J Algebr Comb (2008) 28: 481–493
DOI 10.1007/s10801-007-0112-1

Extended affine Weyl groups of type A_1

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Received: 12 July 2007 / Accepted: 4 December 2007 / Published online: 13 December 2007

Abstract It is known that elliptic Weyl groups, extended affine Weyl groups of nullity 2, have a finite presentation called the generalized Coexter presentation. Similar to the finite and affine case this presentation is obtained by assigning a Dynkin diagram to the root system. Then there is a prescription to read the generators and relations from the diagram. Recently a similar presentation is given for simply laced extended affine Weyl groups of nullity 3 and rank>1. Employing a new method, we complete this work by giving a similar presentation for nullity 3 extended affine Weyl groups of type A_1.

Keywords Dynkin diagram · Weyl groups · Root system

0 Introduction

In 1985, K. Saito [10] introduced axiomatically the notion of an extended affine root system and considered the classification of extended affine root systems of nullity 2, which are the root systems equipped with a positive semi-definite quadratic form where the radical of the form has dimension two. Since extended affine root systems of nullity 2 are associated to the elliptic singularities, they are also called elliptic root systems.

Extended affine root systems also arise as the root systems of a class of infinite dimensional Lie algebras called *extended affine Lie algebras*. A systematic study of extended affine Lie algebras and their root systems is given in [1], in particular a set

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of axioms, different from those given by Saito [10], is extracted from algebras for the corresponding root systems. In [3], the relation between axioms of [10] and [1] for extended affine root systems is clarified.

Let R be an extended affine root system of nullity ν and $\mathcal{V}$ be the real span of R which by definition is equipped with a positive semi-definite bilinear form. Then the Weyl group of R is the group generated by reflections based on nonisotropic roots of R, considered as a subgroup of the orthogonal group of a hyperbolic extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ (see Section 2). In [11, 12], K. Saito and T. Takebayashi assigned to each elliptic root system a Dynkin diagram and described from the viewpoint of a generalization of Coxeter groups the generators and their relations of elliptic Weyl groups. Recently, T. Takebayashi [13] has extended this result to nullity 3 simply laced extended affine root systems (except type A_1). In this work we consider type A_1 and complete Takebayashi's result for simply laced nullity 3 extended affine Weyl groups. To achieve this, we have employed a new method using a recently given finite presentation for reduced extended affine Weyl groups (see [6] and [7]). This approach considers nullities $\nu \leq 3$ at once, in particular it provides a new proof for the elliptic Weyl group of type A_1.

We should mention that the study of extended affine Weyl groups was initiated by R. V. Moody and Z. Shi in the article [9], in which the authors studied the structure of those extended affine Weyl groups which arise as the Weyl groups of toroidal Lie algebras (see also Remark 2.2).

We refer the interested reader to [2–4] for the study of extended affine Weyl groups in general and to [8] for the extended affine Weyl groups of type A_1.

1 Extended affine root systems of type A_1 and nullity $\nu \leq 3$

Throughout this work we assume R is an extended affine root system of type A_1 and nullity $\nu \leq 3$. That is, R is a spanning subset of a $(\nu + 1)$-dimensional real vector space $\mathcal{V} = \mathbb{R}\epsilon + \sum_{r=1}^{\nu} \mathbb{R}\sigma_r$ of the form

$$R = R(A_1, S) = (S + S) \cup (\pm\epsilon + S) \tag{1.1}$$

where $\mathcal{V}$ is equipped with the positive semi-definite bilinear form determined by $(\epsilon, \epsilon) = 2$ and $(\epsilon, \sigma_r) = (\sigma_r, \sigma_s) = 0$, $1 \leq r, s \leq \nu$, and S is a semilattice of rank ν in $\mathcal{V}^0 := \sum_{r=1}^{\nu} \mathbb{R}\sigma_r$. For the details about extended affine root systems (and semilattices) we refer the reader to [1, Chapter II], in particular, we will use the notation and concepts introduced there without further explanation. It is known that we may assume, $\mathcal{B} = \{\sigma_1, \dots, \sigma_\nu\} \subseteq S$. Let $\Lambda = \sum_{r=1}^{\nu} \mathbb{Z}\sigma_r$. As it is shown in [6, §2], there is a unique set, denoted $\mathrm{supp}(S)$, consisting of subsets of $\{1, \dots, \nu\}$ such that $\emptyset \in \mathrm{supp}(S)$ and

$$S = \bigcup_{J \in \mathrm{supp}(S)} (\tau_J + 2\Lambda), \quad \text{where} \quad \tau_J := \sum_{r \in J} \sigma_r. \tag{1.2}$$

(If $J = \emptyset$ we set by convention $\sum_{r \in J} \sigma_r = 0$). Since $\mathcal{B} \subseteq S$, we have $\{r\} \in \mathrm{supp}(S)$ for all $1 \leq r \leq \nu$. The collection $\mathrm{supp}(S)$ is called the *supporting class of S* (with

Table 1 Supporting classes of semilattices for $\nu \leq 3$

ν	index	supp(S)
0	0	$\{\emptyset\}$
1	1	$\{\emptyset, \{1\}\}$
2	2	$\{\emptyset, \{1\}, \{2\}\}$
	3	$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
3	3	$\{\emptyset, \{1\}, \{2\}, \{3\}\}$
	4	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$
	5	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$
	6	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$
	7	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

respect to $\mathcal{B}$). We call the integer $\mathrm{ind}(S) := |\mathrm{supp}(S)| - 1$, *index* of S. By [1, Proposition II.4.2 and Table II.4.5] and [5, Proposition 1.12], we may assume (and we do) that R is of the form $R(A_1, S)$ where S is one of the semilattices of rank ν in Λ given in Table 1, according to their supporting classes.

We attach to R a Dynkin diagram $\Gamma(R)$ as follows:

(**i**) The set of nodes of $\Gamma(R)$ is the set

$$|\Gamma(R)| := \{n_J \epsilon + \tau_J \mid J \in \mathrm{supp}(S)\}, \tag{1.3}$$

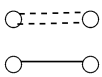
where

$$n_J = \begin{cases} -1 & \text{if } 1 \in J \\ 1 & \text{if } 1 \notin J. \end{cases}$$

For simplicity we write n_r instead of n_J if $J = \{r\}$, so $n_1 = -1$ and $n_2 = n_3 = 1$. For example if $R = R(A_1, S)$ where S has nullity 3 and index 6, then from Table 1 we have

$$|\Gamma(R)| = \{-\epsilon + \sigma_1, \epsilon + \sigma_2, \epsilon + \sigma_3, -\epsilon + \sigma_1 + \sigma_2, -\epsilon + \sigma_1 + \sigma_3, \epsilon + \sigma_2 + \sigma_3\}.$$

(**ii**) Bonds among two distinct nodes α, β of $\Gamma(R)$ are inserted according to:



if $(\alpha, \beta) = 2$

if $(\alpha, \beta) = -2$

2 Extended affine Weyl groups of type A_1

We keep all the notation as in the previous section. Let $\tilde{\mathcal{V}} := \mathbb{R}\epsilon \oplus \sum_{r=1}^{\nu} \mathbb{R}\sigma_r \oplus \sum_{r=1}^{\nu} \mathbb{R}\lambda_r$, a ν-dimensional extension of $\mathcal{V}$. Now we extend the form $(\cdot, \cdot)$ on $\mathcal{V}$ to a non-degenerate form, denoted again by $(\cdot, \cdot)$, on $\tilde{\mathcal{V}}$ as follows:

- $(\epsilon, \lambda_r) = (\lambda_r, \lambda_s) := 0, \quad 1 \leq r, s \leq \nu,$
- $(\sigma_r, \lambda_s) := \delta_{r,s}, \quad 1 \leq r, s \leq \nu.$

For a subset T of $\tilde{\mathcal{V}}$, we denote by $T^\times$ the set of $\alpha \in T$ with $(\alpha,\alpha) \neq 0$. Let $\mathrm{O}(\tilde{\mathcal{V}})$ be the group of orthogonal transformations on $\tilde{\mathcal{V}}$ with respect to $(\cdot,\cdot)$. Then the *extended affine Weyl group* $\mathcal{W}$ of R is the subgroup of $\mathrm{O}(\tilde{\mathcal{V}})$ generated by reflections w_α, $\alpha \in R^\times$, defined by

$$w_\alpha(u) = u - (u,\alpha)\alpha, \quad (u \in \tilde{\mathcal{V}}).$$

For $\alpha \in \mathcal{V}$ and $\sigma \in \mathcal{V}^0$ we define a linear map $T_\alpha^\sigma \in \mathrm{End}(\tilde{\mathcal{V}})$ by

$$T_\alpha^\sigma(u) := u + (\alpha,u)\sigma - (\sigma,u)\alpha - \frac{(\alpha,\alpha)}{2}(\sigma,u)\sigma \quad (u \in \tilde{\mathcal{V}}). \tag{2.1}$$

Then from [6, Lemma 1.1], for $w \in \mathrm{O}(\tilde{\mathcal{V}})$, $\beta,\gamma \in \mathcal{V}$, $\sigma,\delta \in \mathcal{V}^0$ and $\alpha \in \mathcal{V}^\times$, we have

$$w_\alpha^2 = 1 \quad \text{and} \quad w w_\alpha w^{-1} = w_{w(\alpha)}, \tag{2.2}$$

and

$$\begin{aligned} &T_\alpha^\sigma = w_{\alpha+\sigma} w_\alpha, \qquad w T_\beta^\delta w^{-1} = T_{w(\beta)}^\delta, \quad T_\sigma^\delta \in Z(\mathrm{O}(\tilde{\mathcal{V}})),\\ &[T_\alpha^\sigma, T_\beta^\delta] = T_\sigma^{(\alpha,\beta)\delta}, \quad T_{\beta+\gamma}^\sigma = T_\beta^\sigma T_\gamma^\sigma \quad \text{and} \quad T_\beta^{\sigma+\delta} = T_\beta^\sigma T_\beta^\delta T_\delta^{(\beta,\beta)\sigma/2}. \end{aligned} \tag{2.3}$$

(Here $[x,y]$ denotes the commutator $x^{-1}y^{-1}xy$ of two group elements x, y, and $Z(\cdot)$ denotes the center of a group.)

Lemma 2.1 *If $\alpha,\beta \in R^\times$ and $\sigma,\delta \in \mathcal{V}^0$ such that $\alpha+\sigma, \alpha+\delta, \beta+\sigma \in R$, then the following elements of $\mathcal{W}$ are central*:

(*i*) $w_\alpha w_{\alpha+\sigma} w_\beta w_{\beta+\sigma}$, *and* $w_{\alpha+\sigma} w_\alpha w_{\beta+\sigma} w_\beta$ *with* $\alpha+\beta \in \mathcal{V}^0$,

(*ii*) $w_{\alpha+\sigma} w_\alpha w_\beta w_{\beta+\sigma}$, *with* $\alpha-\beta \in \mathcal{V}^0$,

(*iii*) $(w_\alpha w_{\alpha+\sigma} w_{\alpha+\delta})^2$.

Moreover, $[w_{\alpha+\delta} w_\alpha, w_{\alpha+\sigma} w_\alpha] = (w_\alpha w_{\alpha+\sigma} w_{\alpha+\delta})^2 = (w_\alpha w_{-\alpha+\sigma} w_{-\alpha+\delta})^2$.

Proof Using (2.3) we have

$$\begin{aligned} w_{\alpha+\sigma} w_\alpha w_{\beta+\sigma} w_\beta &= T_\alpha^\sigma T_\beta^\sigma = T_{\alpha+\beta}^\sigma,\\ w_{\alpha+\sigma} w_\alpha w_\beta w_{\beta+\sigma} &= T_\alpha^\sigma (T_\beta^\sigma)^{-1} = T_{\alpha-\beta}^\sigma,\\ w_\alpha w_{\alpha+\sigma} w_\beta w_{\beta+\sigma} &= (T_\alpha^\sigma)^{-1}(T_\beta^\sigma)^{-1} = T_{-\alpha-\beta}^\sigma = (T_{\alpha+\beta}^\sigma)^{-1} \end{aligned}$$

and

$$\begin{aligned} (w_\alpha w_{\alpha+\sigma} w_{\alpha+\delta})^2 &= (w_\alpha w_{\alpha+\sigma} w_{\alpha+\delta})(w_\alpha w_{\alpha+\sigma} w_{\alpha+\delta})\\ &= (w_\alpha w_{\alpha+\sigma})(w_{\alpha+\delta} w_\alpha)(w_{\alpha+\sigma} w_\alpha)(w_\alpha w_{\alpha+\delta})\\ &= T_{-\alpha}^\sigma T_\alpha^\delta T_\alpha^\sigma T_{-\alpha}^\delta = [T_\alpha^\sigma, T_{-\alpha}^\delta] = T_\delta^{(\alpha,\alpha)\sigma}. \end{aligned}$$

Thus (i)-(iii) hold. ∎

Lemma 2.2 *Suppose that $\sigma,\delta\in\mathcal{V}^0$, $\alpha_i\in R^\times$, $1\le i\le 8$ with $\alpha_2-\alpha_1=\alpha_6-\alpha_5=\delta$ and $\alpha_4-\alpha_1=\alpha_2-\alpha_3=\alpha_8-\alpha_5=\alpha_6-\alpha_7=\sigma$. Then*

$$w_{\alpha_1}w_{\alpha_4}w_{\alpha_2}w_{\alpha_3}=w_{\alpha_5}w_{\alpha_8}w_{\alpha_6}w_{\alpha_7}\in Z(\mathcal{W}).$$

Proof Using (2.2) and (2.3) we have

$$\begin{aligned} w_{\alpha_1}w_{\alpha_4}w_{\alpha_2}w_{\alpha_3} &= w_{\alpha_1}w_{\alpha_1+\sigma}w_{\alpha_2}w_{\alpha_2-\sigma}\\ &= T^{\sigma}_{-\alpha_1}T^{\sigma}_{\alpha_2}=T^{\sigma}_{\alpha_2-\alpha_1}=T^{\sigma}_{\alpha_6-\alpha_5}=T^{\sigma}_{-\alpha_5}T^{\sigma}_{\alpha_6}\\ &= w_{\alpha_5}w_{\alpha_5+\sigma}w_{\alpha_6}w_{-\alpha_6+\sigma}\\ &= w_{\alpha_5}w_{\alpha_8}w_{\alpha_6}w_{\alpha_7}. \end{aligned}$$

The above equalities also show that $w_{\alpha_1}w_{\alpha_4}w_{\alpha_2}w_{\alpha_3}=T^{\sigma}_{\delta}\in Z(\mathcal{W})$ (see (2.3)). □

To simplify our notation we write $\{r<s\}$ for the set $\{r,s\}$ with $1\le r<s\le\nu$. For any $1\le r,s\le\nu$ and $J\subseteq\{1,\dots,\nu\}$, we set

$$t_r:=T^{\sigma_r}_{\epsilon}=w_{\epsilon+\sigma_r}w_{\epsilon},\qquad c_{r,s}:=T^{\sigma_s}_{\sigma_r},\tag{2.4}$$

and

$$z_J:=\begin{cases}\prod_{\{r,s\in J\mid r<s\}}c_{r,s}, & \text{if } J\in\operatorname{supp}(S),\\ c^2_{r,s}, & \text{if } J=\{r<s\}\notin\operatorname{supp}(S),\\ 1, & \text{otherwise.}\end{cases}\tag{2.5}$$

(Here we interpret the product on an empty index set to be 1). Note that from (2.4) and (2.3), it follows that $c_{r,s}$ and z_J are central elements of $\mathrm{O}(\tilde{\mathcal{V}})$.

For $1\le r,s\le\nu$ we set

$$\delta(r,s):=\begin{cases}\begin{cases}1, & \text{if } r\le s,\\ -1, & \text{if } s<r,\end{cases} & \text{if } \{r,s\}\in\operatorname{supp}(S),\\ \begin{cases}2, & \text{if } r<s,\\ -2, & \text{if } s<r,\end{cases} & \text{if } \{r,s\}\notin\operatorname{supp}(S).\end{cases}\tag{2.6}$$

Then using (2.2), (2.3) and (2.4) we have

$$t_r=w_{\epsilon}w_{-\epsilon+\sigma_r},\quad w_{\epsilon}t_rw_{\epsilon}=t_r^{-1}\quad\text{and}\quad [t_r,t_s]=z^{2\delta(r,s)^{-1}}_{\{r,s\}}.\tag{2.7}$$

Considering that a Weyl group is a group generated by elements of order 2 the following lemma, which will be used frequently in the sequel, can be verified easily using the standard facts from group theory:

Lemma 2.3 *Suppose that G is a group and $b_1,b_2,\dots,b_r\in G$. Suppose that $b_1^2=b_2^2=\cdots=b_r^2=1$. Then for each $n\in\mathbb{Z}$,* (*i*) $[b_i, (b_1b_2\cdots b_r)^n]=1$ *for all* $1\le i\le r$ *if and only if*

$$(b_1b_2\cdots b_r)^n=(b_j\cdots b_rb_1b_2\cdots b_{j-1})^n,$$

for all $2\le j\le r$,

(*ii*) *if* $[b_i, b_1b_2\cdots b_r]=1$ *for all* $1\le i\le r$, *then*

$$(b_1b_2\cdots b_r)^2=(b_{j+1}\cdots b_rb_1b_2\cdots b_{j-1})^2=(b_1b_2\cdots b_{j-1}b_{j+1}\cdots b_r)^2,$$

for all $1\le j\le r-1$.

Lemma 2.4 (*i*) *If* $\{r<s\}\in supp(S)$, *then*

$$\begin{aligned} z_{\{r,s\}} &= w_{-\epsilon+\sigma_r+\sigma_s}w_{-\epsilon+\sigma_r}w_{\epsilon+\sigma_s}w_\epsilon = w_{\epsilon+\sigma_s+\sigma_r}w_{\epsilon+\sigma_r}w_\epsilon w_{\epsilon+\sigma_s}\\ &= w_\epsilon w_{\epsilon+\sigma_r+\sigma_s}t_rt_s. \end{aligned}$$

(*ii*) *If* $\{r<s\}\notin supp(S)$, *then*

$$z_{\{r,s\}}=(w_{-\epsilon+\sigma_r}w_{\epsilon+\sigma_s}w_\epsilon)^2=(w_{\epsilon+\sigma_s}w_{\epsilon+\sigma_r}w_\epsilon)^2.$$

(*iii*) *If* $\{1,2,3\}\in supp(S)$, *then*

$$z_{\{1,2,3\}}=z_{\{1,2\}}z_{\{1,3\}}z_{\{2,3\}}=w_\epsilon w_{\epsilon+\sigma_1+\sigma_2+\sigma_3}t_1t_2t_3,$$

and

$$w_{\epsilon+\sigma_3}w_\epsilon w_{\epsilon+\sigma_2}w_{\epsilon+\sigma_2+\sigma_3}=w_{-\epsilon+\sigma_1+\sigma_3}w_{-\epsilon+\sigma_1}w_{-\epsilon+\sigma_1+\sigma_2}w_{-\epsilon+\sigma_1+\sigma_2+\sigma_3}.$$

Proof (i) By the fact that $z_{\{r,s\}}$ is a central element of $\mathrm{O}(\tilde{\mathcal{V}})$ and using (2.3), (2.4), (2.5), (2.2) we obtain

$$\begin{aligned} z_{\{r,s\}} &= c_{r,s}=T_{\sigma_r}^{\sigma_s}=T_{\epsilon+\sigma_r}^{\sigma_s}T_{-\epsilon}^{\sigma_s}\\ &= (w_{\epsilon+\sigma_r+\sigma_s}w_{\epsilon+\sigma_r})(w_{-\epsilon+\sigma_s}w_\epsilon)\\ &= (w_{\epsilon+\sigma_r+\sigma_s}w_{\epsilon+\sigma_r})w_\epsilon(w_\epsilon w_{-\epsilon+\sigma_s}w_\epsilon)\\ &= w_{\epsilon+\sigma_r+\sigma_s}w_{\epsilon+\sigma_r}w_\epsilon w_{\epsilon+\sigma_s}\\ &= (w_\epsilon w_{\epsilon+\sigma_r+\sigma_s}w_\epsilon)(w_\epsilon w_{\epsilon+\sigma_r}w_\epsilon)w_{\epsilon+\sigma_s}w_\epsilon\\ &= w_{-\epsilon+\sigma_r+\sigma_s}w_{-\epsilon+\sigma_r}w_{\epsilon+\sigma_s}w_\epsilon\\ &= w_\epsilon z_{\{r,s\}}w_\epsilon=w_\epsilon w_{\epsilon+\sigma_r+\sigma_s}w_{\epsilon+\sigma_r}w_\epsilon w_{\epsilon+\sigma_s}w_\epsilon\\ &= w_\epsilon w_{\epsilon+\sigma_r+\sigma_s}t_rt_s. \end{aligned}$$

(ii) From (2.4), (2.7) and the fact that $\delta(r,s)=2$, we have

$$\begin{aligned} z_{\{r,s\}} &= w_\epsilon z_{\{r,s\}}w_\epsilon=w_\epsilon[t_r,t_s]w_\epsilon\\ &= w_\epsilon[w_{\epsilon+\sigma_r}w_\epsilon, w_{\epsilon+\sigma_s}w_\epsilon]w_\epsilon=(w_{\epsilon+\sigma_r}w_\epsilon w_{\epsilon+\sigma_s})^2 \end{aligned}$$

$$= w_\epsilon z_{\{r,s\}} w_\epsilon = w_\epsilon (w_{\epsilon+\sigma_r} w_\epsilon w_{\epsilon+\sigma_s})^2 w_\epsilon$$
$$= (w_\epsilon w_{\epsilon+\sigma_r} w_\epsilon w_{\epsilon+\sigma_s} w_\epsilon)^2 = (w_{-\epsilon+\sigma_r} w_{\epsilon+\sigma_s} w_\epsilon)^2.$$

The second equality in (ii) follows from Lemma 2.3(i).

(iii) From Table 1 we see that if $\{1,2,3\} \in \mathrm{supp}(S)$, then $\{r,s\} \in \mathrm{supp}(S)$ for all $1 \le r < s \le 3$. Thus using (i), (2.2), (2.5), (2.3), (2.4), (2.6) and Lemma 2.3(i), we have

$$\begin{aligned} z_{\{1,2,3\}} &= c_{1,2} c_{1,3} c_{2,3} = z_{\{1,2\}} z_{\{1,3\}} z_{\{2,3\}} = T_{\sigma_1}^{\sigma_2} T_{\sigma_1}^{\sigma_3} T_{\sigma_2}^{\sigma_3} \\ &= T_\epsilon^{-\sigma_3} T_\epsilon^{-\sigma_2} T_\epsilon^{-\sigma_1} T_{-\sigma_1}^{-\sigma_2} T_{-\sigma_1}^{-\sigma_3} T_{-\sigma_2}^{-\sigma_3} T_\epsilon^{\sigma_1} T_\epsilon^{\sigma_2} T_\epsilon^{\sigma_3} \\ &= T_\epsilon^{-\sigma_1-\sigma_2-\sigma_3} T_\epsilon^{\sigma_1} T_\epsilon^{\sigma_2} T_\epsilon^{\sigma_3} = (T_\epsilon^{\sigma_1+\sigma_2+\sigma_3})^{-1} T_\epsilon^{\sigma_1} T_\epsilon^{\sigma_2} T_\epsilon^{\sigma_3} \\ &= w_\epsilon w_{\epsilon+\sigma_1+\sigma_2+\sigma_3} T_\epsilon^{\sigma_1} T_\epsilon^{\sigma_2} T_\epsilon^{\sigma_3} = w_\epsilon w_{\epsilon+\sigma_1+\sigma_2+\sigma_3} t_1 t_2 t_3. \end{aligned}$$

The last equality follows immediately from Lemma 2.2. □

Proposition 2.1 *(i)* $\mathcal{W} = \langle w_\epsilon, t_r, z_{\{r,s\}} \mid 1 \le r < s \le \nu \rangle$.

(ii) $Z(\mathcal{W}) = \langle z_{\{r,s\}} \mid 1 \le r < s \le \nu \rangle$.

(iii) Each element $w \in \mathcal{W}$ has a unique expression in the form

$$w = w_\epsilon^n \prod_{r=1}^{\nu} t_r^{m_r} \prod_{1 \le r < s \le \nu} z_{\{r,s\}}^{m_{r,s}}, \quad (n \in \{0,1\},\ m_r, m_{r,s} \in \mathbb{Z}).$$

Proof Consider the fact that S is one of the semilattices listed in Table 1 and use [6, Propositions 2.2, 2.3] and Lemma 2.4. □

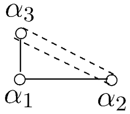
Next, to any subdiagram of $\Gamma(R)$ we attach a relation given as follows:

(0)

α

○ $\qquad \hat{w}_\alpha^2 = 1$

(I)



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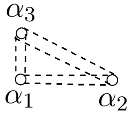
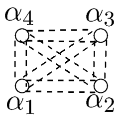


$$(\hat{w}_{\alpha_1} \hat{w}_{\alpha_2} \hat{w}_{\alpha_3})^2 = (\hat{w}_{\alpha_2} \hat{w}_{\alpha_3} \hat{w}_{\alpha_1})^2 = (\hat{w}_{\alpha_3} \hat{w}_{\alpha_1} \hat{w}_{\alpha_2})^2$$

(II)



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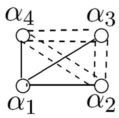
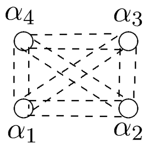


$$\hat{w}_{\alpha_i} (\hat{w}_{\alpha_j} \hat{w}_{\alpha_k} \hat{w}_{\alpha_\ell})^2 = (\hat{w}_{\alpha_j} \hat{w}_{\alpha_k} \hat{w}_{\alpha_\ell})^2 \hat{w}_{\alpha_i} \quad \text{where} \quad \{i,j,k,\ell\} = \{1,2,3,4\}$$

(III)



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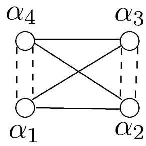
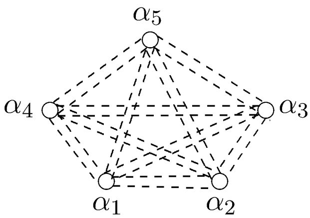


(if $\alpha_4 - \alpha_1 = \alpha_2 - \alpha_3$) (if $\alpha_1 - \alpha_4 = \alpha_2 - \alpha_3$)

$$\hat{w}_{\alpha_1}\hat{w}_{\alpha_4}\hat{w}_{\alpha_2}\hat{w}_{\alpha_3} = \hat{w}_{\alpha_4}\hat{w}_{\alpha_2}\hat{w}_{\alpha_3}\hat{w}_{\alpha_1} = \hat{w}_{\alpha_2}\hat{w}_{\alpha_3}\hat{w}_{\alpha_1}\hat{w}_{\alpha_4} = \hat{w}_{\alpha_3}\hat{w}_{\alpha_1}\hat{w}_{\alpha_4}\hat{w}_{\alpha_2}.$$

(IV)



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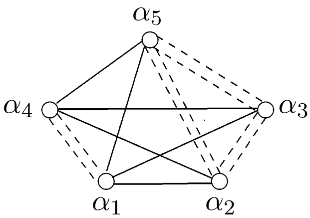
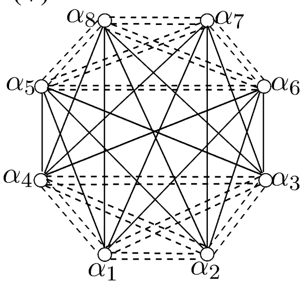


(if $\alpha_4 - \alpha_1 = \alpha_2 - \alpha_3$) (if $\alpha_1 - \alpha_4 = \alpha_2 - \alpha_3$)

$$\hat{w}_{\alpha_5}\hat{w}_{\alpha_1}\hat{w}_{\alpha_4}\hat{w}_{\alpha_2}\hat{w}_{\alpha_3} = \hat{w}_{\alpha_1}\hat{w}_{\alpha_4}\hat{w}_{\alpha_2}\hat{w}_{\alpha_3}\hat{w}_{\alpha_5}.$$

(V)



(if $\alpha_4 - \alpha_1 = \alpha_2 - \alpha_3 = \alpha_8 - \alpha_5 = \alpha_6 - \alpha_7$ and $\alpha_2 - \alpha_1 = \alpha_6 - \alpha_5$)

$$\hat{w}_{\alpha_1}\hat{w}_{\alpha_4}\hat{w}_{\alpha_2}\hat{w}_{\alpha_3} = \hat{w}_{\alpha_5}\hat{w}_{\alpha_8}\hat{w}_{\alpha_6}\hat{w}_{\alpha_7}.$$

Let $\hat{\mathcal{W}}$ be the group defined by generators $\hat{w}_\alpha, \alpha \in |\Gamma(R)|$ and relations attached to the subdiagrams of $\Gamma(R)$ given by (0)–(IV) above. To study $\widehat{\mathcal{W}}$ we need to introduce some notations.

For $1 \le j \le 3$ set

$$\hat{t}_j := (\hat{w}_{n_j\epsilon+\sigma_j}\hat{w}_\epsilon)^{n_j} = \begin{cases} \hat{w}_\epsilon\hat{w}_{-\epsilon+\sigma_1}, \text{ if } j=1, \\ \hat{w}_{\epsilon+\sigma_j}\hat{w}_\epsilon, \quad \text{if } j=2,3. \end{cases} \tag{2.8}$$

Also for $1 \le r < s \le 3$ set

$$\hat{z}_{r,s} := \begin{cases} \hat{w}_{n_r\epsilon+\sigma_r+\sigma_s}\hat{w}_{n_r\epsilon+\sigma_r}(\hat{w}_\epsilon \hat{w}_{\epsilon+\sigma_s})^{n_r}, & \text{if } \{r,s\} \in \operatorname{supp}(S), \\ \left((\hat{w}_{\epsilon+\sigma_s}\hat{w}_{n_r\epsilon+\sigma_r})^{n_r}\hat{w}_\epsilon\right)^2, & \text{if } \{r,s\} \notin \operatorname{supp}(S). \end{cases} \tag{2.9}$$

Lemma 2.5 *(i)* $\hat{\mathcal{W}} = \langle \hat{w}_\epsilon,\ \hat{t}_j,\ \hat{z}_{r,s} \mid 1 \le j \le 3, 1 \le r < s \le 3\rangle$.

(ii) $\hat{w}_\epsilon \hat{t}_j \hat{w}_\epsilon = \hat{t}_j^{-1}$, *for* $1 \le j \le 3$.

(iii) $\hat{z}_{r,s} \in Z(\hat{\mathcal{W}})$, *for* $1 \le r < s \le 3$.

(iv) $[\hat{t}_r, \hat{t}_s] = \hat{z}_{r,s}^{2\delta(r,s)^{-1}}$, *for* $1 \le r < s \le 3$.

(v) Any $\hat{w} \in \hat{\mathcal{W}}$ *can be written in the form*

$$\hat{w} = \hat{w}_\epsilon^n \prod_{r=1}^{3} \hat{t}_r^{m_r} \prod_{1 \le r < s \le 3} \hat{z}_{r,s}^{m_{r,s}}, \tag{2.10}$$

where $n \in \{0, 1\}$ *and* $m_r, m_{r,s} \in \mathbb{Z}$.

Proof (i) Let T be the group in the right hand side of the statement and $\alpha \in |\Gamma(R)|$. If $\alpha = \pm\epsilon + \sigma_j$ then from the way $\hat{t}_j$ is defined it is clear that $\hat{w}_\alpha \in T$. If α is of the form $\alpha = \pm\epsilon + \sigma_r + \sigma_s$, then $\{r,s\} \in \operatorname{supp}(S)$ and so $\hat{w}_\alpha \in T$ by the way $\hat{t}_r$'s and $\hat{z}_{r,s}$'s are defined. Finally suppose $\alpha = -\epsilon + \sigma_1 + \sigma_2 + \sigma_3$ (note from Table 1, that this happens only if S is a lattice). From relations of the form (V) we have

$$\hat{w}_{\epsilon+\sigma_3}\hat{w}_\epsilon \hat{w}_{\epsilon+\sigma_2}\hat{w}_{\epsilon+\sigma_2+\sigma_3} = \hat{w}_{-\epsilon+\sigma_1+\sigma_3}\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{-\epsilon+\sigma_1+\sigma_2}\hat{w}_{-\epsilon+\sigma_1+\sigma_2+\sigma_3}$$

and so we see that the generator $\hat{w}_\alpha$ can be expressed in terms of generators which we considered earlier in our argument. Thus $\hat{w}_\alpha \in T$.

(ii) Using (**0**) and (2.8) we have

$$\hat{w}_\epsilon \hat{t}_j \hat{w}_\epsilon = \hat{w}_\epsilon (\hat{w}_{n_j\epsilon+\sigma_j}\hat{w}_\epsilon)^{n_j}\hat{w}_\epsilon = (\hat{w}_\epsilon \hat{w}_{n_j\epsilon+\sigma_j})^{n_j} = \hat{t}_j^{-1}.$$

(iii) Let $1 \le r < s \le 3$. Let $\Gamma_{r,s}$ be the set of $\alpha \in R$ such that $\hat{w}_\alpha$ appears in the definition of $\hat{z}_{r,s}$ (see (2.9)). Since $\Gamma_{r,s} \subseteq |\Gamma(R)|$, $\hat{z}_{r,s} \in \hat{\mathcal{W}}$. So it remains to show that $\hat{z}_{r,s}$ commutes with all $\hat{w}_\alpha$, $\alpha \in |\Gamma(R)|$. First suppose that $\{r,s\} \in \operatorname{supp}(S)$. Then the 4-nodes appearing in $\Gamma_{r,s}$ generate a subdiagram of the form (III) and so the corresponding relations hold in $\hat{\mathcal{W}}$. Therefore by Lemma 2.3(i), $\hat{z}_{r,s}$ commutes with all $\hat{w}_\alpha$ with $\alpha \in \Gamma_{r,s}$. If $\alpha \in |\Gamma(R)| \setminus \Gamma_{r,s}$, then the 5-nodes $\{\alpha\} \cup \Gamma_{r,s}$ generate a subdiagram of the form (IV) and so the corresponding relation guarantees that $\hat{z}_{r,s}$ commutes with $\hat{w}_\alpha$. This completes the argument for the case $\{r,s\} \in \operatorname{supp}(S)$.

Next suppose $\{r,s\} \notin \operatorname{supp}(S)$. Then $\Gamma_{r,s}$ consists of 3-nodes which generate a subdiagram of the form (I). Then the relations imposed by this diagram together with Lemma 2.3(i) show that $\hat{z}_{r,s}$ commutes with $\hat{w}_\alpha$, $\alpha \in \Gamma_{r,s}$. If $\alpha \in |\Gamma(R)| \setminus \Gamma_{r,s}$, then the 4-nodes $\{\alpha\} \cup \Gamma_{r,s}$ generate a subdiagram of the form (II). Then the relations imposed by this diagram guarantee that $\hat{z}_{r,s}$ commutes with $\hat{w}_\alpha$. This completes the proof of part (iii).

(iv) If $1 \leq r < s \leq 3$, then we have

$$
\begin{aligned}
[\hat{t}_r, \hat{t}_s] &= \begin{cases} [\hat{w}_\epsilon \hat{w}_{-\epsilon+\sigma_1}, \ \ \hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon], \text{ if } r=1, \\ [\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon, \ \ \hat{w}_{\epsilon+\sigma_3}\hat{w}_\epsilon], \quad \text{if } r=2, \end{cases} \\
(\text{using (0)}) \ &= \begin{cases} (\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon)^2, \qquad \text{if } r=1, \\ \hat{w}_\epsilon(\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon\hat{w}_{\epsilon+\sigma_3})^2\hat{w}_\epsilon, \text{ if } r=2, \end{cases} \\
(\text{using Lemma 2.3(i) and (I)}) \ &= \begin{cases} (\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon)^2, \text{ if } r=1, \\ (\hat{w}_{\epsilon+\sigma_3}\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon)^2, \quad \text{if } r=2, \end{cases} \\
(\text{if } \{r,s\} \notin \operatorname{supp}(S)) &= \hat{z}_{r,s}^{2\delta(r,s)^{-1}}.
\end{aligned}
$$

If $\{r,s\} \in \operatorname{supp}(S)$, then we have

$$
\begin{aligned}
[\hat{t}_r, \hat{t}_s] &= \begin{cases} (\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon)^2, \text{ if } r=1, \\ (\hat{w}_{\epsilon+\sigma_3}\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon)^2, \quad \text{if } r=2, \end{cases} \\
(\text{using Lemma 2.3(ii) and (III)}) \ &= \begin{cases} (\hat{w}_{-\epsilon+\sigma_1+\sigma_s}\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon)^2, \text{ if } r=1, \\ (\hat{w}_{\epsilon+\sigma_3}\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon)^2, \qquad\qquad \text{if } r=2, \end{cases} \\
(\text{using Lemma 2.3(ii) and (III)}) \ &= \begin{cases} (\hat{w}_{-\epsilon+\sigma_1+\sigma_s}\hat{w}_{-\epsilon+\sigma_1}\hat{w}_{\epsilon+\sigma_s}\hat{w}_\epsilon)^2, \text{ if } r=1, \\ (\hat{w}_{\epsilon+\sigma_2+\sigma_3}\hat{w}_{\epsilon+\sigma_2}\hat{w}_\epsilon\hat{w}_{\epsilon+\sigma_3})^2, \quad \text{if } r=2, \end{cases} \\
&= \hat{z}_{r,s}^{2\delta(r,s)^{-1}}.
\end{aligned}
$$

(v) This is an immediate consequence of parts (i)-(iv) and the fact that $\hat{w}_\epsilon^2 = 1$. □

We now state our main result.

Theorem 2.1 *Let R be an extended affine root system of type A_1 with nullity $\nu \leq 3$ and let $\mathcal{W}$ be its extended affine Weyl group. Then for $\alpha \in |\Gamma(R)|$, the assignment $\hat{w}_\alpha \longmapsto w_\alpha$ induces an isomorphism $\hat{\mathcal{W}} \cong \mathcal{W}$.*

Proof From (2.7), (1.3), Proposition 2.1(i) Lemmas 2.2 and 2.4, it follows that $\{w_\alpha \mid \alpha \in |\Gamma(R)|\}$ is a set of generators for $\mathcal{W}$ which satisfies the relations attached to $\Gamma(R)$. So the assignment $\hat{w}_\alpha \longmapsto w_\alpha$ for $\alpha \in |\Gamma(R)|$ induces a unique epimorphism $\psi : \hat{\mathcal{W}} \longrightarrow \mathcal{W}$. We now show that ψ is injective. So let $\psi(\hat{w}) = 1$, for some $\hat{w} \in \hat{\mathcal{W}}$. Using Lemma 2.5(v) we may assume that $\hat{w}$ has the form (2.10) for some $n \in \{0,1\}$ and $m_r, m_{r,s} \in \mathbb{Z}$. From (2.7) and Lemma 2.4, it follows that for $1 \leq j \leq 3$ and $1 \leq r < s \leq 3$,

$$\psi(\hat{t}_j) = t_j \quad \text{and} \quad \psi(\hat{z}_{r,s}) = z_{\{r,s\}}. \tag{2.11}$$

Therefore from (2.10) and (2.11) we have

$$1 = \psi(\hat{w}) = w_\epsilon^n \prod_{r=1}^{3} t_r^{m_r} \prod_{1\le r<s\le 3} z_{\{r,s\}}^{m_{r,s}}.$$

Then from Proposition 2.1(iii), it follows that $n = 0$, $m_{r,s} = 0$ for all $1 \le r < s \le \nu$ and $m_r = 0$ for all r. So $\hat{w} = 1$ and ψ is an isomorphism. □

From Proposition 2.1(ii), (2.11) and the fact that $\psi(Z(\widehat{\mathcal{W}})) = Z(\mathcal{W})$ we obtain the following result.

Corollary 2.1 $Z(\hat{\mathcal{W}}) = \langle \hat{z}_{r,s} \mid 1 \le r < s \le 3\rangle$.

If $\nu = 1$ then $|\Gamma(R)| = \{-\epsilon + \sigma_1, \epsilon\}$ and so the Dynkin diagram of R is $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$, so we have

Corollary 2.2 (**Affine case**) *The Weyl group of an affine root system of type* A_1 *is* $(a, b \mid a^2 = b^2 = 1)$.

Let R be an elliptic root system, an extended affine root system of nullity 2. Up to isomorphism there are two elliptic root systems of type A_1, namely $R = R(A_1, S)$ where S is a semilattice of rank two with $\mathrm{ind}(S) \in \{2, 3\}$, see Table 1.

Corollary 2.3 (**Elliptic case**) *Let* $R = R(A_1, S)$ *be an elliptic root system of type* A_1 *with Weyl group* $\mathcal{W}(m)$, $m = ind(S) = 2, 3$. *Then*

$$\mathcal{W}(2) \cong (a, b, c \mid \mathcal{R}(2)) \quad \text{and} \quad \mathcal{W}(3) \cong (a, b, c, d \mid \mathcal{R}(3))$$

where

$$\mathcal{R}(2) := \{a^2 = b^2 = c^2 = 1,\ (abc)^2 = (bca)^2 = (cab)^2\},$$

and

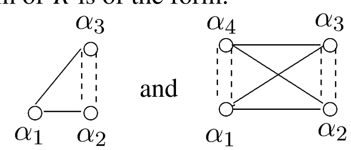
$$\mathcal{R}(3) := \{a^2 = b^2 = c^2 = d^2 = 1,\ abcd = bcda = cdab = dabc\}.$$

In particular, $Z(\mathcal{W}(2)) = \langle (abc)^2\rangle$ *and* $Z(\mathcal{W}(3)) = \langle abcd\rangle$.

Proof We have

$$|\Gamma(R)| = \begin{cases} \{-\epsilon + \sigma_1,\ \epsilon,\ \epsilon + \sigma_2\}, & \text{if } m = 2, \\ \{-\epsilon + \sigma_1,\ \epsilon,\ \epsilon + \sigma_2,\ -\epsilon + \sigma_1 + \sigma_2\}, & \text{if } m = 3, \end{cases}$$

and so the Dynkin diagram of R is of the form:



(if $m = 2$) (if $m = 3$)

Then $\mathcal{W}(m) \cong \hat{\mathcal{W}}(m)$, where $\hat{\mathcal{W}}(m)$ is the presented group given by Theorem 2.1. If $m = 2$, then the relations attached to $\Gamma(R)$ are those listed in $\mathcal{R}(2)$ and so we are done.

If $m = 3$, then the relations attached to $\Gamma(R)$ are those listed in $\mathcal{R}(3)$ plus those of the form $\mathcal{R}(2)$ attached to the triangle subdiagrams of $\Gamma(R)$. But by Lemma 2.3(ii) any relation of the form $\mathcal{R}(2)$ is a consequence of relations of the form $\mathcal{R}(3)$, and so we are done. $\square$

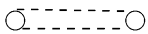
Remark 2.1 We have adapted the set of nodes of $\Gamma(R)$ and the bonds among them in such a way that when nullity is 2 our diagrams coincide with those of [11]. This is done for two reasons. Firstly to complete the presentation given in [13] for simply laced 3-extended affine Weyl groups, and secondly to provide a new proof for the presentation given in [11] for the A_1-type elliptic Weyl groups.

We could choose a more natural set of nodes for $\Gamma(R)$ and a more simple rule for the bonds among them to obtain a generalized Coxeter presentation with a simpler proof in terms of the involved diagrams. In fact we could set:

(**i**) The set of nodes of $\Gamma(R)$ is the set

$$|\Gamma(R)| := \{\epsilon + \tau_J \mid J \in \mathrm{supp}(S)\}.$$

(**ii**) Bonds among any two distinct nodes of $\Gamma(R)$ are inserted as



With this new definition of $\Gamma(R)$ the only bonds among nodes will be double dashed bonds. So the diagrams we obtain are only those from list (0)-(V) (together with their corresponding relations) which contain no single bond. Now for any $1 \leq j \leq \nu$ and $1 \leq r < s \leq \nu$, set

$$\hat{t}_j := \hat{w}_{\epsilon+\sigma_j} \hat{w}_{\epsilon} \tag{2.12}$$

and

$$\hat{z}_{r,s} := \begin{cases} \hat{w}_{\epsilon+\sigma_r+\sigma_s} \hat{w}_{\epsilon+\sigma_r} \hat{w}_{\epsilon} \hat{w}_{\epsilon+\sigma_s}, & \text{if } \{r,s\} \in \mathrm{supp}_{\mathcal{B}}(S), \\ (\hat{w}_{\epsilon+\sigma_s} \hat{w}_{\epsilon+\sigma_r} \hat{w}_{\epsilon})^2, & \text{if } \{r,s\} \notin \mathrm{supp}_{\mathcal{B}}(S). \end{cases} \tag{2.13}$$

Then using (2.12) and (2.13) with a similar argument as in the proof of Lemma 2.5 and Theorem 2.1 one can prove that $\mathcal{W}$ is isomorphic to the group defined by generators $\hat{w}_\alpha$, $\alpha \in \Gamma(R)$ and relations attached to diagrams (0)-(V).

Remark 2.2 In [9], the authors studied *toroidal Weyl groups*, the Weyl groups of *toroidal Lie algebras (or root systems)*. These are in fact the Weyl groups of those simply laced extended affine root systems for which the involved semilattices are lattices. So toroidal root systems are simply laced extended affine root systems of rank > 1 and only a special case of type A_1, namely the one in which the semilattice S is a lattice. The authors assign to each such root system R a *minimal* set Π, called a *basis* and show that, except for type A_1, $\mathcal{W}_R = \mathcal{W}_\Pi$ where $\mathcal{W}_R$ is the Weyl group of R and $\mathcal{W}_\Pi$ is the subgroup of $\mathcal{W}_R$ generated by reflections based on elements of Π. They study the internal semidirect product structure of $\mathcal{W}_R$ and for the special case A_1, they also compare two groups $\mathcal{W}_R$ and $\mathcal{W}_\Pi$. These results are generalized for all extended affine root systems in [2] and [4]. Specially, in [4], a *minimal* set Π is assigned to each extended affine root system of any type, except type BC, and it is shown that for all these types (including type A_1), $\mathcal{W}_R = \mathcal{W}_\Pi$. For simply laced types of rank > 1, the sets Π in [4] and in [9] coincide.

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