Journal of Algebraic Combinatorics 2 (1993), 375–381

Some Extensions and Embeddings of Near Polygons

HANS CUYPERS
Department of Mathematics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

THOMAS MEIXNER
Mathematisches Seminar, University of Giessen, Arndtstrasse 2, D-6300 Giessen, Germany

Received June 29, 1992; Revised April 23, 1993

Abstract. Let $(P, L, *)$ be a near polygon having $s + 1$ points per line, $s > 1$, and suppose k is a field. Let V_k be the k-vector space with basis $\{v_p \mid p \in P\}$. Then the subspace generated by the vectors $v_l = \sum_{p*l} v_p$, where $l \in L$, has codimension at least 2 in V_k.

This observation is used in two ways. First we derive the existence of certain diagram geometries with flag transitive automorphism group, and secondly, we show that any finite near polygon with 3 points per line can be embedded in an affine $GF(3)$-space.

Keywords: near polygon, diagram geometry, affine embedding

1. Introduction

Let $\mathcal{G} = (P, L, *)$ be a finite point-line geometry and k a field, then consider the vector space V_k over k with basis $\{v_p \mid p \in P\}$, and in it the subspace W_k generated by the vectors $v_l = \sum_{p*l} v_p$, where $l \in L$. Denote by $\widetilde{V}_k$ the space V_k/W_k.

For some geometries $\mathcal{G}$, some eigenvalue techniques will provide us with a nontrivial lower bound for the dimension of $\widetilde{V}_k$; in particular, if $\mathcal{G}$ is a near polygon with constant line size $s + 1$, where $s > 1$, we obtain that $dim \widetilde{V}_k \geq 2$.

We apply this information in two different ways. First, using methods developed by the second author in [4, 5] we will derive the existence of certain diagram geometries which are flag transitive extensions of buildings of type C_n and of some generalized polygons over the field with 2 or 3 elements. Secondly we find that finite near polygons with line size 3 embed into an affine $GF(3)$-space.

2. On the dimension of $\widetilde{V}$

Let $\mathcal{G} = (P, L, *)$ be a finite geometry, k a field and V_k the k-vector space with basis $\{v_p \mid p \in P\}$. Suppose N is the incidence matrix of $\mathcal{G}$, i.e., the matrix whose rows and columns are indexed by P respectively L, and with coordinates $N_{p,l}$ equal to 1 if p and l are incident and 0 otherwise. Let W_k be the subspace of V_k generated by the vectors $v_l = \sum_{p*l} v_p$, $l \in L$. Then W_k is the subspace of V_k generated by the column vectors of N. So with $\widetilde{V}_k = V_k/W_k$ we have

$$dim\widetilde{V}_k = dimV_k - dimW_k = |P| - rank_k N \geq |P| - rank_Q N.$$

Thus to obtain a lower bound for the dimension of $\widetilde{V}_k$ we can look for an upper bound for the $\mathbb{Q}$-rank of N. The following proposition will be helpful for finding such an upper bound for some geometries $\mathcal{G}$. See [7].

PROPOSITION 2.1. (Smith [7]). *Let $\mathcal{G} = (P, L, *)$ be a finite geometry with constant line size $s+1$, $s > 0$. Suppose there exists an integer evaluation $e : P \to \mathbb{Z}$ satisfying:*

> *Each line has a unique point q with $e(q)$ minimal*
> *and the other s points have value $e(q) + 1$.*

Then the vector $\mathbf{e} := \sum_{p \in P}(-1/s)^{e(p)} v_p \in V_Q$ is a null vector for N, i.e., $N\mathbf{e} = 0$.

The standard examples of geometries $\mathcal{G}$ admitting such an evaluation e are the near polygons with constant line size $s+1$, $s > 0$. The evaluation e can be chosen to be the distance function from some fixed point p of $\mathcal{G}$. This leads us to the following. (Here d denotes the distance function on the point graph of $\mathcal{G}$.)

COROLLARY 2.1. *Let $\mathcal{G}$ be a finite near polygon with constant line size $s+1$, $s > 1$. Then $dim\widetilde{V}_k \geq 2$.*

Proof. By the above proposition we see that for each point p the vector $\mathbf{e}_p = \sum_{q \in P}(-1/s)^{d(p,q)} v_q \in V_Q$ is a null vector for N. Since $s > 1$ it is easy to see that for $p \neq q$ the vectors $\mathbf{e}_p$ and $\mathbf{e}_q$ are linearly independent. Thus the $\mathbb{Q}$-rank of N is at most $|P| - 2$. So $dim\widetilde{V}_k \geq 2$ for all fields k. □

In [7] one can find some other examples of geometries admitting an integer evaluation as in the hypothesis of the above proposition. For these examples one can use the same arguments as above to find that the dimension of $\widetilde{V}_k$ is at least 2.

Of course the lower bound given in the above corollary is not the best possible. However, for our purposes it is good enough. In [1], one can find more results on the dimension of $\widetilde{V}_k$ in the case where $\mathcal{G}$ is a generalized polygon. Furthermore, in [2] the multiplicities of the eigenvalues of the adjacency matrices of the collinearity graph of the regular near polygons are given. From this information one can deduce the dimension of $\widetilde{V}_Q$.

3. Extensions of diagram geometries

In [4, 5] the second author has described some methods to construct diagram geometries by extending some known geometries. For example the following Lemmas 3.1 and 3.2 can be found in [4, 5]. For the convenience of the reader, we comment on their proofs. (We use the notation of [5].)

LEMMA 3.1. *Let G be a group, $\{X_1, \dots, X_n\}$ a system of minimal parabolic subgroups of G with diagram $\Delta = (\Delta_{ij})_{i,j \leq n}$ and Borel-subgroup $B = \bigcap_{1 \leq i \leq n} X_i$ in G. Suppose V is a G-module, and $v \in V$ such that:*

(1) $[v, X_i] = 1$ for all $i \neq n$;
(2) $C_{X_n}(v) = B$ and $|[v, X_n]| = |X_n : B|$.

Then $\{X_1, \dots, X_n, X_n^v\}$ is a system of minimal parabolic subgroups in the semidirect product $[V, G].G$ with Borel-subgroup B and diagram $\Delta^* = (\Delta^*_{ij})_{i,j \leq n+1}$, where $\Delta^*_{i,j} = \Delta_{i,j}$ for $i, j \leq n$, $\Delta_{i,n+1} = \Delta_{i,n}$ for all $i \leq n$ and $\Delta^*_{n+1,n+1} = 2$.

For a proof of the lemma we have to verify that the chamber systems $\mathcal{C}(\langle X_i, X_n^v\rangle; X_i, X_n^v)$ and $\mathcal{C}(\langle X_i, X_n\rangle; X_i, X_n)$ are isomorphic for $i \leq n-1$, and that $\mathcal{C}(\langle X_n, X_n^v\rangle; X_n, X_n^v)$ is a generalized digon. This follows easily from the hypothesis 1, respectively, 2.

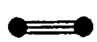
We can apply the results of the previous sections to find the module V needed in the above lemma.

PROPOSITION 3.1. *Let G be one of the following groups:*

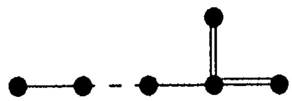
$$PSp_{2n}(q),\ P\Omega_{2n+1}(q),\ U_{2n}(q),\ G_2(q),\ {}^3D_4(q) \text{ or } {}^2F_4(q),$$

where $q = 2$ or 3. Then there is a nontrivial G-module V such that the standard geometry associated to G with diagram

 *respectively*

 *respectively*

extends to a geometry with diagram



respectively

respectively

with group the semidirect product $[V, G].G$.

Proof. Let G be a group as in the hypothesis of the lemma, with Borel subgroup B. Let $X_1, \dots, X_n$ be the minimal parabolic subgroups of G containing B, corresponding to the diagram Δ, such that $|X_n : B| = q + 1$. Set $G_n = \langle X_1, \dots, X_{n-1}\rangle$ and $G_{n-1} = \langle X_1, \dots, X_{n-2}, X_n\rangle$. Then the dual polar space (respectively, generalized polygon) $\mathcal{G}$ associated to G with $q + 1$ points per line can be identified with the cosets of G_n (the points) and G_{n-1} (the lines) in G, such that a point and a line are incident precisely when corresponding cosets have nonempty intersection.

Let the point G_n be called p and the line G_{n-1} be called l. Clearly p and l are incident.

Consider the vector spaces $V = V_k$ and its subspace $W = W_k$ constructed from $\mathcal{G}$, where $k = GF(3)$ if $q = 2$ and $k = GF(2)$ if $q = 3$. Let $v = v_p + W$. Then v is centralized by $X_1, \dots, X_{n-1}$, since they are contained in G_n, while it is not X_n-invariant, since V has dimension at least 2. As B is maximal in X_n, we find $C_{X_n}(v) = B$.

Now consider the submodule $\langle v_p^{X_n}\rangle$ of V. Clearly this is the permutation module for $PGL_2(q)$ of dimension $q + 1$, and its intersection with W contains the vector v_l. Hence we get $|[v, X_n]| = q + 1$ in all cases considered. Application of Lemma 3.1 yields the result. □

A second construction is given in the following lemma.

LEMMA 3.2. *Let G be a finite group with minimal parabolic system, Borel-subgroup B and diagram Δ as in Lemma 3.1. Suppose $K_n \leq B$ is a normal subgroup of X_n such that $X_n/K_n \simeq \Sigma_3$ and $C_{X_n}(K_n) \leq K_n$. If V is an irreducible faithful $\mathbb{C}G$-module satisfying:*

(1) $C_V(B) = C_V(\langle X_1, \dots, X_{n-1}\rangle)$ is 1-dimensional;
(2) $C_V(X_n) = 0$.

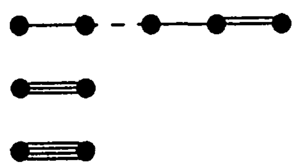
Then there exists an $r \in GL(V)$, such that in $\langle G, G^r\rangle$ the system $X_1, \dots, X_n, X_{n+1} = X_n^r$ has diagram Δ^* as in Lemma 3.1.

We may assume that V is equipped with a positive definite G-invariant hermitian form. Then r can be chosen to be the unitary reflection with reflection hyperplane $v^{\perp}$ for some nonzero vector v in $C_V(\langle X_1, \ldots, X_{n-1}\rangle)$. Then $[r, X_n]$ is a group of order 3 and the hypothesis can be replaced by

(∗) there is a 1-dimensional subspace U of $C_V(\langle X_1, \ldots, X_{n-1}\rangle)$ such that *dim* $\langle U^{X_n}\rangle = 2$.

The following application of the results of Section 2 to find the module V can also be found in [3].

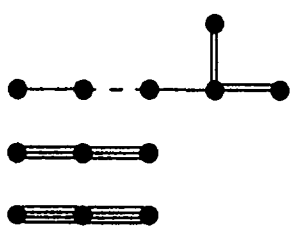
PROPOSITION 3.2. *Let G be one of the groups $PSp_{2n}(2)$, $U_{2n}(2)$, $G_2(2)$ or ${}^2F_4(2)$. Then the standard geometry of minimal parabolic subgroups in G with diagram*



respectively

respectively

extends to a geometry with diagram



respectively

respectively

Proof. Let $\mathcal{G}$ be the dual polar space or generalized polygon associated to G with 3 points per line. Then $V = \widetilde{V}_Q$ is a nontrivial G-module and G and V satisfy the condition (∗), see also [5]. So we can apply Lemma 3.2. □

In the determination of all locally finite classical Tits chamber systems with transitive group of automorphisms and finite chamber stabilizer, the types of diagrams as occuring in the above propositions cannot be ruled out. Existence of such geometries can be proved by showing that the corresponding 'assemblage' exists and then applying [8]. This method is rather abstract and does not give information on finite examples, while our construction in Proposition 3.1 describes some explicit finite examples.

The examples of diagram geometries described above are also described in [5], however, there the modules V needed for the construction, were found by *ad hoc* methods, while here, they are found in a uniform way.

4. Affine embeddings

Let $\mathcal{G} = (P, L, *)$ be a point-line geometry and $\mathbf{A}$ an affine space. Then an embedding of $\mathcal{G}$ into $\mathbf{A}$ is an injective map ϕ from P into the point set of $\mathbf{A}$ such for that every line $l \in L$ the set $\{\phi(p) \mid p * l\}$ is the point set of a line in $\mathbf{A}$.

PROPOSITION 4.1. *A finite near polygon with 3 points per line can be embedded in some affine space over the field $GF(3)$.*

Proof. Let $\mathcal{G}$ be a finite near polygon with 3 points per line. Consider the vector space $V = V_{GF(3)}$ and in it the subspace $W = W_{GF(3)}$. Then the hyperplane $H = \{\sum_{p \in P} a_p v_p \mid \sum_{p \in P} a_p = 0\}$ contains W properly by Corollary 2.1, and hence the points $\langle v_p \rangle$ of $\mathcal{G}$ lie in the affine space $V/W - H/W$. By construction, the three points of a line in $\mathcal{G}$ are just an affine line of this affine space.

The following argument proving injectivity of the map is due to Andries Brouwer. Suppose p and q are two distinct points of P, and l is a line on p in a geodesic from p to q. For each point r of $\mathcal{G}$ let $\pi_l(r)$ be the unique point on l at minimal distance from r. If m is a line of $\mathcal{G}$, then either all points of m have the same distance to l, and π_l maps m onto l, or there is a unique point on l that is closest to all the 3 points on m. Hence the subspace U of V generated by the vectors $v_r - v_{\pi_l(r)}$ and v_l contains W. But $v_p + U \neq v_q + U$. This proves the proposition. □

For generalized quadrangles a complete answer about embeddings into affine spaces is known, see [6, Chapter 7]. Apart from trivial embeddings and embeddings of generalized quadrangles with 3 points per line the only examples of generalized quadrangles that embed in some affine space are the generalized quadrangles $T_2^*(O)$, cf. [6].

If we allow also 'embeddings' for which lines are represented by i-subspaces for some $i \geq 2$, then with the same arguments near polygons with 4 points per line 'embed' in some affine space over $GF(2)$. And considering the modules described in Meixner [5] there are also such affine 'embeddings' of the dual polar spaces related to the groups $PSp_{2n}(7)$.

As noted before, one can find other geometries, see [7], that satisfy the hypothesis of Proposition 2.1. If such a geometry has constant line size 3, then the above construction may yield an embedding into some affine $GF(3)$-space.

References

1. B. Bagchi and N. Sastri, "Codes associated with generalized polygons," *Geom. Dedicata* **27** (1988), 1–8.
2. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, 1989.

3. H. Cuypers, "A construction of geometries that are almost buildings," preprint.
4. T. Meixner, "Chamber systems with extended diagram," *Mitt. Math. Sem. Giessen* **165** (1984), 93–104.
5. T. Meixner, "Tits Kammersysteme mit einer transitiven Automorphismengruppe," *Mitt. Math. Sem. Giessen* **174** (1986).
6. S. Payne and J. Thas, *Finite Generalized Quadrangles*, Pittman Publishing, Boston, 1984.
7. S. Smith, "A geometric condition for incidence-matrix nullvectors," *J. Combin. Theory Series A* **51** (1989), 129–134.
8. J. Tits, "Buildings and group amalgamations," in *Proc. Groups - St. Andrews 1985*, LMS Lecture Notes **121** (1986), 110–127.