aligned}$$

Similarly, we obtain

$$(t/q)^{\theta_j}\frac{(q^{|\theta|-r+1}u_j/tu_k;q)_{\theta_j}}{(q^{|\theta|-r}u_j/u_k;q)_{\theta_j}}=\frac{(tq^{r-|\theta|}u_k/v_j;q)_{\theta_j}}{(q^{r-|\theta|+1}u_k/v_j;q)_{\theta_j}},$$

$$(q/t)^{\theta_j}\frac{(t^2u_j/u_k;q)_{\theta_j}}{(qtu_j/u_k;q)_{\theta_j}}=\frac{(qu_k/t^2v_j;q)_{\theta_j}}{(u_k/tv_j;q)_{\theta_j}}.$$ □

Finally, we have proved the following Pieri formula.

Theorem 3.2 *Let $\lambda=\sum_{i=1}^n\lambda_i\omega_i$ be a dominant weight, and $r\in\mathbb{N}$. Assume $\lambda_k=0$ for some fixed $1\le k\le n$. Define*

$$u_i=\begin{cases}q^{-r}t^{-2}, & i=0,\\ q^{-r+\sum_{j=i}^{k-1}\lambda_j}t^{k-i-1}, & 1\le i\le k-1,\\ q^{-r-\sum_{j=k+1}^{i+1}\lambda_j}t^{k-i-3}, & k\le i\le n-1.\end{cases}$$

We have

$$P_{r\omega_1}P_\lambda=\sum_{\substack{\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}\\|\theta|\le r}}D_\theta(u_0,u_1,\dots,u_{n-1};k,r)\,P_{\lambda+\rho}$$

with

$$\begin{aligned}\rho=&\sum_{i=1}^{k-2}(\theta_i-\theta_{i+1})\omega_i+\theta_{k-1}\omega_{k-1}+\big(r-|\theta|\big)(\omega_k-\omega_{k-1})-\theta_k\omega_{k+1}\\&+\sum_{i=k+2}^{n}(\theta_{i-2}-\theta_{i-1})\omega_i.\end{aligned}$$

Remark For $k=1,2$ (resp. $k=n,n-1$), the first (resp. the last) sum in the above expression of ρ must be understood as zero. This convention will be kept in the next sections.

4 A recurrence formula

Given two multiintegers $\beta = (\beta_1, \dots, \beta_{n-1})$ and $\kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{Z}^{n-1}$, we write $\beta \geq \kappa$ for $\beta_i \geq \kappa_i$ $(1 \leq i \leq n-1)$. We say that an infinite $(n-1)$-dimensional matrix $F = (f_{\beta\kappa})_{\beta,\kappa\in\mathbb{Z}^{n-1}}$ is lower-triangular if $f_{\beta\kappa} = 0$ unless $\beta \geq \kappa$. When all $f_{\kappa\kappa} \neq 0$, there exists a unique lower-triangular matrix $G = (g_{\kappa\gamma})_{\kappa,\gamma\in\mathbb{Z}^{n-1}}$ such that

$$\sum_{\beta\geq\kappa\geq\gamma} f_{\beta\kappa} g_{\kappa\gamma} = \delta_{\beta\gamma}$$

for all $\beta, \gamma \in \mathbb{Z}^{n-1}$, where $\delta_{\beta\gamma}$ is the usual Kronecker symbol. We refer to F and G as mutually inverse.

Such a pair of infinite multidimensional inverse matrices is given in the Appendix, as a corollary of [3, Theorem 2.7] (and, in fact, equivalent to the latter). This result is essential for our purpose.

Given n indeterminates $(u_0, u_1, \dots, u_{n-1})$, $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{N}^{n-1}$, and $k, r \in \mathbb{N}$ with $1 \leq k \leq n$, we define

$$\begin{aligned}
&C_{\theta_1,\dots,\theta_{n-1}}(u_0, u_1, \dots, u_{n-1}; k, r) \\
&\quad = q^{|\theta|} \frac{(t^2 u_0; q)_{|\theta|}}{(qtu_0; q)_{|\theta|}} \prod_{i=1}^{n-1} \frac{(q/t; q)_{\theta_i}}{(q; q)_{\theta_i}} \frac{(qu_i; q)_{\theta_i}}{(qtu_i; q)_{\theta_i}} \\
&\qquad \times \prod_{1\leq i<j\leq n-1} \frac{(qv_i/tv_j; q)_{\theta_j}}{(qv_i/v_j; q)_{\theta_j}} \frac{(tu_i/v_j; q)_{\theta_j}}{(u_i/v_j; q)_{\theta_j}} \\
&\qquad \times \prod_{i=1}^{k-1} \frac{(u_i/tu_0; q)_{\theta_i}}{(qu_i/t^2u_0; q)_{\theta_i}} \frac{(qtu_0/u_i; q)_r}{(t^2u_0/u_i; q)_r} \frac{(tu_0/u_i; q)_r}{(qu_0/u_i; q)_r} \prod_{i=k}^{n-1} \frac{(tu_i/u_0; q)_{\theta_i}}{(qu_i/u_0; q)_{\theta_i}} \\
&\qquad \times \frac{1}{\Delta(v)} \det_{1\leq i,j\leq n-1} \left[v_i^{n-j-1} \left(1 - t^{j-1} \frac{1-tv_i}{1-v_i} \prod_{s=1}^{n-1} \frac{v_i - u_s}{v_i - tu_s} \right) \right]
\end{aligned}$$

with $\Delta(v)$ the Vandermonde determinant $\prod_{1\leq i<j\leq n-1}(v_i - v_j)$. Here is our main result.

Theorem 4.1 *Let $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ be a dominant weight. Assume $\lambda_k = 0$ for some fixed $1 \leq k \leq n$. For any positive integer $r \leq \lambda_{k-1}$, the weight*

$$\lambda^{(r)} = \lambda + r(\omega_k - \omega_{k-1}) = \lambda + r\varepsilon_k$$

is dominant. Define

$$u_i = \begin{cases} q^{-r} t^{-2}, & i = 0, \\ q^{-r+\sum_{j=i}^{k-1} \lambda_j} t^{k-i-1}, & 1 \leq i \leq k-1, \\ q^{-r-\sum_{j=k+1}^{i+1} \lambda_j} t^{k-i-3}, & k \leq i \leq n-1. \end{cases}$$

We have

$$P_{\lambda^{(r)}} = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} C_\theta(u_0,u_1,\ldots,u_{n-1};k,r)\,P_{(r-|\theta|)\omega_1}\,P_{\lambda+\rho}$$

with

$$\rho=\sum_{i=1}^{k-2}(\theta_i-\theta_{i+1})\omega_i+\theta_{k-1}\omega_{k-1}-\theta_k\omega_{k+1}+\sum_{i=k+2}^{n}(\theta_{i-2}-\theta_{i-1})\omega_i.$$

Remark The weight $\lambda+\rho$ has no component on ω_k. Further, similarly as in Theorem 3.1 (see the remark following the proof of that theorem), the condition $\lambda+\rho\in P^+$ is necessarily satisfied in Theorem 4.2 as soon as $C_\theta(u_0,u_1,\ldots,u_{n-1};k,r)\neq 0$. We omit the details which involve a tedious case-by-case analysis.

Proof We make use of the multidimensional matrix inverse given in the Appendix. Let $\beta=(\beta_1,\ldots,\beta_{n-1})$, $\kappa=(\kappa_1,\ldots,\kappa_{n-1})$, $\gamma=(\gamma_1,\ldots,\gamma_{n-1})\in\mathbb{Z}^{n-1}$. If we define

$$f_{\beta\kappa}=C_{\beta_1-\kappa_1,\ldots,\beta_{n-1}-\kappa_{n-1}}\big(q^{|\kappa|}u_0,q^{\kappa_1+|\kappa|}u_1,\ldots,q^{\kappa_{n-1}+|\kappa|}u_{n-1};k,r-|\kappa|\big),$$
$$g_{\kappa\gamma}=D_{\kappa_1-\gamma_1,\ldots,\kappa_{n-1}-\gamma_{n-1}}\big(q^{|\gamma|}u_0,q^{\gamma_1+|\gamma|}u_1,\ldots,q^{\gamma_{n-1}+|\gamma|}u_{n-1};k,r-|\gamma|\big),$$

by this result, the infinite lower-triangular multidimensional matrices $(f_{\beta\kappa})_{\beta,\kappa\in\mathbb{Z}^{n-1}}$ and $(g_{\kappa\gamma})_{\kappa,\gamma\in\mathbb{Z}^{n-1}}$ are mutually inverse.

Now let us replace λ_i in Theorem 3.2 by $\lambda_i+\gamma_i-\gamma_{i+1}$ for $1\le i\le k-2$, λ_{k-1} by $\lambda_{k-1}+\gamma_{k-1}$, λ_{k+1} by $\lambda_{k+1}-\gamma_k$, λ_i by $\lambda_i+\gamma_{i-2}-\gamma_{i-1}$ for $k+2\le i\le n$, and r by $r-|\gamma|$. Then u_0 is replaced by $q^{|\gamma|}u_0$, and u_i by $q^{\gamma_i+|\gamma|}u_i$ for $1\le i\le n-1$. In explicit terms, we are considering the identity

$$\begin{aligned}&P_{(r-|\gamma|)\omega_1}\,P_{\lambda+\tilde\gamma}\\ &\quad=\sum_{\substack{\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le r}} D_\theta\big(q^{|\gamma|}u_0,q^{\gamma_1+|\gamma|}u_1,\ldots,q^{\gamma_{n-1}+|\gamma|}u_{n-1};k,r-|\gamma|\big)P_{\lambda+\tilde\gamma+\rho}\end{aligned}$$

with

$$u_i=\begin{cases}q^{-r}t^{-2}, & i=0,\\ q^{-r+\sum_{j=i}^{k-1}\lambda_j}t^{k-i-1}, & 1\le i\le k-1,\\ q^{-r-\sum_{j=k+1}^{i+1}\lambda_j}t^{k-i-3}, & k\le i\le n-1,\end{cases}$$

and

$$\begin{aligned}\rho=&\sum_{i=1}^{k-2}(\theta_i-\theta_{i+1})\omega_i+\theta_{k-1}\omega_{k-1}+\big(r-|\theta|\big)(\omega_k-\omega_{k-1})-\theta_k\omega_{k+1}\\ &+\sum_{i=k+2}^{n}(\theta_{i-2}-\theta_{i-1})\omega_i,\end{aligned}$$

$$\tilde{\gamma} = \sum_{i=1}^{k-2}(\gamma_i - \gamma_{i+1})\omega_i + \gamma_{k-1}\omega_{k-1} - \gamma_k\omega_{k+1} + \sum_{i=k+2}^{n}(\gamma_{i-2} - \gamma_{i-1})\omega_i.$$

After substituting the summation indices $\theta_i \mapsto \kappa_i - \gamma_i$ for $1 \le i \le n-1$, we obtain exactly

$$\sum_{\kappa\in\mathbb{Z}^{n-1}} g_{\kappa\gamma}\, y_\kappa = w_\gamma \quad \bigl(\gamma \in \mathbb{Z}^{n-1}\bigr)$$

with

$$y_\kappa = P_{\lambda+\tilde{\kappa}}, \qquad w_\gamma = P_{(r-|\gamma|)\omega_1} P_{\lambda+\tilde{\gamma}}$$

and

$$\tilde{\kappa} = \sum_{i=1}^{k-2}(\kappa_i - \kappa_{i+1})\omega_i + \kappa_{k-1}\omega_{k-1} + \bigl(r - |\kappa|\bigr)(\omega_k - \omega_{k-1}) - \kappa_k\omega_{k+1} + \sum_{i=k+2}^{n}(\kappa_{i-2} - \kappa_{i-1})\omega_i.$$

This immediately yields the inverse relation

$$\sum_{\beta\in\mathbb{Z}^{n-1}} f_{\beta\kappa}\, w_\beta = y_\kappa \quad \bigl(\kappa \in \mathbb{Z}^{n-1}\bigr).$$

We conclude by setting $\kappa_i = 0$ for all $1 \le i \le n-1$. □

Finally, by the substitutions $r \to \lambda_k$ and $\lambda_{k-1} \to \lambda_{k-1} + \lambda_k$, we obtain the following very remarkable expansion.

Theorem 4.2 *Let* $\lambda = \sum_{i=1}^{n} \lambda_i\omega_i$ *be a dominant weight, and* $k \in \mathbb{N}$ *fixed with* $1 \le k \le n$. *Define*

$$u_i = \begin{cases} q^{-\lambda_k} t^{-2}, & i = 0, \\ q^{\sum_{j=i}^{k-1}\lambda_j} t^{k-i-1}, & 1 \le i \le k-1, \\ q^{-\sum_{j=k}^{i+1}\lambda_j} t^{k-i-3}, & k \le i \le n-1, \end{cases}$$

and $\mu = \lambda - \lambda_k\,(\omega_k - \omega_{k-1}) = \lambda - \lambda_k\,\varepsilon_k$. *We have*

$$P_\lambda = \sum_{\substack{\theta=(\theta_1,\ldots,\theta_{n-1})\in\mathbb{N}^{n-1} \\ |\theta|\le\lambda_k}} C_\theta(u_0, u_1, \ldots, u_{n-1}; k, \lambda_k)\, P_{(\lambda_k-|\theta|)\omega_1} P_{\mu+\rho}$$

with

$$\rho = \sum_{i=1}^{k-2}(\theta_i - \theta_{i+1})\omega_i + \theta_{k-1}\omega_{k-1} - \theta_k\omega_{k+1} + \sum_{i=k+2}^{n}(\theta_{i-2} - \theta_{i-1})\omega_i.$$

Remark Observe that the weights μ and $\mu+\rho$ have no component on ω_k.

The special case $k=n$ is worth writing out explicitly.

Corollary *Let $\lambda=\sum_{i=1}^n \lambda_i\omega_i$ be a dominant weight. Define $u_0=q^{-\lambda_n}t^{-2}$ and $u_i=q^{\sum_{l=i}^{n-1}\lambda_l}t^{n-i-1}$ $(1\le i\le n-1)$. We have*

$$P_\lambda=\sum_{\substack{\theta=(\theta_1,\dots,\theta_{n-1})\in\mathbb{N}^{n-1}\\ |\theta|\le\lambda_n}} C_\theta(u_0,u_1,\dots,u_{n-1};n,\lambda_n)P_{(\lambda_n-|\theta|)\omega_1}P_\mu$$

with $\mu=\sum_{i=1}^{n-2}(\lambda_i+\theta_i-\theta_{i+1})\omega_i+(\lambda_{n-1}+\lambda_n+\theta_{n-1})\omega_{n-1}$.

The reader may check that this is exactly Theorem 4.1 of [3] (with $n\mapsto n-1$), written for $x_1\cdots x_{n+1}=1$, up to the normalization $Q_\lambda=b_\lambda P_\lambda$ with

$$b_\lambda=\prod_{1\le i\le j\le n}\frac{(q^{\sum_{l=i}^{j-1}\lambda_l}t^{j-i+1};q)_{\lambda_j}}{(q^{1+\sum_{l=i}^{j-1}\lambda_l}t^{j-i};q)_{\lambda_j}}=\prod_{1\le i\le j\le n}\frac{(tu_i/u_j;q)_{\lambda_j}}{(qu_i/u_j;q)_{\lambda_j}},$$

where we set $u_n=1/t$.

5 Examples

In this section we write out the formulas in Theorem 4.2 explicitly for $n=2,3$.

5.1 The root system A_2

For $k=2$, we have $u_0=q^{-\lambda_2}/t^2$, $u_1=q^{\lambda_1}$, and

$$\begin{aligned}C_\theta(u_0,u_1;2,r)&=q^\theta\frac{(t^2u_0;q)_\theta}{(qtu_0;q)_\theta}\frac{(q/t;q)_\theta}{(q;q)_\theta}\frac{(qu_1;q)_\theta}{(qtu_1;q)_\theta}\frac{(u_1/tu_0;q)_\theta}{(qu_1/t^2u_0;q)_\theta}\\&\quad\times\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r}\left(1-\frac{1-tv_1}{1-v_1}\frac{v_1-u_1}{v_1-tu_1}\right).\end{aligned}$$

After some simplifications, we obtain

$$P_{\lambda_1\omega_1+\lambda_2\omega_2}=\sum_{\theta\in\mathbb{N}}C_\theta^{(2)}(\lambda)P_{(\lambda_2-\theta)\omega_1}P_{(\lambda_1+\lambda_2+\theta)\omega_1}$$

with

$$\begin{aligned}C_\theta^{(2)}(\lambda)&=C_\theta(u_0,u_1;2,\lambda_2)\\&=t^\theta\frac{(q^{\lambda_2-\theta+1};q)_\theta}{(tq^{\lambda_2-\theta};q)_\theta}\frac{(1/t;q)_\theta}{(q;q)_\theta}\frac{(q^{\lambda_1+1};q)_\theta}{(tq^{\lambda_1+1};q)_\theta}\end{aligned}$$

$$\times \frac{(tq^{\lambda_1}; q)_{\lambda_2+\theta}}{(q^{\lambda_1+1}; q)_{\lambda_2+\theta}} \frac{(tq^{\lambda_1+1}; q)_{\lambda_2}}{(t^2q^{\lambda_1}; q)_{\lambda_2}} \frac{1-q^{\lambda_1+2\theta}}{1-q^{\lambda_1+\theta}}.$$

This result may be compared with the Jing–Józefiak classical result [1], more precisely, with its restriction to three variables (x_1, x_2, x_3) subject to $x_1x_2x_3 = 1$. Namely, given a partition (μ_1, μ_2), the Macdonald symmetric function $\mathcal{P}_{(\mu_1,\mu_2)}(q,t)$ is given by

$$\mathcal{P}_{(\mu_1,\mu_2)} = \sum_{\theta\in\mathbb{N}} \mathcal{C}_\theta(\mu)\mathcal{P}_{(\mu_2-\theta)}\mathcal{P}_{(\mu_1+\theta)}$$

with

$$\mathcal{C}_\theta(\mu) = \frac{(tq^{\mu_1-\mu_2+1}; q)_{\mu_2}}{(t^2q^{\mu_1-\mu_2}; q)_{\mu_2}} \frac{(q^{\mu_2-\theta+1}; q)_\theta}{(tq^{\mu_2-\theta}; q)_\theta} \frac{(tq^{\mu_1-\mu_2}; q)_{\mu_2+\theta}}{(q^{\mu_1-\mu_2+1}; q)_{\mu_2+\theta}}$$
$$\times t^\theta \frac{(1/t; q)_\theta}{(q; q)_\theta} \frac{(q^{\mu_1-\mu_2+1}; q)_\theta}{(tq^{\mu_1-\mu_2+1}; q)_\theta} \frac{1-q^{\mu_1-\mu_2+2\theta}}{1-q^{\mu_1-\mu_2+\theta}}.$$

Our formula is equivalent to the main result of [1] by the correspondence $\lambda_1 = \mu_1 - \mu_2$, $\lambda_2 = \mu_2$ between dominant weights and partitions, recalled in Sect. 2.

For $k = 1$, we have $u_0 = q^{-\lambda_1}/t^2$, $u_1 = q^{-\lambda_1-\lambda_2}/t^3$, and

$$C_\theta(u_0, u_1; 1, r) = q^\theta \frac{(t^2u_0; q)_\theta}{(qtu_0; q)_\theta} \frac{(q/t; q)_\theta}{(q; q)_\theta} \frac{(qu_1; q)_\theta}{(qtu_1; q)_\theta} \frac{(tu_1/u_0; q)_\theta}{(qu_1/u_0; q)_\theta}$$
$$\times \left(1 - \frac{1-tv_1}{1-v_1} \frac{v_1-u_1}{v_1-tu_1}\right).$$

After some simplifications, we obtain

$$P_{\lambda_1\omega_1+\lambda_2\omega_2} = \sum_{\theta\in\mathbb{N}} C_\theta^{(1)}(\lambda) P_{(\lambda_1-\theta)\omega_1} P_{(\lambda_2-\theta)\omega_2}$$

with

$$C_\theta^{(1)}(\lambda) = C_\theta(u_0, u_1; 1, \lambda_1)$$
$$= t^\theta \frac{(1/t; q)_\theta}{(q; q)_\theta} \frac{(q^{\lambda_1}; 1/q)_\theta}{(tq^{\lambda_1-1}; 1/q)_\theta} \frac{(q^{\lambda_2}; 1/q)_\theta}{(tq^{\lambda_2-1}; 1/q)_\theta} \frac{(t^3q^{\lambda_1+\lambda_2-1}; 1/q)_\theta}{(t^2q^{\lambda_1+\lambda_2-1}; 1/q)_\theta}$$
$$\times \frac{1-t^3q^{\lambda_1+\lambda_2-2\theta}}{1-t^3q^{\lambda_1+\lambda_2-\theta}}.$$

We thus recover exactly the result of Perelomov, Ragoucy, and Zaugg [9, Theorem 1(a)].

5.2 The root system A_3

For $k = 1, 2, 3$, our formulas in Theorem 4.2 write respectively as

$$P_{\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3} = \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(1)}(\lambda) P_{(\lambda_1-i-j)\omega_1} P_{(\lambda_2-i)\omega_2+(\lambda_3+i-j)\omega_3}$$

$$= \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(2)}(\lambda) P_{(\lambda_2-i-j)\omega_1} P_{(\lambda_1+\lambda_2+i)\omega_1+(\lambda_3-j)\omega_3}$$

$$= \sum_{(i,j)\in\mathbb{N}^2} C_{ij}^{(3)}(\lambda) P_{(\lambda_3-i-j)\omega_1} P_{(\lambda_1+i-j)\omega_1+(\lambda_2+\lambda_3+j)\omega_2}.$$

In order to make these expansions explicit, we need to evaluate the determinant of the 2 by 2 matrix A given by

$$A_{kl} = v_k^{2-l}\left(1 - t^{l-1}\frac{1-tv_k}{1-v_k}\frac{v_k-u_1}{v_k-tu_1}\frac{v_k-u_2}{v_k-tu_2}\right)$$

with $v_1 = q^i u_1$, $v_2 = q^j u_2$.

More precisely, we need to compute the quotient of this determinant by the Vandermonde determinant $v_1 - v_2 = q^i u_1 - q^j u_2$. There is no evidence that this quotient may be written in canonical form. Inspired by the explicit result of [2, Theorem 1] (see below), we write this quotient of determinants as

$$\begin{aligned}\frac{\det A}{q^i u_1 - q^j u_2} &= \frac{(t-1)^2}{(t-q^i)(t-q^j)} \\ &\quad\times\left(\frac{1-q^{2i}u_1}{1-q^i u_1}\frac{1-q^{2j}u_2}{1-q^j u_2}\left(1+t^{-1}\frac{1-q^i}{1-q^i u_1/tu_2}\frac{1-q^j}{1-q^j u_2/tu_1}\right)\right. \\ &\quad\left. -\left(q^i u_1 + q^j u_2\right)\frac{1-q^i}{1-q^i u_1}\frac{1-q^j}{1-q^j u_2}\frac{1-q^i/t}{1-q^i u_1/tu_2}\frac{1-q^j/t}{1-q^j u_2/tu_1}\right).\end{aligned}$$

The above identity (which is not trivial) may be easily verified by using any formal calculus software.

Next, for $(i,j)\in\mathbb{N}^2$, we define

$$\begin{aligned}&\nabla_{ij}(u_0,u_1,u_2) \\ &\quad = q^{i+j}\frac{(t^2u_0;q)_{i+j}}{(qtu_0;q)_{i+j}}\frac{(1/t;q)_i}{(q;q)_i}\frac{(u_1;q)_i}{(qtu_1;q)_i}\frac{(1/t;q)_j}{(q;q)_j} \\ &\quad\quad\times\frac{(u_2;q)_j}{(qtu_2;q)_j}\frac{(q^{i-j+1}u_1/tu_2;q)_j}{(q^{i-j+1}u_1/u_2;q)_j}\frac{(tq^{-j}u_1/u_2;q)_j}{(q^{-j}u_1/u_2;q)_j} \\ &\quad\quad\times\left(\frac{1-q^{2i}u_1}{1-u_1}\frac{1-q^{2j}u_2}{1-u_2}\left(1+t^{-1}\frac{1-q^i}{1-q^i u_1/tu_2}\frac{1-q^j}{1-q^j u_2/tu_1}\right)\right. \\ &\quad\quad\left. -\left(q^i u_1 + q^j u_2\right)\frac{1-q^i}{1-u_1}\frac{1-q^j}{1-u_2}\frac{1-q^i/t}{1-q^i u_1/tu_2}\frac{1-q^j/t}{1-q^j u_2/tu_1}\right).\end{aligned}$$

It is readily verified that we have

$$\frac{C_{ij}(u_0,u_1,u_2;1,r)}{\nabla_{ij}(u_0,u_1,u_2)} = \frac{(tu_1/u_0;q)_i}{(qu_1/u_0;q)_i}\frac{(tu_2/u_0;q)_j}{(qu_2/u_0;q)_j},$$

$$\frac{C_{ij}(u_0,u_1,u_2;2,r)}{\nabla_{ij}(u_0,u_1,u_2)} = \frac{(u_1/tu_0;q)_i}{(qu_1/t^2u_0;q)_i}\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r}\frac{(tu_2/u_0;q)_j}{(qu_2/u_0;q)_j},$$

$$\frac{C_{ij}(u_0,u_1,u_2;3,r)}{\nabla_{ij}(u_0,u_1,u_2)} = \frac{(u_1/tu_0;q)_i}{(qu_1/t^2u_0;q)_i}\frac{(qtu_0/u_1;q)_r}{(t^2u_0/u_1;q)_r}\frac{(tu_0/u_1;q)_r}{(qu_0/u_1;q)_r} \times \frac{(u_2/tu_0;q)_j}{(qu_2/t^2u_0;q)_j}\frac{(qtu_0/u_2;q)_r}{(t^2u_0/u_2;q)_r}\frac{(tu_0/u_2;q)_r}{(qu_0/u_2;q)_r}.$$

Now, by Theorem 4.2 the respective recurrence coefficients are determined to be

$$C^{(1)}_{ij}(\lambda) = C_{ij}\big(q^{-\lambda_1}/t^2, q^{-\lambda_1-\lambda_2}/t^3, q^{-\lambda_1-\lambda_2-\lambda_3}/t^4; 1, \lambda_1\big),$$
$$C^{(2)}_{ij}(\lambda) = C_{ij}\big(q^{-\lambda_2}/t^2, q^{\lambda_1}, q^{-\lambda_2-\lambda_3}/t^3; 2, \lambda_2\big),$$
$$C^{(3)}_{ij}(\lambda) = C_{ij}\big(q^{-\lambda_3}/t^2, q^{\lambda_1+\lambda_2}t, q^{\lambda_2}; 3, \lambda_3\big).$$

The cases $k=1,2$ are new. For $k=3$, we recover the first author's earlier result in [2, Theorem 1], more precisely, the restriction of this result to four variables (x_1,x_2,x_3,x_4) subject to $x_1x_2x_3x_4=1$. Namely, given a partition (μ_1,μ_2,μ_3) and $u=q^{\mu_1-\mu_2}$, $v=q^{\mu_2-\mu_3}$, the Macdonald symmetric function $\mathcal{P}_{(\mu_1,\mu_2,\mu_3)}(q,t)$ is given by

$$\mathcal{P}_{(\mu_1,\mu_2,\mu_3)} = \sum_{(i,j)\in\mathbb{N}^2} \mathcal{C}_{ij}(\mu)\mathcal{P}_{(\mu_3-i-j)}\mathcal{P}_{(\mu_1+i,\mu_2+j)}$$

with

$$\begin{aligned}\mathcal{C}_{ij}(\mu) = {} & t^{i+j}\frac{(1/t;q)_i}{(q;q)_i}\frac{(1/t;q)_j}{(q;q)_j}\frac{(tuv;q)_i}{(qt^2uv;q)_i}\frac{(v;q)_j}{(qtv;q)_j}\frac{(q^{-j}t^2u;q)_i}{(q^{-j}tu;q)_i}\frac{(qu;q)_i}{(qtu;q)_i}\\
&\times\frac{(t;q)_{\mu_1-\mu_2+i-j}}{(q;q)_{\mu_1-\mu_2+i-j}}\frac{(t;q)_{\mu_2+j}}{(q;q)_{\mu_2+j}}\frac{(t;q)_{\mu_3-i-j}}{(q;q)_{\mu_3-i-j}}\frac{(q;q)_{\mu_1-\mu_2}}{(t;q)_{\mu_1-\mu_2}}\\
&\times\frac{(q;q)_{\mu_2-\mu_3}}{(t;q)_{\mu_2-\mu_3}}\frac{(q;q)_{\mu_3}}{(t;q)_{\mu_3}}\frac{(q^{i-j}t^2u;q)_{\mu_2+j}}{(q^{i-j+1}tu;q)_{\mu_2+j}}\frac{(qtu;q)_{\mu_2-\mu_3}}{(t^2u;q)_{\mu_2-\mu_3}}\\
&\times\frac{(qt^2uv;q)_{\mu_3}}{(t^3uv;q)_{\mu_3}}\frac{(qtv;q)_{\mu_3}}{(t^2v;q)_{\mu_3}}\frac{1-q^{2i}tuv}{1-tuv}\frac{1-q^{2j}v}{1-v}\\
&\times\left(1+u\frac{1-q^i}{1-q^iu}\frac{1-q^{-j}}{1-q^{-j}t^2u}\left(t-v\big(q^itu+q^j\big)\frac{t-q^i}{1-q^{2i}tuv}\frac{t-q^j}{1-q^{2j}v}\right)\right).\end{aligned}$$

The reader may check our formula is indeed equivalent to [2, Theorem 1] by using the correspondence $\lambda_1=\mu_1-\mu_2$, $\lambda_2=\mu_2-\mu_3$, $\lambda_3=\mu_3$ between dominant weights and partitions.

6 Final remark

The Macdonald polynomial P_λ, $\lambda=\sum_{i=1}^n\lambda_i\omega_i$, is in bijective correspondence with the symmetric function $\mathcal{P}_\mu(x_1,\dots,x_{n+1})$ with $\mu=(\mu_1,\dots,\mu_n)$, $\mu_i=\sum_{j=i}^n\lambda_j$, subject to the condition $x_1\cdots x_{n+1}=1$. Therefore the n recurrence relations that we have obtained for P_λ may be expressed in terms of $\mathcal{P}_\mu(x_1,\dots,x_{n+1})$, subject to $x_1\cdots x_{n+1}=1$.

One may wonder whether this restriction can be removed. Equivalently, given some fixed integer $1\le k\le n$, is it possible to expand the symmetric function $\mathcal{P}_\mu$ in terms of products $\mathcal{P}_{(r)}\mathcal{P}_\rho$ for partitions $\rho=(\rho_1,\dots,\rho_n)$ satisfying $\rho_k=\rho_{k+1}$?

Such a development has been obtained in [3] for $k=n$, in which case $\rho_n=\rho_{n+1}=0$. However, this method cannot be used for other values of k.

Actually, the Pieri expansion of $\mathcal{P}_{(r)}\mathcal{P}_\rho$ involves symmetric functions $\mathcal{P}_\sigma$ with $\sigma-\rho$ a horizontal r-strip. Hence some of these partitions σ have length $l(\sigma)=n+1$. The only exception occurs for $k=n$ since in that case $\rho_n=0$ entails $l(\sigma)\le n$.

Therefore, except for $k=n$, the Pieri multiplication does not conserve the space generated by $\{\mathcal{P}_\kappa, l(\kappa)\le n\}$, and it is not possible to define a Pieri matrix to invert.

This difficulty does not arise in the A_n framework. Then the Pieri matrix can be defined, because the condition $x_1\cdots x_{n+1}=1$ and the property [5, (4.17), p. 325]

$$\mathcal{P}_{(\sigma_1,\dots,\sigma_{n+1})}(x_1,\dots,x_{n+1})=(x_1\cdots x_{n+1})^{\sigma_{n+1}}\mathcal{P}_{(\sigma_1-\sigma_{n+1},\dots,\sigma_n-\sigma_{n+1},0)}(x_1,\dots,x_{n+1})$$

allow us to deal with partitions of length $n+1$.

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Appendix: A multidimensional matrix inverse

The following result (equivalent to one previously given in [3]) is crucial to obtain the recursion formula in Sect. 4.

Lemma *Let $t,u_0,u_1,\dots,u_n$ be indeterminates, and $r,k\in\mathbb{N}$ with $1\le k\le n+1$. Define*

$$\begin{aligned}
f_{\beta\kappa}&=q^{|\beta|-|\kappa|}\frac{(t^2u_0;q)_{|\beta|}}{(qtu_0;q)_{|\beta|}}\frac{(qtu_0;q)_{|\kappa|}}{(t^2u_0;q)_{|\kappa|}}\prod_{i=1}^{n}\frac{(q/t;q)_{\beta_i-\kappa_i}}{(q;q)_{\beta_i-\kappa_i}}\frac{(q^{\kappa_i+|\kappa|+1}u_i;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i+|\kappa|+1}tu_i;q)_{\beta_i-\kappa_i}}\\
&\quad\times\prod_{i=1}^{k-1}\frac{(u_i/tu_0;q)_{\beta_i}}{(qu_i/t^2u_0;q)_{\beta_i}}\frac{(qu_i/tu_0;q)_{\kappa_i}}{(u_i/u_0;q)_{\kappa_i}}\\
&\quad\times\prod_{i=1}^{k-1}\frac{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}\frac{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}}\\
&\quad\times\prod_{i=k}^{n}\frac{(tu_i/u_0;q)_{\beta_i}}{(qu_i/u_0;q)_{\beta_i}}\frac{(qu_i/u_0;q)_{\kappa_i}}{(tu_i/u_0;q)_{\kappa_i}}\\
&\quad\times\prod_{1\le i<j\le n}\frac{(q^{\beta_i-\beta_j+1}u_i/tu_j;q)_{\beta_j-\kappa_j}}{(q^{\beta_i-\beta_j+1}u_i/u_j;q)_{\beta_j-\kappa_j}}\frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_j-\kappa_j}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_j-\kappa_j}}\frac{1}{q^{\beta_i}u_i-q^{\beta_j}u_j}
\end{aligned}$$

$$\times \det_{1\le i,j\le n}\left[\left(q^{\beta_i}u_i\right)^{n-j}\left(1-t^{j-1}\frac{(1-q^{\beta_i+|\kappa|}tu_i)}{(1-q^{\beta_i+|\kappa|}u_i)}\prod_{s=1}^{n}\frac{(q^{\beta_i}u_i-q^{\kappa_s}u_s)}{(q^{\beta_i}u_i-q^{\kappa_s}tu_s)}\right)\right]$$

and

$$\begin{aligned}
g_{\kappa\gamma} = {} & \left(\frac{q}{t}\right)^{|\kappa|-|\gamma|}\frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}}\frac{(qtu_0;q)_{|\gamma|}}{(t^2u_0;q)_{|\gamma|}}\prod_{i=1}^{n}\frac{(t;q)_{\kappa_i-\gamma_i}}{(q;q)_{\kappa_i-\gamma_i}}\frac{(q^{\gamma_i+|\kappa|+1}u_i;q)_{\kappa_i-\gamma_i}}{(q^{\gamma_i+|\kappa|}tu_i;q)_{\kappa_i-\gamma_i}} \\
& \times\prod_{i=1}^{k-1}\frac{(u_i/u_0;q)_{\kappa_i}}{(qu_i/tu_0;q)_{\kappa_i}}\frac{(qu_i/t^2u_0;q)_{\gamma_i}}{(u_i/tu_0;q)_{\gamma_i}} \\
& \times\prod_{i=1}^{k-1}\frac{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}}\frac{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}} \\
& \times\prod_{i=k}^{n}\frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}}\frac{(qu_i/u_0;q)_{\gamma_i}}{(tu_i/u_0;q)_{\gamma_i}} \\
& \times\prod_{1\le i<j\le n}\frac{(q^{\kappa_i-\kappa_j}tu_i/u_j;q)_{\kappa_j-\gamma_j}}{(q^{\kappa_i-\kappa_j+1}u_i/u_j;q)_{\kappa_j-\gamma_j}}\frac{(q^{\gamma_i-\kappa_j+1}u_i/tu_j;q)_{\kappa_j-\gamma_j}}{(q^{\gamma_i-\kappa_j}u_i/u_j;q)_{\kappa_j-\gamma_j}}.
\end{aligned}$$

Then the infinite lower-triangular n-dimensional matrices $(f_{\beta\kappa})_{\beta,\kappa\in\mathbf{Z}^n}$ *and* $(g_{\kappa\gamma})_{\kappa,\gamma\in\mathbf{Z}^n}$ *are mutually inverse.*

Proof Given two nonzero sequences (ξ_κ) and (ζ_κ) and a pair of matrices $(f_{\beta\kappa})$ and $(g_{\kappa\gamma})$ which are mutually inverse, it is easily checked (using the trivial relation $\frac{\xi_\beta}{\xi_\gamma}\delta_{\beta\gamma}=\delta_{\beta\gamma}$) that the matrices $(f_{\beta\kappa}\xi_\beta/\zeta_\kappa)$ and $(g_{\kappa\gamma}\zeta_\kappa/\xi_\gamma)$ are mutually inverse.

We choose

$$\begin{aligned}
\xi_\kappa = {} & \left(\frac{q}{t}\right)^{|\kappa|}\frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}}\prod_{i=1}^{k-1}\frac{(u_i/tu_0;q)_{\kappa_i}}{(qu_i/t^2u_0;q)_{\kappa_i}}\prod_{i=k}^{n}\frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}} \\
& \times\prod_{1\le i<j\le n}\frac{(qu_i/u_j;q)_{\kappa_i-\kappa_j}}{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}\frac{(u_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}}, \\
\zeta_\kappa = {} & \left(\frac{q}{t}\right)^{|\kappa|}\frac{(t^2u_0;q)_{|\kappa|}}{(qtu_0;q)_{|\kappa|}}\prod_{i=1}^{k-1}\frac{(u_i/u_0;q)_{\kappa_i}}{(qu_i/tu_0;q)_{\kappa_i}} \\
& \times\prod_{i=1}^{k-1}\frac{(qu_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(u_i/u_0;q)_{|\kappa|-r+\kappa_i}}\frac{(u_i/tu_0;q)_{|\kappa|-r+\kappa_i}}{(qu_i/t^2u_0;q)_{|\kappa|-r+\kappa_i}} \\
& \times\prod_{i=k}^{n}\frac{(tu_i/u_0;q)_{\kappa_i}}{(qu_i/u_0;q)_{\kappa_i}}\prod_{1\le i<j\le n}\frac{(qu_i/u_j;q)_{\kappa_i-\kappa_j}}{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}\frac{(u_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}},
\end{aligned}$$

together with the pair of mutually inverse matrices $(f_{\beta\kappa})$ and $(g_{\kappa\gamma})$ as defined in [3, Theorem 2.7].

Several elementary manipulations of q-shifted factorials eventually lead to the result in the desired form. To give a sample (concentrating only on the products over $\prod_{1\le i<j\le n}$ of q-shifted factorials), we use the simplification

$$\begin{aligned}
&\frac{(q^{\kappa_i-\kappa_j+1}u_i/tu_j;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i-\kappa_j+1}u_i/u_j;q)_{\beta_i-\kappa_i}}\frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_i-\kappa_i}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_i-\kappa_i}}\\
&\qquad\times\frac{(qu_i/u_j;q)_{\beta_i-\beta_j}}{(tu_i/u_j;q)_{\beta_i-\beta_j}}\frac{(u_i/u_j;q)_{\beta_i-\beta_j}}{(qu_i/tu_j;q)_{\beta_i-\beta_j}}\frac{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}{(qu_i/u_j;q)_{\kappa_i-\kappa_j}}\frac{(qu_i/tu_j;q)_{\kappa_i-\kappa_j}}{(u_i/u_j;q)_{\kappa_i-\kappa_j}}\\
&\quad=\frac{(qu_i/tu_j;q)_{\beta_i-\kappa_j}}{(qu_i/u_j;q)_{\beta_i-\kappa_j}}\frac{(u_i/u_j;q)_{\kappa_i-\beta_j}}{(tu_i/u_j;q)_{\kappa_i-\beta_j}}\frac{(qu_i/u_j;q)_{\beta_i-\beta_j}}{(qu_i/tu_j;q)_{\beta_i-\beta_j}}\frac{(tu_i/u_j;q)_{\kappa_i-\kappa_j}}{(u_i/u_j;q)_{\kappa_i-\kappa_j}}\\
&\quad=\frac{(q^{\beta_i-\beta_j+1}u_i/tu_j;q)_{\beta_j-\kappa_j}}{(q^{\beta_i-\beta_j+1}u_i/u_j;q)_{\beta_j-\kappa_j}}\frac{(q^{\kappa_i-\beta_j}tu_i/u_j;q)_{\beta_j-\kappa_j}}{(q^{\kappa_i-\beta_j}u_i/u_j;q)_{\beta_j-\kappa_j}}
\end{aligned}$$

in the computation of $f_{\beta\kappa}$ in the lemma. □

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