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Quadratic Gröbner bases for smooth 3 × 3 transportation polytopes

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Abstract The toric ideals of 3×3 transportation polytopes $\mathsf{T}_{\mathbf{rc}}$ are quadratically generated. The only exception is the Birkhoff polytope B_3.

If $\mathsf{T}_{\mathbf{rc}}$ is not a multiple of B_3, these ideals even have square-free quadratic initial ideals. This class contains all smooth 3×3 transportation polytopes.

Keywords Toric ideal · Gröbner basis · Quadratic triangulation · Transportation polytope

1 Introduction

1.1 Motivation

Let $P \subset \mathbb{R}^d$ be a lattice polytope (in the lattice $\mathbb{Z}^d$) and $r := |P \cap \mathbb{Z}^d|$ the number of lattice points in P. The normal fan N_P of P defines a projective toric variety X_P. The variety X_P is smooth if and only if every cone in N_P is unimodular. In this case we say that P is *smooth*.

The polytope P defines an ample line bundle $\mathcal{L}_P$ on X_P. It is very ample if P is smooth. It thus defines an embedding $X_P \hookrightarrow \mathbb{P}^{r-1}$ of X_P into $(r-1)$-dimensional projective space. Let I_P be the defining ideal of X_P. The following question was asked explicitly in [6, p. 153], and again in [16, Conjecture 2.9] and [2].

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Question 1 Let P be a smooth lattice polytope. Is the corresponding ideal I_P generated by quadratic binomials?

There are three variations of this question (of strictly increasing strength).

- Is the homogeneous coordinate ring $\mathbb{K}[x_1, \ldots, x_r]/I_P$ of X_P Koszul?
- Does the ideal I_P have a quadratic Gröbner basis?
- Does the ideal I_P have an initial ideal which is generated by square-free quadratic monomials?

The third version has a combinatorial interpretation using only the polytope P. The ideal I_P has a square-free quadratic initial ideal if and only if we can find a special "quadratic" triangulation of P (see Section 1.3 below for a precise definition). In this paper we adopt this convex geometric point of view.

Partial results were obtained by Robert Jan Koelman [10] and by Winfried Bruns, Joseph Gubeladze and Ngô Viêt Trung [4] for surfaces, by Günter Ewald and Alexa Schmeinck [6] for Picard number two, and by Lindsay Piechnik [14] for smooth reflexive 4-polytopes. Rikard Bögvad announced (and later withdrew) a general proof using Frobenius splitting [3].

1.2 Results

Simple transportation polytopes provide a large family of smooth polytopes. Yet, the 3×3 Birkhoff polytope B_3 is a non-simple transportation polytope whose ideal is not generated by quadratic polynomials. In Section 2, we show that in the 3×3 case B_3 is the only (3×3)-transportation polytope whose ideal is not quadratically generated.

Proposition 1 *For any* 3×3 *transportation polytope* $\mathsf{T} \neq B_3$ *the corresponding toric ideal* I_{T} *is quadratically generated.*

If P is a 3×3 transportation polytope which is not a multiple of B_3, we can show in Section 3 that these ideals even have quadratic Gröbner bases. This class contains all smooth 3×3 transportation polytopes.

Theorem 1 *Let* T *be a* 3×3 *transportation polytope. If* T *is not a multiple of* B_3, *then its toric ideal* I_{T} *has a square-free quadratic initial ideal.*

Using a different approach, Lindsay Piechnik and the first author showed that (among other polytopes) even multiples of B_3 have quadratic triangulations [14]. We believe that odd multiples ≥ 3 allow quadratic triangulations as well.

It follows from [4, Theorem 1.4.1] that the ideals of multiples ≥ 3 of B_3 have quadratic Gröbner bases – after a linear coordinate change. Such coordinate changes do, of course, not affect quadratic generation. They do, however, alter the Gröbner bases of the ideal (cf. the discussion in [9, p. 2]).

1.3 Background

Transportation polytopes

Let two vectors $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{Z}^n_{>0}$ and $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{>0}$ with $\sum_{i=1}^n c_i = \sum_{i=1}^m r_i =: s$ be given. The corresponding $(m \times n)$-transportation polytope $\mathsf{T}_{\mathbf{rc}}$ is the set of all non-negative $(m \times n)$-matrices $A = (a_{ij})_{ij}$ satisfying

$$\sum_{i=1}^m a_{ik} = c_k \quad \text{and} \quad \sum_{j=1}^n a_{lj} = r_l$$

for $1 \le k \le n$, $1 \le l \le m$. This is a bounded convex polytope with integral vertices (a lattice polytope for short) in $\mathbb{R}^{mn}$. We number the coordinates of $\mathbb{R}^{mn}$ by a_{ij} for $1 \le i \le m$ and $1 \le j \le n$. The upper $((m-1) \times (n-1))$-minor of a matrix A in the polytope determines all other entries. Hence, the dimension of $\mathsf{T}_{\mathbf{rc}}$ is at most $(m-1)(n-1)$. On the other hand, $a_{ij} = r_i c_j / s$ determines an interior point, so that the dimension is exactly $(m-1)(n-1)$. In what follows, we focus on the case $m = n = 3$.

Toric Ideals

Let $P \subset \mathbb{R}^d$ be a lattice polytope. The point configuration $\mathcal{A} = P \cap \mathbb{Z}^d = \{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ defines a ring homomorphism

$$\begin{aligned} \mathbb{K}[x_1, \dots, x_r] &\longrightarrow \mathbb{K}[t_0^{\pm 1}, \dots t_d^{\pm 1}] \\ x_i &\longmapsto t_0\, \mathbf{t}^{\mathbf{a}_i} := t_0\, t_1^{a_{1i}} \cdot \ldots \cdot t_d^{a_{di}}. \end{aligned}$$

Its kernel is the homogeneous ideal

$$I_P = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \sum u_i \mathbf{a}_i = \sum v_i \mathbf{a}_i, \sum u_i = \sum v_i \rangle.$$

This ideal is called the toric ideal associated to P (see [17, Sect. 4]).

The Birkhoff polytope

The simplest (3×3)-transportation polytope is the Birkhoff polytope B_3 of doubly stochastic matrices, given by $\mathbf{r} = \mathbf{c} = (1, 1, 1)$. The lattice points in B_3 are the six permutation matrices A_σ for $\sigma \in S_3$. If we denote the corresponding variables by x_σ, the toric ideal I_{B_3} is the principal ideal $\langle x_{123}x_{231}x_{312} - x_{132}x_{213}x_{321} \rangle$. So I_{B_3} is not quadratically generated. I_{B_3} has two initial ideals, $\langle x_{123}x_{231}x_{312} \rangle$, and $\langle x_{132}x_{213}x_{321} \rangle$. Geometrically, this corresponds to the fact that $B_3 \cap \mathbb{Z}^9$ is a circuit, i.e., a minimal affinely dependent set. B_3 is the convex hull of the triangle of even permutation matrices together with the triangle of odd permutation matrices. The two triangles meet in their barycenters.

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} + \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}. \qquad (*)$$

This (up to scaling) unique affine relation yields the equation generating I_{B_3}.

Smooth polytopes

For a lattice polytope P, the set of zeros in $\mathbb{P}^{r-1}$ of I_P is the toric variety X_P. This variety is smooth if and only if the edge directions at every vertex of P form a lattice basis. Equivalently, X_P is smooth if and only if the normal fan of P is unimodular (See [7, Sect. 2.1]). In this case we call P a smooth polytope. In particular, smooth polytopes are simple: every vertex belongs to dimension many facets. (So, the Birkhoff polytope is not smooth.) For any set $K \subset [n]$ we denote by K^c the set $[n] \setminus K$.

Lemma 1 *For a transportation polytope* $\mathsf{T}_{\mathbf{rc}}$, *the following are equivalent.*

1. $X_{\mathsf{T}_{\mathbf{rc}}}$ *is smooth.*
2. $\mathsf{T}_{\mathbf{rc}}$ *is smooth.*
3. $\mathsf{T}_{\mathbf{rc}}$ *is simple.*
4. $\sum_{i\in I} r_i \neq \sum_{j\in J} c_j$ *for all sets of indices* $I \subset [m]$, $J \subset [n]$ *that satisfy* $|I|\cdot|J^c|, |I^c|\cdot|J| > 1$.

We have not found a proof in the literature. For completeness, we include one here. (Compare the discussion for general flow polytopes in [1]. Lemma 1 says that in our case, topes and chambers agree.) We are grateful to Matthias Lenz who pointed out and corrected an error in a first version of Lemma 1.

Proof $\underline{(1) \Leftrightarrow (2)}$ by [7, Sect. 2.1]. The implication $\underline{(2) \Rightarrow (3)}$ is valid for all lattice polytopes. The converse, $\underline{(3) \Rightarrow (2)}$ follows from the fact that transportation polytopes arise from a totally unimodular matrix [15, Sect. 19].

$\underline{(4) \Rightarrow (3)}$: All defining inequalities, and hence all facets of $\mathsf{T}_{\mathbf{rc}}$ are of the form $a_{ij} \geq 0$. For any matrix A in $\mathsf{T}_{\mathbf{rc}}$, let Z_A be the set of index pairs corresponding to a facet containing A. Suppose that $\mathsf{T}_{\mathbf{rc}}$ has a vertex A that belongs to $|Z_A| \geq (m-1)(n-1)+1$ facets. Consider the bipartite graph G_A with color classes $\{1,\dots,m\}$ and $\{1,\dots,n\}$, and with edge set the non-facet defining index pairs Z_A^c. This graph has $n+m$ vertices and $\leq n+m-2$ edges. So this graph cannot be connected. Let $I \subset \{1,\dots,m\}$ and $J \subset \{1,\dots,n\}$ be color classes of one component of this graph. All entries of A not corresponding to an edge in G_A are zero. Hence, by the row and column sum condition, $\sum_{i\in I} c_i = \sum_{j\in J} r_j$.

We need to show that the case $|I| = |J^c| = 1$ cannot occur. Assume this were the case and choose a matrix $A \in \mathsf{T}_{\mathbf{rc}}$ with a zero in position (j,i) for $I = \{i\}$ and $J^c = \{j\}$. By construction $c_i = \sum_{k\in J} r_k = \sum_{k\neq j} r_k$, and the entries at (i,k) for $k \neq j$ must be r_j to satisfy the row and column sum condition. Hence, all other entries in the k-th row for $k \neq j$ are 0, and the entry in (j,l) is c_l for $l \neq i$. Thus, A is the only point in $\mathsf{T}_{\mathbf{rc}}$ satisfying $a_{ij} = 0$, so $a_{ij} \geq 0$ does not define a facet. Similarly, the case $|I^c| = |J| = 1$ can be excluded.

For $\underline{(3) \Rightarrow (4)}$ we need some preliminary observations. We use the criterion that an inequality $a_{ij} \geq 0$ defines a facet of $\mathsf{T}_{\mathbf{rc}}$ if and only if there is an $A \in \mathsf{T}_{\mathbf{rc}}$ (not necessarily integral) such that $a_{ij} = 0$ and with all other entries positive.

Now, suppose we are given $I \subset [m]$ and $J \subset [n]$ with $\sum_{i\in I} r_i = \sum_{j\in J} c_j$. Build a matrix $A \in \mathsf{T}_{\mathbf{rc}}$ from a vertex A' of the $I \times J$ transportation polytope, and a vertex A'' of the $I^c \times J^c$ transportation polytope. We abbreviate $m' = |I|$, $m'' = |I^c|$, $n' = |J|$, and $n'' = |J^c|$.

$$\begin{array}{c|c|c|} & I & I^c \\ \hline J & A' & 0 \\ \hline J^c & 0 & A'' \\ \hline \end{array}$$

Claim 1 *The inequalities $a_{ij} \geq 0$ for $(i, j) \in I \times J^c \cup I^c \times J$ define facets of* $\mathsf{T}_{\mathbf{rc}}$.

Proof Say, $(i, j) \in I \times J^c$. Start from all positive A' and A''. Add $\varepsilon m''n'$ to all $I \times J^c$ entries $\neq (i, j)$, and $\varepsilon(m'n'' - 1)$ to all $I^c \times J$ entries. By assumption, $(m'n'' - 1)$ is not zero. Now modify A' and A'' in order to obtain the old row and column sums. This amounts to finding points in two (non-integral) transportation polytopes. For small enough ε, the resulting matrix will have positive entries away from (i, j). □

Claim 2 *If the inequality $a_{ij} \geq 0$ $((i, j) \in I \times J)$ defines a facet of the $I \times J$ transportation polytope, then it also defines a facet of* $\mathsf{T}_{\mathbf{rc}}$.

Proof Let A' be a matrix whose only zero entry is (i, j), and let A'' be all positive. As before, we can subtract suitable constants from A' and A'', and find all positive matrices to counterbalance row and column sums. □

To wrap it up, if A' and A'' are vertices of their transportation polytopes, the block matrix A belongs to at least $(m - 1)(n - 1) + 1$ facets. Hence, $\mathsf{T}_{\mathbf{rc}}$ is not simple. □

The characterization of Lemma 1 is particularly simple in the 3×3 case.

Corollary 1 $\mathsf{T}_{\mathbf{rc}}$ *is smooth if and only if $r_i \neq c_j$ for all i, j.*

Triangulations

In order to show that a toric ideal has a quadratic Gröbner basis, we use the connection to regular triangulations as outlined in [17, Sect. 8]. A subset $F \subseteq P \cap \mathbb{Z}^d$ is a face of a triangulation of P if it is the set of vertices of a simplex of the triangulation; otherwise F is said to be a non-face. Observe that every superset of a non-face is a non-face.

Definition 1 A regular unimodular triangulation whose minimal non-faces have two elements is called a *quadratic triangulation*.[1]

The following characterization is a conglomerate of Corollaries 8.4 and 8.9 in [17].

[1]Simplicial complexes with this non-face property appear in the literature under the names of flag- or clique-complexes.

Theorem 2 *The defining ideal I_P of the projective toric variety $X_P \subset \mathbb{P}^{r-1}$ has a square-free initial ideal if and only if P has a regular unimodular triangulation.*

In that case, the corresponding initial ideal is the Stanley-Reisner ideal of the triangulation: $\mathrm{in}(I_P) = \langle x^F \mid F \text{ minimal non-face} \rangle$.

Here, we abbreviate $x^F := \prod_{i \in F} x_i$. In the example of the Birkhoff polytope, there are two (isomorphic) triangulations of B_3. They are both regular and unimodular. In one of them, the triangle of even permutation matrices is the minimal non-face, in the other one, the triangle of odd permutation matrices is the minimal non-face.

Using this correspondence, Theorem 1 follows from the following theorem which is what we really prove in Section 3.

Theorem 3 *If $\mathsf{T}_{\mathbf{rc}}$ is not a multiple of B_3, then $\mathsf{T}_{\mathbf{rc}}$ has a quadratic triangulation.*

Pulling triangulations

A tool we use in both proofs are pulling refinements of hyperplane subdivisions (compare [11]). Let $\mathcal{S}$ be a subdivision of P, and let $v \in P \cap \mathbb{Z}^d$ be a lattice point in P. Then the *strong pulling refinement* $\mathrm{pull}_v\,\mathcal{S}$ of $\mathcal{S}$ has two kinds of faces.

1. If $F \in \mathcal{S}$, and $v \notin F$, then $F \in \mathrm{pull}_v\,\mathcal{S}$.
2. If $F' \prec F \in \mathcal{S}$, and $v \in F \setminus F'$, then $\mathrm{conv}(v, F') \in \mathrm{pull}_v\,\mathcal{S}$.

If we start with any (regular) subdivision of P and successively pull all lattice points in P, we obtain a (regular) triangulation. If every lattice point in P was a vertex of $\mathcal{S}$, then this triangulation has a nice recursive structure. Suppose we pulled the lattice points of the face $F \in \mathcal{S}$ in the order $\mathbf{a}_1, \ldots, \mathbf{a}_r$. Then F is subdivided into pyramids with apex $\mathbf{a}_1$ and base in the pulling triangulation of facets of F not containing $\mathbf{a}_1$. On the algebraic side, this corresponds to a revlex refinement of a partial term order.

Paco's lemma

We say that a lattice polytope P has facet width 1 if for each of its facets, P lies between the hyperplane spanned by this facet and the next parallel lattice hyperplane. The following fact was first observed by Francisco Santos.

Proposition 2 (Paco's Lemma [13, 18]) *If the lattice polytope P has facet width* 1 *then if every pulling triangulation of P is* (*regular and*) *unimodular.*

2 Quadratic generation

The main tools in the proof of Proposition 1 are a hyperplane subdivision and matrix addition. We will first exhibit a Gröbner basis which consists of quadratic and cubic binomials. Then we go on to show that the cubic elements can be expressed using quadratic members of the ideal. The resulting quadratic generating set will usually fail to be a Gröbner basis.

A transportation polytope has a canonical regular subdivision into polytopes of facet width 1: We slice $\mathsf{T}_{\mathbf{rc}}$ along the hyperplanes $a_{ij} = k$. By Proposition 2, every pulling refinement of this subdivision will be a regular unimodular triangulation. A non-face F of such a triangulation either contains a pair of matrices which differ by ≥ 2 in one entry (a minimal non-face of cardinality 2), or all of F belongs to the same cell of the hyperplane subdivision.

The ideal $I_{\mathsf{T}_{\mathbf{rc}}}$ is generated by a Gröbner basis which is parameterized by the minimal non-faces of the given triangulation. (And the degree of a generator equals the cardinality of the corresponding non-face.) So we need to analyze the cells of the hyperplane subdivision. They have the form

$$\mathbf{Z}_{\mathbf{rc}}(K) = \left\{A \in \mathsf{T}_{\mathbf{rc}} \mid k_{ij} \leq a_{ij} \leq k_{ij} + 1\right\}$$

for some matrix K with row sums $\mathbf{r}'$ and column sums $\mathbf{c}'$. After translation we get

$$\mathbf{Z}_{\mathbf{rc}}(K) - K = \mathbf{Z}_{\mathbf{r}-\mathbf{r}',\mathbf{c}-\mathbf{c}'}(0) =: \mathbf{Z}_{\mathbf{c}-\mathbf{c}'}^{\mathbf{r}-\mathbf{r}'}.$$

Summing the constraints in every row and column, we see that $\mathbf{Z}_{\mathbf{rc}}(K)$ is empty unless $0 \leq r_i - r_i' \leq 3$ and $0 \leq c_j - c_j' \leq 3$ for all i, j. These inequalities have to be strict in order to obtain a full-dimensional cell: for example, if $r_i - r_i' = 3$ for some i, then the entries in row i are fixed to $a_{ij} = k_{ij} + 1$ for all j. Hence, we only need to consider cells where all coefficients of $\mathbf{r} - \mathbf{r}'$ and $\mathbf{c} - \mathbf{c}'$ are 1 or 2. So, up to symmetry, in the (3×3)-case there are only four types of such cells, namely $\mathbf{Z}_{1,1,1}^{1,1,1}$, $\mathbf{Z}_{1,1,2}^{1,1,2}$, $\mathbf{Z}_{1,2,2}^{1,2,2}$, and $\mathbf{Z}_{2,2,2}^{2,2,2}$. In fact, $\mathbf{Z}_{1,1,2}^{1,1,2}$ and $\mathbf{Z}_{1,2,2}^{1,2,2}$ are unimodular simplices, and $\mathbf{Z}_{1,1,1}^{1,1,1} = B_3$ and $\mathbf{Z}_{2,2,2}^{2,2,2}$ are isomorphic as lattice polytopes.

To summarize, $I_{\mathsf{T}_{\mathbf{rc}}}$ is generated by quadratic binomials together with cubic binomials that correspond to affine relations à la $(*)$.

Now let us assume $\mathsf{T}_{\mathbf{rc}} \neq B_3$, and, say, $\mathrm{Z}_{\mathbf{rc}}(K) \cong \mathbf{Z}_{1,1,1}^{1,1,1}$ is a cell in $\mathsf{T}_{\mathbf{rc}}$ giving rise to such a cubic equation. Because $\mathsf{T}_{\mathbf{rc}} \neq B_3$, at least one of the nine adjacent cells $\mathrm{Z}_{\mathbf{rc}}(K - E_{ij})$ has to be in $\mathsf{T}_{\mathbf{rc}}$, where E_{ij} is the $(ij)^{\text{th}}$ unit vector. After translation, we are given the relation $(*)$, and we know that (for $i = j = 1$)

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} \quad \in \quad \mathsf{T}_{\mathbf{rc}} - K.$$

But then, we can use the two quadratic relations

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} + \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

to generate $(*)$. This completes the proof of Proposition 1.

3 Squarefree quadratic initial ideal

In the previous section we have seen that all toric ideals associated to transportation polytopes $\mathsf{T}_{\mathbf{rc}} \neq B_3$ are quadratically generated. Now we head for the stronger result stated in Theorem 1.

We again start by subdividing the polytope into cells by intersection with hyperplanes of the form $a_{ij} = k$, but this time we choose a coarser subdivision to avoid cells isomorphic to the Birkhoff polytope. We show that we can achieve this by taking all hyperplanes $a_{ij} = k$ except for $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$. In a second step we do a pulling triangulation along a vertex order given by a (globally fixed) linear functional. The analysis of the cells was done using the software package `polymake` [8].

3.1 Hyperplane subdivision

Let $\mathsf{T}_{\mathbf{rc}}$ be a transportation polytope with row and column sums $\mathbf{r}$ and $\mathbf{c}$, which is not a multiple of B_3. We order the rows and columns in such a way that $r_1 \geq r_2 \geq r_3$ and $c_1 \geq c_2 \geq c_3$. We can assume that $c_1 \geq r_1$, and thus $c_1 > r_3$ because $\mathsf{T}_{\mathbf{rc}}$ is not a multiple of B_3.

As before, we subdivide $\mathsf{T}_{\mathbf{rc}}$ into cells by cutting with the hyperplanes $\{a_{ij} = k_{ij}\}$ except that we omit the (1, 1)- and the (2, 1)-entries. Hence, our cells are of the form

$$\overline{\mathbf{Z}}_{\mathbf{rc}}(K) = \left\{ A \in \mathsf{T}_{\mathbf{rc}} \,\middle|\, \begin{array}{l} k_{ij} \leq a_{ij} \leq k_{ij} + 1 \\ \text{for } (i, j) = (3, 1) \text{ and } 1 \leq i \leq 3,\ 2 \leq j \leq 3 \end{array} \right\}$$

where $K = (k_{ij})_{ij}$ is a (3×3)-matrix with $k_{11}, k_{21} := 0$. Similar to the previous section, we subtract K from $\mathsf{T}_{\mathbf{rc}}$, arriving at cells of the form

$$\overline{\mathbf{Z}}_{\mathbf{rc}}(K) - K = \overline{\mathbf{Z}}_{\mathbf{r}-\mathbf{r}',\mathbf{c}-\mathbf{c}'}(0).$$

Again, we get $r_3 - r_3', c_2 - c_2', c_3 - c_3' \in \{1, 2\}$. Also, we have

$$c_1 - c_1' = c_1 - k_{31} > r_3 - k_{31} \geq r_3 - r_3'.$$

The projection to the a_{11}-a_{21}-plane is described by the inequalities $r_i - r_i' - 2 \leq a_{i1} \leq r_i - r_i'$ for $i = 1, 2$ and $c_1 - c_1' - 1 \leq a_{11} + a_{21} \leq c_1 - c_1'$. Thus, if $r_1, r_2 \geq 2$, there are four cases (cf. Figure 1 on the left). If $r_1 - r_1' = 1$ or $r_2 - r_2' = 1$, there are three and three more cases (cf. Figure 1 on the right). We cannot have $r_1 - r_1' = r_2 - r_2' = 1$ because $c_1 - c_1' > r_3 - r_3'$.

If we subtract the lower bounds for a_{11} and a_{21}, we obtain the following 20 translation classes of cells in the subdivision. (We list them in the form $(\mathbf{r} - \mathbf{r}')$ $(\mathbf{c} - \mathbf{c}')$.)

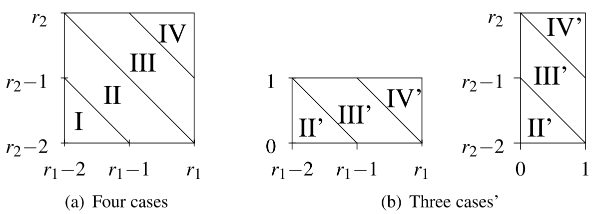

Fig. 1 The projection to the a_{11}-a_{21}-plane

I $(2,2,1)(1,2,2)$
II $(2,2,1)(2,2,1)$, $(2,2,1)(2,1,2)$, $(2,2,2)(2,2,2)$
III $(2,2,1)(3,1,1)$, $(2,2,2)(3,2,1)$, $(2,2,2)(3,1,2)$
IV $(1,1,2)(2,1,1)$

II' $(2,1,1)(1,2,1)$, $(2,1,1)(1,1,2)$, $(2,1,2)(1,2,2)$,
$(1,2,1)(1,2,1)$, $(1,2,1)(1,1,2)$, $(1,2,2)(1,2,2)$
III' $(2,1,1)(2,1,1)$, $(2,1,2)(2,2,1)$, $(2,1,2)(2,1,2)$,
$(1,2,1)(2,1,1)$, $(1,2,2)(2,2,1)$, $(1,2,2)(2,1,2)$
IV' same as IV.

3.2 Triangulating the cells

As before, the cells of the hyperplane subdivision have facet width one. (The facets $a_{11} \geq 0$ and $a_{21} \geq 0$ of P bound only cells of dashed type.) According to Lemma 2, any pulling refinement of our cell decomposition will be unimodular. The subtle part is to devise a pulling order so that the resulting triangulation is flag, i.e., so that the minimal non-faces have cardinality two.

Just as before, a non-face of such a triangulation either contains a non-face of cardinality 2, or it belongs to the same cell of the hyperplane subdivision. Hence, it suffices to guarantee that the induced triangulations of the cells are flag. To achieve this, we order the vertices by decreasing values of the linear functional

$$\mathbf{v} := \begin{bmatrix} 4 & 6 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The induced triangulations of the cells are the pulling triangulations in the induced vertex ordering.

3.2.1 Description of the cells

We give geometric descriptions for all possible cells. For most of them, any triangulation is unimodular and flag. There are two combinatorial types where we have to be careful. We list the vertices of those cells explicitly in Table 1 in the form $[a_{11}\ a_{12}\ a_{21}\ a_{22}]$.

$\overline{\mathbf{Z}}^{1,1,2}_{2,1,1}$, $\overline{\mathbf{Z}}^{1,2,1}_{1,1,2}$, $\overline{\mathbf{Z}}^{1,2,1}_{1,2,1}$, $\overline{\mathbf{Z}}^{1,2,2}_{1,2,2}$, $\overline{\mathbf{Z}}^{2,1,1}_{1,1,2}$, $\overline{\mathbf{Z}}^{2,1,1}_{1,2,1}$, $\overline{\mathbf{Z}}^{2,1,2}_{1,2,2}$, and $\overline{\mathbf{Z}}^{2,2,1}_{1,2,2}$. These cells already are unimodular simplices.

$\overline{\mathbf{Z}}^{2,2,1}_{2,1,2}$, $\overline{\mathbf{Z}}^{2,2,1}_{2,2,1}$, $\overline{\mathbf{Z}}^{2,2,2}_{3,1,2}$, and $\overline{\mathbf{Z}}^{2,2,2}_{3,2,1}$. These cells are pyramids over a triangular prism $\Delta_2 \times \Delta_1$. All six triangulations of such a polytope are unimodular and flag.

$\overline{\mathbf{Z}}^{1,2,2}_{2,1,2}$, $\overline{\mathbf{Z}}^{1,2,2}_{2,2,1}$, $\overline{\mathbf{Z}}^{2,1,2}_{2,2,1}$, and $\overline{\mathbf{Z}}^{2,1,2}_{2,1,2}$. These cells are a join of an edge and a unit square. Both triangulations of such a polytope are unimodular and flag.

$\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$ and $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$. These cells are isomorphic to a Birkhoff polytope B_3 where we have relaxed one facet. Their vertices are listed in Table 1(a) and 1(b).

Table 1 Birkhoff with one relaxed facet and its dual. We record the value of our linear functional in front of each vertex

(a) $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$	(b) $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$	(c) $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$	(d) $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$
$7: [1\ 0\ 0\ 1]$	$11: [2\ 0\ 0\ 1]$	$13: [1\ 1\ 0\ 1]$	$11: [2\ 0\ 0\ 1]$
$6: [1\ 0\ 1\ 1]$	$10: [1\ 1\ 0\ 0]$	$11: [2\ 0\ 0\ 1]$	$10: [2\ 0\ 1\ 1]$
$5: [0\ 1\ 1\ 0]$	$9: [1\ 1\ 1\ 0]$	$9: [1\ 1\ 1\ 0]$	$9: [1\ 1\ 1\ 0]$
$4: [0\ 1\ 2\ 0]$	$8: [2\ 0\ 0\ 0]$	$8: [0\ 1\ 1\ 1]$	$8: [1\ 1\ 2\ 0]$
$3: [1\ 0\ 1\ 0]$	$7: [1\ 0\ 0\ 1]$	$7: [1\ 0\ 0\ 1]$	$7: [2\ 0\ 1\ 0]$
$2: [0\ 0\ 1\ 1]$	$5: [0\ 1\ 1\ 0]$	$6: [1\ 0\ 1\ 1]$	$6: [1\ 0\ 1\ 1]$
$-2: [0\ 0\ 2\ 0]$	$3: [1\ 0\ 1\ 0]$	$5: [0\ 1\ 1\ 0]$	$4: [0\ 1\ 2\ 0]$
		$4: [0\ 1\ 2\ 0]$	$2: [1\ 0\ 2\ 0]$

Table 2 Vertex facet incidences of the interesting cells

(a) $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$	(b) $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$	(c) $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$	(d) $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$
$[v_0\ v_1\ v_2\ v_3\ \ v_4]$	$[v_0\ v_1\ v_2\ v_3]$		
$[v_0\ v_1\ v_2\ v_3\ \ v_5]$	$[v_0\ v_1\ v_3\ v_4]$	$[v_0\ v_1\ v_3\ v_4\ \ v_5]$	$[v_0\ v_1\ v_2\ v_3\ \ v_4]$
$[v_0\ v_1\ v_4\ v_5\ \ v_6]$	$[v_0\ v_1\ v_2\ v_4\ \ v_5]$	$[v_0\ v_1\ v_2\ v_4\ \ v_6]$	$[v_0\ v_1\ v_2\ v_3\ \ v_5\ \ v_6]$
$[v_0\ v_2\ v_4\ v_5\ \ v_6]$	$[v_0\ v_2\ v_3\ v_6]$	$[v_0\ v_3\ v_4\ v_6]$	$[v_0\ v_1\ v_4\ v_5\ \ v_7]$
$[v_1\ v_3\ v_5\ v_6]$	$[v_0\ v_2\ v_4\ v_5\ \ v_6]$	$[v_0\ v_1\ v_2\ v_3\ \ v_5\ \ v_7]$	$[v_0\ v_2\ v_4\ v_5\ \ v_6\ \ v_7]$
$[v_1\ v_3\ v_4\ v_6]$	$[v_0\ v_3\ v_4\ v_6]$	$[v_0\ v_2\ v_3\ v_6\ \ v_7]$	$[v_1\ v_3\ v_4\ v_7]$
$[v_2\ v_3\ v_4\ v_6]$	$[v_1\ v_2\ v_3\ v_5\ \ v_6]$	$[v_1\ v_2\ v_4\ v_5\ \ v_6\ \ v_7]$	$[v_1\ v_3\ v_5\ v_6\ \ v_7]$
$[v_2\ v_3\ v_5\ v_6]$	$[v_1\ v_3\ v_4\ v_5\ \ v_6]$	$[v_3\ v_4\ v_5\ v_6\ \ v_7]$	$[v_2\ v_3\ v_4\ v_6\ \ v_7]$

$\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$ and $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$. These cells are combinatorial duals of the cells in the previous paragraph. Their vertices are in Tables 1(c) and 1(d).

3.2.2 Triangulating the interesting cells

It remains to show that the pulling triangulations of $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$, $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$, $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$, and $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$ given by the order in Table 1 are flag. The vertex facet incidences of these cells are listed in Table 2. We will repeatedly use the following fact.

Lemma 2 *Suppose that all but two facets F_0 and F_1 of the cell Z contain the vertex v_0. Pull v_0, and choose any triangulation of Z refining this subdivision. Then every minimal non-face with more than two elements belongs to F_0 or to F_1.*

Proof The subdivision $\mathrm{pull}_{v_0} Z$ has two maximal cells $Z_0 = \mathrm{conv}(v_0, F_0)$ and $Z_1 = \mathrm{conv}(v_0, F_1)$ which are pyramids with apex v_0. A non-face of a triangulation which refines $\mathrm{pull}_{v_0} Z$ is either contained in one of these two cells, or it contains points $w_0 \in Z_0 \setminus Z_1$ and $w_1 \in Z_1 \setminus Z_0$—a non-face of cardinality two. Because the Z_i are pyramids, minimal non-faces in Z_i belong to F_i. $\square$

Table 3 The vertex facet incidences of the interesting triangulations

(a) $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$	(b) $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$	(c) $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$	(d) $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$
$[v_0\ v_1\ v_3\ v_4\ v_6]$	$[v_0\ v_1\ v_4\ v_5\ v_6]$	$[v_0\ v_1\ v_2\ v_6\ v_7]$	$[v_0\ v_1\ v_3\ v_4\ v_7]$
$[v_0\ v_1\ v_3\ v_5\ v_6]$	$[v_0\ v_1\ v_3\ v_4\ v_6]$	$[v_0\ v_3\ v_4\ v_5\ v_7]$	$[v_0\ v_1\ v_3\ v_6\ v_7]$
$[v_0\ v_2\ v_3\ v_4\ v_6]$	$[v_0\ v_1\ v_2\ v_5\ v_6]$	$[v_0\ v_3\ v_4\ v_6\ v_7]$	$[v_0\ v_1\ v_5\ v_6\ v_7]$
$[v_0\ v_2\ v_3\ v_5\ v_6]$	$[v_0\ v_1\ v_2\ v_3\ v_6]$	$[v_0\ v_1\ v_4\ v_5\ v_7]$	$[v_0\ v_2\ v_3\ v_4\ v_7]$
		$[v_0\ v_1\ v_4\ v_6\ v_7]$	$[v_0\ v_2\ v_3\ v_6\ v_7]$

Triangulation of the cell $\overline{\mathbf{Z}}^{1,2,1}_{2,1,1}$ Looking at the vertex facet incidences in Table 2(a), we see that all facets opposite vertex v_0 are simplices. Hence, after pulling at vertex v_0 we obtain a simplicial complex consisting of the 4 simplices listed in Table 3(a). Pulling at the other vertices does not change the complex anymore, and its only minimal missing faces are the edges (v_1, v_2) and (v_4, v_5).

Triangulation of the cell $\overline{\mathbf{Z}}^{2,1,1}_{2,1,1}$ The two facets opposite vertex v_0 are pyramids over the square (v_1, v_3, v_5, v_6) with apex v_2 and v_4 respectively. According to Lemma 2, any refinement of the pulling of v_0 will be flag.

In our case, we obtain four facets, which are given in Table 3(b). The minimal non-faces are the edges (v_2, v_4) and (v_3, v_5).

Triangulation of the cell $\overline{\mathbf{Z}}^{2,2,1}_{3,1,1}$ The vertex facet incidences of this cell are in Table 2(c). There are again only two facets opposite vertex v_0, which are a square pyramid S and a prism over a triangle P. So again, after Lemma 2, we are home.

The facets of our triangulation are listed in Table 3(c). The minimal non-faces are the five edges (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_2, v_5), and (v_5, v_6).

Triangulation of the Cell $\overline{\mathbf{Z}}^{2,2,2}_{2,2,2}$ The vertex facet incidences of this cell are in Table 2(d). This time there are three facets S, P_1 and P_2 opposite the vertex v_0. $S = (v_1, v_3, v_4, v_7)$, is a simplex, $P_1 := (v_1, v_3, v_5, v_6, v_7)$ is a square pyramid with apex v_7 and $P_2 := (v_2, v_3, v_4, v_6, v_7)$ is a square pyramid with apex v_3. Pulling at vertex v_0 gives us a decomposition into three cells which are pyramids over S, P_1 and P_2. The vertex v_1 is contained in the base of P_1, hence pulling at v_1 decomposes F_1 into two simplices, while F_2 is not affected. Pulling at v_2 decomposes P_2 into two simplices, and we obtain the simplicial complex given in Table 3(d). Pulling at the remaining vertices does not change this complex anymore. This leaves us with the five minimal non-faces (v_1, v_2), (v_2, v_5), (v_3, v_5), (v_4, v_5) and (v_4, v_6).

Hence, after pulling at all vertices we obtain a flag triangulation, and, we have proven the theorems stated in the introduction:

Theorem 4 *If* $\mathsf{T}_{\mathbf{rc}}$ *is not a multiple of* B_3, *then* $\mathsf{T}_{\mathbf{rc}}$ *has a quadratic triangulation.*

Theorem 5 *If* $\mathsf{T}_{\mathbf{rc}}$ *is not a multiple of* B_3, *then* $I_{\mathsf{T}_{\mathbf{rc}}}$ *has a square-free quadratic initial ideal.*

4 Outlook

In some ways, these results come as a little bit of a disappointment. Seeing that the toric ideal of the Birkhoff polytope is *not* quadratically generated, we started this project in the hope to find a counterexample to the conjectures among 3×3 transportation polytopes.

Using the methods presented here, we now think it is conceivable to prove quadratic generation in the smooth case. The same seems substantially harder for the triangulation/Gröbner basis result. In any case, the techniques can be used to improve known degree bounds for other toric ideals, be it for sets of generators or for Gröbner bases: it is sufficient to bound the degrees within the cells.

Matthias Lenz [12] was able to adapt the proof of Proposition 1 to show that the toric ideals of all flow polytopes, smooth or not, are generated in degree three. (Flow polytopes are the natural generalization of transportation polytopes. They can be realized as faces of transportation polytopes.) In the case of the Birkhoff polytope this was conjectured by Diaconis and Eriksson [5, Conj. 7].

However, the present results have not converted us into strong believers in the quadratic Gröbner basis conjecture.

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