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A q-Analog of the Hook Walk Algorithm for Random Young Tableaux

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Abstract. A probabilistic algorithm, called the *q-hook walk*, is defined. For a given Young diagram, it produces a new one by adding a random box with probabilities, depending on a positive parameter q. The corresponding Markov chain in the space of infinite Young tableaux is closely related to the knot invariant of Jones, constructed via traces of Hecke algebras. For $q = 1$, the algorithm is essentially the *hook walk* of Greene, Nijenhuis, and Wilf. The q-hook formula and a q-deformation of Young graph are also considered.

Keywords: Young diagram, random Young tableau, hook formula, q-analog, Hecke algebra

1. Introduction

In the papers [1, 2] Greene, Nijenhuis, and Wilf suggested a probabilistic algorithm, the so-called *hook walk* and used it for two purposes. The first was a new simple proof of the hook formula of Frame et al. [3] for the number f_λ of Young tableaux of a given shape λ. The second one was a procedure for efficient simulation of random Young tableaux with probabilities $M_\lambda = f_\lambda/|\lambda|!$, depending only on their shape λ. The aim of the present paper is both to extend and generalize these results.

By the extension we mean that essentially the same hook walk can be used to generate random Young tableaux with no more than m rows with probabilities

$$M_\lambda = m^{-|\lambda|} \prod_{b \in \lambda} \frac{m + c(b)}{h(b)}. \tag{1}$$

We use the notation and terminology of [4]. In particular, $|\lambda| = \lambda_1 + \cdots + \lambda_r$ is the number of boxes in a Young diagram λ with row lengths $\lambda_1, \ldots, \lambda_r$, $h(b)$ is the hook length of a box $b \in \lambda$, and $c(b)$ is the content of b. In the limit $m \to \infty$ the Plancherel measure arises:

$$M_\lambda = \prod_{b \in \lambda} h^{-1}(b) = f_\lambda/|\lambda|!. \tag{2}$$

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Our version of the hook walk is slightly different from the original one; it was inspired by that of Pittel [5].

All these results have natural q-analogs. We shall define the *q-hook walk* procedure and show how it can be used to generate random Young tableaux with probabilities

$$M_\lambda = [m]^{-|\lambda|} \prod_{b \in \lambda} \frac{[m + c(b)]}{[h(b)]} \tag{3}$$

and

$$M_\lambda = \prod_{b \in \lambda} [h(b)]^{-1}. \tag{4}$$

Here, by definition, $[k] = (1-q^k)/(1-q)$ for $k \in \mathbf{N}$. In fact, we shall deal with the more general family of random Young tableaux, depending on two parameters q, t. It is worth remarking that exactly the same family arises in the description of Markov traces for Hecke algebra generating the 2-variable Jones invariant of knots and links [6].

Let us now describe the q-analog of the usual hook formula

$$f_\lambda = \frac{|\lambda|!}{\prod_{b \in \lambda} h(b)}. \tag{5}$$

Consider the Young graph (lattice) Y. By definition, its vertices are Young diagrams, and edges are those pairs of diagrams (ν, μ), for which $\nu \subset \mu$ and $|\mu| = |\nu| + 1$. In other words, ν is obtained from μ by erasing a specified box $a \in \mu$. We are going to define the multiplicities $m_{\nu\mu}(q)$ of the corresponding edges (ν, μ) in the Young graph. To this end, we shall associate an integer

$$r(b) = c(b_\alpha) - c(b') + c(b_\omega) - c(b'')$$

to each box $b \in \mu$. Here b' and b'' are the end boxes of the arm and the leg of the hook of $b; b_\alpha$ and b_ω are the last boxes of the first row and of the first column of μ (in Figure 5 below the diagram μ is overturned). Let us recall that the content $c(b)$ of a box $b \in \mu$ is defined as $c(b) = j - i$, where i, j are the row and column numbers of the box.

We are ready now to write down the multiplicities:

$$m_{\nu\mu}(q) = \frac{q^{r(a)}[n]}{\sum_{b \in \mu} q^{r(b)}}. \tag{6}$$

Example 1. On Figure 1 the numbers $r(b)$ are inscribed into corresponding boxes, e.g., for the box $b = (1, 2)$ we have $b' = (1, 4)$, $b'' = (2, 2)$ and $b_\alpha = (1, 4)$, $b_\omega = (3, 1)$ so that $r(b) = -2$. It follows, that for the diagram $\nu = (4, 2, 1)$ obtained by erasing the box $a = (2, 3)$ the multiplicity is

$$m_{\nu\mu}(q) = q^{-1} \frac{1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7}{q^{-5} + q^{-3} + q^{-2} + q^{-1} + 2 + q^2 + q^5}.$$

0	-2	-3	-5
2	0	-1	
5			

Figure 1. $\mu = (4, 3, 1)$.

We define the q-analog $f_\mu(q)$ of the numbers f_μ recurrently : $f_\emptyset(q) = 1$ for the empty diagram $\mu = \emptyset$ and

$$f_\mu(q) = \sum_\nu f_\nu(q) m_{\nu\mu}(q) \tag{7}$$

where ν runs over diagrams, immediately preceeding μ in the Young graph. It is clear, that for $q = 1$ we have $m_{\nu\mu}(1) = 1$ and $f_\mu(1) = f_\mu$.

THEOREM 1. *There is a hook formula for $f_\mu(q)$:*

$$f_\mu(q) = q^{n(\mu)} \frac{[n]!}{\prod_{b \in \mu}[h(b)]}. \tag{8}$$

Here $[n]! = \prod_{k=1}^n [k]$ and $n(\mu) = \sum_{k=1}^r (k-1)\mu_k$ for $\mu = (\mu_1, \dots, \mu_r)$, cf. [4]. Since multiplicities $m_{\nu\mu}(q)$ are rational in q, it is a bit surprising, that the functions $f_\mu(q)$ are in fact polynomials. We shall prove the theorem, using a q-hook walk procedure.

Example 2. For a Young diagram $\mu = (r_1 + r_2, r_2)$ with $n = r_1 + 2r_2$ boxes let

$$\sigma_\mu(q) = \frac{(1 + q + \cdots + q^{n-1})}{(1 + \cdots + q^{r_1-1}) + q^{r_1+1}(1 + q^{r_1+1})(1 + \cdots + q^{r_2-1})}.$$

Then $m_{\nu,\mu}(q) = \sigma_\mu(q)$ for $\nu = (r_1 + r_2 - 1, r_2)$ and $m_{\tau,\mu}(q) = q^{2r_1+2}\sigma_\mu(q)$ for $\tau = (r_1 + r_2, r_2 - 1)$. The truncated part of the Young graph, consisting of two-row diagrams and the corresponding multiplicities is indicated in Figure 2.

2. Main formulae

The main result of the paper is completely elementary. Fix a pair of sequences of reals $\{x_i\}_{i=1}^d$ and $\{y_i\}_{i=0}^d$ such that

$$y_0 < x_1 < y_1 < x_2 < \cdots < x_d < y_d \tag{9}$$

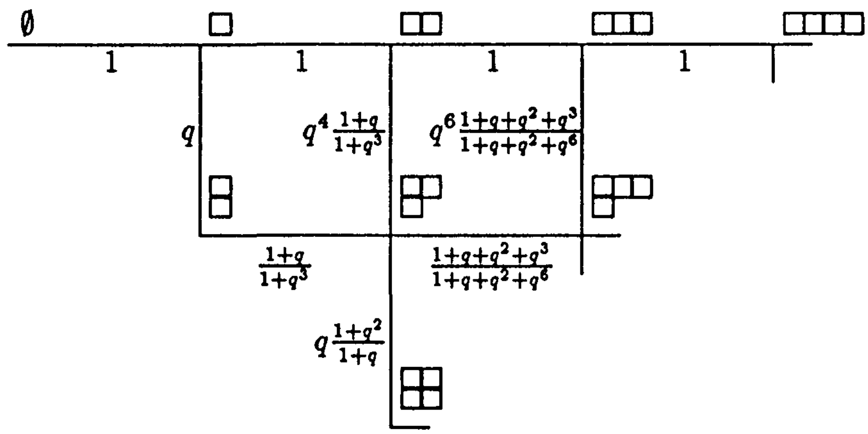

Figure 2. q-branching for two-row diagrams.

and let $z = \sum_{i=1}^{d} x_i - \sum_{i=1}^{d-1} y_i$. Consider the following families of rational functions of a variable q (each family contains d functions, indexed by $k = 1, \ldots, d$):

$$\pi'_k(q) = \prod_{i=1}^{k-1} \frac{1 - q^{x_k - y_i}}{1 - q^{x_k - x_i}} \prod_{i=k+1}^{d} \frac{1 - q^{x_k - y_{i-1}}}{1 - q^{x_k - x_i}} \tag{10}$$

$$\pi_k(q) = \prod_{i=1}^{k-1} \frac{q^{y_i} - q^{x_k}}{q^{x_i} - q^{x_k}} \prod_{i=k+1}^{d} \frac{q^{x_k} - q^{y_{i-1}}}{q^{x_k} - q^{x_i}} \tag{11}$$

$$r_k(q) = \frac{(q^{y_0} - q^{x_k})}{(q^{y_0} - q^{z})} \pi_k(q) \tag{12}$$

$$c_k(q) = \frac{(q^{x_k} - q^{y_d})}{(q^{z} - q^{y_d})} \pi_k(q) \tag{13}$$

$$p_k(q) = \frac{(q^{y_0} - q^{x_k})(q^{x_k} - q^{y_d})}{S(q)} \pi_k(q) \tag{14}$$

where

$$S(q) = \sum_{1 \le i \le j \le d} (q^{y_{i-1}} - q^{x_i})(q^{x_j} - q^{y_j}).$$

THEOREM 2. *The sum in each family is identically equal to one:*

$$\sum_k \pi_k(q) = 1 \tag{15}$$

$$\sum_k \pi'_k(q) = 1 \tag{16}$$

$$\sum_k r_k(q) = 1 \tag{17}$$

$$\sum_k c_k(q) = 1 \tag{18}$$

$$\sum_k p_k(q) = 1. \tag{19}$$

Specializing q in (10)–(14) to be one, we get the sequences (with $k = 1, \ldots, d$)

$$\pi_k = \prod_{i=1}^{k-1} \frac{(x_k - y_i)}{(x_k - x_i)} \prod_{i=k+1}^{d} \frac{(x_k - y_{i-1})}{(x_k - x_i)} \tag{20}$$

$$r_k = \frac{(x_k - y_0)}{(z - y_0)} \pi_k \tag{21}$$

$$c_k = \frac{(y_d - x_k)}{(y_d - z)} \pi_k \tag{22}$$

$$p_k = \frac{(y_d - x_k)(x_k - y_0)}{S} \pi_k \tag{23}$$

where $S = \sum_{1 \le i \le j \le d} (x_i - y_{i-1})(y_j - x_j)$.

COROLLARY 1.

$$\sum \pi_k = 1 \tag{24}$$

$$\sum r_k = 1 \tag{25}$$

$$\sum c_k = 1 \tag{26}$$

$$\sum p_k = 1. \tag{27}$$

The identities (24) and (27) were implicit in [1, 2]. They were first stated explicitly and proved algebraically by Vershik [7]. Kirillov [8] had proved (24)–(27) and announced (15).

An easy proof of Theorem 2 is given in Section 5. Another proof in Section 4 is based on the q-hook walk algorithm, which is the main subject of the paper.

3. q-hook walk

Consider a rectangle in the plane with vertices (u_o, v_o), (u_o, v_d), (u_d, v_d), (u_d, v_o), (in Figure 3 u-axis is directed downwards and v-axis to the right). For a point T with coordinates (u, v), we denote by $c(T) = v - u$ its *content*.

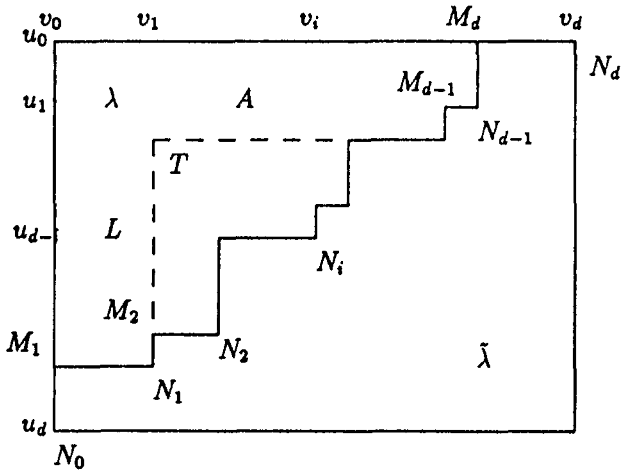

Figure 3. Complementary diagrams and a hook.

Let λ be a Young diagram, inscribed in the rectangle and $\mu = \tilde{\lambda}$–the complementary Young diagram (on Figure 3 it is overturned). We assume that $v_{d-1} < v_d$ and $u_{d-1} < u_d$, where v_{d-1} and u_{d-1} are lengths of the first row and of the first column of λ.

Let $N_0, M_1, N_1, \ldots, M_d, N_d$, be the consecutive vertices on the line, separating diagrams λ and $\tilde{\lambda}$. Denote by (u_{d-i}, v_i) the coordinates of N_i, $i = 0, 1, \ldots, d$ and let $x_i = c(M_i) = v_{i-1} - u_{d-i}$, $y_i = c(N_i) = v_i - u_{d-i}$ be the contents of the points M_i, N_i. It is clear that the condition (9) is satisfied and $z = \sum_{i=1}^{d} x_i - \sum_{i=1}^{d-1} y_i = v_0 - u_0 = c(O)$, where O is the upper left corner of the rectangle. For a true Young diagram λ the numbers u_i, v_i, x_i, y_i will be integers, but we shall use this assumption in Sections 5, 6 only.

Consider now a pair of integers s, t with $1 \leq s \leq t \leq d$. We describe an (s, t)-*hook* $H(s, t)$ with the corner in a point T with coordinates (u_{d-t}, v_{s-1}). By definition, $H(s, t) = A \cup L$ is a union of two intervals: the *arm* of the hook $A = \{(u_{d-t}, v) : v_{s-1} \leq v \leq v_{t-1}\}$, joining the vertex T with M_t and the *leg* $L = \{(u, v_{s-1}) : u_{d-t} \leq u \leq u_{d-s}\}$, joining T with M_s. The *length* of the hook $H(s, t)$ is equal to $x_t - x_s$.

We are in a position now to define the Markov chain which we call *q-hook walk*. Its state space is the finite set $X = \{(s, t) \in \mathbf{N} \times \mathbf{N} : 1 \leq s \leq t \leq d\}$. The dynamics of the chain will depend on a fixed parameter $q > 0$.

For the current state (s, t) consider the hook $H(s, t)$ and note that a point $Q \in H(s, t)$ with coordinates (u, v) is completely determined by its content $h = c(Q) = v - u$. Choose a point $Q \in H(s, t)$ at random, with probability $Cq^h\, dh$, where $C = (\ln q)/(q^{x_t} - q^{x_s})$ is the normalization constant (for $q = 1$ we use the limiting homogeneous distribution $dh/(x_t - x_s)$). We construct the new

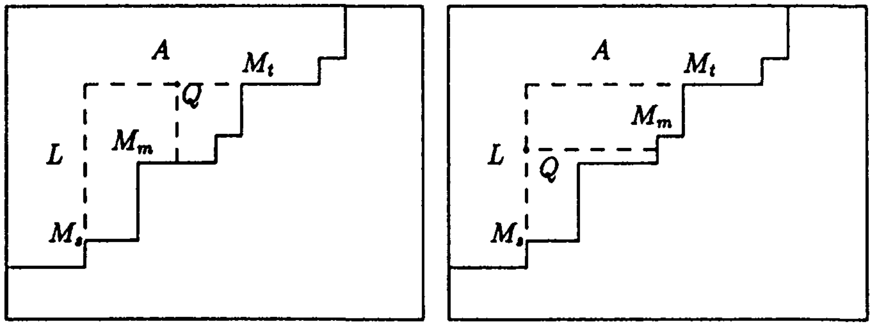

Figure 4. Constructing new state.

state (s', t'), depending on Q. There are two possibilities (Figure 4):

(A) for $Q \in A$ and $v_{m-1} < v \leq v_m$ we set $(s', t') = (s, m)$
(L) if $Q \in L$ and $u_{d-m} \leq u < u_{d-m+1}$, then $(s', t') = (m, t)$.

Note that the difference $(t - s)$ is decreasing a.s., while $s < t$. After at most d moves, the q-hook walk will reach the subset $X_0 = \{(k, k) : k = 1, \dots, d\}$ and will remain in the final state (k, k) thereafter. We shall show in the next section that the numbers (10)–(14) are exactly the probabilities for the q-hook walk to stop in a final state (k, k) for appropriate initial distribution on X. Let us now describe these distributions.

Let $\Pi(s, t) = \{(u, v) : u_{d-s} < u < u_{d-s+1}, v_{t-1} < v < v_t\}$ be a rectangle inside the diagram $\widetilde{\lambda}$ (see Figure 5). We call

$$S_{s,t}(q) = \frac{q^{y_{(s-1)}} - q^{x_s}}{1-q} \cdot \frac{q^{x_t} - q^{y_t}}{1-q}$$

the *q-area* of $\Pi(s, t)$ and consider it as statistical weight of a state $(s, t) \in X$ (Figure 5). For any subset $Y \subset X$ we have a distribution P_Y on Y with probabilities

$$P_Y(s, t) = S_{s,t}(q)/S_Y, \quad (s, t) \in Y,$$

where $S_Y = \sum_{(i,j)\in Y} S_{i,j}(q)$. We are interested in four particular cases:

$$Y_\pi = \{(1, d)\}, \qquad P_\pi = 1 \tag{28}$$

$$Y_r = \{(s, d) : 1 \leq s \leq d\}, P_r(s, d) = (q^{y_{s-1}} - q^{x_s})/S_{Y_r} \tag{29}$$

$$Y_c = \{(1, t) : 1 \leq t \leq d\}, \quad P_c(1, t) = (q^{x_t} - q^{y_t})/S_{Y_c} \tag{30}$$

$$Y_p = X, \qquad P_p = S_{s,t}(q)/S_X \tag{31}$$

Remark, that S_X is the same as $S(q)$ in (14).

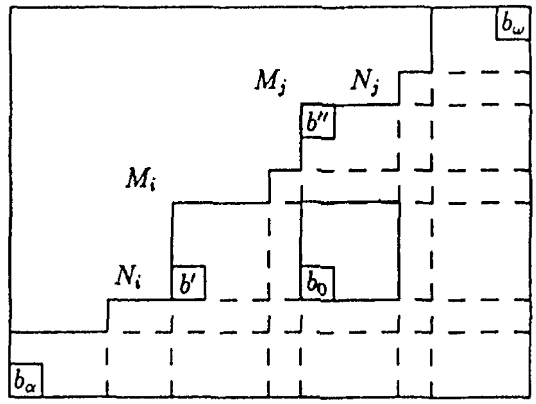

Figure 5. Partition of $\mu = \bar{\lambda}$ into rectangles.

4. Proof of the main theorem

In this section we shall prove Theorem 2, using q-hook walk algorithm. It follows immediately from:

THEOREM 3. *Assume that the initial distribution for the q-hook walk is one of (28)–(31). Then the corresponding probabilities of final states (k, k) are given by the formulae (11)–(14).*

Proof. Denote by $\pi_k^{(i,j)}(q)$, $r_k^{(i,j)}(q)$, $c_k^{(i,j)}(q)$, $p_k^{(i,j)}(q)$ the probabilities of the final state (k, k) for q-hook walk with initial distributions on the subsets

$$
\begin{aligned}
Y_\pi^{(i,j)} &= \{(i, j)\} \\
Y_r^{(i,j)} &= \{(s, j) : i \le s \le j\} \\
Y_c^{(i,j)} &= \{(i, t) : i \le t \le j\} \\
Y_p^{(i,j)} &= \{(s, t) : i \le s \le t \le j\}
\end{aligned}
$$

correspondingly. We are going to prove the identities $\pi_k(q) = \pi_k^{(1,d)}(q)$, $r_k(q) = r_k^{(1,d)}(q)$, $c_k(q) = c_k^{(1,d)}(q)$, $p_k(q) = p_k^{(1,d)}(q)$. □

LEMMA 1. *The following recurrence relations hold:*

$$\pi_k^{(i,j)}(q) = \frac{q^{x_i} - q^{z_{i,j}}}{q^{x_i} - q^{x_j}} \cdot r_k^{(i+1,j)}(q) + \frac{q^{z_{i,j}} - q^{x_j}}{q^{x_i} - q^{x_j}} \cdot c_k^{(i,j-1)}(q) \tag{32}$$

$$r_k^{(i,j)}(q) = \frac{q^{y_{i-1}} - q^{x_i}}{q^{y_{i-1}} - q^{z_{i,j}}} \cdot \pi_k^{(i,j)}(q) + \frac{q^{x_i} - q^{z_{i,j}}}{q^{y_{i-1}} - q^{z_{i,j}}} \cdot r_k^{(i+1,j)}(q) \tag{33}$$

$$c_k^{(i,j)}(q) = \frac{q^{z_{i,j}} - q^{x_j}}{q^{z_{i,j}} - q^{y_j}} \cdot c_k^{(i,j-1)}(q) + \frac{q^{x_j} - q^{y_j}}{q^{z_{i,j}} - q^{y_j}} \cdot \pi_k^{(i,j)}(q) \tag{34}$$

where $z_{i,j} = v_{i-1} - u_{d-j}$.

Proof. All the identities follow from the Bayes' formula. For instance, the random point Q in the hook $H(i, j)$ belongs to its leg with probability $(q^{x_i} - q^{z_{i,j}})/(q^{x_i} - q^{x_j})$ and to its arm with probability $(q^{z_{i,j}} - q^{x_j})/(q^{x_i} - q^{x_j})$. One can easily see that the conditional probabilities to finish the q-hook walk in the state (k, k) are $r_k^{(i+1,j)}(q)$ and $c_k^{(i,j-1)}(q)$ correspondingly. In the same way, the formula (33) corresponds to the partition $Y_r^{(i,j)} = \{(i, j)\} \cup Y_r^{(i+1,j)}$, (34) corresponds to the partition $Y_c^{(i,j)} = \{(i, j)\} \cup Y_c^{(i,j-1)}$, and the formula (35) below corresponds to the partition $Y_p^{(i,j)} = Y_p^{(1,j-1)} \cup Y_r^{(1,j)}$.

Using the induction by $(j - i)$, one can easily verify that

$$\pi_k^{(i,j)}(q) = q^{z_{i,j} - x_k} \cdot \prod_{s=i}^{k-1} \frac{q^{y_s} - q^{x_k}}{q^{x_s} - q^{x_k}} \cdot \prod_{t=k+1}^{j} \frac{q^{x_k} - q^{y_{t-1}}}{q^{x_k} - q^{x_t}}$$

$$r_k^{(i,j)}(q) = \frac{q^{y_{i-1}} - q^{x_k}}{q^{y_{i-1}} - q^{z_{i,j}}} \cdot \pi_k^{(i,j)}(q) = \frac{q^{x_{i-1}} - q^{x_k}}{q^{x_{i-1}} - q^{z_{i-1,j}}} \pi_k^{(i-1,j)}(q)$$

$$c_k^{(i,j)}(q) = \frac{q^{x_k} - q^{y_j}}{q^{z_{i,j}} - q^{y_j}} \cdot \pi_k^{(i,j)}(q) = \frac{q^{x_k} - q^{x_{j+1}}}{q^{z_{i,j+1}} - q^{x_{j+1}}} \pi_k^{(i,j+1)}(q)$$

and the first three identities (15), (17), and (18) follow.

To prove the identity (19), we shall use the recurrence relation

$$p_k^{(1,j)}(q) = \frac{S_j'(q)}{S_j(q)} \cdot p_k^{(1,j-1)}(q) + \frac{(q^{y_0} - q^{z_{1,j}})(q^{x_j} - q^{y_j})}{S_j(q)} \cdot r_k^{(1,j)}(q) \tag{35}$$

where $S_j(q) = S_{Y_j}$, with $Y_j = \{(s, t) : 1 \le s \le t \le j\}$, and $S_j'(q)$ is defined by $q^{z_{1,j}} \cdot S_{j+1}'(q) = q^{z_{1,j+1}} \cdot S_j(q)$. The identity (35) is nothing other than the Bayes' theorem applied to the partition of the set $Y_p^{(1,j)}$ into two "hypotheses": $Y_p^{(1,j)} = Y_p^{(1,j-1)} \cup Y_r^{(1,j)}$. Assume, that

$$p_k^{(1,j-1)}(q) = \frac{(q^{y_0} - q^{x_k})(q^{x_k} - q^{y_{j-1}})}{S_{j-1}(q)} \cdot \pi_k^{(1,j-1)}(q).$$

It is easy to check that this is equal to

$$p_k^{(1,j-1)}(q) = \frac{(q^{y_0} - q^{x_k})(q^{x_k} - q^{y_j})}{S_j'(q)} \cdot \pi_k^{(1,j)}(q).$$

Substituting this expression in (35), we find out that

$$p_k^{(1,j)}(q) = \frac{(q^{y_0} - q^{x_k})(q^{x_k} - q^{y_j})}{S_j(q)} \cdot \pi_k^{(1,j)}(q).$$

and the latter formula is proved by induction argument. For $j = d$ the right-hand side coincides with (14) and the theorem is proved. □

5. Random Young tableaux

In this section we describe explicitly random Young tableaux with transition probabilities (11) and (12). To this end we need a few facts from the symmetric function theory.

Let $A = \mathbf{R}[p_1, p_2, \ldots]$ be the algebra of polynomials in infinite number of variables $p_1, p_2, \ldots$ For Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\rho = (1^{m_1}, 2^{m_2}, \ldots)$ with the same number of boxes n let $p_\rho = p_1^{m_1} p_2^{m_2} \ldots, z_\rho = 1^{m_1} m_1! 2^{m_2} m_2! \ldots$ and define the Schur function by Frobenius identity

$$s_\lambda = \sum_\rho \chi_\rho^\lambda \cdot \frac{p_\rho}{z_\rho} \tag{36}$$

where χ_ρ^λ is the value of the irreducible character χ^λ of the symmetric group S_n on a permutation with m_k cycles of length k, $k = 1, 2, \ldots$. We shall use the well-known identity

$$s_{(1)} \cdot s_\nu = \sum_\mu s_\mu \tag{37}$$

where μ runs over all Young diagrams arising from ν by adding a box.

Consider two sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ of noincreasing nonnegative real numbers and assume that $\gamma = 1 - \sum \alpha_i - \sum \beta_i$ is also nonnegative. Substituting $p_1 = 1$ and

$$p_n(\alpha;\beta) = \sum_i \alpha_i^k + (-1)^{k+1} \sum_i \beta_i^k, \; n \geq 2$$

in (36) defines the *extended Schur function* $s_\lambda(\alpha;\beta)$. If $\gamma = 0$, the functions $s_\lambda(\alpha;\beta)$ are the same as those defined in [4]. It is easy to check, that $s_\lambda(\alpha;\beta) \geq 0$ for all λ. Since $s_{(1)}(\alpha;\beta) = p_1(\alpha;\beta) = 1$, it follows from (37) that

$$\sum_\mu \frac{s_\mu(\alpha;\beta)}{s_\nu(\alpha;\beta)} = 1$$

and we can define Markov measure $M^{(\alpha;\beta)}$ on the space T of infinite Young tableaux $t = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ by transition probabilities

$$M^{(\alpha;\beta)}(\lambda_{n+1} = \mu | \lambda_n = \nu) = \frac{s_\mu(\alpha;\beta)}{s_\nu(\alpha;\beta)}.$$

See [6] for the connection with describing characters and factor representations of the infinite symmetric group and Hecke algebra. Here we are interested only in several particular cases.

LEMMA 2. (Macdonald [4], 1.3, ex. 1–3). *If* $\alpha = \{q^k(1-q)/(1-q^m)\}_{k=0}^{m-1}$, $\beta \equiv 0$, *then*

$$s_\lambda(\alpha;\beta) = [m]^{-|\lambda|} q^{n(\lambda)} \prod_{b\in\lambda} \frac{[m+c(b)]}{[h(b)]} \tag{38}$$

If $\alpha = \{(1-q)q^k\}_{k=0}^{\infty}$, $\beta \equiv 0$, *then*

$$s_\lambda(\alpha;\beta) = q^{n(\lambda)} \prod_{b\in\lambda} [h(b)]^{-1} \tag{39}$$

If $\alpha = \{(1-t)/(1-q)q^k\}_{k=0}^{\infty}$, $\beta = \{t(1-q)q^k\}_{k=0}^{\infty}$, *then*

$$s_\lambda(\alpha;\beta) = q^{n(\lambda)} \prod_{b\in\lambda} \frac{(1-t)+tq^{c(b)}}{1-q^{h(b)}} \tag{40}$$

If $\alpha \equiv 0$, $\beta = \{q^k(1-q)/(1-q^m)\}_{k=0}^{m-1}$, *then*

$$s_\lambda(\alpha;\beta) = [m]^{-|\lambda|} q^{n(\lambda')} \prod_{b\in\lambda} \frac{[m-c(b)]}{[h(b)]} \tag{41}$$

COROLLARY 2. *Let* λ *be the Young diagram of Section 3 and* Λ *be obtained from* λ *by adding a box, adjacent to the vertex* M_k. *The transition probabilities*

$$p_{\lambda\Lambda} = \frac{s_\Lambda(\alpha;\beta)}{s_\lambda(\alpha;\beta)}$$

in cases (38)–(41) are equal to $r_k(q), \pi_k(q), (1-t)\pi_k(q)+t\pi'_k(q), c_k(q)$ *correspondingly.*

The corollary gives an independent proof of formulae (16)–(18), provided all $\{x_i\}$, $\{y_i\}$ in (9) are integers. Since the functions (11)–(12) are rational, the formulae (16)–(18) for the general case also follow.

6. q-hook formula

In this section we show that Theorem 1 is a consequence of (19). Let us denote by μ the complementary diagram $\bar{\lambda}$ (with $n = |\mu|$ boxes) in Figure 5 and let ν be the diagram, obtained from μ by erasing a box with a corner in the point M_k, $1 \le k \le d$. Note that all the diagrams below are true Young diagrams, with integer values of x_j, y_j for all j.

For the moment, let us denote the right-hand side of (8) by $g_\mu(q)$; we have to show that $g_\mu(q) = f_\mu(q)$, where $f_\mu(q)$ was defined recurrently in Section 1. The first step is computation of the quotient $g_\nu(q)/g_\mu(q)$.

LEMMA 3

$$\frac{g_\nu(q)}{g_\mu(q)} = q^{u_{d-k}-u_d+1}\frac{[y_d - x_k][x_k - y_0]}{[n]}\prod_{i=1}^{k-1}\frac{[x_k - y_i]}{[x_k - x_i]}\prod_{j=k+1}^{d}\frac{[y_{j-1} - x_k]}{[x_j - x_k]}$$

Proof. By definition of $n(\mu)$ (See [4]), $n(\nu) - n(\mu) = u_{d-k} - u_d + 1$. All factors of the type $[h(b)]$ in the left-hand side cancel, exept those sitting in the row and in the column of μ containing the box to be erased. The quotient for the row is equal to

$$\frac{[y_k - x_k]}{[1]}\frac{[y_{k+1} - x_k]}{[x_{k+1} - x_k]}\cdots\frac{[y_d - x_k]}{[x_d - x_k]}$$

and for the column it is

$$\frac{[x_k - y_{k-1}]}{[1]}\frac{[x_k - y_{k-2}]}{[x_k - x_{k-1}]}\cdots\frac{[x_k - y_0]}{[x_k - x_1]},$$

so the lemma follows.

Since $\sum_{i=1}^{k-1}(y_i - x_i) = v_{k-1} - v_0$, one can easily see that

$$\prod_{i=1}^{k-1}\frac{1 - q^{x_k - y_i}}{1 - q^{x_k - x_i}}\prod_{i=k+1}^{d}\frac{1 - q^{y_{i-1} - x_k}}{1 - q^{x_i - x_k}} = q^{v_0 - v_{k-1}}\pi_k(q).$$

Since $y_0 = u_d - v_0$ and $x_k = v_{k-1} - u_{d-k}$ we can rewrite Lemma 3 as

$$\frac{g_\nu(q)}{g_\mu(q)} = q^{-2x_k+1}\frac{q^{y_0} - q^{x_k}}{1 - q}\frac{q^{x_k} - q^{y_d}}{1 - q^n}\pi_k(q)$$

and hence

$$p_k(q) = q^{2x_k - 1}\frac{(1-q)(1-q^n)}{S(q)}\frac{g_\nu(q)}{g_\mu(q)}. \tag{42}$$

□

LEMMA 4. *Let $m_{\nu\mu}(q)$ be the multiplicities defined in (6). Then*

$$m_{\nu\mu}(q) = q^{2x_k - 1}\frac{(1-q)(1-q^n)}{S(q)}$$

Proof. The formula follows from the identity

$$q^{-2x_k+1}S_{i,j}(q) = \sum_{b \in \Pi(i,j)} q^{r(b)-r(a)}. \tag{43}$$

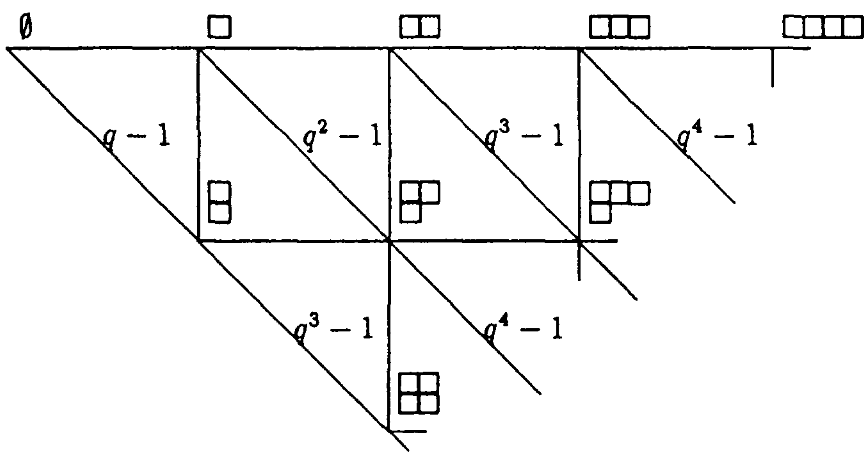

Figure 6. Another q-branching for two-row diagrams.

To prove the latter, note that

$$\sum_{b\in \Pi(i,j)} q^{r(b)-r(a)} = q^{r(b_0)-r(a)}\frac{1-q^{y_j-x_j}}{1-q}\frac{1-q^{x_i-y_{i-1}}}{1-q}$$

where b_0 is the box in the lower left corner of the rectangle $\Pi(i, j)$. For the box b_0 we have $c(b_\alpha) - c(b') = y_{i-1} - y_0$, $c(b_\omega) - c(b'') = x_j - (y_d - 1)$ so that $r(b_0) = (y_{i-1} + x_j) - (y_0 + y_d) + 1$. In the same way, for the box a in $\mu\backslash\nu$ we have $c(b_\alpha) - c(b') = x_k - (y_0 + 1)$, $c(b_\omega) - c(b'') = x_k - (y_d - 1)$ and $r(a) = 2x_k - (y_0 + y_d)$. Hence, $r(b_0) - r(a) = (y_{i-1} + y_j) - 2x_k + 1$ and (43) follows. □

COROLLARY 3

$$g_\nu(q)m_{\nu\mu}(q) = g_\mu(q)p_k(q).$$

Now it is clear from (19), that $g_\mu(q)$ satisfy the same recurrent relations (7), as $f_\mu(q)$, hence $g_\mu(q) = f_\mu(q)$ and Theorem 1 is proved.

It is worth remarking that multiplicities (6) provide by no means a unique way of constructing the q-analog of the Young graph. For instance, one can easily check that a q-deformation of the truncated Young triangle, shown in Figure 6, also provides the same q-hook formula (8) as in Theorem 1. In fact, the branching in this figure is the two-row part of a branching providing q-hook formula for any Young diagram.

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