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Two Remarks on Independent Sets

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Abstract. In the first part we generalize the notion of strongly independent sets, introduced in [10] for polynomial ideals, to submodules of free modules and explain their computational relevance. We discuss also two algorithms to compute strongly independent sets that rest on the primary decomposition of squarefree monomial ideals.

Usually the initial ideal $in(I)$ of a polynomial ideal I is worse than I. In [9] the authors observed that nevertheless $in(I)$ is not as bad as one should expect, showing that $in(I)$ is connected in codimension one if I is prime.

In the second part of the paper we add more evidence to that observation. We show that $in(I)$ inherits (radically) unmixedness, connectedness in codimension one and connectedness outside a finite set of points from I and prove the same results also for initial submodules of free modules. The proofs use a deformation from I to $in(I)$.

Keywords: independent set, initial ideal, unmixedness, connectedness in codimension 1

1. Introduction

In this paper we present some results about independent sets with special emphasis on the connection to initial ideals and modules.

Let $S = k[x_v, v \in V]$ be a polynomial ring over a field k, equipped with a noetherian term order $<, I \subset S$ an ideal, and $in(I)$ the *initial ideal* of I, i.e., the ideal generated by the leading terms $lt(f)$ of all $f \in I$.

$\sigma \subseteq V$ is an *independent set modulo I* iff $k[x_v : v \in \sigma] \cap I = (0)$. This definition generalizes a notion well known for prime ideals, see [9] and the references cited therein. $\triangle(I) := \{\sigma \subseteq V : \sigma \text{ is independent } mod\ I\}$ is a simplicial complex for arbitrary ideals $I \subset S$, the *independence complex* of I.

In general it is difficult to find (maximal) independent sets. In their paper [10], the authors therefore investigated the connections between independent sets of I and independent sets of $in(I)$, *strongly independent sets* in their notion. $in(I)$ usually reflects many properties of I but has a comparatively simple computational and combinatorial structure. Hence, one should expect connections between independent and strongly independent sets to be useful for computational purposes. This is indeed the case. More precisely, Kredel and Weispfenning prove in [10] that strongly independent sets of maximal size are independent sets of maximal size equal to $dim\ S/I$. Moreover the authors present an algorithm to construct all strongly independent sets (according to $<$) that can easily be implemented in a Gröbner computation environment.

Computationally submodules of free modules often behave like ideals. Section 2 is devoted to an extension of the notions introduced above to this more general situation. Moreover we discuss two algorithms for computing strongly independent sets that employ the close connection between independent sets and the primary decomposition for (radical) monomial ideals. Such a decomposition can be easily computed, see, e.g., [11] or [2].

In [10] the question arose of whether for prime ideals I all maximal strongly independent sets are of the same size, the *dimension* of I. Reformulated, this means that *rad* $in(I)$ should be unmixed provided I is prime. Meanwhile Kalkbrener and Sturmfels [9] proved this conjecture to be valid. Moreover they showed that for a prime ideal I the simplicial complex $\triangle(in(I))$ is strongly connected.

In the second part we generalize the proof for the fact that $in(I)$ is (radically) unmixed if I is so to $in(M)$ for the submodule M of the free module F. It doesn't use the connectedness theorem [5, Corollary 1] but rests more directly on a deformation from M to $in(M)$. Along the same lines we generalize the connectedness theorem [9, Theorem 1] from prime ideals to ideals and modules connected in codimension one and prove that for homogeneous ideals I the scheme $Proj\ S/in(I)$ and hence $\triangle(in(I))$ is connected provided $Proj\ S/I$ is so. These considerations shed some more light on the main observation of [9] that $in(I)$ is worse than I but not as bad as one should expect.

All examples of prime ideals I considered in [9] satisfy the stronger condition that $\triangle(in(I))$ is even locally strongly connected since $|\triangle|$ is a ball. This is trivially satisfied for $\triangle(I)$ since it is a matroid complex. We give an example which shows that this property need not transfer to $\triangle(in(I))$.

2. Strongly independent sets and primary decomposition of monomial ideals

The first topic of this section is a generalization of the notion of independent sets to submodules of free modules.

Let S be as in the introduction, $Q(S) := S_{(0)}$ its quotient field, and $F = S \otimes_k k^r$ a free S-module with basis $\{e_1, \ldots, e_r\}$. Embed F into the symmetric algebra $S(F) := S \otimes_k S(k^r) = k[x_v, e_i : v \in V, i \in 1 \ldots r]$ and extend $<$ to a noetherian term order on $S(F)$. This defines a *module term order* on F. Any vector $f \in F$ has a unique representation $f = m_1 + m_2 + \ldots + m_k$ with terms $m_1 > m_2 > \ldots > m_k$. Define the leading term $lt(f) := m_1$, for $M \subseteq F$ the leading module $in(M)$ etc. in the usual way. $in(M)$ splits as a submodule of F into the direct sum of r monomial ideals.

$$in_s(M) := \{x^a \ : \ m = x^a \cdot e_s \in in(M)\} \qquad \text{for } s = 1, \ldots, r.$$

Define $\sigma \subseteq V$ to be an *independent set modulo* M iff σ is independent mod $(M \ :\ _sF)$, the annulator of F/M in S and *strongly independent modulo* M iff it is independent modulo $in(M)$.

For $r = 1$ these definitions coincide with the classical ones given in the introduction. Since $f \in M : F$ implies $in(f) \in in(M) : F$, we conclude that again strongly independent sets are independent. Moreover $in(M) : F = \cap_s in_s(M)$ implies that $\sigma \subseteq V$ is strongly independent *mod* M iff it is independent (in the ideal-theoretic sense) $mod\, in_s(M)$ for some s. If we denote the independence complex of M by $\triangle(M)$ this means $\triangle(M) \supseteq \triangle(in(M)) = \cup_s \triangle(in_s(M))$.

Since the dimensions of M and $in(M)$ coincide, we get another way to compute $dim\ F/M = max(dim\ S/in_k(M)\ :\ \forall k)$: It is the maximal size of a strongly independent set modulo M.

In [10] the authors describe an algorithm ***DIMREC*** for the computation of all strongly independent sets of ideals that rests on an inspection of variable sets in a search tree. Below we use the connection between strongly independent sets and primary decompositions of monomial ideals for another approach to the problem.

As explained above we may restrict our attention to squarefree (since $\triangle(I) = \triangle(rad\, I)$) monomial ideal $I \subset S$. For any such ideal there is a simplicial complex $\triangle \subseteq 2^V$ on the vertex set V such that I is generated by the "minimal non faces" of $\triangle$, see, e.g., [13]. Define $A(\sigma) = (x_v : v \notin \sigma)$ for $\sigma \subseteq V$. Then

$$I = I(\triangle) = \bigcap_{\text{max. } \sigma \in \triangle} A(\sigma)$$

is the primary decomposition of the ideal I, and hence $\triangle$ coincides with the collection $\triangle(I)$ of independent sets $mod\, I$.

Such a decomposition can easily be computed, either by induction on the number of generators of I as in [11] or by induction on the number of variables involved as in [2]. Run time experiments, based on REDUCE and the author's research package CALI [8], suggest that both algorithms are better than *DIMREC*. The best performance was obtained with the algorithm from [2].

It is well known, see, e.g., [4], that the codimension problem is NP-hard in n, the number of variables involved. By Sperner's theorem the number of strongly independent sets may indeed grow exponentially in n. On the other hand, the prime decomposition algorithm for monomial ideals in [11], inducting on the number of generators, shows that the number of independent sets is (constructively) bounded by $O(m^d)$ with m the number of generators of I and d an upper degree bound ($\leq n$) for them.

Example. Consider for $0 \leq k \leq n$ the following squarefree monomial ideals:

$$J(k, n) := \left(\prod_{i=j+1}^{j+n} x_i | j = k, \ldots, n \right)$$

and $I(n) := J(0, n)$. An easy induction argument shows that $J(k, n)$ has the following prime decompositon:

$$J(k, n) = \bigcap_{i=1}^{k}(x_{n+i}) \cap \bigcap\{(x_i, x_j) | k < i \leq n, n + k < j \leq n + i\}.$$

Hence $I(n)$ has $\frac{n(n+1)}{2}$ independent sets. On the other hand, both algorithms cited above compute a monomial prime decomposition through the inductive step

$$J(k, n) = \bigcap_{i=k+1}^{n} (x_i + J(i, n)) \cap \bigcap_{i=1}^{k}(x_{n+i}).$$

Counting the monomial prime ideals involved during the algorithm we obtain the following complexity:

$$C_{J(k,n)} = \sum_{i=k+1}^{n} C_{J(i,n)} + k$$

and thus

$$C_{I(n)} = 2^n(n - 1) + 1.$$

Hence on these examples the proposed algorithms are exponential in the number of maximal independent sets produced. A better performance may be achieved by a random ordering of the input monomials as, e.g., proposed in [3]. It's an open question whether there is a deterministic algorithm with linear complexity in the number of maximal independent sets (i.e., that effectively finds each maximal independent set exactly once).

3. Some connections between $in(M)$ and M

Let S, F, M be as in the previous section and $<$ a term order on $S(F)$.

Any such term order can essentially be represented by a weight vector $w \in (\mathbf{R}^W)(W := V \cup [1 \ldots r])$ with nonnegative weights, see [12]. More precisely,

$$a <_w b \quad :\Leftrightarrow \quad w(a) < w(b) \quad \text{for terms } a, b \in S(F)$$

defines a partial order on the set of terms and for any (total) term order $<$ there exists a unique w such that $<$ refines $<_w$.

Since $w \in (\mathbf{N}_+^W)^*$ defines a grading on $S(F)$, every vector $f \in F$ has a unique decomposition $f = f_1 + f_2 + \ldots$ into w-homogeneous components with $deg_w f_1 > deg_w f_2 > \ldots$. Set $lt_w(f) := f_1$ and $in_w(M) := < lt_w(f) : f \in M >$.

Several term orders may define the same initial module $in(M)$. They can be described as refinements of a certain partial term order on $\mathbf{N}^W$ with a finitely generated (pointed) positivity cone, see, e.g., [7]. Since the weight vectors of these refinements form the dual of this positivity cone, there is a term order with positive integer weights $w_0 \in (\mathbf{N}_+^W)^*$ among them belonging to its interior. Hence $in_{w_0}(M) = in(M)$. Moreover if $M_1, M_2, \ldots, M_m$ is a finite collection of submodules of F, there is even a weight vector $w \in (\mathbf{N}_+^W)^*$ such that $in_w(M_k) = in(M_k)$ for all k simultaneously. Indeed, the union of the positivity cones of all M_k is contained in the positivity cone of $<$ and finitely generated. Hence it defines a pointed cone, too.

With $w = w_0$ we can associate a one-parameter deformation from M to $in(M)$ as in [1]. More precisely, assume that the finite set of terms Σ generates $in(M)$ and

$$M = < m - st(m) : m \in \Sigma > \quad \text{with } st(m) = \sum_{n<m} r_{mn} n$$

is the corresponding Gröbner basis. Let t be a new variable. Then

$$\{m - st_t(m) : m \in \Sigma\} \text{ with } \quad st_t(m) := \sum_{n<m} r_{mn} n t^{w(m)-w(n)}$$

is a Gröbner basis of the submodule M_t of $F_t := S_t^r$ generated over $S_t := S[t]$ by this set. Moreover $w(m) > w(n)$ for all m, n such that $r_{mn} \neq 0$ by construction. Hence the fibers $R_c := F_t/M_t \otimes_k k[t]/(t-c)$ are isomorphic to $R = F/M$ for $c \neq 0$ and to $R_0 = F/in(M)$ for $c = 0$.

Alternatively M_t can be described as the homogenization of M with respect to the weight vector w. Define

$$h : F \longrightarrow F_t \quad \text{and} \quad a : F_t \longrightarrow F$$

as the homogenization map and the dehomogenization map respectively, i.e.,

$$h(f) := \sum c_m m t^{d-w(m)} \text{ for } f = \Sigma c_m m \in F \text{ and } d = max\{w(m) : c_m \neq 0\},$$
$$a(f(t)) := f(1) \text{ for } f(t) \in F_t \text{ and more general}$$
$$h(J) := < h(f) : f \in J > \text{ for a submodule } J \text{ of } F.$$

Then $M_t = h(M)$ and $h(M)$ is generated by all (w-)homogeneous vectors $f(t) \in F_t$ such that $f(1) \in M$. This immediately implies that t is a nonzero divisor on F_t/M_t and proves the following

LEMMA 1. *$rk(M) = rk(in(M))$ and $dim\ F/M = dim\ F/in(M) (= dim\ F_t/M_t - 1)$.*

Let $p \in Spec\,S$ be a prime ideal. Recall that $M \subseteq F$ is p-primary iff $Ass\ F/M = \{p\}$ where $p = rad_S(M : F)$ is a prime. More general, for a primary decomposition $M = \cap Q_i$ we obtain

$$rad(M : F) = rad(\bigcap Q_i : F) = \bigcap rad(Q_i : F) = \bigcap_{p \in Ass(F/M)} p.$$

The following lemma is the crucial point in the argument below.

LEMMA 2. *Let $M \subseteq F$ be a submodule of the free module F. Then*

$$rad(in(M) : F) = \bigcap_{p \in Ass(F/M)} rad\, in(p).$$

Hence

$$Supp\, F/in(M) = \bigcup_{p \in Ass(F/M)} Supp\, S/in(p).$$

Proof. Let $M = \cap Q_i$ be the primary decomposition of M with $Q_i \subset F$ p_i-primary. Fix a weight vector $w \in (\mathbf{N}_+^W)^*$ such that $in(M) = in_w(M) in(Q_i) = in_w(Q_i)$, and $in(p_i) = in_w(p_i)$ for all i. Then $M_t = h(M) = \cap h(Q_i)$ is the primary decomposition of M_t where $h(Q_i)$ is $(rad(h(Q_i) : h(F)) = h(p_i))$-primary. Since

$$\Big(\bigcap rad\, in(p_i)\Big) + (t) = \bigcap rad\,(h(p_i), t)$$
$$= rad\,\Big(\Big(\bigcap h(p_i)\Big) + (t)\Big) = rad(h(M) : h(F) + (t)),$$

it remains to prove that the latter expression is equal to $rad_S(in(M) : F) + (t)$.

Assume $f \in (h(M) : h(F) + (t))$, i.e., $(f + at)h(F) \subseteq h(M)$ for some $a \in S_t$. Evaluating at $t = 0$ we conclude that $f(0)F \subseteq in(M)$, i.e., $f(0) \in in(M) : F$. Since $F \equiv f(0)\ (mod\, t)$ it follows that $f \in in(M) : F + (t)$.

Assume $x^a \in in(M) : F$, i.e., $x^a e_i \in in(M) = in(h(M))$ for all i. This implies that for all i there is a representation $x^a e_i = m_i - t(\sum a_{ik} e_k)$ with certain $m_i \in h(M)$. In particular we get a representation

$$(x^a + t a_{rr}) e_r \equiv -t(\sum_{k<r} a_{rk} e_k) \quad (mod\, h(M)).$$

Use this representation to eliminate e_r from all other expressions. We obtain representations

$$(x^a + t a_{rr}) x^a e_i \equiv -t(\sum_{k<r} a_{ik}^{(1)} e_k) \quad (mod\, h(M)).$$

By an induction argument we immediately get for all i

$$(x^a + t a_{rr})(x^a + t a_{r-1,r-1}^{(1)}) \ldots (x^a + t a_{11}^{(r-1)}) e_i \equiv 0 \quad (mod\, h(M))$$

and hence $(x^a)^r \in h(M) : h(F) + (t)$. □

We are now ready to state our results on the connection between M and $in(M)$.

Define M to be *(radically) unmixed* iff all isolated components of *Supp F/M* are of the same dimension. The following result generalizes [9, Corollary 1].

THEOREM 1. *If M is unmixed of dimension d then $in(M)$ is unmixed of dimension d and $\triangle(in(M))$ is pure of dimension $d-1$.*

Proof. The latter assertion follows directly from the decomposition of $\triangle(in(M))$ obtained in Section 2. To prove the former one, by Lemma 2 we can restrict ourselves to the case where M is a prime p of dimension d in S. But then $p_t(=h(p))$ is a homogeneous prime of dimension $d+1$ in S_t and since $in(p)+(t) = p_t+(t)$ with the nonzero divisor t we conclude that $in(p)$ is unmixed of dimension t. □

As a consequence, in difference to the general case discussed above, the dimension of an *unmixed* ideal can be obtained from its Gröbner basis in linear time with respect to the number of variables, since *any* two maximal (with respect to inclusion) strongly independent sets have equal size.

Note that neither (true) unmixedness transfers from I to $in(I)$ nor (radically) unmixedness from $in(I)$ to I:

Macaulay's curve

$$I = (xy - uz,\ x^3 - u^2y,\ x^2z - uy^2,\ y^3 - xz^2)$$

being prime, has an initial ideal

$$in(I) = (xy,\ x^3, x^2z,\ y^3) = (x^3,\ xy,\ y^3,\ z)\bigcap(x^2,\ xy,\ y^3)$$

with embedded prime, whereas

$$I = (x-1)\bigcap(x,\ y) = (x^2 - x,\ xy - y)$$

is radically mixed, but $rad\, in(I) = (x)$ is unmixed.

Define M to be *connected in codimension* 1 iff M is unmixed, say of dimension d, and any two isolated components Q_i and Q_j in the primary decomposition $M = \cap Q_i$ can be connected by a sequence $Q_i = Q_{i_0}, Q_{i_1}, \ldots, Q_{i_m} = Q_j$ of components of M such that $dim\, F/(Q_{i_{k-1}} + Q_{i_k}) = d-1$ for all $k = 1, \ldots, m$.

A simplicial complex $\triangle$ is *strongly connected* iff any two maximal faces $\sigma_i, \sigma_j \in \triangle$ can be connected by a sequence of maximal faces $\sigma_i = \sigma_{i_0}, \sigma_{i_1}, \ldots, \sigma_{i_m} = \sigma_j$ such that $|\sigma_{i_k} \setminus \sigma_{i_{k-1}}| = |\sigma_{i_{k-1}} \setminus \sigma_{i_k}| = 1$ for $k = 1, \ldots, m$, see [9]. This definition is equivalent to the property of $Spec\, S/I(\triangle)$ being connected in codimension one.

Let $\mathbf{m} := (x_v : v \in V)$ be the irrelevant ideal of S. The following result combines connectedness resp. connectedness in codimension 1 of M and $in(M)$. It generalizes [9, Theorem 1] where the same results are proved for prime ideals.

THEOREM 2.

(a) *Let k be algebraically closed. Assume Supp F/M remains connected after removing any finite set of (closed) points. Then (Supp $F/in(M)\backslash\{\mathbf{m}\}$ is connected and $\triangle(in(M))$ is a connected simplicial complex.*
(b) *If M is connected in codimension 1 then $in(M)$ is so and $\triangle(in(M))$ is a strongly connected simplicial complex.*

Proof.

(a) W.l.o.g. we may assume $dim\ S/P > 1$ for all minimal primes of M. Assume $(Supp\ F/in(M))\backslash\{\mathbf{m}\}$ is not connected. Since $(Supp\ S/in(P))\backslash\{\mathbf{m}\}$ is connected for $P \in Ass\ F/M$, [9, Theorem 1], by Lemma 2, M decomposes into two parts $M = A \cap B$ such that $Supp\ F/in(A)$ and $Supp\ F/in(B)$ have only $\mathbf{m}$ in common. Since $in(A) + in(B) \subseteq in(A + B)$, this implies $F/(A + B)$ zero-dimensional, a contradiction.
(b) If $Supp\ F/in(I)$ were not connected in codimension one, we would likewise find a decomposition $M = A \cap B$, such that $dim\, F/(A + B) \leq d - 2$, since $Spec\ S/in(P)$ for $P \in Ass\ F/M$ is connected in codimension one, again by [9, Theorem 1]. □

The restriction given in (a) is essential since $Supp\ S/in(M)$ is always connected. For homogeneous ideals (and modules) (a) can be reformulated:

COROLLARY 1. *Let $I \subset S$ be a homogeneous ideal. If Proj S/I is connected then Proj $S/in(I), \triangle(I)$ and $\triangle(in(I))$ are so.*

Define a simplicial complex $\triangle$ to be *locally strongly connected* iff $link_{\triangle}\sigma$ is connected for all $\sigma \in \triangle$ such that $dim\,\sigma \leq dim\,\triangle - 2$ (including the empty face if $dim\,\triangle > 0$). It follows that $\triangle$ is strongly connected in the sense of [9].

If I is a prime ideal, $\triangle(I)$ is locally strongly connected since it is a matroid complex. Moreover in all examples given in [9], $\triangle(in(I))$ is also locally strongly connected since it is always a quasimanifold in the sense of [6].

Example.

$$I := (x_1x_4 - x_2x_3,\ x_0x_3x_4 - x_0x_4^2 + x_3^3,$$
$$x_0x_2x_3 - x_0x_2x_4 + x_1x_3^2, x_0x_1x_2 - x_0x_2^2 + x_1^2x_3)$$

This is the Gröbner basis with respect to the pure lexicographic term order of the prime ideal with generic point $(s^3, s^2t, stu, su(u - s), u^2(u - s))$, see [11, 8.6.2]. Since $in(I) = (x_0, x_1) \cap (x_0, x_4) \cap (x_1, x_3) \cap (x_2, x_4)$, we get $\triangle(in(I)) = ((234), (123), (024), (013))$. This complex is not locally connected at $\sigma = (0)$.

This example shows that locally connectedness may not transfer to $in(I)$ even for prime ideals.

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