Specht filtrations and tensor spaces for the Brauer algebra

Jun Hu

Abstract Let $m, n \in \mathbb{N}$. In this paper we study the right permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the set of all the Brauer n-diagrams. A new basis for the free $\mathbb{Z}$-module $\mathfrak{B}_n$ spanned by these Brauer n-diagrams is constructed, which yields Specht filtrations for $\mathfrak{B}_n$. For any $2m$-dimensional vector space V over a field of arbitrary characteristic, we give an explicit and characteristic-free description of the annihilator of the n-tensor space $V^{\otimes n}$ in the Brauer algebra $\mathfrak{B}_n(-2m)$. In particular, we show that it is a $\mathfrak{S}_{2n}$-submodule of $\mathfrak{B}_n(-2m)$.

Keywords Brauer algebra · Symmetric group · Tensor space

1 Introduction

Let x be an indeterminate over $\mathbb{Z}$. The Brauer algebra $\mathfrak{B}_n(x)$ over $\mathbb{Z}[x]$ is a unital associative $\mathbb{Z}[x]$-algebra with generators $s_1, \cdots, s_{n-1}, e_1, \cdots, e_{n-1}$ and relations (see [16]):

$$s_i^2 = 1, \qquad e_i^2 = x e_i, \qquad e_i s_i = e_i = s_i e_i, \quad \forall\, 1 \le i \le n-1,$$

$$s_i s_j = s_j s_i, \qquad s_i e_j = e_j s_i, \qquad e_i e_j = e_j e_i, \quad \forall\, 1 \le i < j-1 \le n-2,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad e_i e_{i+1} e_i = e_i, \qquad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad \forall\, 1 \le i \le n-2,$$

$$s_i e_{i+1} e_i = s_{i+1} e_i, \qquad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \quad \forall\, 1 \le i \le n-2.$$

$\mathfrak{B}_n(x)$ is a free $\mathbb{Z}[x]$-module with rank $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$. For any commutative $\mathbb{Z}[x]$-algebra R with x specialized to $\delta \in R$, we define $\mathfrak{B}_n(\delta)_R := R \otimes_{\mathbb{Z}[x]} \mathfrak{B}_n(x)$.

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This algebra was first introduced by Richard Brauer (see [2]) when he studied how the n-tensor space $V^{\otimes n}$ decomposes into irreducible modules over the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, where V is an orthogonal vector space or a symplectic vector space. In Brauer's original formulation, the algebra $\mathfrak{B}_n(x)$ was defined as the complex linear space with basis the set Bd_n of all the Brauer n-diagrams, graphs on $2n$ vertices and n edges with the property that every vertex is incident to precisely one edge. If we arrange the vertices in two rows of n each, the top and bottom rows, and label the vertices in each row of a n-diagram by the indices $1, 2, \cdots, n$ from left to right, then the generator s_i corresponds to the n-diagram with edges connecting vertices i (respectively, $i + 1$) on the top row with $i + 1$ (respectively, i) on bottom row, and all other edges are vertical, connecting vertex k on the top and bottom rows for all $k \neq i, i + 1$. The generator e_i corresponds to the n-diagram with horizontal edges connecting vertices $i, i + 1$ on the top and bottom rows, and all other edges are vertical, connecting vertex k on the top and bottom rows for all $k \neq i, i + 1$. The multiplication of two Brauer n-diagrams is defined as follows. We compose two diagrams D_1, D_2 by identifying the bottom row of vertices in the first diagram with the top row of vertices in the second diagram. The result is a graph, with a certain number, $n(D_1, D_2)$, of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new Brauer n-diagram $D_1 \circ D_2$, the composite diagram. Then we define $D_1 \cdot D_2 = x^{n(D_1, D_2)} D_1 \circ D_2$. In general, the multiplication of two elements in $\mathfrak{B}_n(x)$ is given by the linear extension of a product defined on diagrams. For example, let d be the following Brauer 5-diagram.
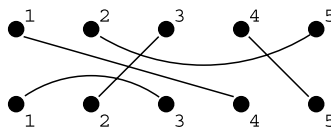
Let d' be the following Brauer 5-diagram.
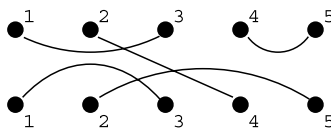
Then dd' is equal to
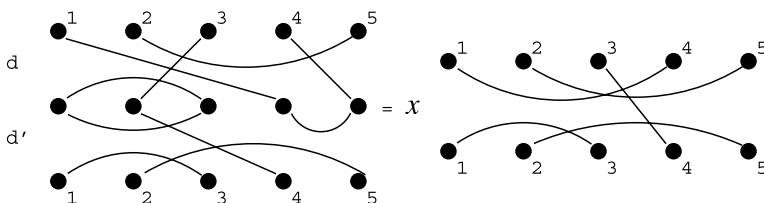
Note that the subalgebra of $\mathfrak{B}_n(x)$ generated by $s_1, s_2, \cdots, s_{n-1}$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_n$ over $\mathbb{Z}[x]$.

The Brauer algebra as well as its quantization (now called the Birman–Wenzl–Murakami algebra) has been studied in a number of papers, e.g., [2–4, 6, 11, 16–18, 22, 23, 29, 35]. The walled Brauer algebra (which is a variant of the Brauer algebra, see [5]) is also studied in the recent preprint [10]. We are mainly interested in the Schur–Weyl duality between symplectic groups and certain specialized Brauer algebras, which we now recall. Let K be an arbitrary infinite field. Let $m, n \in \mathbb{N}$. Let V be a $2m$-dimensional K-vector space equipped with a non-degenerate skew-symmetric bilinear form $(\,,\,)$. Then (see [19], [15, Section 4]) the symplectic similitude group (respectively, the symplectic group) relative to $(\,,\,)$ is

$$GSp(V) := \left\{ g \in GL(V) \,\middle|\, \begin{array}{l} \exists\, d \in K \text{ with } d \neq 0, \text{ such that} \\ (gv, gw) = d(v, w), \ \ \forall\, v, w \in V \end{array} \right\}$$

$$\left(\text{respectively,} \quad Sp(V) := \left\{ g \in GL(V) \,\middle|\, (gv, gw) = (v, w), \ \forall\, v, w \in V \right\} \right).$$

The symplectic similitude group and symplectic group $Sp(V)$ act naturally on V from the left-hand side, and hence on the n-tensor space $V^{\otimes n}$. This left action on $V^{\otimes n}$ is centralized by certain specialized Brauer algebra, which we recall as follows. Let $\mathfrak{B}_n(-2m) := \mathbb{Z} \otimes_{\mathbb{Z}[x]} \mathfrak{B}_n(x)$, where $\mathbb{Z}$ is regarded as $\mathbb{Z}[x]$-algebra by specifying x to $-2m$. Let $\mathfrak{B}_n(-2m)_K := K \otimes_{\mathbb{Z}} \mathfrak{B}_n(-2m)$, where K is regarded as $\mathbb{Z}$-algebra by sending each integer a to $a \cdot 1_K$. Then there is a right action of the specialized Brauer algebra $\mathfrak{B}_n(-2m)_K$ on the n-tensor space $V^{\otimes n}$ which commutes with the above left action of $GSp(V)$. We recall the definition of this action as follows. Let δ_{ij} denote the value of the usual Kronecker delta. For any $1 \leq i \leq 2m$, we set

$$i' := 2m + 1 - i.$$

We fix an ordered basis $\{v_1, v_2, \cdots, v_{2m}\}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}), \ \ (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall\, 1 \leq i, j \leq m.$$

For any $i, j \in \{1, 2, \cdots, 2m\}$, let

$$\epsilon_{i,j} := \begin{cases} 1 & \text{if } j = i' \text{ and } i < j, \\ -1 & \text{if } j = i' \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

The right action of $\mathfrak{B}_n(-2m)$ on $V^{\otimes n}$ is defined on generators by

$$(v_{i_1} \otimes \cdots \otimes v_{i_n})s_j$$
$$:= -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}),$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_n})e_j$$
$$:= \epsilon_{i_j, i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left(\sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}.$$

Let φ, ψ be the following natural K-algebra homomorphisms.

$$\varphi : (\mathfrak{B}_n(-2m)_K)^{op} \to \mathrm{End}_K(V^{\otimes n}),$$
$$\psi : KGSp(V) \to \mathrm{End}_K(V^{\otimes n}).$$

Let k be a positive integer. A composition of k is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \cdots)$ with $\sum_{i \geq 1} \lambda_i = k$. A composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ of k is said to be a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$. In this case, we write $\lambda \vdash k$. The conjugate of λ is defined to be the partition $\lambda' = (\lambda_1', \lambda_2', \cdots)$, where $\lambda_j' := \#\{i | \lambda_i \geq j\}$ for $j = 1, 2, \cdots$. For any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, we use $\ell(\lambda)$ to denote the largest integer t such that $\lambda_t \neq 0$.

Lemma 1.1 ([2–4, 11, 32]) (1) *The natural left action of $GSp(V)$ on $V^{\otimes n}$ commutes with the right action of $\mathfrak{B}_n(-2m)$.*

(2) *if K is an infinite field, then*

(a) $\varphi(\mathfrak{B}_n(-2m)_K) = \mathrm{End}_{KGSp(V)}(V^{\otimes n}) = \mathrm{End}_{KSp(V)}(V^{\otimes n})$,
(b) $\psi(KGSp(V)) = \mathrm{End}_{\mathfrak{B}_n(-2m)_K}(V^{\otimes n})$,

(3) *if K is an infinite field and $m \geq n$ then φ is injective, and hence an isomorphism onto $\mathrm{End}_{KGSp(V)}(V^{\otimes n})$,*

(4) *if $K = \mathbb{C}$, then there is a decomposition of $V^{\otimes n}$ as a direct sum of irreducible $\mathbb{C}GSp(V)$–$\mathfrak{B}_n(-2m)_{\mathbb{C}}$ bimodules*

$$V^{\otimes n} = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\substack{\lambda \vdash n-2f \\ \ell(\lambda) \leq m}} \Delta(\lambda) \otimes D(\lambda'),$$

where $\Delta(\lambda)$ (respectively, $D(\lambda')$) denotes the irreducible $\mathbb{C}GSp(V)$-module (respectively, the irreducible $\mathfrak{B}_n(-2m)_{\mathbb{C}}$-module) corresponding to λ (respectively, corresponding to λ').

Historically, the above results in the case of $K = \mathbb{C}$ are proved in [2, 3] and [4]. For arbitrary infinite field K, 2) and 3) are proved in [11] and [32].

Now there is a natural question, that is, how can one describe the kernel of the homomorphism φ? This question is closely related to invariant theory: see [7]. By [11, Theorem 1.2], we know that the kernel of the homomorphism φ has a rigid

structure in the sense that the dimension of Ker φ does not depend on the choice of the infinite field K, and it is actually defined over $\mathbb{Z}$. Note that in the case of Schur–Weyl duality between general linear group and symmetric group [7, 8, 33, 34], or more generally, between the type A quantum group and the type A Iwahori–Hecke algebra [14, 24], the kernel of the corresponding homomorphism has already been explicitly determined in [14] in terms of the Kazhdan–Lusztig basis and in [20] in terms of the Murphy basis. In this paper, we completely answer the above question by explicitly constructing an integral basis for the kernel of the homomorphism φ. Our description of Ker φ involves a study of the permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the Brauer algebra $\mathfrak{B}_n(x)$. Such a permutation action was previously noted in [17]. We construct a new integral basis for this Brauer algebra, which yields an integral Specht filtration of this Brauer algebra by right $\mathfrak{S}_{2n}$-modules. The kernel of φ is just one of the $\mathfrak{S}_{2n}$-submodules appearing in this filtration. In particular, it turns out that Ker φ is in fact a $\mathfrak{S}_{2n}$-submodule of $\mathfrak{B}_n(-2m)$. The main results of this paper are presented in Theorem 2.11, Theorem 2.13 and Theorem 3.4. It would be interesting to compare the new integral basis we obtained in this paper with the canonical basis for $\mathfrak{B}_n(x)$ constructed in [17]. It would also be interesting to see how the description of Ker φ we give here can be generalized to the quantized case, i.e., the case of Schur–Weyl duality between the quantized enveloping algebra associated to the symplectic Lie algebra sp_{2m} and a certain specialized Birman–Wenzl–Murakami algebra (see [9]).

## 2 The $\mathfrak{S}_{2n}$-action on $\mathfrak{B}_n(x)$

In this section, we shall first recall (cf. [17]) the right permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the set Bd_n. Then we shall construct a new $\mathbb{Z}$-basis for the resulting right $\mathfrak{S}_{2n}$-module, which yields filtrations of $\mathfrak{B}_n(x)$ by right $\mathfrak{S}_{2n}$-modules. Certain submodules occurring in this filtration will play a central role in the next section.

For any fixed-point-free involution σ in the symmetric group $\mathfrak{S}_{2n}$, the conjugate $w^{-1}\sigma w$ of σ by $w \in \mathfrak{S}_{2n}$ is still a fixed-point-free involution. Therefore, we have a right action of the symmetric group $\mathfrak{S}_{2n}$ on the set of all the fixed-point-free involutions in $\mathfrak{S}_{2n}$. Note that the set Bd_n of Brauer n-diagrams can be naturally identified with the set of fixed-point-free involutions in $\mathfrak{S}_{2n}$ as explained below. Hence we get (cf. [17]) a right permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the set Bd_n of all the Brauer n-diagrams. We use "$*$" to denote this right permutation action.

We shall adopt a new labeling of the vertices in each Brauer diagram. Namely, for each Brauer n-diagram D, we shall label the vertices in the top row of D by odd integers $1, 3, 5, \cdots, 2n - 1$ from left to right, and label the vertices in the bottom row of D by even integers $2, 4, 6, \cdots, 2n$ from left to right. This way of labeling is more convenient when studying the permutation action from $\mathfrak{S}_{2n}$. *We shall keep this way of labeling from this section until the end of Section 3, and we shall recover our original way of labeling only in Section 4.* Let us look at an example. Suppose $n = 4$, $s_1 s_7 s_6 = (1, 2)(6, 7, 8)$ is a permutation in $\mathfrak{S}_8$. Let D be the following Brauer 4-diagram.
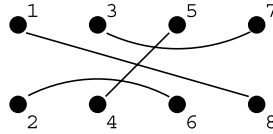
We first identify D with following diagram with 8 vertices.
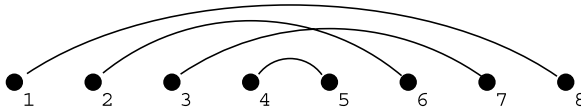
Then $D * (s_1 s_7 s_6)$ can be computed in the following way.
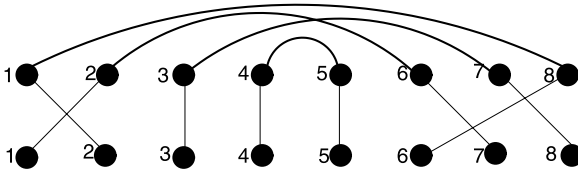
Finally, $D * (s_1 s_7 s_6)$ is equal to the following Brauer 4-diagram.
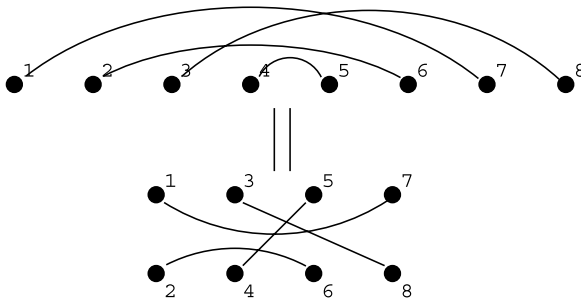
We use β to denote the natural identification of Bd_n with the set of fixed-point-free involutions in $\mathfrak{S}_{2n}$. For any $w \in \mathfrak{S}_{2n}$ and any $D \in \mathrm{Bd}_n$, $D * w = \beta^{-1}(w^{-1}\beta(D)w)$.

For any commutative $\mathbb{Z}$-algebra R, we use $\mathfrak{B}_{n,R}$ to denote the free R-module spanned by Bd_n. Then $\mathfrak{B}_{n,R}$ becomes a right $R[\mathfrak{S}_{2n}]$-module. Let $\mathfrak{B}_n := \mathfrak{B}_{n,\mathbb{Z}}$. Clearly, there is a canonical isomorphism $\mathfrak{B}_{n,R} \cong R \otimes_{\mathbb{Z}} \mathfrak{B}_n$, which is also a right $R[\mathfrak{S}_{2n}]$-module isomorphism. Taking $R = \mathbb{Z}[x]$, we deduce that the Brauer algebra $\mathfrak{B}_n(x)$ becomes a right $\mathbb{Z}[x][\mathfrak{S}_{2n}]$-module. Similarly, the specialized Brauer algebra $\mathfrak{B}_n(-2m)$ becomes a right $K[\mathfrak{S}_{2n}]$-module.

For any integer i with $1 \leq i \leq 2n$, we define

$$\gamma(i) := \begin{cases} i+1, & \text{if } i \text{ is odd,} \\ i-1, & \text{if } i \text{ is even.} \end{cases}$$

Then γ is an involution on $\{1, 2, \cdots, 2n\}$. It is well-known that the subgroup

$$\left\{ w \in \mathfrak{S}_{2n} \mid (\gamma(a))w = \gamma(aw) \text{ for any integer } a \text{ with } 1 \leq a \leq 2n \right\}$$

is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$ of $\mathbb{Z}_2$ and $\mathfrak{S}_n$, which is a Weyl group of type B_n (cf. [21]).

Lemma 2.1 *For any commutative $\mathbb{Z}$-algebra R, there is a right $R[\mathfrak{S}_{2n}]$-module isomorphism*

$$\mathfrak{B}_{n,R} \cong \text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R,$$

where 1_R denotes the rank one trivial representation of $R[\mathbb{Z}_2 \wr \mathfrak{S}_n]$.

Proof Let $1_{\mathfrak{B}_n}$ be the element in Bd_n that connects $2i-1$ to $2i$ for each integer i with $1 \leq i \leq n$. Since $\mathfrak{S}_{2n}$ acts transitively on the set of all the Brauer n-diagrams, it is easy to see that the map ξ_R which send 1_R to $1_{\mathfrak{B}_n}$ extends naturally to a surjective $R[\mathfrak{S}_{2n}]$-module homomorphism from $\text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R$ onto $\mathfrak{B}_{n,R}$.

If R is a field, then we can compare the dimensions of both modules. In that case, we know that the surjection ξ_R must be an injection, and hence be an isomorphism. In general, since there are natural isomorphisms

$$\text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R \cong R \otimes_{\mathbb{Z}} \text{Ind}_{\mathbb{Z}[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{\mathbb{Z}[\mathfrak{S}_{2n}]} 1_{\mathbb{Z}}, \qquad \mathfrak{B}_{n,R} \cong R \otimes_{\mathbb{Z}} \mathfrak{B}_{n,\mathbb{Z}},$$

and ξ_R is naturally identified with $1_R \otimes_{\mathbb{Z}} \xi_{\mathbb{Z}}$, it suffices to show that $\xi_{\mathbb{Z}}$ is an isomorphism. Note also that the short exact sequence

$$0 \to \text{Ker}\, \xi_{\mathbb{Z}} \to \text{Ind}_{\mathbb{Z}[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{\mathbb{Z}[\mathfrak{S}_{2n}]} 1_{\mathbb{Z}} \to \mathfrak{B}_{n,\mathbb{Z}} \to 0$$

splits as $\mathbb{Z}$-modules. It follows that $\text{Ker}\, \xi_R$ is canonically isomorphic to $R \otimes_{\mathbb{Z}} \text{Ker}\, \xi_{\mathbb{Z}}$ for any commutative $\mathbb{Z}$-algebra R. Let $N := \text{Ker}\, \xi_{\mathbb{Z}}$. It is enough to show that $N = 0$. By [1, Proposition 3.8], we only need to show that $N_{(p)} = 0$ for each prime number p. Let $k_p := \mathbb{Z}/(p)$, the residue field at the prime number p. It is clear that $k_p \cong \mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)}$. Note that

$$N_{(p)}/(p)N_{(p)} \cong k_p \otimes_{\mathbb{Z}_{(p)}} N_{(p)} \cong k_p \otimes_{\mathbb{Z}_{(p)}} \text{Ker}\, \xi_{\mathbb{Z}_{(p)}} \cong \text{Ker}\, \xi_{k_p} = 0.$$

Applying Nakayama's lemma ([1, 2.6]), we conclude that $N_{(p)} = 0$. This completes the proof of the lemma. □

For any positive integer k and any composition $\mu = (\mu_1, \cdots, \mu_s)$ of k, the Young diagram of μ is defined to be the set $[\mu] := \{(a,b) \mid 1 \leq a \leq s, 1 \leq b \leq \mu_a\}$. The elements of $[\mu]$ are called nodes of μ. A μ-tableau $\mathfrak{t}$ is defined to be a bijective map

from the Young diagram $[\mu]$ to the set $\{1, 2, \cdots, k\}$. We denote by $\mathfrak{t}^\mu$ the μ-tableau in which the numbers $1, 2, \cdots, k$ appear in order along successive rows. The row stabilizer of $\mathfrak{t}^\mu$, denoted by $\mathfrak{S}_\mu$, is the standard Young subgroup of $\mathfrak{S}_k$ corresponding to μ. For example, if $k = 6$, $\mu = (2, 3, 1)$, then

$$\mathfrak{t}^\mu = \begin{matrix} 1\,2 \\ 3\,4\,5 \\ 6 \end{matrix}, \qquad \mathfrak{S}_\mu = \text{the subgroup of } \mathfrak{S}_6 \text{ generated by } \{s_1, s_3, s_4\}.$$

We define

$$x_\mu = \sum_{w \in \mathfrak{S}_\mu} w, \qquad y_\mu = \sum_{w \in \mathfrak{S}_\mu} (-1)^{\ell(w)} w,$$

where $\ell(-)$ is the length function in $\mathfrak{S}_k$. If μ is a partition of k, we denote by $\mathfrak{t}_\mu$ the μ-tableau in which the numbers $1, 2, \cdots, k$ appear in order along successive columns. Let $w_\mu \in \mathfrak{S}_k$ be such that $\mathfrak{t}^\mu w_\mu = \mathfrak{t}_\mu$. For example, if $k = 8$, $\mu = (3, 3, 1, 1)$, then

$$\mathfrak{t}^\mu = \begin{matrix} 1\,2\,3 \\ 4\,5\,6 \\ 7 \\ 8 \end{matrix}, \qquad \mathfrak{t}_\mu = \begin{matrix} 1\,5\,7 \\ 2\,6\,8 \\ 3 \\ 4 \end{matrix}, \qquad w_\mu = (2, 5, 6, 8, 4)(3, 7).$$

For any partition μ of $2n$, we define the associated Specht module S^μ to be the right ideal of the group algebra $\mathbb{Z}[\mathfrak{S}_{2n}]$ generated by $y_{\mu'} w_{\mu'} x_\mu$. In particular, $S^{(2n)}$ is the one-dimensional trivial representation of $\mathfrak{S}_{2n}$, while $S^{(1^{2n})}$ is the one dimensional sign representation of $\mathfrak{S}_{2n}$. By [13, Theorem 3.5] and [31, 5.3], our S^μ is isomorphic to the dual Specht module $\widetilde{S}^\mu$ introduced in [31, Section 5]. For any commutative $\mathbb{Z}$-algebra R, we write $S^\mu_R := R \otimes_{\mathbb{Z}} S^\mu$. Then $\{S^\mu_{\mathbb{Q}} | \mu \vdash 2n\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{Q}[\mathfrak{S}_{2n}]$-modules.

For any composition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of n, let $2\lambda := (2\lambda_1, \cdots, 2\lambda_s)$, which is a composition of $2n$. We use $\mathcal{P}_n$ to denote the set of partitions of n. We define $2\mathcal{P}_n := \{2\lambda \mid \lambda \in \mathcal{P}_n\}$.

Lemma 2.2 *There is an isomorphism of right $\mathbb{Q}[\mathfrak{S}_{2n}]$-modules*:

$$\mathfrak{B}_{n,\mathbb{Q}} \cong \bigoplus_{\lambda \in 2\mathcal{P}_n} S^\lambda_{\mathbb{Q}}.$$

Proof This follows from Lemma 2.1 and [27, Chapter VII, (2.4)]. □

Let a be an integer with $0 \le a \le n$. Let $1 \le i_1, \cdots, i_a, j_1, \cdots, j_a \le 2n$ be $2a$ distinct integers. Let

$$I := \{1, 2, \cdots, 2n\} \setminus \{i_1, \cdots, i_a, j_1, \cdots, j_a\}.$$

Let $\mathfrak{S}_I$ be the symmetric group on the set I. Let $\underline{i} := (i_1, \cdots, i_a)$, $\underline{j} := (j_1, \cdots, j_a)$. Let

$$\mathrm{Bd}_n(\underline{i}, \underline{j}) := \left\{ D \in \mathrm{Bd}_n \mid D \text{ connects } i_s \text{ with } j_s \text{ for each } 1 \le s \le a \right\}.$$

Lemma 2.3 *With the notations as above, for any $w \in \mathfrak{S}_I$, we have*

$$\left(\sum_{D \in \mathrm{Bd}_n(\underline{i}, \underline{j})} D \right) * w = \sum_{D \in \mathrm{Bd}_n(\underline{i}, \underline{j})} D.$$

Proof For any $D \neq D' \in \mathrm{Bd}_n(\underline{i}, \underline{j})$, it is clear that

$$D * w \neq D' * w \in \mathrm{Bd}_n(\underline{i}, \underline{j}).$$

Therefore, the lemma follows easily from a counting argument. $\qquad\square$

Definition 2.4 For any non-negative even integers a, b with $a + b \leq 2n$, we define

$$\mathrm{Bd}_{(b)}^{(a)} := \left\{ D \in \mathrm{Bd}_n \left| \begin{array}{c} \text{the vertex labeled by } i \text{ is connected with} \\ \text{the vertex labeled by } \gamma(i) \text{ whenever} \\ i \leq a \text{ or } i > a + b \end{array} \right. \right\},$$

$$X_{(b)}^{(a)} := \sum_{D \in \mathrm{Bd}_{(b)}^{(a)}} D.$$

For any even integer k with $0 \leq k \leq 2n$, let $X_{(k)} := X_{(k)}^{(0)}$.

Definition 2.5 Let $\lambda = (\lambda_1, \cdots, \lambda_s)$ be a composition of $2n$ such that λ_i is even for each i, and define

$$X_\lambda := X_{(\lambda_1)}^{(0)} X_{(\lambda_2)}^{(\lambda_1)} \cdots X_{(\lambda_s)}^{(\lambda_1 + \cdots + \lambda_{s-1})} \in \mathfrak{B}_n.$$

Note that X_λ is nothing but a sum of all the Brauer n-diagram D which satisfies the following condition: for each integer $1 \leq i \leq s$ and each integer a with $\sum_{j=1}^{i-1} \lambda_j + 1 \leq a \leq \sum_{j=1}^{i} \lambda_j$, the vertex labeled by a in D can only be connected with a vertex labeled by b for some integer b with $\sum_{j=1}^{i-1} \lambda_j + 1 \leq b \leq \sum_{j=1}^{i} \lambda_j$ and $b \neq a$.

Corollary 2.6 *Let $\lambda = (\lambda_1, \cdots, \lambda_s)$ be a composition of $2n$ such that λ_i is even for each i. Then, for any $w \in \mathfrak{S}_\lambda$, we have that*

$$X_\lambda * w = X_\lambda.$$

Proof For any non-negative even integers a, b with $a + b \leq 2n$, the set $\mathrm{Bd}_{(b)}^{(a)}$ is just a special case of the set $\mathrm{Bd}_n(\underline{i}, \underline{j})$ we defined before. Therefore, by Lemma 2.3, for any $w \in \mathfrak{S}_{\{a+1, a+2, \cdots, a+b\}}$, we have

$$X_{(b)}^{(a)} * w = X_{(b)}^{(a)}.$$

Now we note that the elements $X_{(\lambda_1)}^{(0)}, X_{(\lambda_2)}^{(\lambda_1)}, \cdots, X_{(\lambda_s)}^{(\lambda_1 + \cdots + \lambda_{s-1})}$ pairwise commute with each other. Hence the corollary follows at once. $\qquad\square$

Let k be a positive integer and μ be a composition of k. A μ-tableau $\mathfrak{t}$ is called *row standard* if the numbers increase along rows. We use $\mathrm{RowStd}(\mu)$ to denote the set of all the row-standard μ-tableaux. Suppose μ is a partition of k. Then $\mathfrak{t}$ is called *column standard* if the numbers increase down columns, and *standard* if it is both row and column standard. In this case, it is clear that both $\mathfrak{t}^\mu$ and $\mathfrak{t}_\mu$ are standard μ-tableaux. We use $\mathrm{Std}(\mu)$ to denote the set of all the standard μ-tableaux. Let $\lambda \in 2\mathcal{P}_n$. For any $\mathfrak{t} \in \mathrm{RowStd}(\lambda)$, let $d(\mathfrak{t}) \in \mathfrak{S}_{2n}$ be such that $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$. Let $X_{\lambda,\mathfrak{t}} := X_\lambda * d(\mathfrak{t})$. For any commutative $\mathbb{Z}$-algebra R, we define

$$\mathcal{M}_R^\lambda := R\text{-Span}\left\{ X_{\nu,\mathfrak{t}} \,\middle|\, \mathfrak{t} \in \mathrm{Std}(\nu), \lambda \trianglelefteq \nu \in 2\mathcal{P}_n \right\},$$

where "$\trianglerighteq$" is the usual dominance order, defined for example in [31]. We write $\mathcal{M}^\lambda = \mathcal{M}_{\mathbb{Z}}^\lambda$. We are interested in the module $\mathcal{M}_R^\lambda$. In the remaining part of this paper, we shall see that this module is actually a right $\mathfrak{S}_{2n}$-submodule of $\mathfrak{B}_{n,R}$, and it shares many properties with the permutation module $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$. In particular, it also has a Specht filtration, and it is stable under base change, i.e., $R \otimes_\mathbb{Z} \mathcal{M}^\lambda \cong \mathcal{M}_R^\lambda$ for any commutative $\mathbb{Z}$-algebra R.

For our purpose, we need to recall some results in [31] and [28] on the Specht filtrations of permutation modules over the symmetric group $\mathfrak{S}_{2n}$. Let λ, μ be two partitions of $2n$. A μ-tableau of type λ is a map $\mathrm{S} : [\mu] \to \{1, 2, \cdots, 2n\}$ such that each i appears exactly λ_i times. S is said to be semistandard if each row of S is weakly increasing and each column of S is strictly increasing. Let $\mathcal{T}_0(\mu, \lambda)$ be the set of all the semistandard μ-tableaux of type λ. Then $\mathcal{T}_0(\mu, \lambda) \neq \emptyset$ only if $\mu \trianglerighteq \lambda$. For each standard μ-tableau $\mathfrak{s}$, let $\lambda(\mathfrak{s})$ be the tableau which is obtained from $\mathfrak{s}$ by replacing each entry i in $\mathfrak{s}$ by r if i appears in row r of $\mathfrak{t}^\lambda$. Then $\lambda(\mathfrak{s})$ is a μ-tableau of type λ.

For each standard μ-tableau $\mathfrak{t}$ and each semistandard μ-tableau S of type λ, we define

$$x_{\mathrm{S},\mathfrak{t}} := \sum_{\mathfrak{s} \in \mathrm{Std}(\mu), \lambda(\mathfrak{s}) = \mathrm{S}} d(\mathfrak{s})^{-1} x_\mu d(\mathfrak{t}).$$

Then by [31, Section 7], the set

$$\left\{ x_{\mathrm{S},\mathfrak{t}} \,\middle|\, \mathrm{S} \in \mathcal{T}_0(\mu, \lambda), \mathfrak{t} \in \mathrm{Std}(\mu), \lambda \trianglelefteq \mu \vdash 2n \right\}$$

form a $\mathbb{Z}$-basis of $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$. Furthermore, for any commutative $\mathbb{Z}$-algebra R, the canonical surjective homomorphism $R \otimes_\mathbb{Z} x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}] \twoheadrightarrow x_\lambda R[\mathfrak{S}_{2n}]$ is an isomorphism.

For each partition μ of $2n$ and for each semistandard μ-tableau S of type λ, according to the results in [31, Section 7] and [28], both the following $\mathbb{Z}$-submodules

$$M_{\mathrm{S}}^\lambda := \mathbb{Z}\text{-Span}\Big(\left\{ x_{\mathrm{S},\mathfrak{s}} \,\middle|\, \mathfrak{s} \in \mathrm{Std}(\mu) \right\}$$

$$\cup \left\{ x_{\mathrm{T},\mathfrak{t}} \,\middle|\, \mathrm{T} \in \mathcal{T}_0(\nu, \lambda), \mathfrak{t} \in \mathrm{Std}(\nu), \mu \vartriangleleft \nu \vdash 2n \right\} \Big),$$

$$M_{\mathrm{S},\triangleright}^\lambda := \mathbb{Z}\text{-Span}\left\{ x_{\mathrm{T},\mathfrak{t}} \,\middle|\, \mathrm{T} \in \mathcal{T}_0(\nu, \lambda), \mathfrak{t} \in \mathrm{Std}(\nu), \mu \vartriangleleft \nu \vdash 2n \right\},$$

are $\mathbb{Z}[\mathfrak{S}_{2n}]$-submodules, and the quotient of M_S^λ by $M_{S,\rhd}^\lambda$ is canonically isomorphic to S^μ so that the image of the elements $x_{S,\mathfrak{s}}$, where $\mathfrak{s} \in \mathrm{Std}(\mu)$, forms the standard $\mathbb{Z}$-basis of S^μ. In other words, it gives rise to the Specht filtrations of $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$, each semistandard μ-tableau of type λ yields a factor which is isomorphic to S^μ so that $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ has a series of factors, ordered by $\trianglelefteq$, each isomorphic to some S^μ, $\mu \trianglerighteq \lambda$, the multiplicity of S^μ being the number of semistandard μ-tableaux of type λ.

We write $\lambda = (\lambda_1, \lambda_2, \cdots) = (a_1^{k_1}, a_2^{k_2}, \cdots,)$, where $a_1 > a_2 > \cdots$, $k_i \in \mathbb{N}$ for each i, where $a_i^{k_i}$ means that a_i repeats k_i times. Let $\widetilde{\mathfrak{S}}_\lambda$ be the subgroup of $\mathfrak{S}_{\lambda'}$ consisting of all the elements w satisfying the following condition: for any integers i, j with $\lambda_i = \lambda_j$, and any integers a, b with $1 \leq a, b \leq \lambda_i$,

$$(\mathfrak{t}_\lambda(i, a))w = \mathfrak{t}_\lambda(j, a) \quad \text{if and only if} \quad (\mathfrak{t}_\lambda(i, b))w = \mathfrak{t}_\lambda(j, b). \qquad (2.1)$$

Let $\widetilde{D}_\lambda$ be a complete set of right coset representatives of $\widetilde{\mathfrak{S}}_\lambda$ in $\mathfrak{S}_{\lambda'}$. By convention, we set $0! = 1$.

Lemma 2.7 *For any partition $\lambda \in 2\mathcal{P}_n$, let*

$$n_\lambda := \prod_{i \geq 1}(\lambda_i' - \lambda_{i+1}')!, \qquad h_\lambda := \sum_{w \in \widetilde{D}_\lambda}(-1)^{\ell(w)}w.$$

Then $|\widetilde{\mathfrak{S}}_\lambda| = n_\lambda$, and

$$X_\lambda * \big(w_\lambda y_{\lambda'}\big) = n_\lambda\big(X_\lambda * (w_\lambda h_\lambda)\big),$$

and for any commutative $\mathbb{Z}$-algebra R, $1_R \otimes_{\mathbb{Z}} (X_\lambda * (w_\lambda h_\lambda)) \neq 0$ in $\mathfrak{B}_{n,R}$.

Proof By definition,

$$y_{\lambda'} = \sum_{w \in \mathfrak{S}_{\lambda'}}(-1)^{\ell(w)}w = \Big(\sum_{w \in \widetilde{\mathfrak{S}}_\lambda}(-1)^{\ell(w)}w\Big)h_\lambda.$$

Note that the condition $\lambda \in 2\mathcal{P}_n$ implies that for any $w \in \widetilde{\mathfrak{S}}_\lambda$, $\ell(w)$ is an even integer. Therefore,

$$y_{\lambda'} = \Big(\sum_{w \in \widetilde{\mathfrak{S}}_\lambda} w\Big)h_\lambda.$$

Since for each $w \in \widetilde{\mathfrak{S}}_\lambda$,

$$X_\lambda * (w_\lambda w) = X_\lambda * (w_\lambda w w_\lambda^{-1}) * w_\lambda = X_\lambda * w_\lambda,$$

it follows that

$$X_\lambda * \Big(w_\lambda \sum_{w \in \widetilde{\mathfrak{S}}_\lambda} w\Big) = n_\lambda X_\lambda * w_\lambda,$$

from which the first statement of this lemma follows. Let d be the Brauer n-diagram in which the vertex labeled by $\mathfrak{t}_\lambda(i, 2j - 1)$ is connected with the vertex labeled by

$t_\lambda(i, 2j)$ for any $1 \leq i \leq \lambda'_1, 1 \leq j \leq \lambda_i/2$. Then it is easy to see that d appears with coefficient 1 in the expression of $X_\lambda * (w_\lambda h_\lambda)$ as linear combinations of basis of Brauer n-diagrams. It follows that for any commutative $\mathbb{Z}$-algebra R, $1_R \otimes_\mathbb{Z} (X_\lambda * (w_\lambda h_\lambda)) \neq 0$ in $\mathfrak{B}_{n,R}$, as required. $\square$

Let t be a tableau with entries in $\{1, 2, \cdots, k\}$. For each integer a with $1 \leq a \leq k$, we define $\text{res}_t(a) = j - i$ if $t(i, j) = a$. Following [30], we define the Jucys-Murphy operators of $\mathbb{Z}[\mathfrak{S}_{2n}]$.

$$\begin{cases} L_1 := 0, \\ L_a := (a-1, a) + (a-2, a) + \cdots + (1, a), \quad a = 2, 3, \cdots, 2n. \end{cases}$$

Then for each partition λ of $2n$, and each integer $1 \leq a \leq 2n$, we have (by [13, (3.14)])

$$\left(x_\lambda w_\lambda y_{\lambda'}\right) L_a = \text{res}_{t_\lambda}(a)\left(x_\lambda w_\lambda y_{\lambda'}\right).$$

For each standard λ-tableau t, we define

$$\Theta_t := \prod_{i=1}^{n} \prod_{\substack{u \in \text{Std}(\lambda) \\ \text{res}_u(i) \neq \text{res}_t(i)}} \frac{L_i - \text{res}_u(i)}{\text{res}_t(i) - \text{res}_u(i)}.$$

For each partition $\lambda \in 2\mathcal{P}_n$, by Corollary 2.6 and Frobenius reciprocity, there is a surjective right $\mathbb{Z}[\mathfrak{S}_{2n}]$-module homomorphism π_λ from $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ onto $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ which is defined by $x_\lambda h \mapsto X_\lambda * h, \forall h \in \mathbb{Z}[\mathfrak{S}_{2n}]$. In particular, by Lemma 2.7,

$$\left(X_\lambda * w_\lambda h_\lambda\right) * L_a = \text{res}_{t_\lambda}(a)\left(X_\lambda * (w_\lambda h_\lambda)\right).$$

Proposition 2.8 *For any partition $\lambda \in 2\mathcal{P}_n$, we have that*

$$[X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}] : S_\mathbb{Q}^\lambda] = 1.$$

Proof By Lemma 2.2, we have

$$\mathfrak{B}_{n,\mathbb{Q}} \cong \bigoplus_{\mu \in 2\mathcal{P}_n} S_\mathbb{Q}^\mu.$$

It is well-known that each $S_\mathbb{Q}^\mu$ has a basis $\{v_t\}_{t \in \text{Std}(\mu)}$ satisfying

$$v_t L_i = \text{res}_t(i) v_t, \quad \forall 1 \leq i \leq n.$$

Let λ be a fixed partition in $2\mathcal{P}_n$. Since $X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}] \subseteq \mathfrak{B}_{n,\mathbb{Q}}$, we can write

$$X_\lambda * (w_\lambda h_\lambda) = \sum_{\mu \in 2\mathcal{P}_n} \sum_{t \in \text{Std}(\mu)} A_t v_t,$$

where $A_t \in \mathbb{Q}$ for each t.

For each $\mu \in 2\mathcal{P}_n$ and each $\mathfrak{t} \in \mathrm{Std}(\mu)$, we apply the operator $\Theta_\mathfrak{t}$ on both sides of the above identity and use Lemma 2.7 and the above discussion. We get that $A_\mathfrak{t} \neq 0$ if and only if $\mu = \lambda$ and $\mathfrak{t} = \mathfrak{t}_\lambda$. In other words, $X_\lambda * (w_\lambda h_\lambda) = A_{\mathfrak{t}_\lambda} v_{\mathfrak{t}_\lambda}$ for some $0 \neq A_{\mathfrak{t}_\lambda} \in \mathbb{Q}$. This implies that the projection from $X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}]$ to $S_\mathbb{Q}^\lambda$ is nonzero. Hence, $[X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}] : S_\mathbb{Q}^\lambda] = 1$, as required. $\qquad\square$

For each partition $\lambda \in 2\mathcal{P}_n$, by the natural surjective $\mathbb{Z}[\mathfrak{S}_{2n}]$-module homomorphism π_λ from $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ onto $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$, we know that the elements $\pi_\lambda(x_{S,\mathfrak{t}})$, where $S \in \mathcal{T}_0(\mu, \lambda), \mathfrak{t} \in \mathrm{Std}(\mu), \lambda \trianglelefteq \mu \vdash 2n$, span $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ as $\mathbb{Z}$-module.

Proposition 2.9 *For any two partitions λ, μ of $2n$, and for any $S \in \mathcal{T}_0(\mu, \lambda)$, we have that $\pi_\lambda(M_S^\lambda) \subseteq \mathcal{M}^\lambda$. In particular, $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}] \subseteq \mathcal{M}^\lambda$.*

Proof We first prove a weak version of the claim in this proposition. That is, for any two partitions λ, μ of $2n$, and for any $S \in \mathcal{T}_0(\mu, \lambda)$,

$$\pi_\lambda(M_S^\lambda) \subseteq \mathcal{M}_\mathbb{Q}^\lambda.$$

We consider the dominance order "$\trianglelefteq$" and use induction on λ. We start with the partition $(2n)$, which is the unique maximal partition of $2n$ with respect to "$\trianglelefteq$". In this case, $x_{(2n)}\mathbb{Z}[\mathfrak{S}_{2n}] = \mathbb{Z}x_{(2n)}$, and $X_{(2n)}\mathbb{Z}[\mathfrak{S}_{2n}] = \mathbb{Z}X_{(2n)}$, it is easy to see the claim in this proposition is true for $\lambda = (2n)$.

Now let $\lambda \lhd (2n)$ be a partition of $2n$. Assume that for any partition ν of $2n$ satisfying $\nu \rhd \lambda$, the claim in this proposition is true. We now prove the claim for the partition λ.

Let $\mu \trianglerighteq \lambda$ be a partition of $2n$ with $\mathcal{T}_0(\mu, \lambda) \neq \emptyset$. We consider again the dominance order "$\trianglelefteq$" and use induction on μ. Since $\mathcal{T}_0((2n), \lambda)$ contains a unique element $S_\star$, $\mathrm{Std}((2n)) = \{\mathfrak{t}^{(2n)}\}$ and

$$\pi_\lambda(x_{S_\star, \mathfrak{t}^{(2n)}}) = \pi_\lambda(x_{(2n)}) = X_{(2n)} \in \mathcal{M}^\lambda.$$

So in this case the claim of this proposition is still true.

Now let $\mu \trianglerighteq \lambda$ be a partition of $2n$ with $\mathcal{T}_0(\mu, \lambda) \neq \emptyset$ and $\mu \lhd (2n)$. Assume that for any partition ν of $2n$ satisfying $\mathcal{T}_0(\nu, \lambda) \neq \emptyset$ and $\nu \rhd \mu$, and for any $S \in \mathcal{T}_0(\nu, \lambda)$,

$$\pi_\lambda(M_S^\lambda) \subseteq \mathcal{M}_\mathbb{Q}^\lambda.$$

Let $S \in \mathcal{T}_0(\mu, \lambda)$. The homomorphism π_λ induces a surjective map from $M_S^\lambda / M_{S, \rhd}^\lambda$ onto

$$\left(\pi_\lambda(M_S^\lambda)\right) / \left(\pi_\lambda(M_{S, \rhd}^\lambda)\right).$$

Hence it also induces a surjective map $\widetilde{\pi}_\lambda$ from

$$\left(\mathbb{Q} \otimes_\mathbb{Z} M_S^\lambda\right) / \left(\mathbb{Q} \otimes_\mathbb{Z} M_{S, \rhd}^\lambda\right) \cong \mathbb{Q} \otimes_\mathbb{Z} \left(M_S^\lambda / M_{S, \rhd}^\lambda\right) \cong S_\mathbb{Q}^\mu$$

onto

$$\mathbb{Q} \otimes_\mathbb{Z} \left(\pi_\lambda(M_S^\lambda) / \pi_\lambda(M_{S, \rhd}^\lambda)\right).$$

Since $S_{\mathbb{Q}}^{\mu}$ is irreducible, the above map is either a zero map or an isomorphism. If it is a zero map, then (by induction hypothesis)

$$\pi_{\lambda}(M_S^{\lambda}) \subseteq \pi_{\lambda}(M_{S,\rhd}^{\lambda})_{\mathbb{Q}} \subseteq \mathcal{M}_{\mathbb{Q}}^{\lambda}.$$

It remains to consider the case where $\widetilde{\pi}_{\lambda}$ is an isomorphism. In particular,

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left(\pi_{\lambda}(M_S^{\lambda})/\pi_{\lambda}(M_{S,\rhd}^{\lambda}) \right) \cong S_{\mathbb{Q}}^{\mu}.$$

Applying Lemma 2.2, we know that $\mu \in 2\mathcal{P}_n$.

On the other hand, the homomorphism π_{μ} also induces a surjective map from $x_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]/M_{S_0,\rhd}^{\mu}$ onto

$$\left(\pi_{\mu}(x_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]) \right)/\left(\pi_{\mu}(M_{S_0,\rhd}^{\mu}) \right) = X_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]/\left(\pi_{\mu}(M_{S_0,\rhd}^{\mu}) \right),$$

where S_0 is the unique semistandard μ-tableau in $\mathcal{T}_0(\mu,\mu)$. Hence it also induces a surjective map $\widetilde{\pi}_{\mu}$ from

$$\left(\mathbb{Q} \otimes_{\mathbb{Z}} x_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}] \right)/\left(\mathbb{Q} \otimes_{\mathbb{Z}} M_{S_0,\rhd}^{\mu} \right) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \left(x_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]/M_{S_0,\rhd}^{\mu} \right) \cong S_{\mathbb{Q}}^{\mu}$$

onto

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left(X_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]/\pi_{\mu}(M_{S_0,\rhd}^{\mu}) \right) \cong \left(X_{\mu}\mathbb{Q}[\mathfrak{S}_{2n}] \right)/\left(\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\mu}(M_{S_0,\rhd}^{\mu}) \right).$$

By the Specht filtration of $M_{\mathbb{Q}}^{\mu}$, we know that $S_{\mathbb{Q}}^{\mu}$ does not occur as composition factor in $\mathbb{Q} \otimes_{\mathbb{Z}} M_{S_0,\rhd}^{\mu}$. Hence $S_{\mathbb{Q}}^{\mu}$ does not occur as composition factor in $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\mu}(M_{S_0,\rhd}^{\mu})$. By Proposition 2.8, $S_{\mathbb{Q}}^{\mu}$ occurs as composition factor with multiplicity one in $X_{\mu}\mathbb{Q}[\mathfrak{S}_{2n}]$. Therefore, $X_{\mu}\mathbb{Q}[\mathfrak{S}_{2n}] \neq \mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\mu}(M_{S_0,\rhd}^{\mu})$. It follows that $\widetilde{\pi}_{\mu}$ must be an isomorphism. Hence

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left(X_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]/\pi_{\mu}(M_{S_0,\rhd}^{\mu}) \right) \cong S_{\mathbb{Q}}^{\mu}.$$

We write $A = \pi_{\lambda}(M_S^{\lambda})$, $B = X_{\mu}\mathbb{Z}[\mathfrak{S}_{2n}]$. Since $S_{\mathbb{Q}}^{\mu}$ appears only once in $\mathfrak{B}_{n,\mathbb{Q}}$, it follows that $S_{\mathbb{Q}}^{\mu}$ must occur as composition factor in the module

$$\left(\mathbb{Q} \otimes_{\mathbb{Z}} A \right) \cap \left(\mathbb{Q} \otimes_{\mathbb{Z}} B \right) = \mathbb{Q} \otimes_{\mathbb{Z}} (A \cap B).$$

Hence $S_{\mathbb{Q}}^{\mu}$ can not occur as composition factor in the module

$$\left(\mathbb{Q} \otimes_{\mathbb{Z}} A \right)/\left(\mathbb{Q} \otimes_{\mathbb{Z}} (A \cap B) \right) \cong \mathbb{Q} \otimes_{\mathbb{Z}} (A/A \cap B).$$

Therefore, the image of the canonical projection $\mathbb{Q} \otimes_{\mathbb{Z}} A \to \mathbb{Q} \otimes_{\mathbb{Z}} (A/A \cap B)$ must be contained in the image of $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_{\lambda}(M_{S,\rhd}^{\lambda})$. However, by induction hypothesis, both $\pi_{\lambda}(M_{S,\rhd}^{\lambda})$ and B are contained in the $\mathbb{Q}$-span of $\{X_{\alpha,\mathfrak{u}} | \mathfrak{u} \in \mathrm{Std}(\alpha), \lambda \trianglelefteq \alpha \in 2\mathcal{P}_n\}$. It follows that

$$\pi_{\lambda}(M_S^{\lambda}) \subseteq \mathcal{M}_{\mathbb{Q}}^{\lambda},$$

as required.

Now we begin to prove $\pi_\lambda(M_S^\lambda) \subseteq \mathcal{M}^\lambda$. Suppose that

$$\pi_\lambda(M_S^\lambda) \nsubseteq \mathcal{M}^\lambda.$$

Then (by the $\mathbb{Z}$-freeness of $\mathfrak{B}_n$) there exist an element $x \in M_S^\lambda$, some integers $a, a_{\mathfrak{u}}$, and a prime divisor $p \in \mathbb{N}$ of a, such that

$$a\pi_\lambda(x) = \sum_{\lambda \trianglelefteq \alpha \in 2\mathcal{P}_n} \sum_{\mathfrak{u} \in \text{Std}(\alpha)} a_{\mathfrak{u}} X_\alpha * d(\mathfrak{u}),$$

and

$$\Sigma_p := \{\alpha \in 2\mathcal{P}_n \mid \lambda \trianglelefteq \alpha, \ p \nmid a_{\mathfrak{u}}, \text{ for some } \mathfrak{u} \in \text{Std}(\alpha)\} \neq \emptyset.$$

We take an $\alpha \in \Sigma_p$ such that α is minimal with respect to "$\trianglelefteq$". Then we take an $u \in \text{Std}(\alpha)$ such that $p \nmid a_{\mathfrak{u}}$ and $\ell(d(\mathfrak{u}))$ is maximal among the elements in the set $\{\mathfrak{u} \in \text{Std}(\alpha) \mid p \nmid a_{\mathfrak{u}}\}$. Let $\sigma_{\mathfrak{u}}$ be the unique element in $\mathfrak{S}_{2n}$ such that $d(\mathfrak{u})\sigma_{\mathfrak{u}} = w_\alpha$ and $\ell(w_\alpha) = \ell(d(\mathfrak{u})) + \ell(\sigma_{\mathfrak{u}})$. We consider the finite field $\mathbb{F}_p$ as a $\mathbb{Z}$-algebra. By [12, (4.1)], we know that for any partitions β, γ of $2n$, and element $w \in \mathfrak{S}_{2n}$,

$$x_\beta w y_{\gamma'} \neq 0 \text{ only if } \gamma \trianglerighteq \beta; \text{ while } x_\beta w y_{\beta'} \neq 0 \text{ only if } w \in \mathfrak{S}_\beta w_\beta.$$

Hence by Lemma 2.7 and the homomorphism π_λ,

$$X_\beta * (wh_\gamma) \neq 0 \text{ only if } \gamma \trianglerighteq \beta; \ X_\beta * (wh_\beta) \neq 0 \text{ only if } w \in \mathfrak{S}_\beta w_\beta.$$

Using Lemma 2.7 again, we get

$$0 = 1_{\mathbb{F}_p} \otimes_{\mathbb{Z}} \left(a\pi_\lambda(x) * (\sigma_{\mathfrak{u}} h_\alpha)\right) = 1_{\mathbb{F}_p} \otimes_{\mathbb{Z}} \left(a_{\mathfrak{u}} X_\alpha * (w_\alpha h_\alpha)\right) \neq 0,$$

which is a contradiction. This proves that $\pi_\lambda(M_S^\lambda) \subseteq \mathcal{M}^\lambda$. $\qquad \square$

Corollary 2.10 *For any partition $\lambda \in 2\mathcal{P}_n$ and any commutative $\mathbb{Z}$-algebra R, $\mathcal{M}_R^\lambda$ is a right $\mathfrak{S}_{2n}$-submodule of $\mathfrak{B}_{n,R}$.*

Proof This follows directly from Proposition 2.9. $\qquad \square$

Theorem 2.11 *For any partition $\lambda \in 2\mathcal{P}_n$ and any commutative $\mathbb{Z}$-algebra R, the canonical map $R \otimes_{\mathbb{Z}} \mathcal{M}^\lambda \to \mathcal{M}_R^\lambda$ is an isomorphism, and the set*

$$\left\{ X_{v,\mathfrak{t}} \mid \mathfrak{t} \in \text{Std}(v), \lambda \trianglelefteq v \in 2\mathcal{P}_n \right\}$$

forms an R-basis of $\mathcal{M}_R^\lambda$. In particular, the set

$$\left\{ X_{\lambda,\mathfrak{t}} \mid \mathfrak{t} \in \text{Std}(\lambda), \lambda \in 2\mathcal{P}_n \right\}$$

forms an R-basis of $\mathfrak{B}_{n,R}$.

Proof We take $\lambda = (2^n)$, then $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}] = \mathfrak{B}_n$. It is well-known that $\mathfrak{B}_{n,R} \cong R \otimes_{\mathbb{Z}} \mathfrak{B}_n$ for any commutative $\mathbb{Z}$-algebra R. Applying Proposition 2.9 and counting the dimension, we get that for any $\mathbb{Z}$-algebra R which is a field, the set

$$\left\{ X_{\lambda,\mathfrak{t}} \,\middle|\, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in 2\mathcal{P}_n \right\}$$

must form an R-basis of $\mathfrak{B}_{n,R}$. Since for any commutative $\mathbb{Z}$-algebra R, the natural map $r \otimes_{\mathbb{Z}} d \mapsto rd, \forall r \in R, d \in \mathfrak{B}_n$, defines an R-module isomorphism $R \otimes_{\mathbb{Z}} \mathfrak{B}_n \cong \mathfrak{B}_{n,R}$, it follows that the above set is still an R-basis of $\mathfrak{B}_{n,R}$ for any commutative $\mathbb{Z}$-algebra R.

By the R-linear independence of the elements in this set and Corollary 2.10, we also get that, for any partition $\lambda \in 2\mathcal{P}_n$, the set

$$\left\{ X_{\nu,\mathfrak{t}} \,\middle|\, \mathfrak{t} \in \mathrm{Std}(\nu), \lambda \trianglelefteq \nu \in 2\mathcal{P}_n \right\}$$

must form an R-basis of $\mathcal{M}_R^\lambda$. Therefore, for any commutative $\mathbb{Z}$-algebra R, the canonical map $R \otimes_{\mathbb{Z}} \mathcal{M}^\lambda \to \mathcal{M}_R^\lambda$ is an isomorphism.   □

Remark 2.12 Note that for any partition $\lambda \in 2\mathcal{P}_n$, $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}] \subseteq \mathcal{M}^\lambda$. But $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ is not necessarily equal to $\mathcal{M}^\lambda$ in general. For example, one sees easily that

$$X_{(6,2)} \notin X_{(4,4)} \mathbb{Z}[\mathfrak{S}_8].$$

In fact, if this is not the case, then we can write

$$X_{(6,2)} + \sum_i a_i X_{(4,4)} * w_i = \sum_j b_j X_{(4,4)} * w'_j,$$

for some positive integers a_i, b_j and some elements $w_i, w'_j \in \mathfrak{S}_{2n}$. However, if we express both sides into linear combinations of Brauer 4-diagrams and count the number of terms, we find that this is impossible (as the equation $15 + 9a = 9b$ has no solutions in $\mathbb{Z}$).

Theorem 2.13 *For any partition $\lambda \in 2\mathcal{P}_n$ and any commutative $\mathbb{Z}$-algebra R, we define*

$$\mathcal{M}_R^{\rhd\lambda} := R\text{-Span}\left\{ X_{\nu,\mathfrak{t}} \,\middle|\, \mathfrak{t} \in \mathrm{Std}(\nu), \lambda \lhd \nu \in 2\mathcal{P}_n \right\}.$$

Then $\mathcal{M}_R^{\rhd\lambda}$ is a right $R[\mathfrak{S}_{2n}]$-submodule of $\mathcal{M}_R^\lambda$, and there is a $R[\mathfrak{S}_{2n}]$-module isomorphism

$$\mathcal{M}_R^\lambda / \mathcal{M}_R^{\rhd\lambda} \cong S_R^\lambda.$$

In particular, $\mathfrak{B}_{n,R}$ has a multiplicity free Specht filtration.

Proof It suffices to consider the case where $R = \mathbb{Z}$. We first show that

$$\mathcal{M}_{\mathbb{Q}}^\lambda \cong \oplus_{\lambda \trianglelefteq \mu \in 2\mathcal{P}_n} S_{\mathbb{Q}}^\mu, \quad \mathcal{M}_{\mathbb{Q}}^{\rhd\lambda} \cong \oplus_{\lambda \lhd \mu \in 2\mathcal{P}_n} S_{\mathbb{Q}}^\mu.$$

For each $\mu \in 2\mathcal{P}_n$, we use ρ_μ^λ to denote the composition of the embedding $\mathcal{M}_{\mathbb{Q}}^\lambda \hookrightarrow \mathfrak{B}_{n,\mathbb{Q}}$ and the projection $\mathfrak{B}_{n,\mathbb{Q}} \twoheadrightarrow S_{\mathbb{Q}}^\mu$. Suppose that $\rho_\mu^\lambda \neq 0$. Then ρ_μ^λ must be a surjection. We claim that $\mu \unrhd \lambda$. In fact, if $\mu \ntrianglerighteq \lambda$, then for any $\lambda \trianglelefteq \nu \in 2\mathcal{P}_n$, $\mu \ntrianglerighteq \nu$, and $x_\nu \mathbb{Z}[\mathfrak{S}_{2n}] w_{\mu'} x_\mu w_\mu y_{\mu'} = 0$, hence $X_{\nu,\mathfrak{t}} * (w_{\mu'} x_\mu w_\mu y_{\mu'}) = 0$ for any $\mathfrak{t} \in \mathrm{Std}(\nu)$. It follows that $\mathcal{M}_{\mathbb{Q}}^\lambda (w_{\mu'} x_\mu w_\mu y_{\mu'}) = 0$. Therefore, $S_{\mathbb{Q}}^\mu (w_{\mu'} x_\mu w_\mu y_{\mu'}) = 0$. On the other hand, since $S_{\mathbb{Q}}^\mu \cong x_\mu w_{\mu'} y_{\mu'} \mathbb{Q}[\mathfrak{S}_{2n}]$, and by [25, Lemma 5.7],

$$x_\mu w_{\mu'} y_{\mu'} (w_{\mu'} x_\mu w_\mu y_{\mu'}) = \Big(\prod_{(i,j) \in [\mu]} h_{i,j}^\mu \Big) x_\mu w_{\mu'} y_{\mu'} \neq 0,$$

where $h_{i,j}^\mu$ is the (i,j)-hook length in $[\mu]$, we get a contradiction. Therefore, $\rho_\mu^\lambda \neq 0$ must imply that $\mu \unrhd \lambda$. Now counting the dimensions, we deduce that $\mathcal{M}_{\mathbb{Q}}^\lambda \cong \oplus_{\lambda \trianglelefteq \mu \in 2\mathcal{P}_n} S_{\mathbb{Q}}^\mu$. In a similar way, we can prove that $\mathcal{M}_{\mathbb{Q}}^{\rhd\lambda} \cong \oplus_{\lambda \lhd \mu \in 2\mathcal{P}_n} S_{\mathbb{Q}}^\mu$. It follows that $\mathcal{M}_{\mathbb{Q}}^\lambda / \mathcal{M}_{\mathbb{Q}}^{\rhd\lambda} \cong S_{\mathbb{Q}}^\lambda$.

We now consider the natural map from $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ onto $\mathcal{M}^\lambda / \mathcal{M}^{\rhd\lambda} \cong S^\lambda$. Since $\mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathfrak{S}_0, \rhd}^\lambda$ does not contain $S_{\mathbb{Q}}^\lambda$ as a composition factor, it follows that (by Proposition 2.9) the image of $M_{\mathfrak{S}_0, \rhd}^\lambda$ must be contained in $\mathcal{M}^{\rhd\lambda}$. Therefore we get a surjective map from S^λ onto $\mathcal{M}^\lambda / \mathcal{M}^{\rhd\lambda} \cong S^\lambda$. This map sends the standard basis of S^λ to the canonical basis of $\mathcal{M}^\lambda / \mathcal{M}^{\rhd\lambda}$. So it must be injective as well, as required. $\qquad\square$

3 The *n*-tensor space $V^{\otimes n}$

In this section, we shall use the results obtained in Section 2 and in [11] to give an explicit and characteristic-free description of the annihilator of the *n*-tensor space $V^{\otimes n}$ in the Brauer algebra $\mathfrak{B}_n(-2m)$.

Let K be an arbitrary infinite field. Let $m, n \in \mathbb{N}$. Let V be a $2m$-dimensional symplectic K-vector space. Let $Sp(V)$ be the corresponding symplectic group, acting naturally on V, and hence on the *n*-tensor space $V^{\otimes n}$ from the left-hand side. As we mentioned in the introduction, this left action on $V^{\otimes n}$ is centralized by the specialized Brauer algebra $\mathfrak{B}_n(-2m)_K := K \otimes_{\mathbb{Z}} \mathfrak{B}_n(-2m)$, where K is regarded as a $\mathbb{Z}$-algebra by sending each integer a to $a \cdot 1_K$. The Brauer algebra $\mathfrak{B}_n(-2m)_K$ acts on the *n*-tensor space $V^{\otimes n}$ from the right-hand side. Let φ be the natural K-algebra homomorphism

$$\varphi : (\mathfrak{B}_n(-2m)_K)^{op} \rightarrow \mathrm{End}_K (V^{\otimes n}).$$

By the discussion in [11, Section 3], $V^{\otimes n}$ is a tilting module over $K Sp(V)$. By [14, (4.4)], the dimension of $\mathrm{End}_{K Sp(V)}(V^{\otimes n})$ is independent of the choice of the infinite field K. Therefore, the dimension of $\mathrm{Ker}\, \varphi := \{ y \in \mathfrak{B}_n(-2m)_K \mid \varphi(y) = 0 \}$ is also independent of the choice of the infinite field K. That is, the dimension of the annihilator of the *n*-tensor space $V^{\otimes n}$ in the Brauer algebra $\mathfrak{B}_n(-2m)_K$ is independent of the choice of the infinite field K.

Lemma 3.1 *With the notations as above, we have that*

$$\dim(\operatorname{Ker}\varphi) = \sum_{\substack{\lambda \in 2\mathcal{P}_n \\ \lambda_1 > 2m}} \dim S^\lambda.$$

Proof By Lemma 2.2 and Lemma 1.1, it suffices to consider the case where $K = \mathbb{C}$ and to show that

$$\dim \operatorname{End}_{\mathbb{C}Sp_{2m}(V)}\bigl(V^{\otimes n}\bigr) = \sum_{\substack{\lambda \in 2\mathcal{P}_n \\ \lambda_1 \leq 2m}} \dim S^\lambda.$$

Note that $\dim S^\lambda = \dim S^{\lambda'}$, and

$$\operatorname{End}_{\mathbb{C}Sp_{2m}(V)}\bigl(V^{\otimes n}\bigr) \cong \Bigl(\bigl(V^{\otimes n}\bigr) \otimes \bigl(V^{\otimes n}\bigr)^*\Bigr)^{Sp(V)} \cong \bigl(V^{\otimes 2n}\bigr)^{Sp(V)}.$$

Therefore, it suffices to show that

$$\dim\bigl(V^{\otimes 2n}\bigr)^{Sp(V)} = \sum_{\substack{\lambda \in 2\mathcal{P}_n \\ \lambda_1 \leq 2m}} \dim S^{\lambda'}.$$

By the well-known Schur–Weyl duality between the general linear group $GL(V)$ and the symmetric group $\mathfrak{S}_{2n}$ on the tensor space $V^{\otimes 2n}$, we know that there is a $(GL(V), \mathfrak{S}_{2n})$-bimodules decomposition

$$V^{\otimes 2n} \cong \bigoplus_{\substack{\lambda \vdash 2n \\ \ell(\lambda) \leq 2m}} \widetilde{\Delta}_\lambda \otimes S^\lambda,$$

where $\widetilde{\Delta}_\lambda$ denotes the irreducible Weyl module with highest weight λ over GL(V). Here we identify λ with $\lambda_1 \varepsilon_1 + \cdots + \lambda_{2m} \varepsilon_{2m}$, ε_i is the weight of $GL(V)$ defined by

$$\varepsilon_i\bigl(\operatorname{diag}\{t_1, t_2, \cdots, t_{2m}\}\bigr) = t_i, \ \forall t_1, \cdots, t_{2m} \in K^\times,$$

for $i = 1, 2, \cdots, 2m$. It follows that

$$\dim\bigl(V^{\otimes 2n}\bigr)^{Sp(V)} = \sum_{\substack{\lambda \vdash 2n \\ \ell(\lambda) \leq 2m}} \dim\Bigl((\widetilde{\Delta}_\lambda \downarrow_{Sp(V)})^{Sp(V)}\Bigr) \dim(S^\lambda).$$

By the branching law (see [26, Proposition 2.5.1]) from $GL(V)$ to $Sp(V)$, we know that

$$\dim\Bigl((\widetilde{\Delta}_\lambda \downarrow_{Sp(V)})^{Sp(V)}\Bigr) = 1$$

if $\lambda' \in 2\mathcal{P}_n$; and 0 otherwise. This proves that

$$\dim\bigl(V^{\otimes 2n}\bigr)^{Sp(V)} = \sum_{\substack{\lambda \in 2\mathcal{P}_n \\ \lambda_1 \leq 2m}} \dim S^{\lambda'},$$

as required.   □

Let a, b be two integers such that $0 \leq a, b \leq n$ and $a + b$ is even. Let

$$I_a^{\mathrm{odd}} := \{1, 3, 5, \cdots, 2a - 1\}, \quad I_b^{\mathrm{even}} := \{2, 4, 6, \cdots, 2b\}.$$

If $a \geq b$, we define $\mathrm{Bd}_n(a, b)$ to be the set of all Brauer n-diagrams D such that:

- for each integer s with $a + 1 \leq s \leq n$, D connects the vertex labeled by $2s - 1$ with the vertex labeled by $2s$;
- for each integer s with $1 \leq s \leq (a - b)/2$, D connects the vertex labeled by $2b + 4s - 2$ with the vertex labeled by $2b + 4s$.
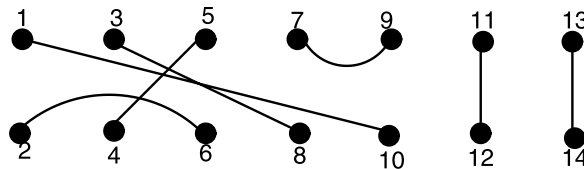
If $b \geq a$, we define $\mathrm{Bd}_n(a, b)$ to be the set of all Brauer n-diagrams D such that:

- for each integer $b + 1 \leq s \leq n$, D connects the vertex labeled by $2s - 1$ with the vertex labeled by $2s$;
- for each integer $1 \leq s \leq (b - a)/2$, d connects the vertex labeled by $2a + 4s - 3$ with the vertex labeled by $2a + 4s - 1$.

For example, let D_1 be the following Brauer 7-diagram
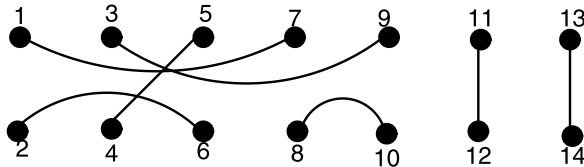
Then $D_1 \in \mathrm{Bd}_7(5, 3)$. Let D_2 be the following Brauer 7-diagram

Then $D_2 \in \mathrm{Bd}_7(3, 5)$.

Lemma 3.2 *Let a, b be two integers such that $0 \leq a, b \leq n$ and $a + b$ is even. Suppose that $a + b \geq 2m + 2$, then*

$$\sum_{D \in \mathrm{Bd}_n(a,b)} D \in \mathrm{Ker}\, \varphi.$$

The proof of Lemma 3.2 is somewhat complicated and will be given in Section 4.

Given any two subsets $A^{(1)} \subseteq \{1, 3, \cdots, 2n - 1\}$, $A^{(2)} \subseteq \{2, 4, \cdots, 2n\}$ with $|A^{(1)}| + |A^{(2)}|$ is even, we set $2n_0 = |A^{(1)}| + |A^{(2)}|$, and

$$\{a_1, a_2, \cdots, a_{2n-2n_0}\} := \{1, 2, \cdots, 2n\} \setminus (A^{(1)} \cup A^{(2)}).$$

Let $(i_1, j_1, i_2, j_2, \cdots, i_{n-n_0}, j_{n-n_0})$ be a fixed permutation of

$$\{a_1, a_2, \cdots, a_{2n-2n_0}\}.$$

Let

$$\mathbf{i} := (i_1, i_2, \cdots, i_{n-n_0}), \qquad \mathbf{j} := (j_1, j_2, \cdots, j_{n-n_0}).$$

We define $\mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})$ to be the set of all Brauer n-diagrams D such that: for each integer s with $1 \leq s \leq n - n_0$, D connects the vertex labeled by i_s with the vertex labeled by j_s. Note that the set $\mathrm{Bd}_n(a, b)$ we defined before is a special case of the set $\mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})$ we defined here.

Corollary 3.3 *With the notations as above and suppose that $2n_0 = |A^{(1)}| + |A^{(2)}| \geq 2m + 2$, then we have*

$$\sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D \in \mathrm{Ker}\, \varphi.$$

Proof Let $n_1 = |A^{(1)}|, n_2 = |A^{(2)}|$. If $n_1 \geq n_2$, then for any Brauer diagram $D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})$, there exist at least $\frac{n_1 - n_2}{2}$ bottom horizontal edges between the vertices labeled by the integers in the following set

$$\{2, 4, 6, \cdots, 2n\} \setminus A^{(2)}.$$

As a result, we deduce that there exist elements $\sigma_{A^{(1)}} \in \mathfrak{S}_{(1,3,\cdots,2n-1)}$, $\sigma_{A^{(2)}} \in \mathfrak{S}_{(2,4,\cdots,2n)}$ and a Brauer diagram D_1, such that

(1) for any integer a with $1 \leq a \leq n_1$, D_1 connects the vertex labeled by $2a - 1$ with the vertex labeled by $2a$.

(2)

$$\sigma_{A^{(1)}} \Bigg(\sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D \Bigg) \sigma_{A^{(2)}} = D_1 \Bigg(\sum_{D \in \mathrm{Bd}_n(|A^{(1)}|, |A^{(2)}|)} D \Bigg).$$

In this case, since both $\varphi(\sigma_{A^{(1)}})$ and $\varphi(\sigma_{A^{(2)}})$ are invertible, it follows directly from Lemma 3.2 that $\sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D \in \mathrm{Ker}\, \varphi$.

If $n_1 \leq n_2$, then for any Brauer diagram $D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})$, there exist at least $\frac{n_2 - n_1}{2}$ top horizontal edges between the vertices labeled by the integers in the following set

$$\{1, 3, 5, \cdots, 2n - 1\} \setminus A^{(1)}.$$

As a result, we deduce that there exist elements $\sigma_{A^{(1)}} \in \mathfrak{S}_{(1,3,\cdots,2n-1)}$, $\sigma_{A^{(2)}} \in \mathfrak{S}_{(2,4,\cdots,2n)}$ and a Brauer diagram D_2, such that

(3) for any integer a with $1 \leq a \leq n_2$, D_2 connects the vertex labeled by $2a - 1$ with the vertex labeled by $2a$.

(4)

$$\sigma_{A^{(1)}} \Bigg(\sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D \Bigg) \sigma_{A^{(2)}} = D_2 \Bigg(\sum_{D \in \mathrm{Bd}_n(|A^{(1)}|, |A^{(2)}|)} D \Bigg).$$

By the same argument as before, we deduce that $\sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D \in \mathrm{Ker}\,\varphi$ in this case. This completes the proof of the corollary. □

The following is the main result of this section, which gives an explicit and characteristic-free description of the annihilator of the n-tensor space $V^{\otimes n}$ in the Brauer algebra $\mathfrak{B}_n(-2m)$.

Theorem 3.4 *With the notations as in Lemma* 1.1 *and Lemma* 3.1, *we have that*

$$\mathrm{Ker}\,\varphi = \mathcal{M}_K^{(2m+2, 2^{n-m-1})},$$

where $(2m+2, 2^{n-m-1}) := (2m+2, \underbrace{2, \cdots, 2}_{n-m-1 \; copies})$, $\mathcal{M}_K^{(2m+2, 2^{n-m-1})}$ *is the right* $K[\mathfrak{S}_{2n}]$-*module associated to* $(2m+2, 2^{n-m-1})$ *as defined in Section* 2. *In particular,* $\mathrm{Ker}\,\varphi$ *is a* $\mathfrak{S}_{2n}$-*submodule.*

Proof It is easy to see that for any partition $\mu \in 2\mathcal{P}_n$, $\mu \trianglerighteq (2m+2, 2^{n-m-1})$ if and only if $\mu_1 > 2m$. Therefore,

$$\dim \mathcal{M}_K^{(2m+2, 2^{n-m-1})} = \sum_{\substack{\lambda \in 2\mathcal{P}_n \\ \lambda_1 > 2m}} \dim S^\lambda.$$

Applying Lemma 1.1 and Lemma 3.1, we see that to prove this theorem, it suffices to show that $\mathcal{M}_K^{(2m+2, 2^{n-m-1})} \subseteq \mathrm{Ker}\,\varphi$. Equivalently, it suffices to show that for any partition $\lambda = (\lambda_1, \cdots, \lambda_s) \in 2\mathcal{P}_n$ satisfying $\lambda_1 > 2m$, and any $w \in \mathfrak{S}_{2n}$, $\varphi(X_\lambda * w) = 0$.

By the definition of the element X_λ, the action "$*$" and the multiplication rule of Brauer diagrams, we deduce that

$$X_\lambda * w = \sum_{\mathbf{i},\mathbf{j}} \sum_{D \in \mathrm{Bd}_n^{\mathbf{i},\mathbf{j}}(A^{(1)}, A^{(2)})} D,$$

where

$$A^{(1)} := \left\{ (i)w \mid i = 1, 2, 3, \cdots, \lambda_1 \right\} \cap \left\{ 1, 3, 5, \cdots, 2n-1 \right\},$$

$$A^{(2)} := \left\{ (i)w \mid i = 1, 2, 3, \cdots, \lambda_1 \right\} \cap \left\{ 2, 4, 6, \cdots, 2n \right\},$$

and $|A^{(1)}| + |A^{(2)}| = 2n_0 = \lambda_1$, and $\mathbf{i} := (i_1, i_2, \cdots, i_{n-n_0})$, $\mathbf{j} := (j_1, j_2, \cdots, j_{n-n_0})$ such that $(i_1, j_1, i_2, j_2, \cdots, i_{n-n_0}, j_{n-n_0})$ is a permutation of the integers in $\{1, 2, \cdots, 2n\} \setminus (A^{(1)} \cup A^{(2)})$. We now apply Corollary 3.3. It follows immediately that $\varphi(X_\lambda * w) = 0$ as required. This completes the proof of the theorem. □

Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with basis $\{v_1, v_2, \cdots, v_{2m}\}$. For any commutative $\mathbb{Z}$-algebra R, we define $V_R := R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$. We have an action of the algebra $\mathfrak{B}_n(-2m)$ on $V_{\mathbb{Z}}^{\otimes n}$ which is obtained by restricting the earlier action, and hence an action of

$\mathfrak{B}_n(-2m)_R$ on $V_R^{\otimes n}$. Let $S_R^{sy}(m,n)$ (see [11, Section 2] and [32]) be the symplectic Schur algebra over R. If R is a field, then $S_R^{sy}(m,n)$ is a quasi-hereditary algebra over R, and $V_R^{\otimes n}$ is a tilting module over $S_R^{sy}(m,n)$. Applying [14, (4.4)], we know that, for any commutative $\mathbb{Z}$-algebra R, there is a canonical isomorphism

$$R \otimes_{\mathbb{Z}} \text{End}_{S_{\mathbb{Z}}^{sy}(m,n)}\left(V_{\mathbb{Z}}^{\otimes n}\right) \cong \text{End}_{S_R^{sy}(m,n)}\left(V_R^{\otimes n}\right). \tag{3.1}$$

Note that $\varphi\left(\mathfrak{B}_n(-2m)_{\mathbb{C}}\right) = \text{End}_{S_{\mathbb{C}}^{sy}(m,n)}\left(V_{\mathbb{C}}^{\otimes n}\right)$. By restriction, we get that

$$\varphi\left(\mathfrak{B}_n(-2m)_{\mathbb{Z}}\right) \subseteq \text{End}_{S_{\mathbb{Z}}^{sy}(m,n)}\left(V_{\mathbb{Z}}^{\otimes n}\right).$$

Hence for any commutative $\mathbb{Z}$-algebra R,

$$\varphi\left(\mathfrak{B}_n(-2m)_R\right) \subseteq \text{End}_{S_R^{sy}(m,n)}\left(V_R^{\otimes n}\right).$$

By the main result in [11], we know that the above inclusion "$\subseteq$" can be replaced by "=" when $R = K$ is an infinite field. Counting dimensions, we see that the above inclusion "$\subseteq$" can be replaced by "=" when $R = K$ is an arbitrary field.

Corollary 3.5 *For any commutative $\mathbb{Z}$-algebra R,*

$$\varphi\left(\mathfrak{B}_n(-2m)_R\right) = \text{End}_{S_R^{sy}(m,n)}\left(V_R^{\otimes n}\right).$$

In particular, Theorem 3.4 is always true if we replace the infinite field K by any commutative $\mathbb{Z}$-algebra R.

Proof By (3.1), it suffices to show that

$$\varphi\left(\mathfrak{B}_n(-2m)_{\mathbb{Z}}\right) = \text{End}_{S_{\mathbb{Z}}^{sy}(m,n)}\left(V_{\mathbb{Z}}^{\otimes n}\right).$$

Let $N := \left(\text{End}_{S_{\mathbb{Z}}^{sy}(m,n)}\left(V_{\mathbb{Z}}^{\otimes n}\right)\right)/\varphi\left(\mathfrak{B}_n(-2m)_{\mathbb{Z}}\right)$. It is enough to show that $N = 0$. By [1, Proposition 3.8], it suffices to show that $N_{(p)} = 0$ for each prime number p. Let $k_p := \mathbb{Z}/(p)$, the residue field at the prime number p. It is clear that $k_p \cong \mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)}$. Note that

$$N_{(p)}/(p)N_{(p)} \cong k_p \otimes_{\mathbb{Z}_{(p)}} N_{(p)} \cong k_p \otimes_{\mathbb{Z}} N$$

$$\cong \left(\text{End}_{S_{k_p}^{sy}(m,n)}\left(V_{k_p}^{\otimes n}\right)\right)/\varphi\left(\mathfrak{B}_n(-2m)_{k_p}\right) = 0.$$

Applying Nakayama's lemma ([1, 2.6]), we conclude that $N_{(p)} = 0$. This completes the proof of the corollary.   □

4 Proof of Lemma 3.2

We shall first fix some notations and conventions. Since the element $\sum_{D\in\mathrm{Bd}_n(a,b)} D$ in Lemma 3.2 actually lies in $\mathfrak{B}_n$, we can choose to work inside the Brauer algebra $\mathfrak{B}_n(-2m)_{\mathbb{C}}$ in this section. Furthermore, throughout this section, we shall recover our original way of labeling of vertices in each Brauer n-diagram. That is, the vertices in each row of a Brauer n-diagram will be labeled by the indices $1, 2, \cdots, n$ from left to right. This way of labeling is more convenient when we need to express each Brauer diagram in terms of the standard generators s_i, e_i for $1 \leq i \leq n-1$ and to consider the action of Brauer diagrams on the n-tensor space $V^{\otimes n}$.

Let f be an integer with $0 \leq f \leq [n/2]$, where $[n/2]$ is the largest non-negative integer not bigger than $n/2$. Define

$$\mathfrak{D}_f := \left\{ d \in \mathfrak{S}_n \left| \begin{array}{l} (2j-1)d < (2j)d \text{ for } 1 \leq j \leq f, \\ 1d < 3d < \cdots < (2f-1)d, \\ (2f+1)d < (2f+2)d < \cdots < nd \end{array} \right. \right\}.$$

For each partition λ of $n - 2f$, we denote by $\mathrm{Std}_{2f}(\lambda)$ the set of all the standard λ-tableaux with entries in $\{2f+1, \cdots, n\}$. The initial tableau $\mathfrak{t}^\lambda$ in this case has the numbers $2f+1, \cdots, n$ in order along successive rows. Again, for each $\mathfrak{t} \in \mathrm{Std}_{2f}(\lambda)$, let $d(\mathfrak{t})$ be the unique element in $\mathfrak{S}_{\{2f+1,\cdots,n\}} \subseteq \mathfrak{S}_n$ with $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$. Let $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$ and $d_1, d_2 \in \mathfrak{D}_f$. Then $d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2$ corresponds to the Brauer n-diagram where the top horizontal edges connect $(2i-1)d_1$ and $(2i)d_1$, the bottom horizontal edges connect $(2i-1)d_2$ and $(2i)d_2$, for $i = 1, 2, \cdots, f$, and the vertical edges connect $(j)d_1$ with $(j)\sigma d_2$ for $j = 2f+1, 2f+2, \cdots, n$.

Lemma 4.1 ([11, Corollary 3.3]) *With the above notations, the set*

$$\left\{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \mid 0 \leq f \leq [n/2], \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}, d_1, d_2 \in \mathfrak{D}_f \right\}$$

is a basis of the Brauer algebra $B_n(x)_R$, which coincides with the natural basis given by Brauer n-diagrams.

Given an element $d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2$ as above, let D be its representing Brauer n-diagram. Let $v_{\underline{i}} := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ be a simple n-tensor in $V^{\otimes n}$.

Lemma 4.2 *With the notations as above, we have that*

$$v_{\underline{i}} D = (-1)^{\ell(d_1^{-1}\sigma d_2)} \big(v_{\underline{i}} \circ D \big),$$

where $v_{\underline{i}} \circ D$ can be described as follows:

(1) *If $(j)d_1^{-1}\sigma d_2 = (k)$ for $j \in \big\{ (2f+1)d_1, (2f+2)d_1, \cdots, (n)d_1 \big\}$, then the kth position of $v_{\underline{i}} \circ D$ is v_{i_j};*

(2) *For each $1 \leq j \leq f$, the $((2j-1)d_2, (2j)d_2)$th position of $v_{\underline{i}} \circ D$ is the following sum*:

$$\epsilon_{i_{(2j-1)d_1}, i_{(2j)d_1}} \sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}).$$

Remark 4.3 Intuitively, the action of the Brauer n-diagram D on $v_{\underline{i}}$ can be thought as follows. Let $(a_1, b_1), \cdots, (a_f, b_f)$ be the set of all the horizontal edges in the top row of D, where $a_s < b_s$ for each s and $a_1 < a_2 < \cdots < a_f$. Let $(c_1, d_1), \cdots, (c_f, d_f)$ be the set of all the horizontal edges in the bottom row of D, where $c_s < d_s$ for each s and $c_1 < c_2 < \cdots < c_f$. Then for each $1 \le j \le f$, the (c_j, d_j)th position of $v_{\underline{i}} \circ D$ is the following sum:

$$\epsilon_{i_{a_j}, i_{b_j}} \sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}).$$

We list those vertices in the top row of D which are not connected with horizontal edges from left to right as $i_{k_{2f+1}}, i_{k_{2f+2}}, \cdots, i_{k_n}$. Then, for each integer j with $2f + 1 \le j \le n$, the $(j \sigma d_2)$th position of $v_{\underline{i}} \circ D$ is $v_{i_{k_j}}$.

Given an arbitrary element $d_2 \in \mathfrak{D}_f$, we define

$$\mathrm{Bd}^{(f)}(n; d_2) := \left\{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \mid d_1 \in \mathfrak{D}_f, \sigma \in \mathfrak{S}_{\{2f+1, 2f+2, \cdots, n\}} \right\}.$$

Note that $\mathrm{Bd}^{(f)}(n; d_2)$ consists of all the Brauer n-diagrams whose bottom horizontal edges are

$$((1)d_2, (2)d_2), ((3)d_2, (4)d_2), \cdots, ((2f-1)d_2, (2f)d_2).$$

Lemma 4.4 *Let f be an integer with $0 \le f \le [n/2]$. Let $d_2 \in \mathfrak{D}_f$. Then for any $\sigma \in \mathfrak{S}_n$,*

$$\sigma \left(\sum_{D \in \mathrm{Bd}^{(f)}(n; d_2)} D \right) = \sum_{D \in \mathrm{Bd}^{(f)}(n; d_2)} D.$$

Proof It suffices to show that for each integer $1 \le i < n$,

$$s_i \left(\sum_{D \in \mathrm{Bd}^{(f)}(n; d_2)} D \right) = \sum_{D \in \mathrm{Bd}^{(f)}(n; d_2)} D. \tag{4.1}$$

In fact, for $D, D' \in \mathrm{Bd}^{(f)}(n; d_2)$ with $D \ne D'$, it is clear that $s_i D \ne s_i D'$, and both $s_i D$ and $s_i D'$ are still lie in $\mathrm{Bd}^{(f)}(n; d_2)$. Now counting the number of Brauer n-diagrams occurring in both sides, we prove (4.1) and hence also prove the lemma. $\square$

Similarly, given an arbitrary element $d_1 \in \mathfrak{D}_f$, we define

$$\mathrm{Bd}^{(f)}(d_1; n) := \left\{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \mid d_2 \in \mathfrak{D}_f, \sigma \in \mathfrak{S}_{\{2f+1, 2f+2, \cdots, n\}} \right\}.$$

Then $\mathrm{Bd}^{(f)}(d_1; n)$ consists of all the Brauer n-diagrams whose top horizontal edges are

$$((1)d_1, (2)d_1), ((3)d_1, (4)d_1), \cdots, ((2f-1)d_1, (2f)d_1).$$

The following result can be proved in the same way as Lemma 4.4.

Lemma 4.5 *Let f be an integer with $0 \leq f \leq [n/2]$. Let $d_1 \in \mathfrak{D}_f$. Then for any $\sigma \in \mathfrak{S}_n$,*

$$\left(\sum_{D \in \mathrm{Bd}^{(f)}(d_1;n)} D \right) \sigma = \sum_{D \in \mathrm{Bd}^{(f)}(d_1;n)} D.$$

Let $\underline{i} = (i_1, i_2, \cdots, i_n)$, where $1 \leq i_j \leq 2m$ for each j. An ordered pair (s, t) $(1 \leq s < t \leq n)$ is called a *symplectic pair* in $\underline{i}$ if $i_s = i'_t$. Two ordered pairs (s, t) and (u, v) are called disjoint if $\{s, t\} \cap \{u, v\} = \emptyset$. We define the *symplectic length* $\ell_s(\underline{i})$ to be the maximal number of disjoint symplectic pairs (s, t) in $\underline{i}$. Let

$$I(2m, n) := \left\{ \underline{i} = (i_1, \cdots, i_n) \mid 1 \leq i_j \leq 2m, i_j \in \mathbb{N}, \forall j \right\}.$$

For an arbitrary element $v \in V^{\otimes n}$, we say the simple tensor $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ is involved in v, if $v_{\underline{i}}$ has nonzero coefficient in writing v as linear combination $\sum_{\underline{j} \in I(2m,n)} k_{\underline{j}} v_{\underline{j}}$ of the basis $\{v_{\underline{j}} \mid \underline{j} \in I(2m, n)\}$ of $V^{\otimes n}$. We now consider a special case of Lemma 3.2.

Proposition 4.6 *We have that*

$$\sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D \in \mathrm{Ker}\, \varphi.$$

Proof By the above discussion and the definition of $\mathrm{Bd}_n(m + 1, m + 1)$, any Brauer diagram $D \in \mathrm{Bd}_n(m + 1, m + 1)$ only acts on the first $m + 1$ components of any simple n-tensor $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \in V^{\otimes n}$. Therefore, to show that $\sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D \in \mathrm{Ker}\, \varphi$, we can assume without loss of generality that $n = m + 1$. Note that

$$\sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D = \sum_{0 \leq f \leq [n/2]} \sum_{d_2 \in \mathfrak{D}_f} \sum_{D \in \mathrm{Bd}^{(f)}(n;d_2)} D$$

$$= \sum_{0 \leq f \leq [n/2]} \sum_{d_1 \in \mathfrak{D}_f} \sum_{D \in \mathrm{Bd}^{(f)}(d_1;n)} D.$$

Suppose that $\sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D \notin \mathrm{Ker}\, \varphi$. Then there exists a simple n-tensor $v_{\underline{i}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \in V^{\otimes n}$, such that

$$v_{\underline{i}} \sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D \neq 0.$$

Suppose that $\ell_s(\underline{i}) = f$ for some integer $0 \leq f \leq [n/2]$. By Lemma 4.4, for any $\sigma \in \mathfrak{S}_n$,

$$\sigma^{-1} \sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D = \sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D.$$

Therefore, replacing $v_{\underline{i}}$ by $v_{\underline{i}\sigma}$ if necessary, we can assume without loss of generality that $i_{2s-1} = i'_{2s} < i_{2s}$ for each integer $1 \leq s \leq f$, $i_1 \leq i_3 \leq \cdots \leq i_{2f-1}$ and $i_{2f+1} \leq$

$i_{2f+2} \leq \cdots \leq i_n$. Furthermore, if $i_j = i_k$ for some integers $j \neq k$, then

$$v_{\underline{i}} \sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D = v_{\underline{i}} \frac{1+s_{(j,k)}}{2} \sum_{D \in \mathrm{Bd}_n(m+1,m+1)} D = 0,$$

where $s_{(j,k)}$ denotes the transposition (j,k) in $\mathfrak{S}_n$, and we have used the fact that the length of $s_{(j,k)}$ is an odd integer. Therefore, we can deduce that $i_1, i_2, \cdots, i_n$ are pairwise distinct. Hence, $i_{2s-1} = i'_{2s} < i_{2s}$ for each integer $1 \leq s \leq f$, $i_1 < i_3 < \cdots < i_{2f-1}$ and $i_{2f+1} < i_{2f+2} < \cdots < i_n$. Since $n = m + 1$, $i_1, i_2, \cdots, i_n$ are pairwise distinct implies that f must be bigger than 0. Hence $0 < f \leq [n/2]$.

Note that if $g > f$ and $D \in \mathrm{Bd}^{(g)}(d_1, n)$, then $v_{\underline{i}} D = 0$. Hence we get that

$$v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} D \neq 0. \tag{4.2}$$

By Lemma 4.5, it is easy to see that for any $\sigma \in \mathfrak{S}_n$,

$$v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} D\sigma = v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} D. \tag{4.3}$$

Given any simple n-tensor $v_{\underline{b}} := v_{b_1} \otimes \cdots \otimes v_{b_n}$ which is involved in the left-hand side of (4.2), we claim that $\ell_s(\underline{b}) = f$ and $b_1, \cdots, b_n$ are pairwise distinct. In fact, since $\ell_s(\underline{i}) = f$, it is clear that $\ell_s(\underline{b}) = f$. Suppose that $b_j = b_k$ for some integers $j \neq k$ and $v_{\underline{b}}$ appears with coefficient $A_{\underline{b}} \neq 0$ in the expansion of the left-hand side of (4.2), then $v_{\underline{b}}$ also appears with coefficient $A_{\underline{b}} \neq 0$ in the expansion of

$$v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} Ds_{(j,k)} = v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} D.$$

As a result, $v_{\underline{b}}$ also appears with coefficient $A_{\underline{b}} \neq 0$ in the expansion of

$$v_{\underline{i}} \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} D(1 + s_{(j,k)})/2,$$

which is impossible, since

$$(v_{b_1} \otimes \cdots \otimes v_{b_n})(1 + s_{(j,k)})/2 = 0.$$

By (4.3), for any $\sigma \in \mathfrak{S}_n$, $v_{\underline{b}}$ is involved in the left-hand side of (4.2) if and only if $v_{\underline{b}\sigma}$ is involved in the left-hand side of (4.2). Therefore, we can choose a simple n-tensor $v_{\underline{b}}$ which is involved in the expansion of the left-hand side of (4.2) such that $b_{2s-1} = b'_{2s} < b_{2s}$ for each integer $1 \leq s \leq f$, $b_1 < b_3 < \cdots < b_{2f-1}$, and $b_1, \cdots, b_n$ are pairwise distinct. We now consider the intersection of indices in $\underline{i}$ and $\underline{b}$. By Remark 4.3, the symplectic pairs in $\underline{b} \setminus \underline{i}$ must be produced through the action of contraction operators (i.e., horizontal edges in top row), while the symplectic pairs in $\underline{b} \cap \underline{i}$ may be produced through either the action of contraction operators or the permutation action of vertical edges. Therefore, by the argument at the beginning of

this paragraph, we can further require that there exists an integer $0 \le r \le f$ such that:

(a) $b_t = i_t$ for each integer $2r + 1 \le t \le n$, and
(b) $\{b_1, b_2, \cdots, b_{2r}\} \cap \{i_1, i_2, \cdots, i_n\} = \emptyset$.

Since $n = m + 1$, the above conditions and the fact that $i_1, \cdots, i_n$ are pairwise distinct imply that r must be less than f. Hence $0 \le r < f$.

Let g be an integer with $0 \le g \le f$, $d_1 \in \mathfrak{D}_g$, $D \in \mathrm{Bd}^{(g)}(d_1; n)$, where

$$D = d_1^{-1} e_1 e_3 \cdots e_{2g-1} \sigma d_2, \ \sigma \in \mathfrak{S}_{\{2g+1, 2g+2, \cdots, n\}}, \ d_2 \in \mathfrak{D}_g.$$

We claim that $v_{\underline{b}}$ appears with nonzero coefficient in the expansion of $v_{\underline{i}} D$ if and only if

(1) $g \ge r$, $\sigma = 1$, and
(2) the horizontal edges in the top row of D are of the form

$$(1, 2), (3, 4), \cdots, (2r - 1, 2r), (2a_1 - 1, 2a_1), (2a_2 - 1, 2a_2), \cdots,$$

$$(2a_{g-r} - 1, 2a_{g-r}),$$

where $a_1, \cdots, a_{g-r}$ are some integers satisfying $r + 1 \le a_1 < a_2 < \cdots < a_{g-r} \le f$, and

(3) the horizontal edges in the bottom row of D is the same as those in the top row of D, i.e., $d_2 = d_1$.

In fact, for any Brauer diagram D satisfying the above conditions (1), (2), (3), by Remark 4.3, $v_{\underline{b}}$ does appear with nonzero coefficient in the expansion of $v_{\underline{i}} D$, and the coefficient is $(-1)^g$; on the other hand, suppose that $v_{\underline{b}}$ appears with nonzero coefficient in the expansion of $v_{\underline{i}} D$. By our assumption on $\underline{i}$ and $\underline{b}$, it is easy to see that the tensor factor $v_{b_{2s-1}} \otimes v_{b_{2s}}$ with $1 \le s \le r$ can only be produced through the action of e_{2t-1} for some $1 \le t \le g$. This implies that $g \ge r$. For each integer j with $2g + 1 \le j \le n$, by Remark 4.3, the action of D on $v_{\underline{i}}$ moves the vector in the jd_1th position of $v_{\underline{i}}$ (i.e., $v_{i_{jd_1}}$) to the $(j\sigma d_2)$th position. By our assumption on $\underline{i}, \underline{b}$ again, we deduce that $jd_1 = j\sigma d_2$. But by the definition of $\mathfrak{D}_g$,

$$(2g + 1)d_1 < (2g + 2)d_1 < \cdots < (n)d_1,$$

$$(2g + 1)d_2 < (2g + 2)d_2 < \cdots < (n)d_2.$$

It follows that $\sigma = 1$, and $jd_1 = jd_2$ for any $2g + 1 \le j \le n$. Now the remaining statements of our claim follows easily from the fact that $\sigma = 1$, our assumption on $\underline{i}$ and $\underline{b}$ as well as Remark 4.3.

Therefore, the coefficient of $v_{\underline{b}}$ in the expansion of

$$\sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1; n)} v_{\underline{i}} D$$

is equal to

$$(-1)^g \binom{f - r}{g - r}.$$

Note that $0 \leq r < f$, it follows that the coefficient of $v_{\underline{b}}$ in the left-hand side of (4.2) is

$$\sum_{r \leq g \leq f} (-1)^g \binom{f-r}{g-r} = 0,$$

a contradiction. This completes the proof of the proposition. $\qquad\square$

Next we consider a more general situation than Proposition 4.6, which is still a special case of Lemma 3.2.

Proposition 4.7 *Let a, b be two integers such that $0 \leq a, b \leq n$ and $a + b$ is even. Suppose that $a + b \geq 2m + 2$ and $a \geq b$; then*

$$\sum_{D \in \mathrm{Bd}_n(a,b)} D \in \mathrm{Ker}\, \varphi.$$

Proof By the assumption that $a \geq b$ and the definition of $\mathrm{Bd}_n(a, b)$, any Brauer diagram $D \in \mathrm{Bd}_n(a, b)$ only acts on the first a components of any simple n-tensor in $V^{\otimes n}$. Therefore, we can assume without loss of generality that $n = a$. Also, because of Proposition 4.6, we can assume that $n = a > b$. In particular, $n + b \geq 2m + 2$ and $n = a \geq m + 2$. Suppose that $\sum_{D \in \mathrm{Bd}_n(n,b)} D \notin \mathrm{Ker}\, \varphi$. Then there exists a simple n-tensor $v_{\underline{i}} \in V^{\otimes n}$ such that

$$v_{\underline{i}} \sum_{D \in \mathrm{Bd}_n(n,b)} D \neq 0. \tag{4.4}$$

Applying Lemma 4.4, we know that for any $\sigma \in \mathfrak{S}_n$,

$$\sigma \sum_{D \in \mathrm{Bd}_n(n,b)} D = \sum_{D \in \mathrm{Bd}_n(n,b)} D.$$

Therefore, using the same argument as in the proof of Proposition 4.6, we deduce that $i_1, \cdots, i_n$ are pairwise distinct. Furthermore, let $f = \ell_s(\underline{i})$, we can choose $\underline{i}$ such that $i_{2s-1} = i'_{2s} < i_{2s}$ for each integer $1 \leq s \leq f$, $i_1 < i_3 < \cdots < i_{2f-1}$ and $i_{2f+1} < i_{2f+2} < \cdots < i_n$.

We define

$$\Sigma_f = \left\{ (1, 2), (3, 4), \cdots, (2f - 1, 2f) \right\}.$$

For each $D \in \mathrm{Bd}_n(n, b)$, let $\Sigma^+(D)$ be the set of horizontal edges in the top row of D. Let $c := (n - b)/2$. The assumption (4.4) and the definition of $\mathrm{Bd}_n(n, b)$ imply that $f \geq c$. If $f = c$, then (as $i_1, \cdots, i_n$ are pairwise distinct) we must have $c + b \leq m$, equivalently, $n + b \leq 2m$, which is a contradiction. Therefore, $f \neq c$. So $c < f \leq [n/2]$.

By our choice of $\underline{i}$, it is clear that $v_{\underline{i}} D \neq 0$ only if $\Sigma^+(D) \subseteq \Sigma_f$. Therefore,

$$v_{\underline{i}} \sum_{D \in \mathrm{Bd}_n(n,b)} D = v_{\underline{i}} \sum_{\substack{D \in \mathrm{Bd}_n(n,b) \\ \Sigma^+(D) \subseteq \Sigma_f}} D \neq 0. \tag{4.5}$$

By the definition of $\mathrm{Bd}_n(n,b)$ and Remark 4.3, if the simple n-tensor $v_{\underline{k}} := v_{k_1} \otimes \cdots \otimes v_{k_n}$ is involved in the left-hand side of (4.5), then

$$k_{b+1} = k'_{b+2}, \ k_{b+3} = k'_{b+4}, \ \cdots, \ k_{n-1} = k'_n.$$

For any $\sigma \in \mathfrak{S}_b$, it is easy to see

$$\sum_{D \in \mathrm{Bd}_n(n,b)} D\sigma = \sum_{D \in \mathrm{Bd}_n(n,b)} D.$$

It follows that, for any $\sigma \in \mathfrak{S}_b$, $v_{\underline{k}}$ is involved in the left-hand side of (4.5) if and only if $v_{\underline{k}\sigma}$ is involved in the left-hand side of (4.5). Therefore, we can choose a simple n-tensor $v_{\underline{k}}$ which is involved in the left-hand side of (4.5) such that $k_{2s-1} = k'_{2s} < k_{2s}$ for each integer $1 \leq s \leq f - c$, $k_{2(f-c)+1} < k_{2(f-c)+2} < \cdots < k_b$, and $k_1, \cdots, k_b$ are pairwise distinct. Using the same argument as in the proof of Proposition 4.6, we can further require that there exists an integer $0 \leq r \leq f - c$ such that $k_t = i_t$ for each integer $2r + 1 \leq t \leq b$, and $\{k_1, k_2, \cdots, k_{2r}\} \cap \{i_1, i_2, \cdots, i_n\} = \emptyset$. We claim that $r \neq f - c$. In fact, if $r = f - c$, then the above empty intersection condition on r and our assumption on $\underline{i}$ imply that $f - c + n - 2f + f \leq m$, equivalently, $(n+b)/2 \leq m$, which is impossible. This proves our claim. Hence, $0 \leq r < f - c$.

Let g be an integer with $0 \leq g \leq f$, $d_1 \in \mathfrak{D}_g$, $D \in \mathrm{Bd}^{(g)}(d_1; n)$, where

$$D = d_1^{-1} e_1 e_3 \cdots e_{2g-1} \sigma d_2, \ \sigma \in \mathfrak{S}_{\{2g+1, 2g+2, \cdots, n\}}, \ d_2 \in \mathfrak{D}_g.$$

We claim that $v_{\underline{k}}$ appears with nonzero coefficient in the expansion of $v_{\underline{i}} D$ if and only if

(1) $g \geq r + c$, $\sigma = 1$, and
(2) the horizontal edges in the top row of D are of the form

$$(1, 2), (3, 4), \cdots, (2r - 1, 2r), (2a_1 - 1, 2a_1), (2a_2 - 1, 2a_2), \cdots,$$
$$(2a_{g-r} - 1, 2a_{g-r}),$$

where $a_1, \cdots, a_{g-r}$ are some integers satisfying $r + 1 \leq a_1 < a_2 < \cdots < a_{g-r} \leq f$, and
(3) the horizontal edges in the bottom row of D are of the form

$$(1, 2), (3, 4), \cdots, (2r - 1, 2r), (2a_1 - 1, 2a_1), (2a_2 - 1, 2a_2), \cdots,$$
$$(2a_{g-c-r} - 1, 2a_{g-c-r}), (b + 1, b + 2), (b + 3, b + 4),$$
$$\cdots, (n - 1, n).$$

In fact, for any Brauer diagram D satisfying the above conditions (1), (2), (3), by Remark 4.3, $v_{\underline{k}}$ does appear with nonzero coefficient in the expansion of $v_{\underline{i}} D$, and the coefficient is

$$\left(\prod_{s=1}^{c} \epsilon_{k'_{b+2s-1}, k_{b+2s-1}} \right) (-1)^{g-c}.$$

On the other hand, suppose that $v_{\underline{k}}$ appears with nonzero coefficient in the expansion of $v_{\underline{i}}D$. By our assumption on $\underline{i}$ and $\underline{k}$ and the definition of $\mathrm{Bd}_n(n, b)$, it is easy to see that the tensor factor $v_{k_{2s-1}} \otimes v_{k_{2s}}$ with $1 \leq s \leq r$ and $v_{k_{b+2t-1}} \otimes v_{k_{b+2t}}$ with $1 \leq t \leq c$ can only be produced through the action of e_{2t-1} for some $1 \leq t \leq g$. This implies that $g \geq r + c$. For each integer j with $2g + 1 \leq j \leq n$, by Remark 4.3, the action of D on $v_{\underline{i}}$ moves the vector in the jd_1th position of $v_{\underline{i}}$ (i.e., $v_{i_{jd_1}}$) to the $(j\sigma d_2)$th position. By our assumption on $\underline{i}, \underline{k}$ again, we deduce that $jd_1 = j\sigma d_2$. But by the definition of $\mathfrak{D}_g$,

$$(2g + 1)d_1 < (2g + 2)d_1 < \cdots < (n)d_1,$$

$$(2g + 1)d_2 < (2g + 2)d_2 < \cdots < (n)d_2.$$

It follows that $\sigma = 1$, and $jd_1 = jd_2$ for any $2g + 1 \leq j \leq n$. Now the remaining statements of our claim follows easily from the fact that $\sigma = 1$, our assumption on $\underline{i}$ and $\underline{k}$ as well as Remark 4.3.

Therefore, the coefficient of $v_{\underline{b}}$ in the expansion of

$$\sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \mathrm{Bd}^{(g)}(d_1;n)} v_{\underline{i}}D$$

is equal to

$$\left(\prod_{s=1}^{c} \epsilon_{k'_{b+2s-1}, k_{b+2s-1}}\right)(-1)^{g-c}\binom{f - r}{g - r}.$$

Note that $0 \leq r < f - c$, it follows that the coefficient of $v_{\underline{b}}$ in the left-hand side of (4.2) is

$$\left(\prod_{s=1}^{c} \epsilon_{k'_{b+2s-1}, k_{b+2s-1}}\right) \sum_{c+r \leq g \leq f}(-1)^{g-c}\binom{f - r}{g - r} = 0,$$

a contradiction. This completes the proof of the proposition. $\qquad\square$

Finally, thanks to Proposition 4.7, to complete the proof of Lemma 3.2, we only need to prove the following proposition. We are grateful to the referee for providing the following simple proof.
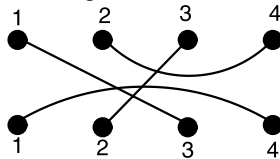
Proposition 4.8 *Let a, b be two integers such that $0 \leq a, b \leq n$ and $a + b$ is even. Suppose that $a + b \geq 2m + 2$ and $b \geq a$; then*

$$\sum_{D \in \mathrm{Bd}_n(a,b)} D \in \mathrm{Ker}\,\varphi.$$

Proof Given a Brauer n-diagram D, we define $\overline{D}$ by reflecting D top to bottom. For example, let D be the following Brauer 4-diagram:
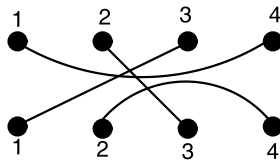
then $\overline{D}$ is the following Brauer 4-diagram:

Extending this linearly to the whole of the Brauer algebra, and it gives an anti-automorphism of the Brauer algebra.

Given this automorphism, we have a contravariant duality on modules: if M is a module for the Brauer algebra $\mathfrak{B}_n(-2m)_{\mathbb{C}}$, then the dual space M^* is also a module, with

$$(fd)(m) := f(m\overline{d}),$$

for $f \in M^*$, $m \in M$, $d \in \mathfrak{B}_n(-2m)_{\mathbb{C}}$. For each integer i with $1 \le i \le 2m$, let $v_i^* \in V^*$ be defined by $v_i^*(v_j) = \delta_{i,j}$, $\forall 1 \le j \le 2m$. For any simple n-tensor $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n} \in V^{\otimes n}$, let $v_{\underline{i}}^* := v_{i_1}^* \otimes \cdots \otimes v_{i_n}^* \in (V^*)^{\otimes n}$. We identify $(V^*)^{\otimes n}$ with $(V^{\otimes n})^*$ in a natural way, i.e., such that

$$v_{\underline{i}}^*(v_{\underline{j}}) = \delta_{\underline{i},\underline{j}}, \ \forall \underline{i}, \underline{j} \in I(2m, n).$$

We claim that the map

$$\theta : \sum_{\underline{i} \in I(2m,n)} a_{\underline{i}} v_{\underline{i}} \mapsto \sum_{\underline{i} \in I(2m,n)} a_{\underline{i}} v_{\underline{i}}^*, \ \forall a_{\underline{i}} \in \mathbb{C},$$

defines a $\mathfrak{B}_n(-2m)_{\mathbb{C}}$-module isomorphism $V^{\otimes n} \cong (V^{\otimes n})^*$. In fact, it suffices to check $\theta(v_{\underline{i}} x) = \theta(x) v_{\underline{i}}$ for each simple n-tensor $v_{\underline{i}} \in V^{\otimes n}$ and each generator x of $\mathfrak{B}_n(-2m)_{\mathbb{C}}$, which is actually trivial. As a result, we get that for any $D \in \mathrm{Bd}_n$, $\underline{i}, \underline{j} \in I(2m, n)$, the coefficient of $v_{\underline{i}}$ in $v_{\underline{j}} D$ is the same as the coefficient of $v_{\underline{j}}$ in $v_{\underline{i}} \overline{D}$. Therefore, $\mathrm{Ker}\,\varphi = \overline{\mathrm{Ker}\,\varphi}$. Now applying Proposition 4.7, we deduce that $\sum_{D \in \mathrm{Bd}_n(a,b)} D \in \mathrm{Ker}\,\varphi$. This completes the proof of the proposition.

$\square$

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