Journal of Algebraic Combinatorics 2 (1993), 57–71

Hilbert Polynomial of a Certain Ladder-Determinantal Ideal

DEVADATTA M. KULKARNI
Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48309-4401; Department of Mathematics, University of Poona, Pune 411007, India

Received October 28, 1991; Revised September 21, 1992

Keywords: Hilbert function, ladder-determinantal ideal, lattice path counting, nonintersecting p-tuple of paths with fixed number of turning points

Abstract. A ladder-shaped array is a subset of a rectangular array which looks like a Ferrers diagram corresponding to a partition of a positive integer. The ideals generated by the p-by-p minors of a ladder-type array of indeterminates in the corresponding polynomial ring have been shown to be hilbertian (i.e., their Hilbert functions coincide with Hilbert polynomials for all nonnegative integers) by Abhyankar and Kulkarni [3, p 53–76]. We exhibit here an explicit expression for the Hilbert polynomial of the ideal generated by the two-by-two minors of a ladder-type array of indeterminates in the corresponding polynomial ring. Counting the number of paths in the corresponding rectangular array having a fixed number of "turning points" above the path corresponding to the ladder is an essential ingredient of the combinatorial construction of the Hilbert polynomial. This gives a constructive proof of the hilbertianness of the ideal generated by the two-by-two minors of a ladder-type array of indeterminates.

1. Introduction

The primality of the ideal generated by the p-by-p minors of a matrix of indeterminates in the corresponding polynomial ring is a well-known result. It is closely connected with the second fundamental theorem of invariant theory in case of vector invariants. The fact that the Hilbert polynomial and Hilbert function of this ideal coincide for all nonnegative integers, i.e., it is a hilbertian ideal, is also proved in different ways. Through the study of singularities of Schubert subvarieties of a flag manifold, Abhyankar came across a question of the primality of the ladder-determinantal ideals. A ladder is a special subset of a rectangle (as shown by shaded part in Figure 1).

In [2], Abhyankar proved the primality of a more general type of determinantal ideals, and gave a generalized second fundamental theorem of invariant theory. A beautiful part of his approach was proving the primality and the hilbertianness in the same breath through the combinatorial techniques. The first success with his technique was to prove the primality and the hilbertianness of a usual

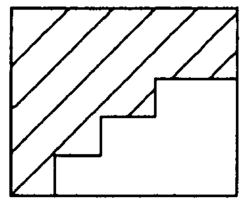

Figure 1. Ladder.

determinantal ideal by giving an explicit polynomial expression for its Hilbert function [1]. In [2], in fact, he succeeds in proving the primality of the ideal generated by the p-by-p minors coming from any "saturated" subset of a rectangle. A subset of a rectangle (a matrix) is said to be saturated if, for each minor whose principal diagonal lies in the set, the whole minor lies in the set. It is easy to see that a ladder is a saturated set. The hilbertianness of these ideals was established by Abhyankar-Kulkarni in [3]. Their methods are purely combinatorial, bringing out the polynomial nature of the Hilbert function by the way of "lattice path counting." The connections between the ladder-determinantal varieties and Schubert subvarieties of a flag manifold are explored in Mulay [5]. Some relations of these ideals with standard monomial theory can also be found in Musili [6].

The question of finding the Hilbert polynomial of a ladder-determinantal ideal is very interesting and it leads to several exciting combinatorial enumerations. In this paper, we give an explicit polynomial expression for the Hilbert function of the ideal generated by the two-by-two minors of a ladder. The question of finding an expression for the Hilbert polynomial of an ideal generated by the p-by-p minors of a ladder for $p > 2$ remains open. An explicit expression for the Hilbert function of a determinantal ideal of a saturated set seems to be a difficult task, partially due to the variety of these saturated sets.

The primality of the determinantal ideal from a saturated set was established by Abhyankar using a special basis of monomials in matrix indeterminates in [2]. Abhyankar defined the index of a monomial in matrix indeterminates to be the maximal size of a minor whose principal diagonal divides the given monomial. If we view a monomial in matrix indeterminates as a multiset of ordered pairs, its index is the length of its longest subsequence so that both components of the ordered pairs in the subsequence are strictly increasing. Abhyankar proved that the set of monomials of index at most p from the saturated set form a free basis for the corresponding quotient ring modulo the ideal generated by the $(p+1)$-by-$(p+1)$ minors from it. In [3], it is shown that a monomial of a matrix indeterminates of index p can be factored into monomials each of index at most one and this factorization comes from the combinatorial method of associating a nonintersecting p-tuple of paths in the rectangular lattice to a monomial of index p. In fact, the polynomial nature of the Hilbert function flows out of the fact that the number of monomials of degree V residing on a nonintersecting p-tuple of paths with total $C + 1$ points and having E nodes (corresponding to the variables which have to appear) is given by a binomial coefficient $\binom{V-E+C}{C}$ which may be viewed as a polynomial in V with rational coefficients. Thus, a recipe for the Hilbert polynomial of the ideal generated by the p-by-p minors of a ladder is to cook a formula for counting the set of nonintersecting p-tuples of paths in a rectangular lattice having a fixed numbers of nodes (the points where the path turns from left to down.)

In this paper, we have essentially used the same recipe. We have given a formula for counting the number of paths above a given ladder having a given number of nodes away from the ladder. Using this we exhibit an explicit expression for the Hilbert polynomial of an ideal generated by the two-by-two minors of a ladder. This of course gives a constructive proof of the hilbertianness of this determinantal ideal. If we have a counting formula for the number of nonintersecting k-tuples of paths with $k \leq p$, above a given ladder having a given number of nodes away from the ladder, we can construct the Hilbert polynomial of the ideal generated by the $(p + 1)$-by-$(p + 1)$ minors of the ladder following the same strategy.

2. Notation and terminology

We put Q for the set of all rational numbers, and Z for the set of all integers. Let N stand for the set of all nonnegative integers and N^* for the set of all positive integers. For A and B in Z, by $[A, B]$ we denote the closed integral segment $\{x \in Z : A \leq x \leq B\}$. For a set X, let card (X) denote the cardinality of X. For p in N^*, let X^p denote the set of all ordered p-tuples with entries from X and $d \in X^p$ can be written as $(d_1, d_2, \ldots, d_p)$.

Monomials, radicals, antichains

For $q = (q_1, q_2) \in N^{*2}$, a closed positive integral rectangle bounded by q_1 and q_2, denoted by rec(q_1, q_2) is a set of all pairs (i, j) such that $i \in [1, q_1]$ and $j \in [1, q_2]$. For a subset S of rec(q_1, q_2) we define the index of S, denoted by ind(S), as

$$\text{ind}(S) = \max\{\ell : \exists (x_1, y_1), \ldots, (x_\ell, y_\ell) \in S \text{ such that } x_1 < x_2 < \cdots < x_\ell \text{ and } y_1 < y_2 < \cdots < y_\ell\}.$$

We may note that $\text{ind}(S) = 0 \Leftrightarrow S = \emptyset$.

Let S be a subset of rec(q_1, q_2) where $q_1, q_2 \in N^*$. By a monomial on a subset S of rec(q_1, q_2) we mean the map $\phi : S \to N$. We denote the set of all monomials on S by mon(S). By support of $\phi \in$ mon(S) we mean the set $\{x \in S : \phi(x) \neq 0\}$ and we denote it by supt ϕ. By the degree of $\phi \in$ mon(S) we mean $\sum_{x \in S} \phi(x)$ and write it as deg(ϕ). We denote the set of all monomials on S of degree V by mon(S, V). For $g \in N$, by monomials on S of index $\leq g$ we mean $\phi \in$ mon(S) with ind(supt ϕ) $\leq g$ and we denote the set of such monomials by mon[S, g] and we put mon(S, V, g) = mon(S, V) $\cap$ mon[S, g].

A subset T of S is called a radical if ind(T) ≤ 1. We note that the support of the monomial of index 1 is a nonempty radical. We denote the set of all radicals in S of cardinality k by rad(S, k), and also rad(S) = $\cup_{k \geq 0}$ rad(S, k). A radical R in rad(S) is called a maximal radical if there does not exist a radical R' in S such that R' contains R properly. We denote the set of all maximal radicals in S by maxrad(S).

Given $q = (q_1, q_2) \in N^{*2}$ and $\ell \in N$, let $\mathcal{A}(q, \ell)$ denote the set of all $a = (a_1, a_2) \in Z^p \times Z^p$ such that

$$1 \leq a_1(1) < a_1(2) < \cdots < a_1(\ell) \leq q_1$$

and

$$q_2 \geq a_2(1) > a_2(2) > \cdots , a_2(\ell) \geq 1.$$

Also, let $\mathcal{A}(q) = \cup_{\ell \geq 0} \mathcal{A}(q, \ell)$. Viewing rec($q_1$, q_2) = $[1, q_1] \times [1, q_2]$ with the product partial order, an antichain T in a set S is such that for any two distinct elements (x_1, y_1) and (x_2, y_2) in T, we have either $x_1 < x_2$ and $y_1 > y_2$, or $x_1 > x_2$ and $y_1 < y_2$. We observe that, in fact, for a in $\mathcal{A}(q, \ell)$, the set $\{(a_1(i), a_2(i)) : 1 \leq i \leq \ell\}$ is an antichain of cardinality ℓ in rec(q_1, q_2); and each antichain in rec(q_1, q_2) is indexed by an element of $\mathcal{A}(q)$. The set of antichains of cardinality ℓ in a set S is denoted by $\mathcal{A}(S, \ell)$ and also $\mathcal{A}(S) = \cup_{\ell \geq 0} \mathcal{A}(S, \ell)$. We note that $\mathcal{A}(S) \subset$ rad(S), and $\mathcal{A}(q) = \mathcal{A}$(rec($q_1$, q_2)).

Ladder

Let $m, n \in N^*$. A ladder in rec(m, n) is an order ideal of rec(m, n) containing $(1, n)$ and $(m, 1)$ under the product partial order. A geometric picture of L will look like the shaded region in Figure 1. We may note that for each ladder L in rec(m, n), we have $h \in N^*$ and $a \in \mathcal{A}(q, h - 1)$ where $q = (m - 1, n - 1)$ such that

$$L = \cup_{i=1}^{h} \text{rec}(a_1(i), a_2(i - 1))$$

where $a_1(h) = m$; $a_1(0) = 1$; $a_2(h) = 1$; and $a_2(0) = n$, and we denote this ladder L by $L((m, n), a, h)$. For the rest of the paper, we set $a_1(i) = m_i$ and $a_2(i) = n_i$ for $i \in [1, h - 1]$. For a ladder L in rec(m, n), we denote

$$L_1\langle i\rangle = \{(i, j) \in L : j \in [1, n]\} \text{ for } i \in [1, m]$$

and

$$L_2\langle i\rangle = \{(i, j) \in L : i \in [1, m]\} \text{ for } j \in [1, n].$$

For a ladder L in rec(m, n), we define for $i \in [1, m]$,

$$\psi_L(i) = \max\{j \in [1, n] : (i, j) \in L_1\langle i\rangle\};$$

and we define the wall of L, denoted by wal(L) as

$$\text{wal}(L) = \cup_{i=1}^{m}\{(i, j) \in L : j \in [\psi_L(i+1), \psi_L(i)]\}$$

where $\psi_L(m+1) = 1$. We define $L^\circ = L\backslash\text{wal}(L)$. L° may be called an unwalled ladder. We note that wal$(L) \in$ maxrad(L). It is easy to see that each maximal radical in L is a path from $(1, n)$ to $(m, 1)$ going either left or down at each vertex of the points in L. It is worth noting that wal(L) is the maximal border strip of L when viewed as a Ferrers diagram.

Binomial coefficients

For $r \in Z$, we define

$$\binom{V}{r} = \begin{cases} \dfrac{V(V-1)\cdots(V-r+1)}{r!} & \text{if } r \in N \\ 0 & \text{if } r \in Z\backslash N \end{cases}$$

and

$$\begin{bmatrix} V \\ r \end{bmatrix} = \begin{cases} \dfrac{(V+1)(V+2)\cdots(V+r)}{r!} & \text{if } r \in N \\ 0 & \text{if } r \in Z\backslash N \end{cases}$$

where V can be taken to be an indeterminate over Q. For V, an indeterminate over Q, we may regard $\binom{V}{r}$ and $\begin{bmatrix} V \\ r \end{bmatrix}$ as polynomials in $Q[V]$. Note that for all $r \in Z$, and V in an overring of Q,

$$\begin{bmatrix} V \\ r \end{bmatrix} = \binom{V+r}{r}.$$

For $V \in Z$ and $r \in Z$ with $V + r \geq 0$, we have

$$\begin{bmatrix} V \\ r \end{bmatrix} = \begin{bmatrix} r \\ V \end{bmatrix}. \tag{1}$$

We also note that for $V \in Z$ and $\ell \in Z$, we have

$$\begin{bmatrix} -V-1 \\ \ell \end{bmatrix} = (-1)^\ell \binom{V}{\ell} \text{ and } \binom{-V-1}{\ell} = (-1)^\ell \begin{bmatrix} V \\ \ell \end{bmatrix}. \tag{2}$$

We use the following two results: For $T_1, T_2 \in Z$ and $A, B \in Z$, we have

$$\sum_{U\in Z}\binom{T_1}{A-U}\binom{T_2}{B+U}=\binom{T_1+T_2}{A+B} \tag{3}$$

where $\left\{U \in Z : \binom{T_1}{A-U}\binom{T_2}{B+U} \neq 0\right\} \subset [-B, A]$. For $T_1, T_2 \in Z$ and $A \in Z$, we have,

$$\begin{bmatrix} T_1 - T_2 \\ A \end{bmatrix} = \sum_{U\in Z}(-1)^U \binom{T_2}{U}\begin{bmatrix} T_1 \\ A-U \end{bmatrix} \tag{4}$$

where $\left\{U \in Z : (-1)^U\binom{T_2}{U}\left[\begin{smallmatrix} T_1 \\ A-U \end{smallmatrix}\right] \neq 0\right\} \subset [0, A]$.

For $h \in N^*$, we denote by $N^h_{\leq}$,

$$N^h_{\leq} = \left\{\tilde{e} \in N^h : \tilde{e} = (e_1, e_2, \ldots, e_h) \text{ such that } e_1 \leq e_2 \leq e_3 \leq \cdots \leq e_h\right\}.$$

For $\tilde{e}, \tilde{f} \in N^h_{\leq}$, we say $\tilde{e} \leq \tilde{f}$ if and only if $e_k \leq f_k$ for $k \in [1, h]$. For $\ell \in N$, let

$$\theta^\ell_0 = (0, 0, \ldots, 0, 0, \ell)$$

and

$$\theta^\ell_\ell = (\ell, \ell, \ldots, \ell, \ell, \ell)$$

be members of $N^h_{\leq}$.

The strategy of our counting is as follows: We count the number of antichains of fixed length ℓ in L° (Theorem 4). We show that there is a map σ from rad (L) onto $\mathcal{A}(L^\circ)$ of which restriction to maxrad(L) gives a bijection with $\mathcal{A}(L^\circ)$ (Theorem 6). We denote the map from $\mathcal{A}(L^\circ)$ onto maxrad(L) by μ and show that $\sigma(\mu(S)) = S$ for $S \in \mathcal{A}(L^\circ)$ (Theorem 5). We note that the number of radicals of length k containing a fixed antichain S in L° and contained in $\mu(S)$, i.e., the unique maximal radical in L such that $\sigma(\mu(S)) = S$, depends only upon k and card(S). Thus varying $\ell =$ card(S), we count the number of radicals of length k in L using counting of antichains of fixed length ℓ in L° (Theorem 9). We note that the support of all monomials on L having index less than or equal to one is a radical in L and the number of monomials of degree V having support in the fixed radical R in L depend only on V and card(R). Using counting of radicals of length k in L and varying k, we count mon$(L, V, 1)$ (Theorem 11).

3. Counting antichains

LEMMA 1. *Let $\ell \in N$ and $\tilde{e} = (e_1, e_2, \ldots, e_h) \in N^h_{\leq}$ with $e_h = \ell$. For a sequence of integers $m_0 = 1 < m_1 < m_2 < \cdots < m_h$, if $A_{\tilde{e}}$ is the set of $\tilde{a} = (a_1, a_2, \ldots, a_\ell) \in N^{*\ell}$ such that* (*i*) $a_i < a_{i+1}$ *for* $i \in [1, \ell - 1]$

and

(*ii*) $1 \leq a_i \leq m_h - 1$ *for* $i \in [1, \ell]$

and

(*iii*) $\text{card}(\{i : a_i < m_k\}) = e_k$ *for* $k \in [1, h]$;

then

$$\text{card}(A_{\bar{e}}) = \prod_{k=1}^{h} \binom{m_k - m_{k-1}}{e_k - e_{k-1}}$$

where $e_0 = 0$

Proof. We note that the cardinality of the set of strictly increasing maps from $[e_{k-1} + 1, e_k]$ into $[m_{k-1}, m_k - 1]$ is $\binom{m_k - m_{k-1}}{e_k - e_{k-1}}$ for $k \in [1, h]$. For each $a \in A_{\bar{e}}$, we can associate the unique h-tuple $(a^{[k]})_{1 \leq k \leq h}$ where for $1 \leq k \leq h$,

$$a^{[k]} : [e_{k-1} + 1, e_k] \to [m_{k-1}, m_k - 1]$$

is a strictly increasing map given by $a^{[k]}(i) = a_i$. Thus, the result follows from the initial observation. □

LEMMA 2. *Let* $\ell \in N$ *and* $\bar{f} = (f_1, f_2, \ldots, f_h) \in N^h_{\leq}$ *with* $f_h = \ell$. *For a sequence of integers* $n_0 > n_1 > \cdots > n_h = 1$, *if* $B_{\bar{f}}$ *is the set of* $\bar{b} = (b_1, b_2, \ldots, b_\ell) \in N^{*\ell}$ *such that*

(*i*) $b_i > b_{i+1}$ *for* $i \in [1, \ell - 1]$

and

(*ii*) $1 \leq b_i \leq n_0 - 1$ *for* $i \in [1, \ell]$

and

(*iii*) $\text{card}(\{i : b_i \geq n_k\}) = f_k$ *for* $k \in [1, h]$;

then

$$\text{card}(B_{\bar{f}}) = \prod_{k=1}^{h} \binom{n_{k-1} - n_k}{f_k - f_{k-1}}$$

where $f_0 = 0$.

Proof. We note that the cardinality of the set of strictly decreasing maps from $[f_{k-1}+1, f_k]$ into $[n_k, n_{k-1}-1]$ is $\binom{n_{k-1}-n_k}{f_k-f_{k-1}}$ for $k \in [1, h]$. For each $b \in B_{\tilde{f}}$, we can associate the unique h-tuple $(b^{[k]})_{1\le k\le h}$ where for $1 \le k \le h$

$$b^{[k]} : [f_{k-1}+1, f_k] \to [n_k, n_{k-1}-1]$$

is a strictly decreasing map given by $b^{[k]}(i) = b_i$. Thus, the result follows from the initial observation. □

THEOREM 3. *Under the assumptions of Lemma 1, for* $\tilde{a} = (a_1, a_2, \ldots, a_\ell)$ *in* $A_{\tilde{e}}$, *we define*

$$S_1\langle\tilde{a}\rangle = \{x = (x_1, x_2) \in \mathcal{A}(L^\circ, \ell) : x_1(i) = a_i \text{ for } i \in [1, \ell]\}.$$

Then

$$\operatorname{card}(S_1\langle\tilde{a}\rangle) = \sum_{\{\tilde{f}\in N^h_{\le}:\,\theta^\ell_0\le\tilde{f}\le\tilde{e}\}} \prod_{k=1}^{h} \binom{n_{k-1}-n_k}{f_k-f_{k-1}}. \tag{5}$$

Proof. We note that for $x \in S_1\langle\tilde{a}\rangle$,

$$x \in \mathcal{A}(L^\circ, \ell) \Leftrightarrow \operatorname{card}(\{j : x_2(j) \ge n_k\}) \le e_k \text{ for } k \in [1, h]. \tag{6}$$

If for some $k_0 \in [1, h]$, $\operatorname{card}(\{j : x_2(j) \ge n_{k_0}\}) > e_{k_0}$, then $x_2(e_{k_0}+1) \ge n_{k_0}$ and $x_1(e_{k_0}+1) \ge m_{k_0}$, so $(x_1(e_{k_0}+1), x_2(e_{k_0}+1)) \notin L^\circ$ which violates the assumption that $x \in \mathcal{A}(L^\circ, \ell)$. Thus (6) is true.

We define $F' : S_1\langle\tilde{a}\rangle \to N^h$ given by

$$F'(x) = (F'_1(x), F'_2(x), \ldots, F'_h(x))$$

where for $k \in [1, h]$,

$$F'_k(x) = \operatorname{card}(\{j : x_2(j) \ge n_k\}). \tag{7}$$

From (6), it follows that

$$F'(S_1\langle\tilde{a}\rangle) \subseteq \{\tilde{f} \in N^h_{\le} : \theta^\ell_0 \le \tilde{f} \le \tilde{e}\}$$

giving us

$$S_1\langle\tilde{a}\rangle = \coprod_{\{\tilde{f}\in N^h_{\le}:\,\theta^\ell_0\le\tilde{f}\le\tilde{e}\}} F'^{-1}(\tilde{f}).$$

The proof is completed by using the Lemma 2 and the definition of F'.

The region of the sum in (5) is contained in $\{\tilde{e} \in N^h : e_1 + e_2 + \cdots + e_h \le \ell h\}$ for $\ell \in N$, and hence $S(L^\circ, \ell)$ is always an integer-valued function. □

For $\ell \in N$, and a ladder L, we define

$$S(L^\circ, \ell) = \sum_{\{\tilde{e} \in N^h_{\leq} : \theta^\ell_0 \leq \tilde{e} \leq \theta^\ell_\ell\}} \prod_{i=1}^{h} \binom{m_i - m_{i-1}}{e_i - e_{i-1}} \left\{ \sum_{\{\tilde{f} \in N^h_{\leq} : \theta^\ell_0 \leq \tilde{f} \leq \tilde{e}\}} \prod_{j=1}^{h} \binom{n_{j-1} - n_j}{f_j - f_{j-1}} \right\}. \tag{8}$$

THEOREM 4. *For $\ell \in N$, and a ladder L,*

$$\mathrm{card}(\mathcal{A}(L^\circ, \ell)) = S(L^\circ, \ell).$$

Proof. Let

$$C = \{\tilde{a} \in N^\ell_{\leq} : 1 \leq a_i \leq m_h - 1 \text{ for } i \in [1, \ell] \text{ and } a_i < a_{i+1} \text{ for } i \in [1, \ell - 1]\}.$$

Let us define the function $F^*(x)$ for each $x = (x_1, x_2)$ in $\mathcal{A}(L^\circ, \ell)$ to be $(x_1(1), x_1(2), \ldots, x_1(\ell))$ which is an element of C. It is clear that for $\tilde{a}$ in C, $F^{*^{-1}}(\tilde{a}) = S_1\langle\tilde{a}\rangle$, by the definition of $S_1\langle\tilde{a}\rangle$ in Theorem 3. This gives us

$$\mathcal{A}(L^\circ, \ell) = \coprod_{\tilde{a} \in A} S_1\langle\tilde{a}\rangle. \tag{9}$$

We define the function $E^* : C \to N^h_{\leq}$ given by $E^*(\tilde{a}) = (E^*_1(\tilde{a}), E^*_2(\tilde{a}), \ldots, E^*_h(\tilde{a}))$ where

$$E^*_k(\tilde{a}) = \mathrm{card}(\{i : a_i < m_k\}) \text{ for } k \in [1, h]$$

for $\tilde{a} = (a_1, a_2, \ldots, a_\ell)$. It is clear that

$$E^*(C) \subseteq \{\tilde{e} \in N^h_{\leq} : \theta^l_0 \leq \tilde{e} \leq \theta^\ell_\ell\} \tag{10}$$

and for $\tilde{e}$ in $E^*(C)$, we have

$$E^{*-1}(\tilde{e}) = A_{\tilde{e}} \tag{11}$$

by the definition of $A_{\tilde{e}}$ in Lemma 1.

It follows from (9), (10), and (11) that

$$|\mathcal{A}(L^\circ, \ell)| = \sum_{\{\tilde{e} \in N^h_{\leq} : \theta^\ell_0 \leq \tilde{e} \leq \theta^\ell_\ell\}} \sum_{\tilde{a} \in A_{\tilde{e}}} |S_1\langle\tilde{a}\rangle|$$

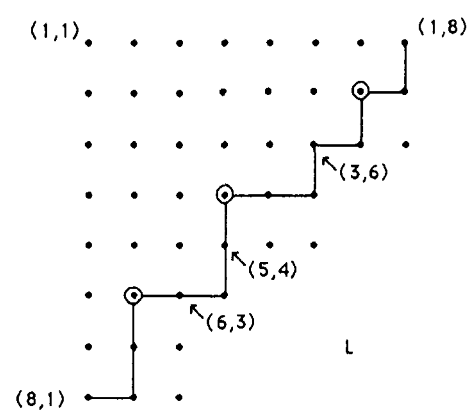

Figure 2. Ladder determined by $a = ((3, 5, 6), (6, 4, 3))$ and $(m, n) = (8, 8)$

and the proof can be completed using Lemma 1 and Theorem 3. □

4. Counting radicals

We define a map $\sigma : \text{rad}(L) \to \mathcal{A}(L^\circ)$ by $\sigma(T)$ to be the set of minimal elements of the poset $T \cap L^\circ$ (under the product order) for a radical T in L. We note that if a radical T is contained in $\text{wal}(L)$ then $\sigma(T) = \emptyset$. It is clear that $\sigma(T)$ is an antichain contained in T and $\sigma(T) = T$ for $T \in \mathcal{A}(L^\circ)$. If T' is a subset of a radical T in L and $\sigma(T) \subseteq T'$, then it follows immediately that $\sigma(T) \subseteq \sigma(T')$.

For an antichain S in L°, let $L_{\geq S}$ denote the filter in L generated by S; i.e., the set of points of L which lie above some point of S, i.e., all points (x, y) in L such that there is an (x', y') in S with $x' \leq x$, $y' \leq y$. Given a filter F in L, let the lower wall of F, denoted by $\text{lw}(F)$, be the set of points (a, b) in F such that

$$(a-1, b) \in F \text{ and } (a, b-1) \in F \Rightarrow (a-1, b-1) \notin F.$$

It is clear that for a filter F in L, $\text{lw}(F)$ is a radical in L. Further, for $S \in \mathcal{A}(L)$, $\text{lw}(L_{\geq S})$ will contain S. In particular, if (a, b) is in a filter F in L and $a = 1$ or $b = 1$ then $(a, b) \in \text{lw}(F)$. We now define a map $\mu : \mathcal{A}(L^\circ) \to \text{maxrad}(L)$ given by $\mu(S) = \text{lw}(\text{wal}(L) \cup (L_{\geq S}))$ for S in $\mathcal{A}(L^\circ)$. It is clear that $\mu(S)$ is a maximal radical in L, and $S \subset \mu(S)$ for S in $\mathcal{A}(L^\circ)$.

Example. This is a ladder L (see Figure 2) determined by $a = ((3, 5, 6), (6, 4, 3))$ and $(m, n) = (8, 8)$. Here $h = 4$. The wall of L is given by $\{(1, 8), (2, 8), (3, 8), (3, 7), (3, 6), (4, 6), (5, 6), (5, 5), (5, 4), (6, 4), (6, 3), (7, 3), (8, 3), (8, 2), (8, 1)\}$. The encircled points illustrate an antichain $S = \{(2, 7), (4, 4), (6, 2)\}$ in L°. The path from $(1, 8)$ to $(8, 1)$ shown in Figure 2 gives the maximal radical $\mu(S)$. It is clear that the minimal elements of $\mu(S)$ in L° are exactly the elements of S, i.e., $\sigma(\mu(S)) = S$.

THEOREM 5. *For S in $\mathcal{A}(L^\circ)$, we have*

$$\sigma(\mu(S)) = S.$$

Proof. We note that $\mu(S) \cap L^\circ = \text{lw}(L_{\geq S} \cap L^\circ)$ for S in $\mathcal{A}(L^\circ)$. By the definition of $L_{\geq S}$, it is clear that the minimal elements of $\text{lw}(L_{\geq S} \cap L^\circ)$ are contained in S, i.e., $\sigma(\mu(S)) \subseteq S$. Since S is contained in L° and a minimal element in $\mu(S) \cap L^\circ$ cannot come from the $\text{wal}(L)$, we have $S \subseteq \sigma(\mu(S))$. □

THEOREM 6. *Let σ^* be the restriction of σ to* $\text{maxrad}(L)$. $\sigma^* : \text{maxrad}(L) \to \mathcal{A}(L^\circ)$ *is a bijection.*

Proof. If R is a maximal radical in $L, L_{\geq R} = L_{\geq \sigma(R)}$, since $\sigma(R)$ contains only minimal elements of R. This gives us that $\mu(\sigma(R)) = \mu(R) = R$. The theorem follows from Theorem 5. □

THEOREM 7. *Given $S \in \mathcal{A}(L^\circ)$, there is a unique maximal radical $\mu(S)$ in L such that*

$$\sigma(\mu(S)) = S.$$

Proof. It follows immediately from Theorem 6. □

LEMMA 8. *For $k \in N$ and $S \in \mathcal{A}(L^\circ)$, let $R_S(k) = \{T \in \text{rad}(L, k) : \sigma(T) = S\}$. We have*

$$\text{card}(R_S(k)) = \binom{m_h + n_0 - 1 - \text{card}(S)}{k - \text{card}(S)}.$$

Proof. From Theorem 7, it follows that

$$R_S(k) = \{R \in \text{rad}(L, k) : S \subseteq R \subset \mu(S)\}.$$

We know that a maximal radical $\mu(S)$ in L contains $m_h + n_0 - 1$ elements. One may see this by establishing one-to-one correspondence between $\mu(S)$ and $\text{wal}(L)$ by associating $(x, y) \in \text{wal}(L)$ to $(x', y') \in \mu(S)$ if and only if $y - x = y' - x' = t$ for $t \in [1 - m_h, n_0 - 1]$. By an obvious counting principle, the lemma follows. □

THEOREM 9. *For a ladder L and $k \in N$, we have*

$$\text{card}(\text{rad}(L, k)) = \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \binom{m_h + n_0 - 1 - \ell}{k - \ell}.$$

Proof. Using the definitions of σ and $R_S(k)$ for S in $\mathcal{A}(L^\circ)$ as in Lemma 8, we have

$$\operatorname{rad}(L, k) = \coprod_{l=0}^{\infty} \coprod_{S\in\mathcal{A}(L^\circ,\ell)} R_S(k).$$

Using Lemma 8 and Theorem 4, we get

$$\begin{aligned}\operatorname{card}(\operatorname{rad}(L, k)) &= \sum_{\ell=0}^{\infty} \sum_{S\in\mathcal{A}(L^\circ,\ell)} \binom{m_h + n_0 - 1 - \ell}{k-\ell}.\\ &= \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \binom{m_h + n_0 - 1 - \ell}{k-\ell}.\end{aligned}$$

It is easy to note that the set of those ℓ which give a nonzero contribution in the summation is contained in $[0, \min\{k, m_h - 1, n_0 - 1\}]$. □

5. Counting mon $(L, V, 1)$

LEMMA 10. *For $R \in$* rad(L) *and $V \in N$, we define*

$$\operatorname{mon}_R[L, V, 1] = \{\phi \in \operatorname{mon}(L, V, 1) : \operatorname{supt} \phi = R\}.$$

We have

$$\operatorname{card}(\operatorname{mon}_R[L, V, 1]) = \begin{bmatrix} \operatorname{card}(R) - 1 \\ V - \operatorname{card}(R) \end{bmatrix}.$$

Proof. Immediate. □

For $V \in N$, and a ladder L, we define

$$\underline{F}(L, V) = \sum_{k=0}^{\infty} \left\{ \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \binom{m_h + n_0 - 1 - \ell}{k-\ell} \right\} \begin{bmatrix} k-1 \\ V-k \end{bmatrix}. \tag{12}$$

For V in an overring of Q and a ladder L, we define

$$\overline{F}(L, V) = \sum_{u=0}^{\infty} (-1)^u \left\{ \sum_{\ell=0}^{\infty} \binom{\ell}{u} S(L^\circ, \ell) \right\} \binom{V}{m_h + n_0 - 2 - u}. \tag{13}$$

The values of k, ℓ or u in (12) or (13) giving nonzero contribution in summation are contained in $[0, m_h + n_0 - 1]$. We also note that $\overline{F}(L, V)$ can be viewed as a polynomial in V with integer coefficients.

THEOREM 11. *For $V \in N$ and a ladder L, we have*

$$\text{card}(\text{mon}(L, V, 1)) = \underline{F}(L, V).$$

Proof. Since the support of each monomial in $\text{mon}(L, V, 1)$ is a radical in L, using the definition in Lemma 10, we get

$$\text{mon}(L, V, 1) = \coprod_{k=0}^{\infty} \coprod_{R \in \text{rad}(L,k)} \text{mon}_R[L, V, 1].$$

Using Lemma 10, we have

$$\text{card}(\text{mon}(L, V, 1)) = \sum_{k=0}^{\infty} \text{card}(\text{rad}(L, k)) \begin{bmatrix} k-1 \\ V-k \end{bmatrix}.$$

The theorem follows from Theorem 9 and the definition of $\underline{F}(L, V)$. □

THEOREM 12. *For $V \in N$ and a ladder L, we have*

(i) $\underline{F}(L, V) = \overline{F}(L, V)$; *and*
(ii) $\text{card}(\text{mon}(L, V, 1)) = \overline{F}(L, V)$.

Proof. Since the summations in $\underline{F}(L, V)$ have finite support, changing the order of summations in (12), we get

$$\underline{F}(L, V) = \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \left\{ \sum_{k=0}^{\infty} \binom{m_h + n_0 - 1 - \ell}{k - \ell} \begin{bmatrix} k-1 \\ V-k \end{bmatrix} \right\}.$$

From (3), we have

$$\underline{F}(L, V) = \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \binom{m_h + n_0 - 2 - \ell + V}{V - \ell}. \tag{14}$$

For $V \in N$ and $\ell \in [0, \min\{m_h - 1, n_0 - 1\}]$, since

$$m_h + n_0 - 2 - \ell + V = V + [(m_h - 1) + (n_0 - 1) - \ell] \geq 0,$$

it follows from (2) that

$$\binom{m_h + n_0 - 2 - \ell + V}{V - \ell} = \binom{m_h + n_0 - 2 - \ell + V}{m_h + n_0 - 2}.$$

In fact, for $\ell \in N$, we have

$$S(L^\circ, \ell) \binom{m_h + n_0 - 2 - \ell + V}{V - \ell} = S(L^\circ, \ell) \binom{m_h + n_0 - 2 - \ell + V}{m_h + n_0 - 2}. \tag{15}$$

Thus from (14) and (1), we get

$$\underline{F}(L, V) = \sum_{\ell=0}^{\infty} S(L^\circ, \ell) \begin{bmatrix} V - \ell \\ m_h + n_0 - 2 \end{bmatrix}. \tag{16}$$

From (16) and (4) for $V \in N$, we have

$$\underline{F}(L, V) = \sum_{u=0}^{\infty} (-1)^u \left\{ \sum_{\ell=0}^{\infty} \binom{\ell}{u} S(L^\circ, \ell) \right\} \begin{bmatrix} V \\ m_h + n_0 - 2 - u \end{bmatrix},$$

and by the definition of $\overline{F}(L, V)$, we get

$$\underline{F}(L, V) = \overline{F}(L, V).$$

From Theorem 11, it follows that for $V \in N$,

$$\text{card}(\text{mon}(L, V, 1)) = \overline{F}(L, V).$$

□

6. I_2^L is hilbertian

For $h \in N^*$, $(m, n) \in N^{*2}$ and $a \in \mathcal{A}(L^\circ)$, let $L((m, n), a, h) = L$ be a ladder in rec(m, n) as we have been using from §2. Let k be a field and let $X = (X_{ij})_{(i,j) \in L}$ denote the family of indeterminates over a field k. Let $A = k[X]$ be a polynomial ring in indeterminates X_{ij} over a field k where $(i, j) \in L$. For any $p \in N^*$, let I_p^L denote an ideal generated by the p-by-p minors of L in A (by minors here we mean the determinants of the square p-by-p submatrices of ladder-type array of indeterminates X).

For a homogeneous ideal I in A, we recall that the Hilbert function of I, denoted by $h_I : N \to N$, is given by $h_I(V) = \dim_k(A/I)_V$ for $V \in N$ where $A/I = \oplus_{V \in N}(A/I)_V$ is a graded decomposition of A/I. For such an I in A, by Hilbert's theorem, for V in an overring of Q, there exists a polynomial $H_I(V) \in Q[V]$ and $v_0 \in N$ such that $H_I(V) = h_I(V)$ for $V \in N$ with $V \geq v_0$. $H_I(V)$ is called the Hilbert polynomial of I. We define an ideal I in A to be *hilbertian* if for all $V \in N$, $h_I(V) = H_I(V)$.

For $L = \text{rec}(m, n)$ and $\mathcal{P} \in N^*$, Abhyankar, in [1], has given a formula for the Hilbert function of I_{p+1}^L in A and has proved that I_{p+1}^L is hilbertian. In fact, he has proved the following theorem for any ladder L in [2]:

ABHYANKAR'S THEOREM. For a ladder L and $p \in N^*$ and an ideal I_p^L in A, mon$(L, V, p-1)$ forms a free k-basis for $(A/I_p^L)_V$, i.e, for $V \in N$, $h_{I_p^L}(V) =$ card(mon$(L, V, p-1)$).

We prove the following theorem using this:

THEOREM 13. *For a ladder L and I_2^L in A, I_2^L is hilbertian.*

Proof. We know that for V in an overring of Q, by Hilbert's theorem, there exists $H_{I_2^L}(V) \in Q[V]$, i.e., the Hilbert polynomial of I_2^L such that

$$h_{I_2^L}(V) = H_{I_2^L}(V) \text{ for } V \in N \text{ and } V \geq v_0 \tag{17}$$

for some $v_0 \in N$. By Abhyankar's theorem and Theorem 12,

$$h_{I_2^L}(V) = \overline{F}(L, V) \text{ for all } V \in N. \tag{18}$$

From (17) and (18), we have,

$$H_{I_2^L}(V) = \overline{F}(L, V) \text{ for } V \in N \text{ and } V \geq v_0$$

and this forces

$$H_{I_2^L}(V) = \overline{F}(L, V) \text{ in } Q[V] \tag{19}$$

as $H_{I_2^L}(V)$ and $\overline{F}(L, V)$ are polynomials in one indeterminate V over a ring Q. Thus, from (18) and (19), we have

$$h_{I_2^L}(V) = H_{I_2^L}(V) \text{ for all } V \in N.$$

This establishes that I_2^L is hilbertian. □

Acknowledgments

Part of this work was done under the guidance of Professor Shreeram Abhyankar for the Ph.D. dissertation at Purdue University. The author wishes to thank Professor Abhyankar for his constant support and encouragement. The author would like to express deep gratitude towards the referees who helped generously with improving the exposition.

References

1. Abhyankar S., *Combinatorie des Tableaux de Young, Varietes Determinantielles et Calcul de Fonctions de Hilbert*, Nice Lecture Notes Rendiconti del Seminario Matematico 42, A. Galligo Universitá E. Polytecnico di Torino (1984), 65–88.
2. Abhyankar S., *Enumerative Combinatorics of Young Tableaux*, Marcel Dekker, New York, 1988.
3. Abhyankar S. and Kulkarni D.M., "On Hilbertian ideals," *Linear Algebra and Its Applications* **116** (1989), 53–76.
4. Kulkarni D.M., thesis, Purdue University, West Lafayette, IN (1985).
5. Mulay S.B., "Determinantal loci and the flag variety," *Advances in Mathematics* **74** (1989), 1–30.
6. Musili C., "Applications of standard monomial theory," preprint.