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Highly Symmetric Subgraphs of Hypercubes

A.E. BROUWER
Eindhoven Univ. of Technology, Eindhoven, Netherlands

I.J. DEJTER
Univ. of Puerto Rico, Rio Piedras, PR, 00931, Puerto Rico

C. THOMASSEN
Technical Univ. of Denmark, Lyngby, Denmark

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Abstract. Two questions are considered, namely (i) How many colors are needed for a coloring of the n-cube without monochromatic quadrangles or hexagons? We show that four colors suffice and thereby settle a problem of Erdös. (ii) Which vertex-transitive induced subgraphs does a hypercube have? An interesting graph has come up in this context: If we delete a Hamming code from the 7-cube, the resulting graph is 6-regular, vertex-transitive and its edges can be two-colored such that the two monochromatic subgraphs are isomorphic, cubic, edge-transitive, nonvertex-transitive graphs of girth 10.

Keywords: edge-coloring, hypercube, vertex-transitive subgraph

1. The 7-cube minus a Hamming code

If I is a set, we denote by 2^I its power set. If $x, y \in 2^I$, we denote their symmetric difference by $x + y$, and consider 2^I as a vector space over $\mathbb{F}_2$. The set 2^I carries a natural graph structure: x and y are neighbors (which we write $x \sim y$) if and only if $|x + y| = 1$. For $|I| = n$ this graph is called the *n-cube*, and for unspecified n a *hypercube*.

Now let $I = \mathbb{Z}_7$ (the integers modulo 7) and let $H \subseteq 2^I$ consist of $\emptyset$, the seven sets $\{1 + i, 2 + i, 4 + i\}\,(i \in I)$, and the complements of these eight sets. Then H is a subspace of 2^I. In fact, H is a perfect 1-error-correcting code in the 7-cube: The vertices of H are pairwise nonadjacent, and each vertex outside H has precisely one neighbor in H. (Since H is also a subspace, this means that H is a Hamming code.) It follows that the subgraph Γ of the 7-cube induced by $X = 2^I \backslash H$ has 112 vertices and is regular of valency 6. Since H is a subspace and is invariant under a cyclic permutation of I, it is clear that Γ is vertex-transitive. Our aim is to color the edges of Γ red and white in such a way that the two monochromatic graphs thus obtained (both with vertex set X) are isomorphic cubic graphs of girth 10. (Then it follows that the edges of 2^I can be colored with three colors such that there are no monochromatic g-gons for $g < 10$.) Consider an edge xy of Γ, where x has odd weight (cardinality).

Define $i, j \in I$ by $x + \{i\} \in H$ and $y = x + \{j\}$. Now color the edge xy red if $j - i \in \{1, 2, 4\}$ and white otherwise. Denote the red subgraph of Γ by Γ_R and the white subgraph by Γ_W.

(1) $\Gamma_R \cong \Gamma_W$.
Indeed, if $u \in H$ has odd weight, then $x \mapsto x + u$ is an isomorphism.

(2) $\mathrm{Aut}(\Gamma_R) = \mathrm{Aut}(\Gamma_W)$ *is solvable of order 168, acts (sharply) transitively on the edges of both* Γ_R *and* Γ_W, *and has two orbits on their vertex set* X.
Indeed, Let H_0 be the even-weight subcode of H. The group of order 168 generated by the translations over members of H_0, the cyclic shifts of the set I of coordinates, and the permutation $i \mapsto 2i$ of I is sharply edge-transitive and has two orbits on X (because it preserves the parity of the weight). We shall see below that this group really is the full automorphism group of our two graphs.

(3) Γ_R *has diameter 8; for any vertex* x *of odd weight there is a unique antipode* $x + \{i, i + 3, i + 5, i + 6\}$ *at distance 8, where* i *is determined by* $x + \{i\} \in H$; *no two vertices of even weight have distance 8.*
In order to verify this it is sufficient to grow the distance classes from $x = \{1\}$ and $x = \{0, 1\}$. It follows that $\mathrm{Aut}\Gamma_R$ has two orbits on X, and then it is easily seen that $\mathrm{Aut}\Gamma_R$ is no larger than the group found under (2).

Let us mention some other interesting properties of Γ and Γ_R.

(4) *Any quadrangle in* Γ *has three edges of one color and one edge with the other color. If* xy *is a white edge, then* x *and* y *have distance 3 in* Γ_R, *and there is a unique path* $x \sim u \sim v \sim y$ *in* Γ_R *joining them.*

(5) Γ_R *has girth 10.*

(6) Γ_R *is an 8-cover of the Heawood graph, the point-line incidence graph of the Fano plane.*
Indeed, if we identify two vertices of Γ_R when they differ by an element of H_0, we obtain a graph Δ isomorphic to the Heawood graph (namely, defined on the vertex set $I \cup I'$ by $i - j'$ iff $i - j \in \{1, 2, 4\}$).

The graph discussed in this section was constructed in Dejter and Guan [7]; it may well be the same graph as the one constructed by R.M. Foster according to Bouwer [2].

2. Vertex-transitive subgraphs of the n-cube

The graph Γ of Section 1 is a vertex-transitive induced subgraph of the 7-cube. Which other graphs are vertex-transitive induced subgraphs of some n-cube? Unfortunately there seem to be too many to classify. An example of a nice

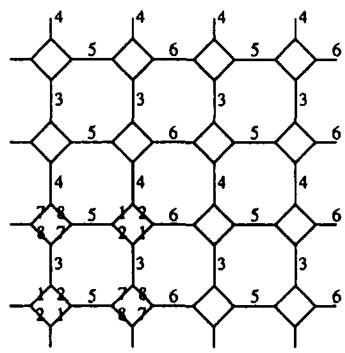

Figure 1. Vertex-transitive cubic graph on 64 vertices.

vertex-transitive cubic graph on 64 vertices that is an induced subgraph of the 8-cube is given in Figure 1.

The graph is drawn on a torus, i.e., left and right sides are identified, and so are top and bottom. Label i on an edge means that it joins vertices of the form x and $x + \{i\}$. (If we change labels 7, 8 to 1, 2, respectively, we see that the graph is also a subgraph of the 6-cube.) Without proof we mention that changing the size of this picture yields cubic graphs on $16m_1m_2$ vertices (with $m_1, m_2 \geq 2$), and also changes in dimension are possible. Similarly, one has, e.g., a quotient on 32 vertices of the hexagonal tiling of the plane that is an induced subgraph of the 6-cube – indeed, this graph occurs as the subgraph of the points at maximal distance from a given point in one of the two generalized hexagons of order (2, 2). Again many variations are possible. (For pictures, cf. e.g., Cohen and Tits [4] and Coxeter, Frucht and Powers [6, p. 130].)

Now the above graphs were vertex-transitive, but not necessarily for the group induced by the automorphism group of the hypercube. Examples of the latter are sub m-cubes of an n-cube, the 'doubled Odd' or 'revolving-door' graphs induced by the m-sets and the $(m+1)$-sets in a $(2m+1)$-cube, and the complement of a Hamming code in a $(2^m - 1)$-cube. (It is easy to see that the latter can be chosen so as to be invariant under a cyclic permutation of I, cf. [10, Chapter 9, §3].) Generalizing the first example, one may remark that the collection of vertex-transitive induced subgraphs of hypercubes is closed under taking Cartesian products. Generalizing the second example, one may remark that if G is an arbitrary permutation group on the set I of $n = 2m+1$ coordinate positions such that G moves some m-set M to a set disjoint from it, then the G-orbit M^G of M together with the orbit $\overline{M}^G$ of its complement $\overline{M} = I \backslash M$ induces a not totally disconnected vertex-transitive graph. Generalizing the second and third example, one may remark that the set of all vectors at maximal distance from a perfect binary linear code induces a connected vertex-transitive graph. (Thus, starting with the binary Golay code (see [10]), we find a graph on $\binom{23}{3}.2^{12}$ vertices, regular of valency 20.) Generalizing all previous examples, one may remark that if G is an arbitrary permutation group on the set I of n coordinate positions, and C is any binary linear code of word length n, and u is any binary vector of length n, then the union X of the G-orbits of the vectors $c+u$ (for $c \in C$) induces a vertex-transitive graph. Other types of examples exist. For example, a $2m$-gon is found as an induced subgraph of an m-cube by taking the orbit of the origin $\mathbf{0}$ under the cyclic group generated by $g : (u_1, \ldots, u_m) \mapsto (u_2, \ldots, u_m, u_1+1)$. Thus, even with the more strict requirement that the group must be induced by the automorphism group of the hypercube, there seem to be too many examples to classify.

3. Coloring the edges of a hypercube

It is easy to color the edges of the n-cube in two colors such that there is no monochromatic quadrangle. Just give the edge xy color $i\,(i = \pm 1)$ if $|x|$ is even and $|y| = |x| + i$. Moreover, the edges of an n-cube can be colored in 4 colors such that there is no monochromatic quadrangle or hexagon. Indeed, we can color the edges of the subgraph induced by the m-sets and the $(m+1)$-sets in two colors such that there is no monochromatic hexagon: Fix a total order on the set I of coordinate positions, and color the edge xy (where $y = x \cup \{j\}$) white whenever $|\{i \in x | i > j\}|$ is even, and red otherwise. (For $n \geq 5$ this yields color classes with girth 8; a monochromatic 8-gon is given by $13 \sim 134 \sim 34 \sim 234 \sim 23 \sim 235 \sim 35 \sim 135 \sim 13$.)

Clearly, the n-cube has $n2^{n-1}$ edges. Erdös [8] conjectured that, for each $\varepsilon > 0$ and n sufficiently large, every subgraph of the n-cube with $\varepsilon n 2^{n-1}$ edges contains a hexagon. The above 4-coloring shows that this is false for $\varepsilon \leq \frac{1}{4}$.

In Section 1 we saw that it is possible to find a 3-coloring of the edges of the n-cube without monochromatic quadrangle or hexagon, when $n \leq 7$. We don't know whether this can be done for larger n.

Remarks added in proof. In the meantime, Fan Chung wrote [3], where she also solves Erdös' conjecture. Marston Conder answered our question above in [5] by constructing a three-coloring of the edges of the n-cube without monochromatic quadrangle or hexagon, somewhat similar to the construction above. The graph constructed in Section 1 is different from the unique trivalent graph on 112 vertices with girth 10 in Foster's census ([1, 9]) since ours is not vertex-transitive.

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