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Random Walks in Weyl Chambers and the Decomposition of Tensor Powers

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Abstract. We consider a class of random walks on a lattice, introduced by Gessel and Zeilberger, for which the reflection principle can be used to count the number of k-step walks between two points which stay within a chamber of a Weyl group. We prove three independent results about such "reflectable walks": first, a classification of all such walks; second, many determinant formulas for walk numbers and their generating functions; third, an equality between the walk numbers and the multiplicities of irreducibles in the kth tensor power of certain Lie group representations associated to the walk types. Our results apply to the defining representations of the classical groups, as well as some spin representations of the orthogonal groups.

Keywords: random walk, representation of Lie group, tensor power, Weyl group, hyperbolic Bessel function

1. Introduction

The *ballot problem*, a classical problem in random walks, asks how many ways there are to walk from the origin to a point $(\lambda_1, \dots, \lambda_n)$, taking k unit-length steps in the positive coordinate directions while staying in the region $x_1 \geq x_2 \geq \cdots \geq x_n$. The solution is known in terms of the hook-length formula for Young tableaux; a combinatorial proof, using a reflection argument, is given in [16, 18].

In [5], Gessel and Zeilberger consider a more general question, for which some of the same techniques apply. For certain "reflectable" walk types, we want to count the number of k-step walks between two points of a lattice, staying within a chamber of a Weyl group. The steps must have certain allowable lengths and directions.

In this paper, we show that this is equivalent to decomposing into irreducibles the kth tensor power of certain representations of reductive Lie groups. We classify the reflectable walk types and their corresponding representations. For many cases, we derive determinant formulas for the number of walks, or equivalently, for the multiplicities of irreducibles in tensor powers. In particular, our formulas apply to the defining representations of the classical groups, as well

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as some spin representations of the orthogonal groups. Our results are closely related to those obtained by Proctor [11].

2. Reflectable random walks

2.1. *Definitions*

A walk type is defined by a lattice L, a set S of allowable steps between lattice points, and a polygonal cone C to which the walks are confined. Without affecting the walk problems, we may restrict L and C to the linear span of S, so that L, S, and C have the same linear span. (We may weight the steps with the relative probabilities of choosing each, but this will make little difference in what follows.)

We will assume C is a *Weyl chamber*. That is, $L, S, C \subset \mathbf{R}^n$; C is defined by a system of simple roots $\Delta \subset \mathbf{R}^n$ as

$$C = \{\vec{x} \in \mathbf{R}^n \,|\, (\alpha, \vec{x}) \geq 0 \text{ for all } \alpha \in \Delta\}\,; \tag{1}$$

the orthogonal reflections $r_\alpha : \vec{x} \mapsto \vec{x} - \frac{2(\alpha,\vec{x})}{(\alpha,\alpha)}\alpha$ preserve L and S for all α in Δ; and the r_α generate a finite group W of linear transformations, the *Weyl group*.

Example. In the ballot problem, $L = \mathbb{Z}^n, S = \{e_1 = (1,0,\ldots,0),\ldots,e_n = (0,\ldots,0,1)\}, C$ is defined by the simple roots $\Delta = \{e_i - e_{i+1}, 1 \leq i \leq n-1\}$, and W is the symmetric group S_n permuting the n coordinates.

Definition. A walk type (L, S, C) is *reflectable* if the following equivalent conditions hold:

(1) Any step $s \in S$ from any lattice point in the interior of C will not exit C.
(2) For each simple root α_i, there is a real number k_i such that $(\alpha_i, s) = \pm k_i$ or 0 for all steps $s \in S$ and (α_i, λ) is an integer multiple of k_i for all $\lambda \in L$.

The reflectability condition guarantees that a walk cannot exit C without landing on a wall of C at some step.

Example. The walk type $L = \mathbb{Z}^2, S = \{\pm e_1 \pm e_2\}, C = \{(x_1, x_2) \mid x_2 > x_1 > 0\}$ is *not* reflectable. However, it becomes reflectable if we let C be a coordinate quadrant, or if we restrict L to be the lattice points (x_1, x_2) with $x_1 + x_2$ even.

2.2. *The theorem of Gessel and Zeilberger*

In a reflectable random walk problem, we want to compute $b_{\eta\lambda,k}$, the number of walks from η to λ of length k which stay in the *interior* of a Weyl chamber.

(The ballot problem can be converted to this form by starting at the point $(n-1, n-2, \ldots, 0)$ instead of the origin, and requiring the coordinates to remain strictly ordered.)

Let $\chi(\vec{u}) = \sum_{s\in S} \vec{u}^s$, the generating function for the steps in the formal monomials $\vec{u}^{(x_1,\ldots,x_n)} = u_1^{x_1}\cdots u_n^{x_n}$. (We call this χ because it will later correspond to a character, with weights equal to the permitted steps.) Let $c_{\gamma,k}$ denote the number of random walks of length k, with steps in S, from the origin to γ, but *unconstrained* by a chamber. Then we have

$$c_{\gamma,k} = \chi(\vec{u})^k \,|_{\vec{u}^\gamma},$$

where $|_{\vec{u}^\gamma}$ denotes the coefficient of $\vec{u}^\gamma$ in the polynomial.

The fundamental result of Gessel and Zeilberger [5] is:

THEOREM 1. *If the walk from η to λ is reflectable, then*

$$b_{\eta\lambda,k} = \sum_{w\in W} \operatorname{sgn}(w) c_{\lambda-w(\eta),k}. \tag{2}$$

Proof. Every walk from any $w(\eta)$ to λ which does touch at least one wall has some last step j at which it touches a wall; let the wall be the hyperplane perpendicular to α_i, choosing the largest i if there are several choices [11]. Reflect all steps of the walk up to step j across that hyperplane; the resulting walk is a walk from $w_{\alpha_i}w(\eta)$ to λ which also touches wall i at step j. This clearly gives a pairing of walks, and since w_{α_i} has sign -1, these two walks cancel out in (2). The only walks which do not cancel in these pairs are the walks which stay within the Weyl chamber, and since $w(\eta)$ is inside the Weyl chamber only if w is the identity, this is the desired number of walks. □

2.3. *Generating functions*

It is often natural to study these walks by studying their exponential generating functions. If the generating function for unconstrained random walks is $h_\gamma(x) = \sum_{k=0}^{\infty} c_{\gamma k} x^k/k!$, then we have

$$h_\gamma(x) = \exp(x\chi(\vec{u}))\,|_{\vec{u}^\gamma}.$$

Let $g_{\eta\lambda}(x) = \sum_{k=0}^{\infty} b_{\eta\lambda,k} x^k/k!$ be the corresponding generating function for random walks in the Weyl chamber. Then we have:

COROLLARY 1. *With hypotheses as in Theorem 1,*

$$g_{\eta\lambda}(x) = \sum_{w\in W} \operatorname{sgn}(w) h_{\lambda-w(\eta)}(x). \tag{3}$$

As an illustration of the usefulness of exponential generating functions, suppose the set S of steps can be partitioned into two subsets S_1 and S_2 orthogonal to each other, and $W = W_1 \times W_2$ with W_i acting only on S_i and fixing the steps of the other subset, $i = 1, 2$. Then we can use the corollary and the properties of the exponential to factor the exponential generating function: $g_{\eta\lambda}(x) = g_{1,\eta_1\lambda_1}(x) g_{2,\eta_2\lambda_2}(x)$, where $g_{i,\eta_i\lambda_i}(x)$ is the generating function for the walk with steps S_i and $\eta = \eta_1 + \eta_2$, $\lambda = \lambda_1 + \lambda_2$, with $\eta_i, \lambda_i \in \mathrm{Span}_{\mathbb{R}} S_i$.

In particular, if $S_1 = \{0\}$ with W_1 trivial, we have $g_{\eta\lambda}(x) = e^x g_{2,\eta\lambda}(x)$. That is, adding the step 0 to the allowable steps for any random walk corresponds to multiplying the exponential generating function by e^x.

3. Decomposition of tensor powers

3.1. Characters

An important combinatorial problem in Lie theory is to determine the number of times each irreducible representation of a group or algebra occurs in the k-fold tensor power of a given finite-dimensional representation V. That is, we wish to determine the positive integers $a_{U,k}$ for which $V^{\otimes k} \cong \bigoplus_U a_{U,k} U$, where U runs over irreducible representations. We may let V be a virtual representation (a formal difference of representations). We study the case of a complex reductive group such as $GL_n(\mathbb{C})$, a compact real Lie group such as $U(n)$, or $O(n)$, or the Lie algebra of such a group. (see [1, 3, 6]).

For convenience, we will discuss $\mathfrak{g}$, a reductive Lie algebra over $\mathbb{C}$, and a finite-dimensional virtual representation V. We recall some standard facts [6]. We know $\mathfrak{g}$ possesses a maximal abelian subalgebra, its *Cartan subalgebra* $\mathfrak{h}$*; a root system*, a certain finite set in $\mathfrak{h}^*$ (the linear functionals on $\mathfrak{h}$); and a *weight lattice* Λ in $\mathfrak{h}^*$. A Weyl group W defined by the root system acts on $\mathfrak{h}$ and $\mathfrak{h}^*$. We choose a fundamental Weyl chamber of *dominant weights* in the weight lattice.

We define an integrable character of $\mathfrak{h}$ to be an element of $\mathbb{C}[\mathfrak{h}^*]$, the formal $\mathbb{C}$-linear combinations of symbols $\vec{u}^\lambda$ for λ in the weight lattice.

Example. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}), \mathfrak{h}$ is the set of all diagonal matrices; the root system is $\{e_i - e_j, 1 \le i, j \le n\}$, where e_k gives the kth coordinate of a diagonal matrix; and the weight lattice is $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$. W is the symmetric group S_n permuting the n diagonal entries.

A representation V of such a $\mathfrak{g}$ is defined up to isomorphism by its character $\chi_V = \sum_\lambda m_{V,\lambda} \in \mathbb{C}[\Lambda]$, where

$$m_{V,\lambda} = \dim_{\mathbb{C}}\{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Characters of $\mathfrak{g}$ are invariant under the Weyl group, and span the space of invariants $\mathbb{C}[\Lambda]^W$. In fact the irreducible representations of $\mathfrak{g}$ are indexed by dominant weights (or orbits of W on the weight lattice), and their characters form a basis of $\mathbb{C}[\Lambda]^W$. A direct sum (or tensor product) of representations corresponds to an ordinary sum (resp. ordinary product) of their characters.

Thus, our problem of decomposing $V^{\otimes k}$ reduces to finding integers $a_{\mu,k}$ such that

$$\chi_V^k = \sum_\mu a_{\mu,k}\chi_\mu, \tag{4}$$

where χ_μ is the character of the irreducible representation of $\mathfrak{g}$ with highest weight μ.

The case $\mu = 0$ corresponds to the trivial representation, so $a_{0,k}$ will be the dimension of invariants in the kth tensor power of V.

3.2. *Weyl's character formula*

The character χ_μ is given by the Weyl character formula:

$$\chi_\mu = \frac{\sum_{w\in W} \operatorname{sgn}(w)\vec{u}^{w(\rho+\mu)}}{\delta}, \tag{5}$$

where $\rho \in \Lambda$ is the half-sum of the positive roots, and the Weyl denominator δ is

$$\sum_{w\in W} \operatorname{sgn}(w)\vec{u}^{w(\rho)},$$

Example. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, we have $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+1}{2})$, and

$$\delta(\vec{u}) = \sum_\sigma \operatorname{sgn}(\sigma)\prod_{i=1}^n u_i^{\frac{n+1}{2}-\sigma(i)} = \det_{n\times n}\left|u_i^{\frac{n+1}{2}-i}\right|, \tag{6}$$

a Vandermonde determinant. (We denote $u_1^{\lambda_1}\cdots u_n^{\lambda_n}$ by $\vec{u}^{(\lambda_1,\dots,\lambda_n)}$.)

Now, $\delta\chi_\mu$ is essentially a monomial, i.e., there is only one dominant weight λ for which $\vec{u}^\lambda$ appears in this expression, and in fact $\lambda = \mu + \rho$. Thus, multiplying (4) by δ, we get

$$a_{\mu,k} = \delta\chi_V(\vec{u})^k\big|_{\vec{u}^\mu} \tag{7}$$

where $\chi|_{\vec{u}^\lambda}$ denotes the coefficient of $\vec{u}^\lambda$ in the element $\chi \in \mathbb{C}[\mathfrak{h}^*]$. Multiplying out by the terms of δ, we obtain

$$a_{\mu,k} = \sum_{w \in W} \operatorname{sgn}(w) \chi_V(\vec{u})^k \big|_{\vec{u}^{\rho+\mu-w(\rho)}}. \tag{8}$$

Forming an exponential generating function, we have

$$f_\mu(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{a_{\mu,k}}{k!} x^k$$
$$= \sum_{w \in W} \operatorname{sgn}(w) \exp(x\chi_V(\vec{u}))\big|_{\vec{u}^{\rho+\mu-w(\rho)}}. \tag{9}$$

3.3. *Equivalence of tensor powers and random walks*

The right-hand sides of these equations are the same sums of unconstrained walks as in (2) and (3), with $\eta = \rho, \lambda = \rho + \mu$. This gives us a correspondence between random walks in a Weyl chamber and the decomposition of tensor powers. In particular, equating the right sides of (8) and (2), and likewise of (9) and (3), we have the following result.

THEOREM 2. *Let V be a finite-dimensional representation of a reductive complex Lie algebra $\mathfrak{g}$. Let C be a Weyl chamber, S the set of weights of V, and L some lattice containing S and ρ, the half-sum of the positive roots.*

If (L, S, C) defines a reflectable walk type, then the number $b_{\rho,\rho+\mu,k}$ of random walks with k steps from ρ to $\rho + \mu$ which stay strictly within the principal Weyl chamber is equal to the multiplicity $a_{\mu,k}$ of the irreducible with highest weight μ in the kth tensor power of V; and the corresponding exponential generating functions $g_{\rho,\rho+\mu}$ and f_μ are equal.

The statement remains valid if we replace $\mathfrak{g}$ by a connected Lie group which is reductive or compact.

Specific cases of the theorem are implicitly known. With allowed steps $e_1, \dots, e_n$ and Weyl group $A_{n-1} = S_n$ (V being the defining representation of Sl_n or GL_n), the walks correspond to Young tableaux with at most n rows. Likewise, the steps $\pm e_1, \dots, \pm e_n$ and the Weyl group B_n (V a representation of the symplectic group), correspond to up-down tableaux [14]. For relations with orthogonal tableaux, see [10, 11, 12].

4. Classification

We outline a procedure to list all reflectable walks in a Weyl chamber, summarizing our results in Subsection 4.5.

4.1. Maximal lattices

Given a reflectable walk type (L, S, C) in $\mathbb{R}^n$, with C defined by a system of simple roots Δ, we can embed L in a "maximal" lattice $L_{S,C}$ as follows. Let π_0 be the orthogonal projection of $\mathbb{R}^n$ onto $\Delta^\perp$, and let $(\alpha_i, s) = \pm k_i$ or 0 for $\alpha_i \in \Delta, s \in S$. Then

$$L_{S,C} = \{\vec{x} \in \mathbb{R}^n \mid (\alpha_i, \vec{x}) \in k_i\mathbb{Z} \text{ for all } \alpha_i \in \Delta, \text{ and } \pi_0(\vec{x}) \in \pi_0(L)\} . \tag{10}$$

(This is maximal among all lattices L' for which (L', S, C) is a reflectable walk type and for which $\pi_0(L') = \pi_0(L)$.) Counting the walks for (L, S, C) is clearly a special case of the problem for $L_{S,C}$, so we shall assume $L = L_{S,C}$, choosing an arbitrary lattice for $\pi_0(L)$.

4.2. Classification of chambers

The simple roots Δ defining C and W may be partitioned into minimal subsets each orthogonal to the others: $\Delta = \Delta_1 \amalg \cdots \amalg \Delta_r$, with $\Delta_j \perp \Delta_k$ for $j \neq k$. We may then write

$$C = \mathbb{R}^{n_0} \times C_1 \times \cdots \times C_r \subset \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_r}, \tag{11}$$

where $\mathbb{R}^{n_j} = \mathrm{Span}_{\mathbb{R}}\Delta_j$, and $\mathbb{R}^{n_0} = \Delta^\perp$. Now, according to the classification of Weyl groups [2, 6], the irreducible factors $\Delta_j \subset \mathbb{R}^{n_j}$ and the reflection group W_j which they generate must be one of the classical types $A_n, B_n = C_n, D_n$ or the exceptional types E_6, E_7, E_8, F_4, G_2 (the subscript indicating the rank n_j).

4.3. Compatible steps

Given a Weyl group W and chamber C in $\mathbb{R}^n$, we will say that two steps $s_1, s_2 \in \mathbb{R}^n$ are *compatible* if: for each simple root α_i, (α_i, s_1) and (α_i, s_2) have the same absolute value k_i, or one of them is 0; and the projections $\pi_0(s_1), \pi_0(s_2) \in \Delta^\perp$ generate a discrete lattice. All the steps in S are compatible with each other if and only if there is a lattice L such that (L, S, C) is a reflectable walk.

Let π^j be the orthogonal projection from $\mathbb{R}^n$ to the irreducible component $\mathbb{R}^{n_j}$. Then (L, S, C) is reflectable if and only if all the projections $(\pi_j(L), \pi_j(S), \pi_j(C)), j = 0, \ldots, r$, are reflectable. This is clear from the compatibility characterization of reflectability and the discussion of maximal lattices. Thus, it suffices to classify pairs (S, C), where C is a chamber of one of the irreducible Weyl groups listed above, and S is a W-invariant set of mutually compatible steps. (Note that in the component $\mathbb{R}^{n_0}$ with trivial Weyl group, any walk is reflectable.)

Example. The Weyl group $A_{n-1} = S_n$ acts on $\mathbb{Z}^n$ by permutations of the coordinates. The roots of A_{n-1} span the hyperplane H of points whose coordinates sum to 0, the orthogonal complement of which is $\mathbb{R}(e_1 + \cdots + e_n)$. Thus, a walk will be reflectable if the projections of the steps onto H give a reflectable walk and the sums of the coordinates of the steps generate a discrete subgroup of $\mathbb{R}$.

Now, all $s \in S$ must be compatible with their images ws for $w \in W$. (If this holds, we say s is *self-compatible*.) This is the main constraint on the possible S. To see this, we examine the W-images of an arbitrary step.

The most general form of the W-action is as follows. We fix the lengths of α_i in one of the standard ways, and let $\{\check{\omega}_i\} \subset \mathbb{R}^n$ be the dual basis to $\{\alpha_i\} = \Delta \subset \mathbb{R}^n$, so that $s = \sum_{i=1}^n (\alpha_i, s)\check{\omega}_i$. Note that the reflection r_i fixes $\check{\omega}_j : r_i(\check{\omega}_j) = \check{\omega}_j$ for $i \neq j$; and the coefficients of $r_j(\check{\omega}_j)$ are

$$(\alpha_i, r_j(\check{\omega}_j)) = \delta_{ij} - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)}. \tag{12}$$

Thus, the coefficients of $r_j(\check{\omega}_j)$ are the jth column of the identity matrix minus the Cartan matrix. That is, $\check{\omega}_j$ is transformed under r_j by the rule

$$r_j(\check{\omega}_j) = -\check{\omega}_j + \sum_{i \neq j} c_{i,j}\check{\omega}_i, \tag{13}$$

where $c_{i,j}$ is the number of links connecting the nodes i and j in the Dynkin diagram of W, provided the arrow is pointed from i to j; and $c_{i,j} = 1$ otherwise.

4.4. *Classification of steps*

We now find the self-compatible W-orbits of steps for each irreducible Weyl group. The reflection law above gives some general restrictions. For instance: for each W-orbit, consider the representative s_{dom}, which lies in the principal Weyl chamber (i.e., all the $\check{\omega}_i$-coefficients $(a_i, s_{dom}) \geq 0$). Only *one* of the coefficients can be nonzero, since otherwise we can easily find a chain of reflections generating incompatible steps from s_{dom}.

If s is any self-compatible step, then the coefficients of $\check{\omega}_i$ for i in any parabolic subgroup of W must define a self-compatible step for that sub-group. This allows some use of induction on the rank of W. Finally, if s is any self-compatible step in the case of a Dynkin diagram with a node of order 3, the only i's such that (α_i, s) is nonzero must lie in a parabolic subgroup whose diagram is linear.

For the classical Weyl groups we supplement the general description of the W-action by the usual description in terms of permutations and sign changes in the e_i basis.

Example. The symmetric group A_{n-1} acts on $\mathbb{R}^{n-1} \cong \{(x_1, \ldots, x_n) \mid \sum_i x_i = 0\} \subset \mathbb{R}^n$ by permuting the n coordinate vectors e_i; $\Delta = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$

and the $\check{\omega}_i$ coefficients of the W-orbit of $s = (x_1, \ldots, x_n)$ are $(\alpha_i, s) = x_{\sigma(i)} - x_{\sigma(i+1)}, \sigma \in W$. Since, for each i, these coefficients are to stay within $\{k_i, 0, -k_i\}$ as σ varies, we conclude that at most two values may appear among the x_i. Assuming $s = s_{dom}, x_1 \geq \cdots \geq x_n$, we find that s must be a scalar multiple of one of the fundamental weights (or coweights)

$$\check{\omega}_i = \sum_{j=1}^{i} e_j - \frac{i}{n} \sum_{j=1}^{n} e_j, 1 \leq i \leq n-1. \tag{14}$$

We may check that these self-compatible W-orbits are compatible with each other (provided they are scaled the same), so we have concluded the classification in this case: up to a uniform dilation, S is any union of the W-orbits of the fundamental weights.

For the exceptional Weyl groups, we use the $\check{\omega}_i$ basis and the general reflection law to determine the self-compatible weights. The restrictions above make the computations easy for G_2 and F_4; we used the program SimpLie to exhaust the remaining possibilities for the E series.

4.5. Results of classification

A walk type (L, S, C) in $\mathbb{R}^n$ is reflectable if and only if its orthogonal projections $(\pi_j(L), \pi_j(S), \pi_j(C))$ to the irreducible factors of C are reflectable.

The walk types with irreducible Weyl chamber C and maximal lattice $L = L_{S,C}$ are as follows. S must be the W-orbit of a dominant self-compatible step, or a union of such W-orbits which are mutually compatible. We list the dominant self-compatible steps in the $\check{\omega}_i$-basis (the dual of the simple root basis), with step lengths normalized for the most mutual compatibility. We use the Bourbaki numbering of the simple roots [2, 6], and for the Weyl group $B_n = C_n$ we compute in the B_n root system.

The zero-step is always self-compatible dominant, and is compatible with all other steps.

A_n :	$\check{\omega}_1, \ldots, \check{\omega}_n$.	All compatible.
$B_n = C_n$:	$\check{\omega}_1, \check{\omega}_n$.	Not compatible.
D_n :	$\check{\omega}_1, \check{\omega}_{n-1}, \check{\omega}_n$.	All compatible.
E_6 :	$\check{\omega}_1, \check{\omega}_6$.	Compatible.
E_7 :	$\check{\omega}_7$.	
E_8, F_4, G_2 :	None.	

For the representation-theoretic problems corresponding to these reflectable walks, Theorem 2 requires the additional condition that ρ lie in the lattice. (We use the Killing form for which the square length of the long roots is 2, so that the coweights equal the weights for simply-laced root systems.) Except for two cases, the above list gives the unique normalization of steps for which this occurs.

One exceptional case is the weight $\check{\omega}_1$ of the root system B_n. With an additional step of 0 added, this corresponds to the defining representation of SO_{2n+1}. The steps are 0 and $\pm e_1, \ldots, \pm e_n$, and the maximal lattice is $\mathbb{Z}^n$; but $\rho = (\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{1}{2})$ is not in this lattice. Thus, we cannot solve this representation-theoretic problem directly as a reflectable random walk: instead, we must use the indirect technique given in Subsection 5.5.

The other case is the weight $\check{\omega}_n$ of the root system C_n. The steps are the 2^n vectors $\pm e_1 \pm e_2 \cdots \pm e_n$, and the maximal lattice is $2D_n^*$, the sublattice of $\mathbb{Z}^n$ containing points whose coordinates are congruent modulo 2. But $\rho = (n, n-1, \ldots, 1)$ is not in this lattice if $n \geq 2$. Our techniques do not work for the resulting walks. In any case, for $n \geq 3$, the representation-theoretic problem is not interesting; the nth fundamental representation of Sp_{2n} has intermediate weights which violate the reflectability condition, and the virtual representation with weights $\pm e_1 \pm e_2 \cdots \pm e_n$ is a complicated sum of fundamental representations. For $n = 2$, the second fundamental representation has the four weights $\pm e_1 \pm e_2$ and the weight 0, which gives an interesting problem and a walk that could be handled by the technique of Subsection 5.5. However, this problem is equivalent to the problem for the defining representation of SO_5, using the isomorphism of the Lie algebras $\mathfrak{sp}_4$ and $\mathfrak{so}_5$.

5. Computational techniques

The cases in which we can compute the number of random walks, or its exponential generating function, are those cases in which the generating function $\chi(\vec{u})$ for the steps is either a sum or a product of terms in only one variable, and some closely related cases, such as SL_n from the results for GL_n.

In this section, we cover the techniques used to find the formulas. All of the actual formulas, both for random walks and for decompositions, are given in Section 6.

The formulas give generating functions which are determinants of Bessel functions, or individual terms which are determinants of binomial coefficients. Thus, the generating functions are D-finite (that is, each function satisfies a linear homogenous differential equation with polynomial coefficients), or, equivalently, the coefficients are P-recursive [13], satisfying a relation

$$\sum_{i=0}^{r} p_i(k) a_{k+i} = 0$$

for some polynomials p_i.

The Bessel function determinants of this section must clearly be related to the formulas of Gessel [4].

5.1. The determinant technique

All cases use the same basic technique for converting the formulas in (2) and (3) into a determinant, with the determinant coming from the sum over the symmetric group S_n, which is either the whole Weyl group or a subgroup of it.

The basic example is the case of the Weyl group $A_{n-1} = S_n$, with steps allowed in both the positive and negative coordinate directions. In terms of representation theory, this is the direct sum $V \oplus V^*$ of the defining representation of GL_n and its dual. The lattice is $\mathbb{Z}^n$.

Thus, using (3), with the generating function for the steps equal to $\sum(u_i + u_i^{-1})$, the exponential generating function for the number of walks from η to λ which stay within the Weyl chamber is

$$g_{\eta\lambda}(x) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\exp(x(u_i + u_i^{-1})) \Big|_{u_i^{\lambda_i - \eta_{\sigma(i)}}} \right). \tag{15}$$

This sum over σ can be written as a determinant, which gives

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| \exp(x(u + u^{-1})) \Big|_{u^{\lambda_i - \eta_j}} \right| \tag{16}$$

And, finally, we can simplify the terms in this determinant. We have

$$\begin{aligned} \exp(x(u + u^{-1})) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=-k}^{k} \binom{k}{j} u^{k-2j} \\ &= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^k}{k!} \binom{k}{(k+m)/2} \\ &= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^{2k+m}}{k!(k+m)!} \\ &= \sum_{m=-\infty}^{\infty} u^m I_m(2x), \end{aligned}$$

where I_m is the hyperbolic Bessel function of the first kind of order m [17]. Thus the determinant above becomes

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| I_{\lambda_i - \eta_j}(2x) \right|. \tag{17}$$

For the representation-theoretic problem of Theorem 2, we have $\eta = \rho$, where $\rho_i = (n+1)/2 - i$. (If n is even, this is not in our lattice $\mathbb{Z}^n$, but we can translate everything by subtracting $\frac{1}{2}$ from all the coordinates and get an equivalent random walk.) For the representation with highest weight μ, we have $\lambda = \rho + \mu$, which gives the decomposition formula

$$f_\mu(x) = \det_{n\times n}\left|I_{\mu_i-i+j}(2x)\right|. \tag{18}$$

This can also be used to give decomposition formulas for the adjoint representation of GL_n. We know that $V \otimes V^*$ is the direct sum of the adjoint representation with one copy of the trivial. Also, we have

$$(V \otimes V^*)^{\otimes k} = \bigoplus_j \binom{k}{j} V^{\otimes j} \otimes V^{*\otimes(k-j)} \tag{19}$$

The weights of $V^{\otimes j} \otimes V^{*\otimes(k-j)}$ all have total weight $2j-k$, so only representations with $\sum \mu_i = 2j - k$ can appear in these factors. Thus, in particular, the tensor powers of $V \otimes V^*$ appear only in the factor with $k = 2j$, and the jth tensor power appears $\binom{2j}{j}$ times in the $2j$th tensor power of $V \oplus V^*$.

Thus, if we let b_k be the multiplicity of the representation U, whose highest weight has total weight 0, in the kth tensor power of $V \otimes V^*$, we get

$$d_\mu(x^2) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{b_k x^{2k}}{(k!)^2} = \det_{n\times n}\left| I_{\mu_i-i+j}(2x) \right|. \tag{20}$$

We could rewrite this generating function as $d_\mu(x)$. However, it is not a standard exponential generating function, because the denominator is $(k!)^2$ instead of $k!$. This prevents us from directly obtaining the decomposition function for the adjoint representation from this generating function; if we had the exponential generating function, we would just multiply by e^{-x}. We can still calculate the function term by term, using (20) to calculate the first k coefficients of d_μ.

We can also apply the determinant technique to (2). Consider the case in which the steps are all the diagonals in the lattice; that is, the 2^n vectors $\pm\frac{1}{2}e_1 \cdots \pm \frac{1}{2}e_n$. The lattice is thus D_n^*, the weight lattice of D_n, containing points whose coordinates are all integers or all half-integers. The generating function for the steps is

$$\sum_{\epsilon_i=\pm 1} \prod_{i=1}^{n} u_i^{\epsilon_i/2} = \prod_{i=1}^{n} \left(u_i^{1/2} + u_i^{-1/2}\right). \tag{21}$$

In the previous case, with steps in the coordinate directions, the generating function for the steps was a sum of terms in the separate u_i, and thus its exponential was a product of such terms. Here, the function itself is a product of terms in the separate u_i, so there is no need to apply the exponential; instead, we can compute the $b_{\eta\lambda,k}$ explicitly.

We can get the formula for $b_{\eta\lambda,k}$ from (2).

$$b_{\eta\lambda,k} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n}\left(\left(u_i^{1/2}+u_i^{-1/2}\right)^k\Big|_{u_i^{\lambda_i-\eta_{\sigma(i)}}}\right). \tag{22}$$

Again, we write the sum over σ as a determinant. Since the coefficient of u^t in $(u_i^{1/2}+u_i^{-1/2})^k$ is $\binom{k}{(k/2)+t}$, this gives us

$$b_{\eta\lambda,k} = \det_{n\times n}\left|\binom{k}{\frac{k}{2}+\lambda_i-\eta_j}\right| \tag{23}$$

The representation-theoretic problem is not as interesting here, because the representation of GL_n with weights $\prod u_i^{\pm 1}$ is a complicated virtual representation, not a natural one.

5.2. *Projection from $\mathbb{Z}^n$ onto A_{n-1}*

The hook-length formulas (32) and (33) given in Section 6 for walks on $\mathbb{Z}^n$ can also be used for the corresponding walks on the lattice $A_{n-1} = \{(\lambda_1,\ldots,\lambda_n) \mid \sum_i \lambda_i = 0, \lambda_i \equiv \lambda_j (\text{mod } 1)\}$. The steps project to steps with one coordinate $\frac{n-1}{n}$ and the others $-\frac{1}{n}$, the weights of the defining representation of SL_n.

Let $|\mu| = \mu_1+\cdots+\mu_n$ denote the total weight of the partition $\mu = (\mu_1,\ldots,\mu_n)$. If $k = tn + |\lambda| - |\eta|$ for some integer t, then a walk of length k with steps in the coordinate directions, starting at η, can end at $\hat{\lambda} \stackrel{\text{def}}{=} \lambda + (t,t,\ldots,t)$. Thus, $b_{\eta\lambda,k}$ will be equal to the value given for $b_{\eta\hat{\lambda},k}$ by (32). Likewise, if $k = tn + |\mu|$, the multiplicity of the representation with highest weight μ in the kth tensor power for SL_n will be the multiplicity of the representation with highest weight $\mu + (t,t,\ldots,t)$ in the kth tensor power for GL_n, as given by (33).

5.3. *The multilinearity technique*

In other cases, we get a determinant of a sum or difference of terms, because the Weyl group is not just S_n but a semidirect product of S_n and some coordinate changes.

The most natural example is the problem of random walks on $\mathbb{Z}^n$ with the Weyl group $B_n = C_n$ and steps in the positive or negative coordinate directions; this corresponds to the decomposition of tensor powers of the defining representation of Sp_{2n}.

Applying (3) for random walks, we get

$$g_{\eta\lambda}(x) = \sum_{w\in W} \operatorname{sgn}(w)\prod_{i=1}^{n} \exp(x(u_i+u_i^{-1}))\Big|_{\vec{u}^{\lambda-w(\eta)}}. \tag{24}$$

We now write the element w as a product of a σ in the symmetric group and an ϵ which negates some of the coordinates t_i, thus converting u_i to u_i^{-1}. We get

$$g_{\eta\lambda}(x) = \sum_{\sigma\in S_n} \sum_{\epsilon_i=\pm 1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\epsilon_i \exp(x(u_i + u_i^{-1})) \Big|_{u_i^{\lambda_i - \epsilon_i \eta_{\sigma(i)}}} \right). \tag{25}$$

Using the multilinearity of the products in the determinant, we can again write the sum over σ as a determinant, with separate terms for $\epsilon_i = 1$ and $\epsilon_i = -1$ in each entry, and these terms are again the hyperbolic Bessel functions, so we have

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| I_{\lambda_i - \eta_j}(2x) - I_{\lambda_i + \eta_j}(2x) \right|. \tag{26}$$

In the decomposition for Sp_{2n}, we substitute $\eta_i = n + 1 - i$, and $\lambda = \mu + \eta$ as usual.

The same technique also applies, using (2) instead, for the diagonal walk with Weyl group $B_n = C_n$; this corresponds to the spin representation of SO_{2n+1}.

5.4. *The splitting technique*

The Weyl group D_n does not lend itself directly to the multilinearity technique which we used for B_n. We need to use a trick, essentially turning the problem into a sum over B_n.

The random walk on the lattice D_n^* with steps in the coordinate directions has two orbits, the points with all integer coordinates and the points with all half-integer coordinates. The computations are valid if η and λ are in the same orbit; otherwise, the number of walks will obviously be 0. The representation-theory problem is the decomposition of tensor powers of the defining representation of SO_{2n}.

The formula for random walks is again (24), but when we write $w = \sigma\epsilon$, only those ϵ with an even number of sign changes occur. We thus take the sum over all ϵ, but with an additional factor of $(1 + \prod \epsilon_i)/2$; this factor is 1 when there are an even number of sign changes and 0 when there are an odd number. We treat the 1/2 and the $(\prod \epsilon_i)/2$ terms separately, which gives

$$\begin{aligned} g_{\eta\lambda}(x) = \frac{1}{2}\Bigg[&\sum_{\sigma\in S_n} \sum_{\epsilon_i=\pm 1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\epsilon_i \exp(x(u_i + u_i^{-1})) \Big|_{u_i^{\lambda_i - \epsilon_i \eta_{\sigma(i)}}} \right) \\ &+ \sum_{\sigma\in S_n} \sum_{\epsilon_i=\pm 1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\exp(x(u_i + u_i^{-1})) \Big|_{u_i^{\lambda_i - \epsilon_i \eta_{\sigma(i)}}} \right) \Bigg]. \end{aligned} \tag{27}$$

The first term in this sum carries through just as in (25), using the determinant technique. The second term, with no factor of ϵ_i, can be computed by the same method; instead of the minus sign between the two terms in each entry of the determinant (26), we get a plus sign.

A similar argument works for the diagonal walk on D_n^*, corresponding to the direct sum of the two spin representations of SO_{2n}.

5.5. *The subgroup technique*

Although D_n^* is the weight lattice of SO_{2n+1}, the techniques we used for the defining representations of Sp_{2n} and SO_{2n} cannot be applied directly to find an equivalent random walk, because the ρ is not in the maximal lattice $\mathbb{Z}^n$ for the reflectable walk. However, D_n has index 2 in B_n, B_n is generated by D_n and the reflection in the last coordinate, and ρ is now in the maximal lattice D_n^*.

Thus the sum (9) over B_n is equal to

$$f_\mu(x) = \sum_{w \in D_n} \mathrm{sgn}(w) \left[\exp(x\chi_V(\vec{u})\big|_{\vec{u}^{\rho+\mu-w(\rho)}} - \exp(x\chi_V(\vec{u}))\big|_{\vec{u}^{\rho+\mu-w(\rho')}} \right]. \tag{28}$$

where ρ' is obtained from ρ by negating the last coordinate and then applying w. This is a difference of two reflectable random walks; note that χ_V here is $1 + \sum(u_i + u_i^{-1})$, so the exponential generating functions $f_\mu(x)$ will be e^x times the corresponding functions for SO_{2n} with the same lattice. With $\lambda = \mu + \rho$ as usual, we have $f_\mu(x) = g_{\eta\lambda}(x) - g_{\rho'\lambda}(x)$.

We can thus compute the generating function for SO_{2n+1} as a sum of these two functions. However, this is a somewhat indirect argument; we wind up computing a difference of two walks and then adding them together. To actually compute the formulas, it is easier to work directly from (9), not bothering to convert to reflectable random walks in Weyl chambers and then back. We can just use the determinant and multilinearity techniques to get the single determinant,

$$f_\mu(x) = e^x \det_{n\times n} \left| I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) - I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) \right|. \tag{29}$$

The subgroup technique may also be useful for other nonreflectable random walks which become reflectable when we use a smaller Weyl group. For example, the seven-dimensional representation of G_2 does not give a reflectable random walk with Weyl group G_2, but it gives a difference of two such walks with Weyl group A_2. Our methods do not work to analyze the resulting walks.

We could use the subgroup technique by considering A_{n-1} as a subgroup of B_n; this would give us 2^n simple determinants of the from (16), corresponding to the 2^n choices of plus or minus signs in the n columns of (26). We could get similar results for the diagonal walk, or the group D_n.

5.6. *The parity technique for the odd-dimensional orthogonal group*

The decomposition formulas for SO_{2n+1} can be used to give decompositions for O_{2n+1}. Every irreducible representation U_μ of SO_{2n+1} corresponds to two representations $U_\mu^\pm$ of O_{2n+1}, with U_μ^+ taking the transformation -1 to the identity and U_μ^- taking it to -1. Since the defining representation of O_{2n+1} preserves the determinant, the representation U_μ^+ can occur only in even tensor powers, and U_μ^- can occur only in odd tensor powers. Thus we have

$$f_\mu^\pm(x) = \frac{1}{2}(f_\mu(x) \pm f_\mu(-x)).$$

The formula for $f_\mu(-x)$ contains the determinant

$$\det_{n\times n}\left|I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(-2x) - I_{\mu_i+(n+\frac{1}{2}-i)+(n+\frac{1}{2}-j)}(-2x)\right|. \tag{30}$$

Since $I_m(2x)$ is even if m is even and odd if m is odd, we can easily convert this back to a determinant of $I_m(2x)$. If we replace $-2x$ by $2x$, this changes the sign of the second term if $\mu_i + i + j$ is even, and of the first term if $\mu_i + i + j$ is odd. In the resulting matrix, we can then negate column j if j is even, and row i if $\mu_i + i$ is even, getting

$$(-1)^{\sum \mu_i}\det_{n\times n}\left|I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) + I_{\mu_i+(n+\frac{1}{2}-i)+(n+\frac{1}{2}-j)}(2x)\right|. \tag{31}$$

From this, we can get the decomposition formula for O_{2n+1} by adding this to or subtracting it from (29).

The parity argument also works for the spin representations of O_{2n+1}. For the spin representation which preserves the determinant, we again have that odd tensor powers preserve the determinant, while even tensor powers do not. O_{2n+1} also has another spin representation which takes -1 to the identity; all tensor powers of this representation take -1 to the identity.

6. Formulas

We now present the formulas obtained by using the techniques of the previous section, broken down by Weyl group. For each random walk, we list the following information:

The Weyl group W, and corresponding Lie groups G.
The inequalities defining the Weyl chamber C in $\mathbb{R}^n$.
The set S of steps, in the usual basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.
The maximal lattice $L_{S,C}$. The lattices occurring are $\mathbb{Z}^n$, $A_{n-1} = \{(\lambda_1, \ldots, \lambda_n) \mid \sum_i \lambda_i = 0, \lambda_i \equiv \lambda_j (\text{mod } 1)\}$, and $D_n^* = \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$.
The representation V of G whose tensor powers correspond to the random walk.
Formulas for $b_{\eta\lambda,k}$, the number of k-step walks in C from η to λ, and the exponential generating function $g_{\eta\lambda}(x) = \sum_{k=0}^{\infty} \frac{b_{\eta\lambda,k}}{k!} x^k$.

Formulas for $a_{\mu,k} = b_{\rho,\rho+\mu,k}$, the multiplicity of the irreducible μ in the kth tensor power of the representation V of G corresponding whose weights are the steps in S, and the exponential generating function $f_\mu(x) = \sum_{k=0}^{\infty} \frac{a_{\mu,k}}{k!} x^k$. The functions are usually given in terms of the hyperbolic Bessel functions [17]

$$I_m(2x) = \sum_{k=0}^{\infty} \frac{x^{2k+m}}{k!(k+m)!}.$$

The techniques from Section 5 used to produce the formulas.

In some cases, there is another representation of a Lie group which does not lead directly to a reflectable random walk problem but can be reduced to one; such problems are listed as **"Related representation."** In each case, we refer to the specific techniques, which are listed in the previous section with examples.

6.1. Weyl group A_{n-1}, Lie groups GL_n, U_n, SL_n, SU_n

Weyl chamber: $x_1 > x_2 > \cdots > x_n$.

Steps: $e_1, \ldots, e_n$
Lattice: $\mathbb{Z}^n$.

Representation: Defining representation of GL_n or U_n.

Techniques used: Determinant, then use matrix techniques to get the hook-length formulas [3, 9].

Random-walk formula: $b_{\eta\lambda,k}$ = number of standard skew tableaux of shape $\lambda'\backslash\eta'$, where $k = |\lambda| - |\eta|$,

$$\lambda - \lambda' = \eta - \eta' = \frac{1}{2}(n-1, n-3, \cdots, 1-n).$$

The formula is

$$b_{\eta\lambda,k} = S_{\lambda'\backslash\eta'}\big|_{x_1 x_2 \cdots x_k} = k! \prod_{i,j \in \lambda'\backslash\eta'} \frac{1}{h_{ij}}, \tag{32}$$

Decomposition formula: $a_{\mu,k}$ = number of standard Young tableaux of shape μ, where $k = |\mu|$.

$$a_{\mu,k} = S_\mu|_{x_1 x_2 \cdots x_k} = k! \prod_{i,j \in \mu} \frac{1}{h_{ij}}, \tag{33}$$

where h_{ij} is the hook of the square (i, j) in the Young diagram for μ.

Steps: $e_1 - v, \ldots, e_n - v$, where $v = \frac{1}{n}\sum_{j=1}^{n} e_j$

Lattice: A_{n-1}.

Representation: Defining representation of SL_n or SU_n.

Techniques used: Project the lattice $\mathbb{Z}^n$ onto A_{n-1}.

Random-walk formula: $b_{\eta\lambda,k}$ as given by (32) for $b_{\rho+(t,t,\dots,t),\lambda,k}$ where $k = t_n + |\lambda| - |\eta|$.

Decomposition formula: $a_{\mu+(t,t,\dots,t),k}$ as given by (33), where $k = tn + |\mu|$.

Steps: $\pm e_1, \dots, \pm e_n$

Lattice: $\mathbb{Z}^n$.

Representation: Direct sum of defining and dual representations for GL_n or U_n.

Techniques used: Determinant.

Random-walk exponential generating function:

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| I_{\lambda_i-\eta_j}(2x) \right| . \tag{34}$$

Decomposition exponential generating function:

$$f_\mu(x) = \det_{n\times n} |I_{\mu_i-i+j}(2x)| . \tag{35}$$

Related representation: Adjoint representation of GL_n or U_n.

Decomposition doubly-exponential generating function for direct sum of the adjoint and trivial representations (see Subsection 5.1):

$$d_\mu(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{b_k x^k}{(k!)^2} = \det_{n\times n} \left| I_{\mu_i-i+j}(2\sqrt{x}) \right| . \tag{36}$$

Steps: $\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \cdots \pm \frac{1}{2}e_n$ (2^n vectors)

Lattice: D_n^*.

Techniques used: Determinant.

Random-walk formula:

$$b_{\eta\lambda,k} = \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} \right| \tag{37}$$

The representation-theoretic problem is not interesting here.

6.2. *Weyl group $B_n = C_n$, Lie groups Sp_{2n}, SO_{2n+1}, and O_{2n+1}*

Weyl chamber: $x_1 > x_2 > \cdots > x_n > 0$.

Steps: $\pm e_1, \dots, \pm e_n$

Lattice: $\mathbb{Z}^n$.

Representation: Defining representation for Sp_{2n} (see [14] for related results).

Techniques used: Determinant, multilinearity.

Random-walk exponential generating function:

$$g_{\eta\lambda}(x) = \det_{n\times n} \left| I_{\lambda_i-\eta_j}(2x) - I_{\lambda_i+\eta_j}(2x) \right|. \tag{38}$$

Decomposition exponential generating function:

$$f_\mu(x) = \det_{n\times n} \left| I_{\mu_i+(n+1-i)-(n+1-j)}(2x) - I_{\mu_i+(n+1-i)+(n+1-j)}(2x) \right|. \tag{39}$$

Steps: $\pm e_1, \ldots, \pm e_n$

Lattice: D_n^*.

Representations: Defining representations of SO_{2n+1} and O_{2n+1}.

Although the weights and Weyl group are the same in the case above, we do not have ρ in the lattice as required by Theorem 2. We can use the Weyl group D_n to get a reflectable walk, and thus the formula is given in Subsection 6.3.

Steps: $\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \cdots \pm \frac{1}{2}e_n$

Lattice: D_n^*.

Techniques used: Determinant, multilinearity.

Representation: Spin representational of SO_{2n+1}.

Random-walk formula:

$$b_{\eta\lambda,k} = \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \lambda_i - \eta_j} - \binom{k}{\frac{k}{2} + \lambda_i + \eta_j} \right|. \tag{40}$$

Decomposition formula:

$$a_{\mu,k} = \det_{n\times n} \left| \binom{k}{\frac{k}{2} + \mu_i + (n+\frac{1}{2}-i) - (n+\frac{1}{2}-j)} - \binom{k}{\frac{k}{2} + \mu_i + (n+\frac{1}{2}-i) + (n+\frac{1}{2}-j)} \right|. \tag{41}$$

Related representation: Spin representations of O_{2n+1}.

Additional technique used: Parity.

Decomposition formula: For the spin representation which takes -1 to the identity, the formula above is valid if the representation μ takes -1 to the identity. For the spin representation which takes -1 to itself, the above formula is valid if the representation μ takes -1 to itself for k odd, and to the identity for k even. In the other cases, $a_{\mu,k} = 0$.

6.3. Weyl group D_n Lie group SO_{2n}, defining representations of SO_{2n+1} and O_{2n+1}

Weyl chamber: $x_1 > x_2 > \cdots > x_n,\ x_{n-1} > -x_n$

Steps: $\pm e_1, \ldots, \pm e_n$
Lattice: D_n^*.

Techniques used: Determinant, multilinearity, splitting.

Representation: Defining representation of SO_{2n} (see [7, 8] for related results).

Random-walk exponential generating function (for $\lambda_i \equiv \mu_i \pmod 1$; clearly 0 otherwise):

$$g_{\eta\lambda}(x) = \frac{1}{2}\Big[\det_{n\times n}|I_{\lambda_i-\eta j}(2x) - I_{\lambda_i+\eta j}(2x)| + \det_{n\times n}|I_{\lambda_i-\eta j(2x)} + I_{\lambda_i+\eta j}(2x)|\Big]. \tag{42}$$

Decomposition exponential generating function (for $\mu_i \in \mathbb{Z}$):

$$f_\mu(x) = \frac{1}{2}\det_{n\times n}|I_{\mu_i+(n-i)-(n-j)}(2x) + I_{\mu_i+(n-i)+(n-j)}(2x)|. \tag{43}$$

(The first column of the other determinant is 0.)

Related representation: Defining representation of SO_{2n+1} (see [7, 8, 10, 15] for related results). This requires that 0 be added to the list of steps, since it is a weight of the representation.

Additional technique used: Subgroup (or work directly from (9), don't use reflectable random walks, and use determinant and multilinearity techniques).

Decomposition exponential generating function:

$$f_\mu(x) = e^x \det_{n\times n}\left|I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) - I_{\mu_i+(n+\frac{1}{2}-1)-(n+\frac{1}{2}-j)}(2x)\right|. \tag{44}$$

Related representation: Defining representation of O_{2n+1}.

Additional technique used: Parity. (See Subsection 5.6 for the $f_\mu^\pm$ notation.)

Decomposition exponential generating function:

$$\begin{aligned} f_\mu^\pm(x) &= \frac{1}{2}(f_\mu(x) \pm f_\mu(-x)) \\ &= \frac{1}{2}\Big[e^x \det_{n\times n}|I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) - I_{\mu_i+(n+\frac{1}{2}-i)+(n+\frac{1}{2}-j)}(2x)| \\ &\quad \pm(-1)^{\sum \mu_i} e^{-x}\det_{n\times n}|I_{\mu_i+(n+\frac{1}{2}-i)-(n+\frac{1}{2}-j)}(2x) \\ &\quad + I_{\mu_i+(n+\frac{1}{2}-j)+(n+\frac{1}{2}-j)}(2x)|\Big]. \end{aligned} \tag{45}$$

Steps: $\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \cdots \pm \frac{1}{2}e_n$
Lattice: D_n^*.

Techniques used: Determinant, multilinearity, splitting.

Representation: Sum of the two spin representations of SO_{2n}.

Random-walk formula:

$$b_{\eta\lambda,k} = \frac{1}{2}\left[\det_{n\times n}\left|\binom{k}{\frac{k}{2}+\lambda_i-\eta_j} - \binom{k}{\frac{k}{2}+\lambda_i+\eta_j}\right| + \det_{n\times n}\left|\binom{k}{\frac{k}{2}+\lambda_i-\eta_j} + \binom{k}{\frac{k}{2}+\lambda_i+\eta_j}\right|\right]. \tag{46}$$

Decomposition formula:

$$a_{\mu,k} = \frac{1}{2}\det_{n\times n}\left|\binom{k}{\frac{k}{2}+\mu_i+(n-i)-(n-j)} + \binom{k}{\frac{k}{2}+\mu_i+(n-i)+(n-j)}\right|. \tag{47}$$

(The first column of the other determinant is 0.)

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