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# On $m$-regular systems on $\mathcal{H}(5, q^2)$

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**Abstract** The notion of $m$-regular system on the Hermitian variety $\mathcal{H}(n, q^2)$ was introduced by B. Segre (Ann. Math. Pura Appl. 70:1–201, 1965). Here, three infinite families of hemisystems on $\mathcal{H}(5, q^2)$, $q$ odd, are constructed.

**Keywords** Hermitian variety · Commuting polarities · Regular system · Hemisystem

## 1 Introduction and basics

In [11], regular systems of $\mathcal{H}(n, q^2)$ were introduced. A *regular system of order m* of $\mathcal{H}(n, q^2)$ is a set $\mathcal{R}$ of generators of $\mathcal{H}(n, q^2)$ with the property that every point lies on exactly $m$ generators of $\mathcal{R}$, $0 < m < \alpha$, where $\alpha$ denotes the number of generators of $\mathcal{H}(n, q^2)$ passing through a point of $\mathcal{H}(n, q^2)$. A regular system having the same order as its complement (in the set of all generators) is said to be a *hemisystem*. A regular system of $\mathcal{H}(n, q^2)$ of order $m$ has size $tm$, where $t$ is the ovoid number of $\mathcal{H}(n, q^2)$, so that $t = q^e + 1$, where $e = n$ if $n$ is odd, and $e = n + 1$ if $n$ is even.

In [11], Segre proved that, if $n = 3$ and $q$ is odd, a regular system of order $m$ must be a *hemisystem* on $\mathcal{H}(3, q^2)$, and hence $m = (q + 1)/2$. He also constructed a hemisystem on $\mathcal{H}(3, 9)$ admitting the linear group $PSL(3, 4)$. See also [6] for an alternative construction. Thus, the study of regular systems allowed Segre to demonstrate that the singular points of a Hermitian surface $\mathcal{H}(3, q^2)$ cannot be partitioned by totally singular lines: it has no spread.

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In [1], the nonexistence of regular systems of $\mathcal{H}(3,q^2)$ for $q$ even was established. A simple proof that a regular system of $\mathcal{H}(3,q^2)$ is a hemisystem (and so $q$ is odd) was given by Thas in [13], by showing that the concurrency graph of the lines of a regular system on $\mathcal{H}(3,q^2)$ of order $m$ is a strongly regular graph $srg(v,k,\lambda,\mu)$, with $v=(q^3+1)(q+1)-m$, $k=(q^2+1)(q-m)$, $\lambda=q-m-1$ and $\mu=q^2+1-m(q+1)$, and by applying the fact that in an $srg(v,k,\lambda,\mu)$, $(v-k-1)\mu=k(k-\lambda-1)$. Even more general was the work of Cameron-Goethals-Seidel [3], who defined a hemisystem of a generalized quadrangle of order $(s,s^2)$, $s$ odd, to be a set of points meeting every line in $(s+1)/2$ points and showed that the collinearity graph of such a set is strongly regular. Finally, in [14], the conjecture that there are no hemisystems on $\mathcal{H}(3,q^2)$ for $q>3$ was made. In [5], counterexamples to this conjecture are constructed on $\mathcal{H}(3,q^2)$, for all odd prime powers $q$, admitting $P\Omega^-(4,q)$, and giving Segre's example for $q=3$. All of this is motivated by the study of partial quadrangles introduced by Cameron [2], as each hemisystem of $\mathcal{H}(3,q^2)$ gives rise to a partial quadrangle.

In this paper we construct three infinite families of hemisystems on the Hermitian variety $\mathcal{H}(5,q^2)$, $q$ odd, admitting the orthogonal groups $P\Omega_6^\epsilon(q)$, $\epsilon=\pm$, and $P\Omega_5(q)$, respectively. We also present a technique of obtaining regular systems of $\mathcal{H}(n,q^2)$ starting from a regular system in higher dimensions with certain special properties.

With the aid of a computer we also found regular systems of $\mathcal{H}(5,q^2)$ for some low values of $q$ that are not hemisystems. This proves that there does not exist in higher dimensions an analogue of the result that a regular system of $\mathcal{H}(3,q^2)$ is a hemisystem.

## 2 Orthogonal polarities commuting with a unitary polarity

The notion of commuting polarities was introduced by J. Tits in 1955 [15]. Ten years later, B. Segre in his outstanding paper [11], completely devoted to the geometry of Hermitian varieties, developed the theory of polarities commuting with a non-degenerate unitary polarity defined on a finite projective space giving rise to one of the most beautiful and deeper "chapters" of finite geometries.

In this Section we describe the geometry arising from commuting orthogonal and unitary polarities.

For the reader's convenience, some facts about the Hermitian variety $\mathcal{H}(5,q^2)$ are summarized below.

In PG$(5,q^2)$ a *non-singular Hermitian variety* is defined to be the set of all absolute points of a non-degenerate unitary polarity, and is denoted by $\mathcal{H}(5,q^2)$.

A Hermitian variety $\mathcal{H}\cong\mathcal{H}(5,q^2)$ has the following properties, for which [11], [8], are excellent sources.

1. The number of points on $\mathcal{H}$ is $(q^5+1)(q^4+q^2+1)$.
2. Generators of $\mathcal{H}$ are planes and there are $(q+1)(q^3+1)(q^5+1)$ of them.
3. Through every point $P$ of $\mathcal{H}$ there pass exactly $(q+1)(q^3+1)$ generators.

From now on, we assume that $q$ is a power of an odd prime.

Let $\mathcal{B}$ be an orthogonal polarity commuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}(5,q^2)$. Set $\mathcal{V}=\mathcal{B}\mathcal{U}=\mathcal{U}\mathcal{B}$. Then $\mathcal{V}$ is a non-linear collineation and from [11], the fixed points of $\mathcal{V}$ on $\mathcal{H}(5,q^2)$ form a non-degenerate quadric $\mathcal{Q}$. In particular, $\mathcal{Q}=\mathcal{H}(5,q^2)\cap\Sigma_0$, where $\Sigma_0$ is a suitable subgeometry of $\Sigma$ isomorphic to PG$(5,q)$.

Hence, the projective orthogonal group $PGO_6^{\epsilon}(q)$, $\epsilon=\pm$, associated with $\mathcal{V}$ is seen to be a subgroup of the projective unitary group $PGU_6(q^2)$ associated with $\mathcal{U}$.

In terms of forms, let us assume that $(V,g)$ is a 6-dimensional unitary space over $F=GF(q^2)$. Let $K$ be the subfield of index two of $F$. Choose a basis $B=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ of $V$ such that $g(v_i,v_j)\in K$ for all $i$, $j$ and let $W$ denote the $K$-span of $B$. It turns out that the restriction $\bar{g}$ of $g$ to $W$ is a non-degenerate symmetric bilinear form. If $B$ is an orthonormal basis, then the discriminant of $\bar{g}$ is a square. By replacing $v_1$ by $\omega v_1$, where $\omega$ is a generator of $GF(q^2)^*$, the discriminant of $\bar{g}$ is a non-square. Therefore, we obtain embeddings $O_6^{\epsilon}<GU_6(q^2)$ for both $\epsilon=+$ and $\epsilon=-$. It follows that $PGO_6^{\pm}(q)$ is a subgroup of $PGU_6(q^2)$.

Now, let $\mathcal{H}=\mathcal{H}(5,q^2)$ be the Hermitian surface of $PG(5,q^2)$, $q$ odd, with equation $X_0^{q+1}+X_1^{q+1}+\cdots+X_4^{q+1}+X_5^{q+1}=0$, where $X_0$,$X_1$,$X_2$,$X_3$, $X_4$,$X_5$ are homogeneous coordinates in $PG(5,q^2)$. Let $\{Q_a\mid a\in GF(q^2)\setminus\{0\},\ a^{q+1}=1\}$ denote a family of $q+1$ quadrics of $PG(5,q^2)$, where $Q_a$ has equation $aX_0^2+X_1^2+X_2^2+X_3^2+X_4^2+X_5^2=0$. Straightforward computations show that each of these quadrics is hyperbolic and any two of them intersect in the parabolic quadric $\mathcal{P}$, given by the equation $X_1^2+X_2^2+X_3^2+X_4^2+X_5^2=0$, lying in the hyperplane $\bar{\pi}$ with equation $X_0=0$. Let $\pi$ denote the Baer hyperplane of $\bar{\pi}$ whose normalized point coordinates lie in the subfield $GF(q)$, and let $\mathcal{P}=\bar{\mathcal{P}}\cap\pi$ denote the associated subquadric of $\bar{\mathcal{P}}$ in $\pi$. Furthermore, let $U=\mathcal{H}\cap\bar{\pi}\cong\mathcal{H}(4,q^2)$ be the Hermitian variety, given by equation $X_1^{q+1}+X_2^{q+1}+X_3^{q+1}+X_4^{q+1}+X_5^{q+1}=0$, that one obtains by intersecting the Hermitian variety $\mathcal{H}$ with the hyperplane $\bar{\pi}$.

From [11, p. 146] each quadric $Q_a$ is permutable with $\mathcal{H}$. In particular, $(q+1)/2$ of them, say $Q_{a_1},\ldots,Q_{a_{(q+1)/2}}$, are such that $\mathcal{H}\cap Q_{a_i}$ is an elliptic quadric $\mathcal{E}_i$ embedded in a Baer subgeometry $B_i\cong PG(5,q)$ of $PG(5,q^2)$, for $i=1,2,\ldots,(q+1)/2$ and with $a_i^{(q+1)/2}=-1$. Also, from [11, Section 75] $Q_{a_i}\cap\mathcal{H}=B_i\cap\mathcal{H}=\mathcal{E}_i$ for each $i$. The remaining $(q+1)/2$ quadrics, say $Q_{b_1},\ldots,Q_{b_{(q+1)/2}}$, are such that $\mathcal{H}\cap Q_{b_i}$ is a hyperbolic quadric $\mathcal{I}_i$ embedded in a Baer subgeometry $S_i\cong PG(5,q)$ of $PG(5,q^2)$, for $i=1,2,\ldots,(q+1)/2$ and with $b_i^{(q+1)/2}=1$. Notice that all Baer subgeometries $B_i$ and $S_i$, where $i\in\{1,\ldots,(q+1)/2\}$, share the Baer hyperplane $\pi$ containing the Baer parabolic quadric $\mathcal{P}$ and the point $T=\bar{\pi}^{\mathcal{U}}=(1,0,0,0,0,0)$.

Assume that $\mathcal{Q}=\mathcal{H}\cap\Sigma_0$ is elliptic and let $G_1$ denote the stabilizer of $\mathcal{Q}$ in $PSU_6(q^2)$. From [4, Proposition 2.2] $G_1=PGO_6^{+}(q^2)\cap PSU_6(q^2)=PSO_6^{-}(q)\cdot 2$.

First of all we give some information on the orbits of $G_1$ on singular points of $\mathcal{H}$. To this aim, we will need the following result due to M. Sved [12].

**Lemma 2.1** *Let $\Sigma_0$ be a Baer subgeometry of the projective space* PG$(5,q^2)$. *For a hyperplane $H$ of $PG(5,q^2)$, $H\cap\Sigma_0$ is either a hyperplane of $\Sigma_0$ or a solid of $\Sigma_0$.*

Next, we prove the following proposition.

**Proposition 2.2** *The group $G_1$ has four orbits on singular points of $\mathcal{H}$.*

*Proof* It is sufficient to show that $G_1$ has three orbits on points of $\mathcal{H} \setminus \mathcal{Q}$. An orbit, say $I$, consists of points lying on $GF(q^2)$-extended lines of $\mathcal{Q}$. Let $P$ be a point of $\mathcal{H} \setminus \mathcal{Q} \cup I$. From Lemma 2.1, $P^\perp \cap \Sigma$, where $\Sigma \simeq PG(5,q)$, is not a PG$(4,q)$, as $P \notin \Sigma$, thus it is a PG$(3,q)$, say $\mathcal{W}$. It turns out that $\mathcal{W}$ cannot be tangent to $\mathcal{Q}$, as $\mathcal{W}$ extends to a tangent solid $\bar{\mathcal{W}}$ to $\mathcal{H}$, and $\bar{\mathcal{W}}^\perp$ is also tangent to $\mathcal{H}$, and $\bar{\mathcal{W}} \cap \mathcal{Q} = \bar{\mathcal{W}}^\perp \cap \mathcal{Q}$, so no point of $\mathcal{H} \setminus \mathcal{Q} \cup I$ has its polar hyperplane containing $\mathcal{W}$. Two orbits of $G$ correspond to the two cases $\mathcal{W}$ is of $-$-type or of $+$-type. By Witt's theorem, the isometry group of $\mathcal{Q}$ is transitive both on 3-dimensional $-$-sections and $+$-sections of $\mathcal{Q}$, and these isometries extend to $\mathcal{H}$ via the embedding $O_6^-(q) \le U_6(q^2)$. Hence, it is sufficient to show that the stabilizer of a 3-dimensional section $\mathcal{W} \cap \mathcal{H}$ in $PSO_6^-(q)$ is transitive on points $P$ of $\mathcal{H} \setminus \mathcal{Q} \cup I$ polar to $\mathcal{W} \cap \mathcal{H}$, in both cases. If $\mathcal{W}$ is of $+$-type, then $\bar{\mathcal{W}}^\perp \cap \mathcal{W}$ is an external line to $\mathcal{Q}$, and $|\bar{\mathcal{W}}^\perp \cap \mathcal{H}| = q+1$. Here, the $^-$notation denotes the extension of the line $\mathcal{W}^\perp$ to a line over $GF(q^2)$. The cyclic group of order $q+1$, $C_{q+1} \le PSO_6^-(q)_{\mathcal{W}}$, is faithful on $\mathcal{W}^\perp$, so faithful on $\bar{\mathcal{W}}^\perp$. But $PSU_6(q^2)_{\bar{\mathcal{W}}^\perp}$ acts on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$ as $PSL_2(q)$ on PG$(1,q)$; hence $C_{q+1}$ is regular on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$. If $\mathcal{W} \cap \mathcal{Q}$ is of $-$-type, then $\bar{\mathcal{W}}^\perp \cap \mathcal{Q}$ is secant to $\mathcal{Q}$ and $|\bar{\mathcal{W}} \cap \mathcal{H}| = |\bar{\mathcal{W}}^\perp \cap \mathcal{H}| = q+1$. The cyclic group of order $q-1$, $C_{q-1} \le PSO_6^-(q)_{\mathcal{W}}$, is faithful on $\mathcal{W}^\perp$, so faithful on $\bar{\mathcal{W}}^\perp$. Again $PSU_6(q^2)_{\bar{\mathcal{W}}^\perp}$ acts on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$ as $PSL_2(q)$ on PG$(1,q)$. Hence $C_{q-1}$ is regular on $(\bar{\mathcal{W}}^\perp \cap \mathcal{H}) \setminus \mathcal{Q}$.

The $G_1$-orbits have sizes: $(q+1)(q^3+1)$, $(q^2+1)(q^3+1)(q^2-q)$, $q^4(q^3+1)(q^2-1)/2$, $q^4(q^3+1)(q^2+1)/2$: they are points on $\mathcal{Q}$, points on $GF(q^2)$-extended lines of $\mathcal{Q}$ but not on $\mathcal{Q}$, points on generators of $\mathcal{H}$ meeting $\mathcal{Q}$ at a line and complement of these in $\mathcal{H}$. □

**Proposition 2.3** *$G_1$ has three orbits on planes of $\mathcal{H}$.*

*Proof* First of all we observe that no generator of $\mathcal{H}$ can be disjoint from $\mathcal{Q}$. Indeed an easy counting argument shows that a generator of $\mathcal{H}$ either meets $\mathcal{Q}$ at a line or at a point or it is disjoint from $\mathcal{Q}$ and meets $I$ at a non-degenerate conic. The three orbits have size $(q+1)(q^3+1)(q^3+1)$, $(q+1)(q^3+1)(q^4-q)$ and $(q+1)(q^3+1)(q^5-q^4-q^2+q)$, respectively. □

Assume now that $\mathcal{Q} = \mathcal{H} \cap \Sigma_0$ is hyperbolic. Let $G_2$ denote the stabilizer of $\mathcal{Q}$ in $PSU_6(q^2)$. From [4, Proposition 2.2], it turns out that $G_2 = PGO_6^+(q^2) \cap PSU_6(q^2) = PSO_6^+(q) \cdot 2$.

**Proposition 2.4** *The group $G_2$ has four orbits on singular points of $\mathcal{H}$.*

*Proof* It is sufficient to show that $G_2$ has three orbits on singular points of $\mathcal{H} \setminus \mathcal{Q}$. An orbit, say $I$, consists of points lying on $GF(q^2)$-extended planes of $\mathcal{Q}$. Let $P$ be a point of $\mathcal{H} \setminus \mathcal{Q} \cup I$. From Lemma 2.1, $P^\perp \cap \Sigma$, where $\Sigma \simeq PG(5,q)$, is not a PG$(4,q)$, as $P \notin \Sigma$, thus it is a PG$(3,q)$, say $\mathcal{W}$. It turns out that $\mathcal{W}$ cannot be tangent to $\mathcal{Q}$, as $\mathcal{W}$ extends to a tangent solid $\bar{\mathcal{W}}$ to $\mathcal{H}$, and $\bar{\mathcal{W}}^\perp$ is also tangent to $\mathcal{H}$, and $\bar{\mathcal{W}} \cap \mathcal{Q} = \bar{\mathcal{W}}^\perp \cap \mathcal{Q}$, so no point of $\mathcal{H} \setminus \mathcal{Q} \cup I$ has its polar hyperplane containing $\mathcal{W}$. Two orbits of $G_2$ correspond to the two cases $\mathcal{W}$ is of $-$-type or of $+$-type. By Witt's theorem, the isometry group of $\mathcal{Q}$ is transitive both on 3-dimensional $-$-sections and $+$-sections of $\mathcal{Q}$, and these isometries extend to $\mathcal{H}$ via the embedding $O_6^+(q) \leq U_6(q^2)$. Hence, it is sufficient to show that the stabilizer of a 3-dimensional section $\mathcal{W} \cap \mathcal{H}$ in $PSO_6^+(q)$ is transitive on points $P$ of $\mathcal{H} \setminus \mathcal{Q} \cup I$ polar to $\mathcal{W} \cap \mathcal{H}$, in both cases. If $\mathcal{W}$ is of $+$-type, then $\bar{\mathcal{W}}^\perp \cap \mathcal{W}$ is a secant line to $\mathcal{Q}$, and $|\bar{\mathcal{W}}^\perp \cap \mathcal{H}| = q-1$. Here, the $\bar{}$ notation denotes the extension of the line $\mathcal{W}^\perp$ to a line over $GF(q^2)$. The cyclic group of order $q-1$, $C_{q-1} \leq PSO_6^+(q)_{\mathcal{W}}$, is faithful on $\mathcal{W}^\perp$, so faithful on $\bar{\mathcal{W}}^\perp$. But $PSU_6(q^2)_{\bar{\mathcal{W}}^\perp}$ acts on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$ as $PSL_2(q)$ on PG$(1,q)$; hence $C_{q-1}$ is regular on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$. If $\mathcal{W} \cap \mathcal{Q}$ is of $-$-type, then $\bar{\mathcal{W}}^\perp \cap \mathcal{Q}$ is external to $\mathcal{Q}$ and $|\bar{\mathcal{W}} \cap \mathcal{H}| = |\bar{\mathcal{W}}^\perp \cap \mathcal{H}| = q+1$. The cyclic group of order $q+1$, $C_{q+1} \leq PSO_6^+(q)_{\mathcal{W}}$, is faithful on $\mathcal{W}^\perp$, so faithful on $\bar{\mathcal{W}}^\perp$. Again $PSU_6(q^2)_{\bar{\mathcal{W}}^\perp}$ acts on $\bar{\mathcal{W}}^\perp \cap \mathcal{H}$ as $PSL_2(q)$ on PG$(1,q)$. Hence $C_{q+1}$ is regular on $(\bar{\mathcal{W}}^\perp \cap \mathcal{H}) \setminus \mathcal{Q}$.

The $G_2$-orbits are points on $\mathcal{Q}$, points on $GF(q^2)$-extended planes of $\mathcal{Q}$ but not on $\mathcal{Q}$, points on $\mathcal{H}$ corresponding to elliptic 3-dimensional sections of $\mathcal{Q}$ and points on $\mathcal{H}$ corresponding to hyperbolic 3-dimensional sections of $\mathcal{Q}$. □

**Proposition 2.5** *$G_2$ has four orbits on planes of $\mathcal{H}$.*

*Proof* First of all we observe that no generator of $\mathcal{H}$ can be disjoint from $\mathcal{Q}$. Indeed an easy counting argument shows that a generator of $\mathcal{H}$ either arises from a plane of $\mathcal{Q}$ or meets $\mathcal{Q}$ at a line or at a point or it is disjoint from $\mathcal{Q}$ and meets $I$ at a non-degenerate conic. The three orbits have size $2(q+1)(q^2+1)$, $(q^4-1)(q^2+q+1)$,$q^8+2q^7+q^6-4q^5-10q^4-14q^3-13q^2-8q-3$ and $q^9-2q^7-q^6+4q^5+10q^4+13q^3+12q^2+8q+3$, respectively. □

## 3 A class of hemisystems on $\mathcal{H}$ for all odd prime powers $q$, admitting $P\Omega_6^-(q)$

In this section we provide a construction of a class of hemisystems of $\mathcal{H}$, $q$ odd, admitting the group $P\Omega_6^-(q)$.

**Theorem 3.1** *There exists a hemisystem $H_1$ of $\mathcal{H}$ admitting $P\Omega_6^-(q)$, for all odd prime powers $q$. The full stabilizer of $H_1$ has index 4 in the normalizer of $P\Omega_6^-(q)$ in $P\Gamma U_6(q^2)$, so, in fact, there are 2 such hemisystems, up to equivalence.*

*Proof* As we have seen, the embedding of $P\Omega_6^-(q)$ in $PSU_6(q^2)$, $q$ odd, may be realized by extending a non-degenerate bilinear form $f$ of Witt index 2 on a 6-dimensional vector space over $GF(q)$ (with corresponding quadratic form $Q$) to a non-degenerate Hermitian form $g$ on a 6-dimensional vector space over $GF(q^2)$, with the same Gram matrix. By Proposition 2.2, the group $G_1$ has four orbits on totally isotropic points of $\mathcal{H}$. Under the action of the subgroup $P\Omega_6^-(q) \leq G$ the three $G_1$-plane-orbits given in Proposition 2.3 split into six orbits, two of size

$(q^3+1)(q^2+1)(q+1)/2$, say $O_1$ and $O_2$, two of size $(q^4-q^2)(q+1)(q^3+1)/2$, say $O_3$ and $O_4$, and two of size $(q+1)(q^3+1)(q^5-q^4-q^2+2)/2$, say $O_5$ and $O_6$. The block-tactical decomposition matrix for this orbit decomposition is

$$\begin{bmatrix} q+1 & q+1 & 1 & 1 & 0 & 0 \\ q^2-q & q^2-q & 0 & 0 & q^2+1 & q^2+1 \\ 0 & 0 & \frac{q^4+q^2}{2} & \frac{q^4+q^2}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \\ q^4 & q^4 & \frac{q^4+q^2}{2} & \frac{q^4+q^2}{2} & \frac{q^4+q^2}{2} & \frac{q^4+q^2}{2} \end{bmatrix},$$

and hence the point-tactical decomposition matrix is

$$\begin{bmatrix} \frac{q^3+q^2+q+1}{2} & \frac{q^3+q^2+q+1}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} & 0 & 0 \\ \frac{q+1}{2} & \frac{q+1}{2} & 0 & 0 & \frac{q^4+q^3}{2} & \frac{q^4+q^3}{2} \\ 0 & 0 & \frac{q^3+q^2+q+1}{2} & \frac{q^3+q^2+q+1}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \\ q+1 & q+1 & \frac{(q^2-1)(q+1)}{2} & \frac{(q^2-1)(q+1)}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \end{bmatrix}.$$

It follows that amalgamation of the first, third and fifth orbit yields a $P\Omega_6^-(q)$-invariant hemisystem. In fact, we see $2^3=8$ hemisystems admitting this copy of $P\Omega_6^-(q)$. Take one, $H_1$, of these 8 hemisystems and consider $H_1$, $H_1^c$. The stabilizer $S$ of $\{H_1, H_1^c\}$ in $PSU_6(q^2)$ contains $G_1$ but not $PSU_6(q^2)$. By [4, Theorem 3.14], $G_1$ is a maximal subgroup of $PSU_6(q^2)$. So $S=G_1$. Hence $G_1$ is a normal subgroup of the stabilizer $T$ of $\{H_1, H_1^c\}$ in $P\Gamma U_6(q^2)$. It follows that $P\Omega_6^-(q)$ is normal in $T$, and so in the stabilizer of $H_1$ in $P\Gamma U_6(q^2)$. By [10, Proposition 2.8.2], the normalizer of $P\Omega_6^-(q)$ in $P\Gamma U_6(q^2)$ has 2 orbits of length 4 on the eight hemisystems. □

## 4 A class of hemisystems on $\mathcal{H}$ for all odd prime powers $q$, admitting $P\Omega_6^+(q)$

**Theorem 4.1** *There exists a hemisystem $H_2$ of $\mathcal{H}$ admitting $P\Omega_6^+(q)$, for all odd prime powers $q$.The full stabilizer of $H_2$ has index* 4 *in the normalizer of $P\Omega_6^+(q)$ in $P\Gamma U_6(q^2)$, so, in fact, there are* 4 *such hemisystems, up to equivalence.*

*Proof* By Proposition 2.2, the group $G_2$ has four orbits on totally isotropic points of $\mathcal{H}$. Under the action of the subgroup $P\Omega_6^+(q)\le G_2$ the four $G_2$-plane-orbits given in Proposition 2.3 split into eight orbits, two of size $(q^3+q^2+q+1)$ (extended Latin and Greek planes), say $O_1$ and $O_2$, two of size $(q^2+1)(q^2+q+1)(q^2-1)/2$, say $O_3$ and $O_4$, two of size $(q^2+1)(q^2+q+1)(q^4+q^3-2q^2-5q-3$, say $O_5$ and $O_6$, and two of size $(q^9-2q^7-q^6+4q^5+10q^4+13q^3+12q^2+8q+3)/2$. The block-tactical decomposition matrix for this orbit decomposition is

$$\begin{bmatrix} q^2+q+1 & q^2+q+1 & q+1 & q+1 & 1 & 1 & 0 & 0 \\ q^4-q & q^4-q & q^2-q & q^2-q & 2q^2 & 2q^2 & q^2+1 & q^2+1 \\ 0 & 0 & 0 & 0 & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \\ 0 & 0 & q^4 & q^4 & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} & \frac{q^4+q^2}{2} & \frac{q^4+q^2}{2} \end{bmatrix},$$

and hence the point-tactical decomposition matrix is

$$\begin{bmatrix} q+1 & q+1 & \frac{(q^2-1)(q+1)}{2} & \frac{(q^2-1)(q+1)}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{q^4+q^3+q-1}{2} & \frac{q^4+q^3+q-3}{2} & \frac{q^4+q^3+q-3}{2} & \frac{q^4+q^3+q-3}{2} \\ 0 & 0 & 0 & 0 & \frac{q^3+q^2+q+1}{2} & \frac{q^3+q^2+q+1}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \\ 0 & 0 & q+1 & q+1 & \frac{(q^2-1)(q+1)}{2} & \frac{(q^2-1)(q+1)}{2} & \frac{q^4-q^2}{2} & \frac{q^4-q^2}{2} \end{bmatrix}.$$

It follows that amalgamation of the first, third and fifth and seventh orbit yields a $P\Omega_6^+(q)$-invariant hemisystem. Indeed, we see $2^4 = 16$ hemisystems admitting this copy of $P\Omega_6^+(q)$. Take one, $H_2$, of these 16 hemisystems and consider $H_2$, $H_2^c$. The stabilizer $S$ of $\{H_2, H_2^c\}$ in $PSU(6, q^2)$ contains $G_2$ but not $PSU_6(q^2)$. By [4, Theorem 3.14], $G_2$ is a maximal subgroup of $PSU_6(q^2)$. So $S = G_2$. Hence $G_2$ is a normal subgroup of the stabilizer $T$ of $\{H_1, H_1^c\}$ in $P\Gamma U_6(q^2)$. It follows that $P\Omega_6^-(q)$ is normal in $T$, and so in the stabilizer of $H_1$ in $P\Gamma U_6(q^2)$. By [10, Proposition 2.7.4], the normalizer of $P\Omega_6^-(q)$ in $P\Gamma U_6(q^2)$ has 4 orbits of length 4 on the sixteen hemisystems. □

## 5 A class of hemisystems on $\mathcal{H}$ for all odd prime powers $q$, admitting $P\Omega_5(q)$

In this section we provide a construction of a class of hemisystems of $\mathcal{H}$, $q$ odd, admitting the group $P\Omega_5(q)$.

**Theorem 5.1** *There exists a hemisystem $H_3$ of $\mathcal{H}$ admitting $P\Omega_5(q)$, for all odd prime powers $q$, with full stabilizer of $H_3$ in $P\Gamma U_6(q^2)$ normalizing $P\Omega_5(q)$.*

*Proof* Start from the $P\Omega_6^+(q)$-hemisystem of $\mathcal{H}$. In the geometric setting of Section 2, let $\mathcal{P}$ be a parabolic section of $Q$. Let $J = P\Omega_5(q)$ be the group of $\mathcal{P}$. We have seen that $J$ fixes $(q+1)/2$ hyperbolic quadrics of $PG(5, q^2)$ all commuting with $\mathcal{H}$ such that their intersections with $\mathcal{H}$ are Baer hyperbolic quadrics. Under the action of $J$ the orbit $O_3$ splits into a certain number of orbits $(q-1)/2$ of which have size $q^3 + q^2 + q + 1$ (Greek planes on commuting hyperbolic quadrics). Similarly, the orbit $O_4$ splits into a certain number of orbits $(q-1)/2$ of which have size $q^3 + q^2 + q + 1$ (Latin planes on commuting hyperbolic quadrics).

It turns out that if we replace a $J$-orbit in $O_3$ of size $q^3 + q^2 + q + 1$ with the corresponding $J$-orbit of size $q^3 + q^2 + q + 1$ in $O_4$ (here by corresponding orbits we mean that they cover the same set of points), and leave fixed the other orbits in the $P\Omega_6^+(q)$-invariant hemisystem constructed above, we get again a hemisystem on $\mathcal{H}$ and the new hemisystem is $J$-invariant. Each copy of $P\Omega_5(q)$ in $P\Gamma U_6(q^2)$ is a subgroup of $\frac{q+1}{2}$ subgroups isomorphic to $P\Omega_6^+(q)$ and $\frac{q+1}{2}$ subgroups isomorphic to $P\Omega_6^-(q)$. This is clear from a geometric point of view. From an algebraic point of view, note that the pointwise stabilizer of the quadric $Q(4, q)$ in $P\Gamma U_6(q^2)$ is dihedral of order $2(q+1)$ and contains Baer involutions of both plus and minus type. Each of these $\frac{q+1}{2}$ supergroups isomorphic to $P\Omega_6^+(q)$ has 16 invariant hemisystems, each of which gives $2^{\frac{q-1}{2}}$ hemisystems invariant under this copy of $P\Omega_5(q)$, at most $\frac{q+1}{2}.16$ of which could admit $P\Omega_6^+(q)$ and at most $\frac{q+1}{2}.8$ of which could admit $P\Omega_6^-(q)$. Hence there exists one, $H_3$, that admits neither $P\Omega_6^+(q)$, nor $P\Omega_6^-(q)$. By the results on the maximal subgroups of $P\Gamma U_6(q^2)$ of [9], it follows that the full stabilizer of $H_3$ in $P\Gamma U_6(q^2)$ normalizes $P\Omega_5(q)$. □

## 6 Derivation

In this Section we provide a derivation technique to obtain regular systems of Hermitian varieties starting from a regular system in higher dimension. Assume that $n = 2s + 1$. A regular system of order $m$ $\mathcal{R}$ of $\mathcal{H}(n, q^2)$ is said to be of *index of regularity* $s'$ at a point $P$ of $\mathcal{H}(n, q^2)$, if every subspace of $\mathcal{H}(n, q^2)$ of dimension $s'$ on $P$ lies on a fixed number of generators of $\mathcal{R}$. It is easy to prove that if $\mathcal{R}$ has index of regularity $s' > 0$ at $P$ then it has indices of regularity $0, 1, \ldots, s' - 1$ at $P$.

**Proposition 6.1** *Any regular system $\mathcal{R}$ of $\mathcal{H}(n, q^2)$, $n = 2s + 1 \geq 5$, of index $0 < s' < s$ at $P$ gives rise to a regular system $\mathcal{S}$ of $\mathcal{H}(n-2, q^2)$ of index $s' - 1$.*

*Proof* Let $P^\perp$ denote the tangent hyperplane to $\mathcal{H}(n, q^2)$ at $P$. Choose $\Sigma$ to be any $(n-2)$-subspace of $P^\perp$ not on $P$ such that $\Sigma \cap \mathcal{H}(n, q^2)$ is nonsingular. Then, the set of $(s-1)$-subspaces cut out by members of $\mathcal{R}$ on $\Sigma$ is a regular system $\mathcal{S}$ of $\mathcal{H}(n-2, q^2)$ of index $s' - 1$. □

We now show that both $P\Omega_6^\epsilon(q)$-hemisystems of $\mathcal{H}$ constructed above induce hemisystems of $\mathcal{H}(3, q^2)$.

Let $\mathcal{R}$ be one of the aforementioned hemisystems of $\mathcal{H}$. Let $P$ be a point of $\mathcal{H}$ in the orbit corresponding to elliptic sections of $Q$. Then, from [5], $\mathrm{Stab}_{P\Omega_6^\epsilon(q)}(P)$ contains the group $H = PSL_2(q^2) \simeq P\Omega_4^-(q)$. The group $H$ has two orbits on the set $\mathcal{R}_P$ of generators on $P$, say $R_1$ and $R_2$ of size $(q^2+1)(q+1)/2$ and $(q^3-q)(q+1)/2$, respectively. The group $H$ has three orbits on subgenerators (lines) on $P$, say $S_1$, $S_2$ and $S_3$, of size $q^2+1$, $q^2(q^2+1)(q+1)/2$, $q^2(q^2+1)(q-1)/2$, respectively. If we call points the subgenerators on $P$ and blocks the members of $\mathcal{R}$ then we get the following block-tactical decomposition matrix for this orbit decomposition

$$\begin{bmatrix} 0 & 1 \\ (q^2+1)/2 & 0 \\ (q^2+1)/2 & q^2 \end{bmatrix},$$

and hence the point tactical decomposition is

$$\begin{bmatrix} 0 & (q+1)/2 \\ (q+1)/2 & 0 \\ (q+1)/4 & (q+1)/4 \end{bmatrix}.$$

It follows that $\mathcal{R}$ is also regular of index 1 at $P$ and hence it induces a hemisystem on any solid of $P^\perp$ not on $P$. The induced hemisystem is the hemisystem found in [5] admitting the group $P\Omega_4^-(q)$.

*Remark 6.2* With the aid of MAGMA [7] we also found a number of regular systems of $\mathcal{H}$ for small values of $q$:

- a 10-regular system of $\mathcal{H}(5,4)$ admitting the almost simple group $M_{22}.2$ (it is related to the 2-dimensional dual hyperoval and to a 11-tight set of $\mathcal{H}(5,4)$);
- a 12-regular system of $\mathcal{H}(5,4)$ admitting a group of order 110;
- a 6-regular system of $\mathcal{H}(5,4)$ admitting a group of order 55;
- a 32-regular system of $\mathcal{H}(5,9)$, admitting a soluble group of order $2^3.5.61$.

As a consequence we can say that there is no analogue of the result that a regular system of $\mathcal{H}(3,q^2)$ is a hemisystem for higher dimensions.

We would like to mention that we also found a hemisystem of the Hermitian variety $\mathcal{H}(4,9)$ admitting a metacyclic group of order 5.61 (the normalizer of a Singer cyclic group of $PGU_5(9)$).

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