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On the order of a non-abelian representation group of a slim dense near hexagon

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Abstract In this paper we study the possible orders of a non-abelian representation group of a slim dense near hexagon. We prove that if the representation group R of a slim dense near hexagon S is non-abelian, then R is a 2-group of exponent 4 and $|R| = 2^{\beta}$, $1 + NPdim(S) \leq \beta \leq 1 + dimV(S)$, where $NPdim(S)$ is the near polygon embedding dimension of S and $dimV(S)$ is the dimension of the universal representation module $V(S)$ of S. Further, if $\beta = 1 + NPdim(S)$, then R is necessarily an extraspecial 2-group. In that case, we determine the type of the extraspecial 2-group in each case. We also deduce that the universal representation group of S is a central product of an extraspecial 2-group and an abelian 2-group of exponent at most 4.

Keywords Near polygons · Non-abelian representations · Generalized quadrangles · Extraspecial 2-groups

1 Introduction

A *partial linear space* is a pair $S = (P, L)$ consisting of a non-empty 'point-set' P and a 'line-set' L of subsets of P of size at least two such that any two distinct points x and y are contained in at most one line. Such a line, if it exists, is written as xy

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and the points x and y are said to be *collinear* (notation: $x \sim y$). If x and y are not collinear, we write $x \nsim y$. If each line of S contains exactly three points, then S is called *slim*. For $x \in P$ and $A \subseteq P$, we define

$$x^{\perp} = \{x\} \cup \{y \in P : x \sim y\} \text{ and } A^{\perp} = \bigcap_{x \in A} x^{\perp}.$$

If $P^{\perp}$ is empty, then S is called *non-degenerate*. A subset of P is a *subspace* of S if any line containing at least two of its points is contained in it. For a subset X of P, the *subspace* $\langle X \rangle$ *generated* by X is the intersection of all subspaces of S containing X. A *geometric hyperplane* of S is a subspace of S, different from P, that meets every line non-trivially. The graph $\Gamma(P)$ with vertex set P, two distinct points being *adjacent* if they are collinear in S, is called the *collinearity graph* of S. For $x \in P$ and an integer i, we write

$$\Gamma_i(x) = \{y \in P : d(x, y) = i\},$$
$$\Gamma_{\leq i}(x) = \{y \in P : d(x, y) \leq i\},$$

where $d(x, y)$ denotes the *distance* between x and y in $\Gamma(P)$. The *diameter* of S is the diameter of $\Gamma(P)$. If $\Gamma(P)$ is connected, then S is called a *connected* partial linear space.

1.1 Representation of a partial linear space

Let $S = (P, L)$ be a slim partial linear space. If $x, y \in P$ and $x \sim y$, we define $x * y$ by $xy = \{x, y, x * y\}$.

Definition 1.1 ([9], p. 525) A representation of S is a mapping $\psi : x \mapsto \langle r_x \rangle$ from the point set P of S into the set of subgroups of order 2 of a group R such that the following hold:

(i) R is generated by $Im(\psi)$.
(ii) If $l = \{x, y, x * y\} \in L$, then $\{1, r_x, r_y, r_{x*y}\}$ is a Klein four subgroup of R.

We write (R, ψ) to mean that ψ is a representation of S with *representation group* R and say that (R, ψ) is a representation of S. We set $R_{\psi} = \{r_x : x \in P\}$. The representation (R, ψ) of S is *faithful* if ψ is injective, and is *abelian* or *non-abelian* according as R is abelian or not. Note that, in [9], 'non-abelian representation' means 'the representation group is not necessarily abelian'.

Let S be a connected slim partial linear space. For an abelian representation of S, the representation group can be considered as vector space over F_2, the field with two elements. If S admits at least one abelian representation, then there exists a unique abelian representation ρ_0 of S such that any other abelian representation of S is a composition of ρ_0 and a linear mapping (see [11]). The map ρ_0 is called the *universal abelian representation* of S. The F_2 vector space $V(S)$ underlying the universal abelian representation is called the *universal representation module* of S. Considering

$V(S)$ as an abstract group with the group operation $+$, it has the presentation

$$V(S) = \langle v_x : x \in P;\ 2v_x = 0;\ v_x + v_y = v_y + v_x \text{ for } x, y \in P; \\ \text{and } v_x + v_y + v_{x*y} = 0 \text{ if } x \sim y \rangle$$

and ρ_0 is defined by $\rho_0(x) = \langle v_x \rangle$ for $x \in P$.

A representation (R_1, ψ_1) of S is a *cover* of a representation (R_2, ψ_2) of S if there exists a group homomorphism $\varphi : R_1 \to R_2$ such that $\psi_2(x) = \varphi(\psi_1(x))$ for every $x \in P$. If S admits a non-abelian representation, then there is a *universal representation* $(R(S), \psi_S)$ which is the cover of every other representation of S. The universal representation is unique (see [8], p. 306) and the *universal representation group* $R(S)$ of S has the presentation:

$$R(S) = \langle r_x : x \in P, r_x^2 = 1, r_x r_y r_z = 1 \text{ if } \{x, y, z\} \in L \rangle.$$

Whenever we have a representation of S, the group spanned by the images of the points is a quotient of $R(S)$. Further,

Lemma 1.2 ([9], p. 525) $V(S) = R(S)/[R(S), R(S)]$.

The general notion of a representation group of a finite partial linear space with $p + 1$ points per line for a prime p was introduced by Ivanov [8] in his investigations of Petersen and Tilde geometries (motivated in large measure by questions about the Monster and Baby Monster finite simple groups). A sufficient condition on the partial linear space and on the non-abelian representation of it is given in [12] to ensure that the representation group is a finite p-group. For more on non-abelian representations, we refer to [8], also see ([12], Sections 1 and 2). In this paper, we study the possible orders of a non-abelian representation group of a slim dense near hexagon (Theorem 1.6).

1.2 Near $2n$-gons

A *near* $2n$*-gon* is a connected partial linear space $S = (P, L)$ of diameter n such that for each point-line pair $(x, l) \in P \times L$, l contains a unique point nearest to x. Non-degenerate near 4-gons are precisely *generalized quadrangles* (GQs, for short); that is, non-degenerate partial linear spaces such that for each point-line pair (x, l) with $x \notin l$, x is collinear with exactly one point of l.

Let $S = (P, L)$ be a near $2n$-gon. The sets $S(x) = \Gamma_{\leq n-1}(x)$, $x \in P$, are *special* geometric hyperplanes. A subset C of P is *convex* if every shortest path in $\Gamma(P)$ between two points of C is entirely contained in C. A *quad* is a non-degenerate convex subspace of P of diameter two. Thus a quad carries the structure of a generalized quadrangle. Let $x_1, x_2 \in P$ with $d(x_1, x_2) = 2$ and $|\{x_1, x_2\}^\perp| \geq 2$. If y_1 and y_2 are distinct elements of $\{x_1, x_2\}^\perp$ such that at least one of the lines $x_i y_j$ contains at least three points, then x_1 and x_2 are contained in a unique quad ([13], Proposition 2.5, p. 10). We denote this quad by $Q(x_1, x_2)$.

A near $2n$-gon is called *dense* if each line contains at least three points and any two distinct points at distance two from each other have at least two common neighbours.

In a dense near $2n$-gon, the number of lines through a point is independent of the point ([2], Lemma 19, p. 152). We denote this number by $t+1$. A near $2n$-gon is said to have *parameters* (s,t) if each line contains $s+1$ points and each point is contained in $t+1$ lines. A dense near 4-gon with parameters (s,t) is written as an (s,t)-GQ.

Theorem 1.3 ([13], Proposition 2.6, p. 12) *Let $S=(P,L)$ be a near $2n$-gon and Q be a quad in S. Then, for $x\in P$, either*

(*i*) *there is a unique point $y\in Q$ closest to x (depending on x) and $d(x,z)=d(x,y)+d(y,z)$ for all $z\in Q$; or*
(*ii*) *the points in Q closest to x form an ovoid $\mathcal{O}_x$ of Q.*

The point-quad pair (x,Q) in Theorem 1.3 is called *classical* in the first case and *ovoidal* in the second case. A quad Q in S is *classical* if (x,Q) is classical for each $x\in P$, otherwise it is *ovoidal*.

1.3 Slim dense near hexagons

A near 6-gon is called a *near hexagon*. Let $S=(P,L)$ be a slim dense near hexagon. For $x,y\in P$ with $d(x,y)=2$, we write $|\Gamma_1(x)\cap\Gamma_1(y)|$ as t_2+1 (though this depends on x,y). We have $t_2<t$. We say that a quad Q in S is of *type* $(2,t_2)$ if it is a $(2,t_2)$-GQ. A quad in S is *big* if it is classical. Thus, if Q is a big quad in S, then each point of S has distance at most one from Q.

Theorem 1.4 ([1], Theorem 1.1, p. 349) *Let $S=(P,L)$ be a slim dense near hexagon. Then P is necessarily finite and S is isomorphic to one of the eleven near hexagons with parameters as given below.*

	$\lvert P\rvert$	t	t_2	$dimV(S)$	$NPdim(S)$	a_1	a_2	a_4
(i)	759	14	2	23	22	–	35	–
(ii)	729	11	1	24	24	66	–	–
(iii)	891	20	$4^\star$	22	20	–	–	21
(iv)	567	14	$2,4^\star$	21	20	–	15	6
(v)	405	11	$1,2,4^\star$	20	20	9	9	3
(vi)	243	8	$1,4^\star$	18	18	16	–	2
(vii)	81	5	$1,4^\star$	12	12	5	–	1
$(viii)$	135	6	$2^\star$	15	8	–	7	–
(ix)	105	5	$1,2^\star$	14	8	3	4	–
(x)	45	3	$1,2^\star$	10	8	3	1	–
(xi)	27	2	$1^\star$	8	8	3	–	–

Here, $NPdim(S)$ is the F_2-rank of the matrix $A_3: P\times P\longrightarrow F_2$ defined by $A_3(x,y)=1$ if $d(x,y)=3$ and zero otherwise. We add a star if and only if the corresponding quads are big. The number of quads of type $(2,r)$, $r=1,2,4$, containing a given point of S is indicated by a_r. A '–' in a column means that $a_r=0$.

For a description of the near hexagons $(i)-(iii)$, see [13] and for $(iv)-(xi)$, see [1]. However, the parameters of these near hexagons suffice for our purposes here. We refer to [5] and [6] for other classification results about slim dense near polygons. For more on near polygons, see [4].

1.4 Extraspecial 2-groups

A finite 2-group G is *extraspecial* if its Frattini subgroup $\Phi(G)$, commutator subgroup G' and center $Z(G)$ coincide and have order 2.

An extraspecial 2-group is of exponent 4 and of order 2^{1+2m} for some integer $m \geq 1$ and the maximum of the orders of its abelian subgroups is 2^{m+1} (see [7], Section 20, pp. 78, 79). An extraspecial 2-group G of order 2^{1+2m} is a central product of either m copies of the dihedral group D_8 of order 8 or $m-1$ copies of D_8 with a copy of the quaternion group Q_8 of order 8. In the former case, G possesses a maximal elementary abelian subgroup of order 2^{1+m} and we write $G = 2_+^{1+2m}$. If the latter holds, then all maximal abelian subgroups of G are of the type $2^{m-1} \times 4$ and we write $G = 2_-^{1+2m}$.

Notation 1.5 *For a group G, $G^* = G \setminus \{1\}$.*

1.5 The main result

In this paper, we prove the following.

Theorem 1.6 *Let $S = (P, L)$ be a slim dense near hexagon and (R, ψ) be a non-abelian representation of S. Then*

- (i) *R is a finite 2-group of exponent 4 and order 2^β, where $1 + NPdim(S) \leq \beta \leq 1 + dimV(S)$.*
- (ii) *If $\beta = 1 + NPdim(S)$, then R is an extraspecial 2-group. Further, $R = 2_+^{1+NPdim(S)}$ except for the near hexagon (vi) in Theorem 1.4. In that case, $R = 2_-^{1+NPdim(S)}$.*

Section 2 is about some elementary properties of slim dense near hexagons. In Section 3, we study faithful representations of $(2,t)$-GQs. In Section 4, we study non-abelian representations of slim dense near hexagons. We prove Theorem 1.6 in Section 5.

2 Elementary properties

Let $S = (P, L)$ be a slim dense near hexagon. Since a (2,4)-GQ admits no ovoids, every quad in S of type $(2,4)$ is big (see Theorem 1.3).

Lemma 2.1 ([1], p. 359) *Let Q be a quad in S of type $(2, t_2)$. Then $|P| \geq |Q|(1 + 2(t - t_2))$. Equality holds if and only if Q is big. In particular, if a quad in S of type $(2, t_2)$ is big then so are all quads in S of that type.*

Let Q_1 and Q_2 be two disjoint big quads in S. By Lemma 2.1, Q_1 and Q_2 are of the same type.

Lemma 2.2 ([1], Proposition 4.3, p. 354) *Let π be the map from Q_1 to Q_2 which takes x to z_x, where $x \in Q_1$ and z_x is the unique point in Q_2 at distance one from x. Then*

(*i*) *π is an isomorphism from Q_1 to Q_2.*
(*ii*) *The set $Q_1 * Q_2 = \{x * z_x : x \in Q_1\}$ is a big quad in S disjoint from Q_1 and Q_2.*

Let Y be the subspace of S generated by Q_1 and Q_2. Since Y is the union of Q_1, Q_2 and $Q_1 * Q_2$, it follows that Y is isomorphic to the near hexagon (xi), (x) or (vii) according as Q_1 and Q_2 are of type (2,1), (2,2) or (2,4).

Let $\{i, j\} = \{1, 2\}$. For $x \in P \setminus Y$, we denote by x^j the unique point in Q_j at a distance 1 from x. For $y \in Q_i$, $z_y \in Q_j$ is defined as in Lemma 2.2. The following elementary results are useful for us.

Proposition 2.3 *For $x \in P \setminus Y$, $d(z_{x^i}, x^j) = 1$ and $d(z_{x^1}, z_{x^2}) = d(x^1, x^2) = 2$; that is, $\{x^1, z_{x^1}, x^2, z_{x^2}\}$ is a quadrangle in $\Gamma(P)$.*

Proof Since $x \in \Gamma_1(x^1) \cap \Gamma_1(x^2)$, $d(x^1, x^2) = 2$. Further, $d(x^i, x^j) = d(x^i, z_{x^i}) + d(z_{x^i}, x^j)$. So $d(z_{x^i}, x^j) = 1$ and $d(z_{x^1}, z_{x^2}) = 2$. □

Proposition 2.4 *Let l be a line of S disjoint from Y and $x, y \in l$, $x \neq y$. Then, $x^1y^1 = x^1z_{x^2}$ in Q_1 if and only if $x^2y^2 = x^2z_{x^1}$ in Q_2. In fact, if $x^1y^1 = x^1z_{x^2}$, then $(y^1, y^2) = (z_{x^2}, x^2 * z_{x^1})$ or $(x^1 * z_{x^2}, z_{x^1})$.*

Proof We have $x^jy^j = x^jz_{x^i}$ if and only if $y^j \in \{z_{x^i}, x^j * z_{x^i}\}$. If $y^j = x^j * z_{x^i}$, then $y^i \sim x^i * z_{x^j}$, because $2 = d(y^j, y^i) = d(y^j, x^i * z_{x^j}) + d(x^i * z_{x^j}, y^i)$. Since $y^i \sim x^i$, it follows that y^i is a point in the line $x^iz_{x^j}$ and so $y^i = z_{x^j}$.

If $y^j = z_{x^i}$, then applying the above argument to $(x * y)^j = x^j * z_{x^i}$, we get $(x * y)^i = z_{x^j}$ and so $y^i = x^i * z_{x^j}$. □

Proposition 2.5 *Let l be a line of S disjoint from Y and $x, y \in l$, $x \neq y$. Then $d(z_{x^i}, z_{y^j}) \leq 2$ if and only if $x^iy^i = x^iz_{x^j}$ in Q_i.*

Proof If $x^iy^i = x^iz_{x^j}$ in Q_i, then $x^jy^j = x^jz_{x^i}$ in Q_j (Proposition 2.4) and it follows that $d(z_{x^i}, z_{y^j}) \leq 2$. Conversely, let $x^iy^i \neq x^iz_{x^j}$ in Q_i. Again by Proposition 2.4, $x^jy^j \neq x^jz_{x^i}$ in Q_j. So $y^j \nsim z_{x^i}$. Then $d(x^i, y^j) = d(x^i, z_{x^i}) + d(z_{x^i}, y^j) = 1 + 2 = 3$. This implies that $d(z_{x^i}, z_{y^j}) = 3$. □

Proposition 2.6 *Let Q be a big quad in S disjoint from Y. For $x, y \in Q$ with $x \nsim y$, $\{d(z_{x^1}, z_{y^2}), d(z_{x^2}, z_{y^1})\} = \{2, 3\}$.*

Proof By Lemma 2.2, there exists $w \in \{x, y\}^\perp$ in Q such that $x^1w^1 = x^1z_{x^2}$. By Proposition 2.4, $(w^1, w^2) = (z_{x^2}, x^2 * z_{x^1})$ or $(x^1 * z_{x^2}, z_{x^1})$. Assume that $(w^1, w^2) =$

$(z_{x^2}, x^2 * z_{x^1})$. Then, $d(z_{x^2}, z_{y^1}) = d(w^1, z_{y^1}) = d(w^1, z_{w^1}) + d(z_{w^1}, z_{y^1}) = 2$. Now, $y^2 \sim w^2$ and $y^2 \nsim x^2$ in Q_2 implies that $x^1 \nsim z_{y^2}$. So $d(x^1, z_{y^2}) = 2$ and $d(z_{x^1}, z_{y^2}) = d(z_{x^1}, x^1) + d(x^1, z_{y^2}) = 3$. A similar argument holds if $(w^1, w^2) = (x^1 * z_{x^2}, z_{x^1})$. □

3 Representations of $(2, t)$-GQs

Let $S = (P, L)$ be a $(2, t)$-GQ. Then P is finite and $t = 1, 2$ or 4. For each value of t there exists a unique generalized quadrangle, up to isomorphism ([3], Theorem 7.3, p. 99). A *k-arc* of S is a set of k pair-wise non-collinear points of S. A k-arc is *complete* if it is not contained in a $(k+1)$-arc. A point x is a *center* of a k-arc if x is collinear with every point of it. An *ovoid* of S is a k-arc meeting each line of S non-trivially. A *spread* of S is a set K of lines of S such that each point of S is in a unique member of K. If O (resp., K) is an ovoid (resp., spread) of S, then $|O| = 1 + 2t$ (resp., $|K| = 1 + 2t$).

Since each line contains three points, each pair of non-collinear points of S is contained in a $(2, 1)$-subGQ of S. For $t' < t$, a $(2, t')$-subGQ of S and a point outside it generate a $(2, 2t')$-subGQ in S. The minimal number of points which are necessary to generate a $(2, t)$-GQ is equal to 4 if $t = 1$, 5 if $t = 2$ and 6 if $t = 4$.

3.1 $(2, 2)$-GQ

Let $S = (P, L)$ be a $(2, 2)$-GQ. For any 3-arc T of S, $|T^\perp| = 1$ or 3. Further, $|T^\perp| = 1$ if and only if T is contained in a unique $(2, 1)$-subGQ of S; and $|T^\perp| = 3$ if and only if T is a complete 3-arc. If S admits a k-arc, then $k \leq 5$. Here 5-arcs are ovoids and S contains six ovoids. Each ovoid is determined by any two of its points. Each point of S is in two ovoids and the intersection of two distinct ovoids is a singleton. Any two non-collinear points of S are in a unique ovoid of S and also in a unique complete 3-arc of S. Any incomplete 3-arc of S is contained in a unique ovoid. Any 4-arc of S is not complete and is contained in a unique ovoid. The intersection of two distinct complete 3-arcs of S is empty or a singleton.

A model for the $(2, 2)$-GQ: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. A *factor* of Ω is a set of three pair-wise disjoint 2-subsets of Ω. Let $\mathcal{E}$ be the set of all 2-subsets of Ω and $\mathcal{F}$ be the set of all factors of Ω. Then $|\mathcal{E}| = |\mathcal{F}| = 15$ and the pair $(\mathcal{E}, \mathcal{F})$ is a $(2, 2)$-GQ.

3.2 $(2, 4)$-GQ

Let $S = (P, L)$ be a $(2, 4)$-GQ. If S admits a k-arc, then $0 \leq k \leq 6$. So S has no ovoids. S admits two disjoint 6-arcs. A 5-arc of S is complete if and only if it is contained in a unique $(2, 2)$-subGQ of S. Each incomplete 5-arc has exactly one center and each complete 5-arc of S has exactly two centers. Each 4-arc has two centers and is contained in a unique complete 5-arc and in a unique complete 6-arc. Each 3-arc of S has three centers and is contained in a unique $(2, 1)$-subGQ of S. So any 4-arc of S is contained in a unique $(2, 2)$-subGQ of S.

A model for the $(2,4)$**-GQ:** Let Ω, $\mathcal{E}$ and $\mathcal{F}$ be as in the model of a (2,2)-GQ. Let $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Take

$$P = \mathcal{E} \cup \Omega \cup \Omega'; \; L = \mathcal{F} \cup \{\{i, \{i, j\}, j'\} : 1 \le i \ne j \le 6\}.$$

Then $|P| = 27$, $|L| = 45$ and the pair (P, L) is a (2,4)-GQ.

3.3 Representations

Let $S = (P, L)$ be a $(2, t)$-GQ and (R, ψ) be a representation of S.

Proposition 3.1 *R is an elementary abelian 2-group.*

Proof Let $x, y \in P$ and $x \nsim y$. Let T be a $(2, 1)$-subGQ of S containing x and y. Let $\{x, y\}^{\perp} \cap T = \{a, b\}$. Then $[r_x, r_y] = 1$, because $r_b r_y = r_y r_b$, $r_b r_x = r_x r_b$ and $r_{(a*x)*(b*y)} = r_{(a*y)*(b*x)}$. So R is abelian. □

For the rest of this section we assume that ψ is faithful.

Proposition 3.2 *The following hold*:

(i) $|R| = 2^4$ *if* $t = 1$;
(ii) $|R| = 2^4$ *or* 2^5 *if* $t = 2$, *and both possibilities occur*;
(iii) $|R| = 2^6$ *if* $t = 4$.

Proof Since S is generated by a set of k points where $(t, k) \in \{(1, 4), (2, 5), (4, 6)\}$, F_2-dimension of R is at most k. So $|R| \le 2^k$.

(i) If $t = 1$, then $|R| \ge 2^4$ because $|P| = 9$ and ψ is faithful. So $|R| = 2^4$.

(ii) If $t = 2$, then $|R| \ge 2^4$ because S contains a $(2, 1)$-subGQ. The rest follows from the fact that S has a symplectic embedding in an F_2-vector space of dimension 4 as well as an orthogonal embedding in an F_2-vector space of dimension 5.

(iii) We prove this after Proposition 3.3. □

The following is a partial converse to the fact that $r_x r_y \in R_{\psi}$ for $x, y \in P$ with $x \sim y$. Recall that $R_{\psi} = \{r_x : x \in P\}$.

Proposition 3.3 *Assume that $(t, |R|) \ne (2, 2^4)$. If $r_x r_y \in R_{\psi}$ for distinct $x, y \in P$, then $x \sim y$.*

Proof Let $z \in P$ be such that $r_z = r_x r_y$. If $x \nsim y$, then $T = \{x, y, z\}$ is a 3-arc of S because ψ is faithful. There is no (2,1)-subGQ of S containing T because the subgroup of R generated by the image of such a subGQ is of order 2^4 (Proposition 3.2(i)). Every 3-arc of a $(2, 4)$-GQ is contained in a unique (2,1)-subGQ. So $t = 2$ and T is a complete 3-arc. Let Q be a (2,1)-subGQ of S containing x and y. Then $z \notin Q$ and $P = \langle Q, z\rangle$. Since $r_z \in \langle \psi(Q)\rangle$, $|R| = |\langle \psi(Q)\rangle| = 2^4$, a contradiction to the assumption. □

If $(t, |R|) = (2, 2^4)$, then Proposition 3.3 is not true because in this case $R^* = R_\psi$, so $r_x r_y \in R_\psi$ for non-collinear points x and y.

Proof of Proposition 3.2(iii) If $t = 4$, then there are 16 points of S not collinear with a given point x. By Proposition 3.3, $|R^* \setminus R_\psi| \geq 16$. Thus, $|R| > 2^5$ and so $|R| = 2^6$. This completes the proof. □

Corollary 3.4 *Let $t = 4$ and Q be a $(2,2)$-subGQ of S. Then $|\langle \psi(Q) \rangle| = 2^5$.*

Proof This follows from Proposition 3.2(iii) and the fact that $P = \langle Q, x \rangle$ for $x \in P \setminus Q$. □

Proposition 3.5 *If $t = 2$, then $|R| = 2^4$ if and only if $r_a r_b r_c = 1$ for every complete 3-arc $\{a, b, c\}$ of S.*

Proof Let $T = \{a, b, c\}$ be a complete 3-arc of S and Q be a (2,1)-subGQ of S containing a and b. Then $c \notin Q$ and $P = \langle Q, c \rangle$.

If $r_a r_b r_c = 1$, then $r_c \in \langle \psi(Q) \rangle$ and $|R| = |\langle \psi(Q) \rangle| = 2^4$. Now, assume that $|R| = 2^4$. Let $\{x, y\} = \{a, b\}^\perp \cap Q$. Then $x, y \in T^\perp$, since T is a complete 3-arc. Let z be the point in Q such that $\{x, y, z\}$ is a 3-arc in Q. Then $c \sim z$ and $r_z = (r_a r_x)(r_b r_y)$. Since $H = \langle r_y : y \in x^\perp \rangle$ is a maximal subgroup of R ([10], 4.2.4, p. 68), $|H| = 2^3$. So $r_c = r_a r_b$ or $r_a r_b r_x$, since ψ is faithful. If the latter holds then $r_{c*z} = r_y$, which is not possible because ψ is faithful and $y \neq c * z$. Hence $r_c = r_a r_b$. □

Corollary 3.6 *Assume that $(t, |R|) = (2, 2^4)$. Let $T = \{a, b, c\} \subset P$ be such that $r_a r_b r_c = 1$. Then T is a line or a complete 3-arc.*

Proof Assume that T is not a line. Then, since ψ is faithful, T is a 3-arc. We show that T is complete. Suppose that T is not complete. Let $\{a, b, d\}$ be the complete 3-arc of S containing a and b. Then $r_a r_b r_d = 1$ (Proposition 3.5) and $c \neq d$. So $r_c = r_d$, contradicting the fact that ψ is faithful. □

Lemma 3.7 *If S contains a 3-arc $T = \{a, b, c\}$ such that $r_a r_b r_c \in R_\psi$, then $(t, |R|) = (2, 2^4)$. In particular, T is incomplete.*

Proof Let $x \in P$ be such that $r_x = r_a r_b r_c$. Since ψ is faithful, $x \notin T$. Let $t = 2$. If T is complete, then $|R| = 2^5$ (Proposition 3.5) and x is collinear with at least one point of T, say $x \sim a$. Then $r_b r_c = r_x r_a = r_{x*a} \in R_\psi$, a contradiction to Proposition 3.3. Thus, T is incomplete if $t = 2$.

Let Q_1 be the unique (2,1)-subGQ of S containing T. If $x \in Q_1$, then $\langle \psi(Q_1) \rangle = \langle r_a, r_b, r_c, r_x \rangle$ would be of order 2^3, contradicting Proposition 3.2(i). So $x \notin Q_1$ and $t \neq 1$. Let Q_2 be the (2,2)-subGQ of S generated by Q_1 and x. Then $|\langle \psi(Q_2) \rangle| = 2^4$, and so $t \neq 4$ (Corollary 3.4). Thus $t = 2$ and $|R| = |\langle \psi(Q_2) \rangle| = 2^4$. □

Lemma 3.8 *Let $a, b \in P$ with $a \nsim b$. Set $A = \{r_a r_x : x \nsim a\}$ and $B = \{r_b r_x : x \nsim b\}$. Then $|A \cap B| = t + 2$.*

Proof It is enough to prove that $r_a r_x = r_b r_y$ for $r_a r_x \in A, r_b r_y \in B$ if and only if either $x = b$ and $y = a$ holds or there exists a point c such that $\{c, a, y\}$ and $\{c, b, x\}$ are lines. We need to prove the 'only if' part. Since ψ is faithful, $x \neq b$ if and only if $y \neq a$. Assume that $x \neq b$ and $y \neq a$. For this, we show that $y \sim a$ and $x \sim b$. Then $r_{a*y} = r_a r_y = r_b r_x = r_{b*x}$. Since ψ is faithful, it would then follow that $a * y = b * x$ and this would be our choice of c.

First, assume that $(t, |R|) \neq (2, 2^4)$. Since $a \nsim b$, $r_a r_b \notin R_\psi$ by Proposition 3.3. Since $r_x r_y = r_a r_b$, Proposition 3.3 again implies that $x \nsim y$. Now, $r_a r_b r_y = r_x \in R_\psi$. By Lemma 3.7, $\{a, b, y\}$ is not a 3-arc. This implies that $y \sim a$. By a similar argument, $x \sim b$.

Now, assume that $(t, |R|) = (2, 2^4)$. Suppose that $x \nsim b$. Then $T = \{a, b, x\}$ is a 3-arc of S. By Proposition 3.7, T is incomplete. Let Q be the $(2, 1)$-subGQ in S containing T and let $\{c, d\} = \{a, b\}^\perp \cap Q$. Then $r_x = r_a r_b r_c r_d = r_x r_y r_c r_d$. So $r_y r_c r_d = 1$. By Corollary 3.6, $\{c, d, y\}$ is a complete 3-arc. Since $b \in \{c, d\}^\perp$, it follows that $b \in \{c, d, y\}^\perp$, a contradiction to that $b \nsim y$. So $x \sim b$. A similar argument shows that $y \sim a$. □

Proposition 3.9 *Let $K = R^* \setminus R_\psi$. Each element of K is of the form $r_y r_z$ for some $y \nsim z$ in P, except when $(t, |R|) = (2, 2^5)$. In this case, exactly one element, say α, of K can't be expressed in this way. Moreover, $\alpha = r_u r_v r_w$ for every complete 3-arc $\{u, v, w\}$ of S.*

Proof Since K is empty when $(t, |R|) = (2, 2^4)$, we assume that $(t, |R|) = (1, 2^4)$, $(2, 2^5)$ or $(4, 2^6)$. Fix $a, b \in P$ with $a \nsim b$. Then $r_a r_b \in K$ (Proposition 3.3). Let A and B be as in Lemma 3.8, and set

$$C = \{r_a r_b r_x : \{a, b, x\} \text{ is a 3-arc which is incomplete if } t = 2\}.$$

By Proposition 3.3, $A \subseteq K$ and $B \subseteq K$ and by Lemma 3.7, $C \subseteq K$. Each element of C corresponds to a 3-arc which is contained in a (2,1)-subGQ of S. Let $r_a r_b r_x \in C$ and Q be the (2,1)-subGQ of S containing the 3-arc $\{a, b, x\}$. If $\{a, b\}^\perp \cap Q = \{p, q\}$, then $r_{a*p} r_{b*q} = r_x$ implies that $r_a r_b r_x = r_p r_q$. Thus, every element of C can be expressed in the required form.

By Proposition 3.3, $A \cap C$ and $B \cap C$ are empty. By Lemma 3.8, $|A \cap B| = t + 2$. Then an easy count shows that

$$|A \cup B \cup C| = \begin{cases} 10t - 4 & \text{if } t = 1 \text{ or } 4 \\ 10t - 5 & \text{if } t = 2 \end{cases}.$$

So $K = A \cup B \cup C$ if $t = 1$ or 4, and $K \setminus (A \cup B \cup C)$ is a singleton if $t = 2$. This proves the proposition for $t = 1, 4$ and tells that if $(t, |R|) = (2, 2^5)$, then at most one element of K can't be written in the desired form.

Now, let $(t, |R|) = (2, 2^5)$ and $T = \{u, v, w\}$ be a complete 3-arc of S. By Lemma 3.7, $\alpha = r_u r_v r_w \in K$. Suppose that $\alpha = r_x r_y$ for some $x, y \in P$. Then $x \nsim y$ by Lemma 3.7 and $\{x, y\} \cap T = \Phi$ by Proposition 3.3. Suppose that $x \in T^\perp$ and Q be the $(2, 1)$-subGQ of S generated by $\{x, u, v, y\}$. Since $w \notin Q$ and $r_w = r_u r_v r_x r_y$, it follows that $|R| = 2^4$, a contradiction. So, $x \notin T^\perp$. Similarly, $y \notin T^\perp$. Thus, each of x and y is collinear with exactly one point of T. Let $x \sim u$. Then $y \nsim x * u$, since $x * u \in T^{\perp}$ and $\alpha = r_x r_y$. Let U be the (2,1)-subGQ of S generated by $\{u, x, y, v\}$. Note that $y \sim u$ in U. Let z be the unique point in U such that $\{u, v, z\}$ is a 3-arc of U. Then $r_z = r_x r_y r_u r_v = r_w$. Since $w \neq z$ (in fact, $w \notin U$), this is a contradiction to the faithfulness of ψ. Thus, α can't be expressed as $r_x r_y$ for any x, y in P. This, together with the last sentence of the previous paragraph, implies that α is independent of the complete 3-arc T of S. □

4 Initial results

Let $S = (P, L)$ be a slim dense near hexagon and (R, ψ) be a non-abelian representation of S. For $x \in P$ and $y \in \Gamma_{\leq 2}(x)$, $[r_x, r_y] = 1$: if $d(x, y) = 2$, we apply Proposition 3.1 to the restriction of ψ to the quad $Q(x, y)$. From ([12], Theorem 2.9, see Example 2.2 of [12]) applied to S, we have

Proposition 4.1 *The following hold*:

(*i*) *For* $x, y \in P$, $[r_x, r_y] \neq 1$ *if and only if* $d(x, y) = 3$. *In that case*, $\langle r_x, r_y \rangle$ *is a dihedral group* 2_+^{1+2} *of order* 8.
(*ii*) R *is a finite* 2-*group of exponent* 4, $|R'| = 2$ *and* $R' = \Phi(R) \subseteq Z(R)$.
(*iii*) $r_x \notin Z(R)$ *for each* $x \in P$ *and* ψ *is faithful.*

We write $R' = \langle \theta \rangle$ throughout. Since R' is of order two, Lemma 1.2 implies

Corollary 4.2 $|R| \leq 2^{1+\dim V(S)}$.

Proposition 4.3 *R is a central product $E \circ Z(R)$ of an extraspecial 2-subgroup E of R and $Z(R)$.*

Proof We consider $V = R/R'$ as a vector space over F_2. The map $f : V \times V \longrightarrow F_2$, taking (xZ, yZ) to 0 or 1 accordingly $[x, y] = 1$ or not, is a symplectic bilinear form on V. This form is non-degenerate if and only if $R' = Z(R)$. Let W be a complement in V of the radical of f and E be its inverse image in R. Then E is extraspecial and the proposition follows. □

From Proposition 4.3 it follows that the universal representation group of S is a central product of an extraspecial 2-group and an abelian 2-group of exponent at most 4.

Corollary 4.4 *Let M be an abelian subgroup of R of order 2^m intersecting $Z(R)$ trivially. Then $|R| \geq 2^{2m+1}$. Equality holds if and only if R is extraspecial and M is a maximal abelian subgroup of R intersecting $Z(R)$ trivially.*

The following lemma is useful for us.

Lemma 4.5 *Let $x \in P$ and $Y \subseteq \Gamma_3(x)$. Then $[r_x, \prod_{y \in Y} r_y] = 1$ if and only if $|Y|$ is even.*

Proof Since $R' \subseteq Z(R)$, $[r_x, \prod_{y\in Y} r_y]$ is well-defined (though $\prod_{y\in Y} r_y$ depends on the order of multiplication). Let $y, z \in \Gamma_3(x)$ be distinct. The subgraph of $\Gamma(P)$ induced on $\Gamma_3(x)$ is connected (see [2], Corollary to Theorem 3, p. 156). Let $y = y_0, y_1, \ldots, y_k = z$ be a path in $\Gamma_3(x)$. Then $r_y r_z = \prod r_{y_i * y_{i+1}}$ $(0 \le i \le k-1)$. Since $d(x, y_i * y_{i+1}) = 2$, $[r_x, r_y r_z] = 1$. Now, the result follows from Theorem 4.1(i). □

Notation 4.6 *For a quad Q in S, we denote by M_Q the elementary abelian 2-subgroup of R generated by $\psi(Q)$.*

Proposition 4.7 *Let Q be a quad in S and $M_Q \cap Z(R) \neq \{1\}$. Then Q is of type $(2,2)$, $|M_Q| = 2^5$ and $M_Q \cap Z(R) = \{1, r_a r_b r_c\}$ for every complete 3-arc $\{a, b, c\}$ of S.*

Proof Suppose that $M_Q \cap Z(R) \neq \{1\}$ and $1 \neq m \in M_Q \cap Z(R)$. Then $m \neq r_x$ for each $x \in P$ (Proposition 4.1(iii)). If Q is of type (2,1) or (2,4), then by Proposition 3.9, $m = r_y r_z$ for some $y, z \in Q$, $y \nsim z$. Choose $w \in P \setminus Q$ with $w \sim y$. Then $[r_w, r_z] = [r_w, r_y r_z] = [r_w, m] = 1$. But $d(w, z) = 3$ by Theorem 1.3(i), a contradiction to Proposition 4.1(i).

So Q is a (2,2)-GQ. If $|M_Q| = 2^4$, then $M_Q^* = \{r_x : x \in Q\}$ and $m = r_x \in Z(R)$ for some $x \in Q$, contradicting Proposition 4.1(iii). So $|M_Q| = 2^5$. Now, either $m = r_u r_v$ for some $u, v \in Q, u \nsim v$ or $m = r_a r_b r_c$ for every complete 3-arc $\{a, b, c\}$ of Q (Proposition 3.9). The above argument again implies that the first possibility does not occur. □

Proposition 4.8 *Let Q and Q' be two disjoint big quads in S of type $(2, t_2)$, $t_2 \neq 2$. Then $M_Q \cap M_{Q'} = \{1\}$.*

Proof Suppose that $M_Q \cap M_{Q'} \neq \{1\}$ and $1 \neq m \in M_Q \cap M_{Q'}$. Assume that $m = r_x$ for some $x \in Q$. Choose a point $w \in Q'$ with $d(x, w) = 3$. Then $[r_x, r_w] = [m, r_w] = 1$, since $M_{Q'}$ is abelian. This contradicts Proposition 4.1(i).

So, $m \neq r_x$ for each $x \in P$. Since Q is of type (2,1) or (2,4), $m = r_y r_z$ for some $y, z \in Q$ with $y \nsim z$ (Proposition 3.9). Choose $w \in Q'$ with $w \sim y$. This is possible since Q' is big. Then $d(w, z) = 3$ and $[r_w, r_z] = [r_w, r_y r_z] = [r_w, m] = 1$, again a contradiction to Proposition 4.1(i). □

Proposition 4.9 *Let Q be a quad in S of type $(2,2)$. Then Q is ovoidal if and only if $|M_Q| = 2^5$ and $M_Q \cap Z(R) = \{1\}$.*

Proof First, assume that Q is ovoidal and let $z \in P \setminus Q$ be such that the pair (z, Q) is ovoidal. Let $\mathcal{O}_z = \{x_1, \cdots, x_5\}$ be the ovoid of Q defined as in Theorem 1.3(ii). If $|M_Q| = 2^4$, then for the complete 3-arc $\{x_1, x_2, y\}$ of Q containing x_1 and x_2, $d(y, z) = 3$ and $r_{x_1} r_{x_2} r_y = 1$ (Proposition 3.5). But $[r_z, r_y] = [r_z, r_{x_1} r_{x_2} r_y] = 1$, a contradiction to Proposition 4.1(i). So $|M_Q| = 2^5$. Suppose that $M_Q \cap Z(R) \neq \{1\}$ and $1 \neq m \in M_Q \cap Z(R)$. By Proposition 4.7, $m = r_a r_b r_c$ for every complete 3-arc $\{a, b, c\}$ of Q. In particular, for the complete 3-arc $\{x_1, x_2, y\}$ of Q containing x_1 and x_2, the above argument leads to a contradiction. So $M_Q \cap Z(R) = \{1\}$.

Now, assume that $|M_Q| = 2^5$ and $M_Q \cap Z(R) = \{1\}$. Suppose that Q is classical and let $\{a, b, c\}$ be a complete 3-arc of Q. Then, by Proposition 3.5, $r_a r_b r_c \neq 1$. Since (x, Q) is classical for each $x \in P \setminus Q$, either each of a, b, c is at a distance two from x or exactly two of them are at a distance three from x. In either case, $[r_x, r_a r_b r_c] = 1$ (see Lemma 4.5). So $1 \neq r_a r_b r_c \in M_Q \cap Z(R)$, a contradiction. □

5 Proof of Theorem 1.6

Let $S = (P, L)$ be a slim dense near hexagon and let (R, ψ) be a non-abelian representation of S. By Proposition 4.1(ii), R is a finite 2-group of exponent 4. By Corollary 4.2, $|R| \leq 2^{1+dim V(S)}$. For each of the near hexagons in Theorem 1.4, except (vi), we find an elementary abelian subgroup of R of order 2^ξ, $2\xi = NPdim(S)$, intersecting $Z(R)$ trivially. Then by Corollary 4.4, $|R| \geq 2^{1+2\xi}$ and $R = 2_+^{1+2\xi}$ if equality holds. For the near hexagon (vi) we prove in Subsection 5.3 that $R = 2_-^{1+2\xi}$, thus completing the proof of Theorem 1.6.

5.1 The near hexagons (vii) to (xi)

Let $S = (P, L)$ be one of the near hexagons (vii) to (xi) and Q be a big quad in S. Set $M = M_Q$. Then, by Proposition 4.7, $M \cap Z(R) = \{1\}$ and $|M| = 2^4$ or 2^6 according as Q is of type (2,1) or (2,4). If Q is of type (2,2), then $|M| = 2^4$ or 2^5. Also, if $|M| = 2^5$, then $|M \cap Z(R)| = 2$ because Q is classical (Propositions 4.7 and 4.9). Thus, R has an elementary abelian subgroup of order 2^ξ intersecting $Z(R)$ trivially.

5.2 The near hexagons (i) to (v)

Let $S = (P, L)$ be one of the near hexagons (i) to (v). Fix $a \in P$ and $b \in \Gamma_3(a)$. Let $l_1, \cdots, l_{t+1}$ be the lines containing a, x_i be the point in l_i with $d(b, x_i) = 2$ and $A = \{x_i : 1 \leq i \leq t+1\}$. For a subset X of A, we set $T_X = \{r_x : x \in X\}$, $M_X = \langle T_X \rangle$ and $M = \langle r_b \rangle M_X$. Then M_X and M are elementary abelian 2-subgroups of R.

Proposition 5.1 *Let X be a subset of A such that*

(i) $M_X \cap Z(R) = \{1\}$,
(ii) *T_X is linearly independent.*

Then, $|M| = 2^{|X|+1}$ *and* $M \cap Z(R) = \{1\}$. *In particular,* $|R| \geq 2^{2|X|+3}$.

Proof By (ii), $2^{|X|} \leq |M| \leq 2^{|X|+1}$. If $|M| = 2^{|X|}$, then r_b can be expressed as a product of some of the elements r_x, $x \in X$. Since $[r_a, r_x] = 1$ for $x \in X$, it follows that $[r_a, r_b] = 1$, a contradiction to Proposition 4.1(i). So $|M| = 2^{|X|+1}$. Suppose that $M \cap Z(R) \neq \{1\}$ and $1 \neq z \in M \cap Z(R)$. Let $z = \prod_{y \in X \cup \{b\}} r_y^{i_y}$, $i_y \in \{0, 1\}$. Since $[r_x, z] = 1$, $i_b = 0$ by the above argument. It follows that $z \in M_X$, a contradiction to (i). So $M \cap Z(R) = \{1\}$.

By Corollary 4.4, $|R| \geq 2^{2(|X|+1)+1} = 2^{2|X|+3}$. □

A subset X of A is *good* if (i) and (ii) of Proposition 5.1 hold. In the rest of this Section, we find good subsets of A of size $(\xi - 1)$, thus completing the proof of Theorem 1.6 for the near hexagons (i) to (v). The next Lemma gives a necessary condition for a subset of A to be good.

Lemma 5.2 *Let X be a subset of A which is not good, $\alpha \in M_X \cap Z(R)$ (possibly $\alpha = 1$) and*

$$\alpha = \prod_{x_k \in X} r_{x_k}^{i_k} \tag{1}$$

where $i_k \in \{0, 1\}$. Set $B = \{k : x_k \in X\}$, $B' = \{k \in B : i_k = 1\}$ and $A_{i,j} = \{k \in B' : x_k \in Q(x_i, x_j)\}$ for $1 \leq i \neq j \leq t+1$. Assume that B' is non-empty when $\alpha = 1$. Then

(i) $|B'| \geq 3$,
(ii) *$|B'|$ is even if and only if $|A_{i,j}|$ is even.*

Proof (i) $|B'| \geq 2$ because $r_{x_k} \notin Z(R)$ for each k (Proposition 4.1(iii)). If $|B'| = 2$, then $r_x r_y = \alpha$ for some pair of distinct $x, y \in X$. Since ψ is faithful and r_x, r_y are involutions, $\alpha \neq 1$. For the quad $Q = Q(x, y)$, $1 \neq \alpha \in M_Q \cap Z(R)$. By Proposition 4.7, Q is a $(2,2)$-GQ and $r_a r_b r_c = \alpha$ for each complete 3-arc $\{a, b, c\}$ of Q. In particular, if $\{x, y, w\}$ is the complete 3-arc of Q containing x and y, then $r_x r_y r_w = \alpha$. It follows that $r_w = 1$, a contradiction. So $|B'| \geq 3$.

(ii) Let $w \in Q(x_i, x_j)$ and $w \nsim a$. For each $m \in B'_{i,j} = B' \setminus A_{i,j}$, $x_m \sim a$ and $x_m \notin Q(x_i, x_j)$. So $d(w, x_m) = 3$. Now, $[r_w, \prod_{m \in B'_{i,j}} r_{x_m}] = [r_w, \prod_{m \in B'} r_{x_m}] = [r_w, \alpha] = 1$, since $\alpha \in Z(R)$. So $|B'_{i,j}|$ is even by Lemma 4.5. This implies (ii). □

In what follows, for any subset X of A which is not good, B' is defined relative to an expression as in (1) for an arbitrary but fixed element of $M_X \cap Z(R)$. Any quad Q in S containing the point a is determined by any two distinct points x_i and x_j of A that are contained in Q. In that case we sometimes denote by A_Q the set $A_{i,j}$ defined in Lemma 5.2.

5.2.1 The near hexagon (i)

There are 7 quads in S containing the point $x_1 \in A$. This partitions the 14 points $(\neq x_1)$ of A, say

$$\{x_2, x_3\} \cup \{x_4, x_5\} \cup \{x_6, x_7\} \cup \{x_8, x_9\} \cup \{x_{10}, x_{11}\} \cup \{x_{12}, x_{13}\} \cup \{x_{14}, x_{15}\}.$$

Consider the quad $Q(x_{10}, x_{12})$. We may assume that $Q(x_{10}, x_{12}) \cap A = \{x_{10}, x_{12}, x_{15}\}$. Let $X = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{12}, x_{14}\}$. Then $|X| = 10$. We show that X is a good subset of A.

Assume otherwise. Let $C_1 = \{8, 10, 12, 14\}$ and $C_2 = B \setminus C_1$. For $k \in C_1$, $Q(x_1, x_k) \cap A = \{x_1, x_k, x_{k+1}\}$. So $A_{1,k} \subseteq \{k\}$. By Lemma 5.2(ii), either $C_1 \subseteq B'$

or $C_1 \cap B'$ is empty. If $C_1 \subseteq B'$, then $A_{1,14} = \{14\}$, $A_{10,12} = \{10, 12\}$ and by Lemma 5.2(ii), $|B'|$ would be both odd and even.

So $C_1 \cap B'$ is empty. Then $B' \subseteq C_2$. Since $A_{1,8}$ is empty, $|B'|$ is even. Choose $j \in B'$ (see Lemma 5.2(i)). Observe that there exists a $k \in \{8, \cdots, 15\}$ such that $Q(x_j, x_k) \cap \{x_i : i \in C_2\} = \{x_j\}$. Then $A_{j,k} = \{j\}$ and $|B'|$ is odd also, a contradiction.

5.2.2 *The near hexagon* (ii)

Let $X = \{x_i : 1 \leq i \leq 11\}$. Then $|X| = 11$. Also X is a good subset of A. Otherwise, for some $i, j \in B'$ with $i \neq j$ (see Lemma 5.2(i)), $A_{i,j} = \{i, j\}$ and $A_{i,12} = \{i\}$ and, by Lemma 5.2(ii), $|B'|$ would be both even and odd.

5.2.3 *The near hexagon* (iii)

Let $Q_1, \cdots, Q_5$ be the five (big) quads in S containing x_1 and a. Let

$$\begin{aligned} Q_1 \cap A &= \{x_1, x_2, x_3, x_4, x_5\}, \\ Q_2 \cap A &= \{x_1, x_6, x_7, x_8, x_9\}, \\ Q_3 \cap A &= \{x_1, x_{10}, x_{11}, x_{12}, x_{13}\}, \\ Q_4 \cap A &= \{x_1, x_{14}, x_{15}, x_{16}, x_{17}\}, \\ Q_5 \cap A &= \{x_1, x_{18}, x_{19}, x_{20}, x_{21}\}. \end{aligned}$$

Let $X = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{14}\}$. Then $|X| = 9$. We show that X is a good subset of A. Assume otherwise. Since $Q_5 \cap X$ is empty, A_{Q_5} is empty and, by Lemma 5.2(ii), $|B'|$ and $|A_Q|$ are even for each quad Q in S containing a. Since $A_{Q_3} \subseteq \{10\}$ and $|A_{Q_3}|$ is even, $10 \notin A_{Q_3}$ and so, $10 \notin B'$. This argument with Q_3 replaced by Q_4 shows that $14 \notin B'$. Since $A_{Q_2} \subseteq \{6, 7, 8\}$ and $|A_{Q_2}|$ is even, $j \notin B'$ for some $j \in \{6, 7, 8\}$. Since $|B'| \geq 3$ (Lemma 5.2(i)), $k \in B'$ for some $k \in \{2, 3, 4, 5\}$. Then, $A_{j,k} = \{k\}$, contradicting that $|A_{j,k}|$ is even.

5.2.4 *The near hexagon* (iv)

Let $Q_1, \cdots, Q_6$ be the six big quads in S containing the point a. Any two of these big quads meet in a line through a and any three of them meet only at $\{a\}$. Let

$$\begin{aligned} Q_1 \cap A &= \{x_1, x_2, x_3, x_4, x_5\}, \\ Q_2 \cap A &= \{x_1, x_6, x_7, x_8, x_9\}, \\ Q_3 \cap A &= \{x_2, x_6, x_{10}, x_{11}, x_{12}\}, \\ Q_4 \cap A &= \{x_3, x_7, x_{10}, x_{13}, x_{14}\}, \\ Q_5 \cap A &= \{x_4, x_8, x_{11}, x_{13}, x_{15}\}, \\ Q_6 \cap A &= \{x_5, x_9, x_{12}, x_{14}, x_{15}\}. \end{aligned}$$

Let $X = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{11}\}$. Then $|X| = 9$. We show that X is a good subset of A. Assume otherwise. Since $Q_6 \cap X$ is empty, A_{Q_6} is empty and, by Lemma 5.2(ii), $|B'|$ and $|A_Q|$ are even for every quad Q in S containing a. We first verify that for

$$(i, j, k) \in \{(1, 11, 14), (1, 12, 13), (2, 9, 13), (3, 6, 15), (4, 6, 14), (5, 6, 13)\},$$

$Q(x_i, x_j)$ is of type (2,2) and $Q(x_i, x_j) \cap A = \{x_i, x_j, x_k\}$. Since $A_{1,12} \subseteq \{1\}$ and $|A_{1,12}|$ is even, it follows that $1 \notin B'$. Similarly, considering $A_{2,9}$ and $A_{5,6}$, we conclude that $2 \notin B'$ and $6 \notin B'$. Since $6 \notin B'$, considering $A_{3,6}$ and $A_{4,6}$, we conclude that $3 \notin B'$ and $4 \notin B'$. Since $|B'| \geq 3$ is even, it follows that $B' = \{7, 8, 10, 11\}$ and so $A_{1,11} = \{11\}$, contradicting that $|A_{1,11}|$ is even.

5.2.5 *The near hexagon* (v)

Let Q_1, Q_2, Q_3 be the three big quads in S containing a. Their intersection is $\{a\}$ and any two of these big quads meet in a line through a. We may assume that

$$Q_1 \cap A = \{x_1, x_2, x_3, x_4, x_5\},$$
$$Q_2 \cap A = \{x_1, x_6, x_7, x_8, x_9\},$$
$$Q_3 \cap A = \{x_2, x_6, x_{10}, x_{11}, x_{12}\}.$$

Let $X = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{11}\}$. Then $|X| = 9$. We show that X is a good subset of A. Assume otherwise. We note that the quads $Q(x_r, x_k)$ are of type (2,2) in the following cases:

$$r = 1 \text{ and } k \in \{10, 11, 12\};\ r = 2 \text{ and } k \in \{7, 8, 9\};\ r = 6 \text{ and } k \in \{3, 4, 5\}.$$

Now, $A_{r,s} \subseteq \{r\}$ for $(r, s) \in \{(1, 12), (2, 9), (6, 5)\}$ because $x_s \notin X$. Considering $A_{1,12}$, we conclude that $10, 11 \notin B'$ in view of the following: $A_{1,12} \subseteq \{1\}$, $A_{1,k} \subseteq \{1, k\}$ for $k \in \{10, 11\}$ and the parity of $|B'|$ and $|A_{1,j}|$ are the same for all $j \neq 1$. Similarly, considering $A_{2,9}$ (respectively, $A_{6,5}$) we conclude that $7, 8 \notin B'$ (respectively, $3, 4 \notin B'$). Since $|B'| \geq 3$, it follows that $B' = \{1, 2, 6\}$. But $A_{5,9}$ is empty because $\{x_5, x_9, x_{12}\} \cap X$ and $\{10, 11\} \cap B'$ are empty. So $|B'|$ is even (Lemma 5.2(ii)), a contradiction.

5.3 The near hexagon (vi)

We consider this case separately because the technique of the previous subsection only yields $|R| \geq 2^{17}$ in this case.

Let $S = (P, L)$ be a slim dense near hexagon and Y be a proper subspace of S isomorphic to the near hexagon (vii). Big quads in Y (as well as in S) are of type (2,4). There are three pair-wise disjoint big quads in Y and any two of them generate Y. Fix two disjoint big quads Q_1 and Q_2 in Y. Let (R, ψ) be a non-abelian representation of S. Set $M = \langle \psi(Y) \rangle$ and $M_i = M_{Q_i}$ for $i = 1, 2$. Then $|M_i| = 2^6$ (Proposition 3.2(iii)), $M_i \cap Z(R) = \{1\}$ (Proposition 4.7), $M_1 \cap M_2 = \{1\}$ (Proposition 4.8). Since Y contains pairs of points at distance 3, (M, ψ) is a non-abelian representation of Y (see Proposition 4.1(i)). So, $M = 2_+^{1+12}$ with $M = M_1 M_2 R'$ (Theorem 1.6 for the near hexagon (vii)). Also, $R = M \circ N$, a central product of M and $N = C_R(M)$.

Let $\{i, j\} = \{1, 2\}$. For $x \in P \setminus Y$, we denote by x^j the unique point in Q_j at distance 1 from x. For $y \in Q_i$, let z_y denote the unique point in Q_j at distance 1 from y. For each $x \in P \setminus Y$, we can write $r_x = m_1^x m_2^x n_x$ for some $m_1^x \in M_1$, $m_2^x \in M_2$ and $n_x \in N$.

Proposition 5.3 *For* $x \in P \setminus Y$, $m_i^x = r_{z_{x^j}}$.

Proof Let $H_j = \langle r_w : w \in Q_j \cap x^{j\perp} \rangle \le M_j$. Then H_j is a maximal subgroup of M_j ([10], 4.2.4, p. 68) and $r_x \in C_R(H_1) \cap C_R(H_2)$. For all $h \in H_j$,

$$\big[m_i^x, h\big] = \big[m_1^x m_2^x n_x, h\big] = [r_x, h] = 1.$$

So $m_i^x \in C_{M_i}(H_j)$. Note that $C_{M_i}(H_j) = \langle r_{z_{xj}} \rangle$, a subgroup of order 2. If $m_i^x = 1$, then $r_x = m_j^x n_x$ commutes with every element of M_j. In particular, $[r_x, r_y] = 1$ for every $y \in Q_j \cap \Gamma_3(x)$, a contradiction to Theorem 4.1(i). So $m_i^x = r_{z_{xj}}$. □

Propositions 5.3 implies that n_x is uniquely determined as $n_x = r_x(m_1^x m_2^x)^{-1}$.

Proposition 5.4 *For $x \in P \setminus Y$, n_x is an involution and $n_x \notin Z(R)$. In particular, $r_x \notin M$.*

Proof By Proposition 2.3, $d(z_{x^1}, z_{x^2}) = 2$. So $[m_1^x, m_2^x] = [r_{z_{x2}}, r_{z_{x1}}] = 1$ (Proposition 5.3). Now, $r_x^2 = 1$ implies $n_x^2 = 1$. We show that $n_x \neq 1$ and $n_x \notin Z(R)$. The quad $Q = Q(x^1, x^2)$ in S is of type (2,2) or (2,4) because x^1 and x^2 have at least three common neighbours x, z_{x^1} and z_{x^2}. Let U be the $(2,2)$-GQ in Q generated by $\{x^1, x^2, x, z_{x^1}, z_{x^2}\}$. If Q is of type (2,4), then $\langle \psi(U) \rangle$ is of order 2^5 (Corollary 3.4). If Q is of type (2,2), then $U = Q$ is ovoidal because it is not a big quad. So $\langle \psi(U) \rangle$ is of order 2^5 (Proposition 4.9). Therefore, $r_a r_b r_c \neq 1$ for every complete 3-arc $\{a, b, c\}$ of U (Proposition 3.5). In particular, $n_x = r_x r_{z_{x1}} r_{z_{x2}} \neq 1$ for the complete 3-arc $\{x, z_{x^1}, z_{x^2}\}$ of U. Now, applying Proposition 4.7 (respectively, Proposition 4.9) when Q is of type (2,4) (respectively, of type (2,2)), we conclude that $n_x \notin Z(R)$. □

Proposition 5.5 *Let Q be a big quad in S disjoint from Y and $x, y \in Q$. Then*:

(i) *$[n_x, n_y] = 1$ if and only if $x = y$ or $x \sim y$;*
(ii) *There is a unique line $l_x = \{x, y, x * y\}$ in Q containing x such that $n_{x*y} = n_x n_y$. For any other line $l = \{x, z, x * z\}$ in Q, $n_{x*z} = n_x n_z \theta$.*

Proof (i) Let $x \sim y$. By Propositions 2.5 and 5.3, $[m_2^x, m_1^y] = [m_1^x, m_2^y] = 1$ or θ. Then $[n_x, n_y] = [m_1^x m_2^x n_x, m_1^y m_2^y n_y] = [r_x, r_y] = 1$.

Now, assume that $x \nsim y$. By Propositions 2.6 and 5.3, $\{[m_1^x, m_2^y], [m_2^x, m_1^y]\} = \{1, \theta\}$. Since $[r_x, r_y] = 1$, it follows that $[n_x, n_y] = \theta \neq 1$.

(ii) Let $x \in Q$ and l_x be the line in Q containing x which corresponds to the line $x^j z_{x^i}$ in Q_j. This is possible by Lemma 2.2. For $u, v \in l_x$, $d(z_{u^j}, z_{v^i}) \le 2$ (Proposition 2.5). So $[m_i^u, m_j^v] = 1$. Then $r_{u*v} = (m_1^u m_1^v)(m_2^u m_2^v)(n_u n_v)$. So $n_{u*v} = n_u n_v$. Let l be a line $(\neq l_x)$ in Q containing x. For $y \neq w$ in l, $[m_2^y, m_1^w] = \theta$ because $d(z_{y^1}, z_{w^2}) = 3$ (Proposition 2.5). Then $r_{y*w} = (m_1^y m_2^y n_y)(m_1^w m_2^w n_w) = (m_1^y m_1^w)(m_2^y m_2^w) n_y n_w \theta$ and so $n_{y*w} = n_y n_w \theta$. □

Corollary 5.6 *Let Q be as in Proposition 5.5 and $I_2(N)$ be the set of involutions in N. Define δ from Q to $I_2(N)$ by $\delta(x) = n_x$, $x \in Q$. Then*

(i) *$[\delta(x), \delta(y)] = 1$ if and only if $x = y$ or $x \sim y$.*
(ii) *δ is one-to-one.*

(*iii*) *There exists a spread T in Q such that for $x, y \in Q$ with $x \sim y$,*

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in T \\ \delta(x)\delta(y)\theta & \text{if } xy \notin T \end{cases}.$$

Proof (i) and (iii) follow from Proposition 5.5. We now prove (ii). Let $\delta(x) = \delta(y)$ for $x, y \in Q$. By (i), $x = y$ or $x \sim y$. If $x \sim y$, then $r_{x*y} = r_x r_y = (m_1^x m_1^y)(m_2^x m_2^y)\alpha \in M$, where $\alpha = [m_2^x, m_1^y] \in R'$. But this is not possible as $x * y \notin Y$ (Proposition 5.4). So $x = y$. □

Now, let $S = (P, L)$ be the near hexagon (vi). Then big quads in S are of type (2,4). We refer to ([1], p. 363) for the description of the corresponding Fischer space on the set of 18 big quads in S. This set partitions into two families F_1 and F_2 of size 9 each such that each F_i defines a partition of the point set P of S. Let U_i, $i = 1, 2$, be the linear space whose point set is F_i. If Q_1 and Q_2 are two distinct points of U_i, then the line containing them is $\{Q_1, Q_2, Q_1 * Q_2\}$, where $Q_1 * Q_2$ is defined as in Lemma 2.2. Then U_i is an affine plane of order 3.

Consider the family F_1. Fix a line $\{Q_1, Q_2, Q_1 * Q_2\}$ in U_1 and set $Y = Q_1 \cup Q_2 \cup Q_1 * Q_2$. Then Y is a subspace of S isomorphic to the near hexagon (vii). Fix a big quad Q in U_1 disjoint from Y. Let the subgroups M and N of R be as in the beginning of this subsection. Then $|N| \le 2^7$ because $|R| \le 2^{1+dimV(S)} = 2^{19}$. We show that $N = 2_-^{1+6}$. This would prove Theorem 1.6 in this case.

Let $\{a_1, a_2, b_1, b_2\}$ be a quadrangle in Q, where $a_1 \nsim a_2$ and $b_1 \nsim b_2$. Let δ be as in Corollary 5.6. The subgroup $\langle \delta(a_1), \delta(a_2), \delta(b_1), \delta(b_2)\rangle$ of R is isomorphic to $H = \langle \delta(a_1), \delta(a_2)\rangle \circ \langle \delta(b_1), \delta(b_2)\rangle$. We write $N = H \circ K$ where $K = C_N(H)$. Then $|K| \le 2^3$. There are three more neighbours, say w_1, w_2, w_3, of a_1 and a_2 in Q different from b_1 and b_2. We can write

$$\delta(w_i) = \delta(a_1)^{i_1}\delta(a_2)^{i_2}\delta(b_1)^{j_1}\delta(b_2)^{j_2}k_i$$

for some $k_i \in K$, where $i_1, i_2, j_1, j_2 \in \{0, 1\}$. By Corollary 5.6$(i)$, $[\delta(w_i), \delta(a_r)] = 1$ and $[\delta(w_i), \delta(b_r)] \neq 1$ for $r = 1, 2$. This implies that $i_1 = i_2 = 0$ and $j_1 = j_2 = 1$. So $\delta(w_i) = \delta(b_1)\delta(b_2)k_i$. In particular, k_i is of order 4. Since $[\delta(w_i), \delta(w_j] \neq 1$ for $i \neq j$, it follows that $[k_i, k_j] \neq 1$. Thus, K is non-abelian of order 8 and k_1, k_2, k_3 are three pair-wise distinct elements of order 4 in K. So K is isomorphic to Q_8 and $N = 2_-^{1+6}$.

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