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Commutative combinatorial Hopf algebras

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Abstract We propose several constructions of commutative or cocommutative Hopf algebras based on various combinatorial structures and investigate the relations between them. A commutative Hopf algebra of permutations is obtained by a general construction based on graphs, and its noncommutative dual is realized in three different ways, in particular, as the Grossman–Larson algebra of heap-ordered trees. Extensions to endofunctions, parking functions, set compositions, set partitions, planar binary trees, and rooted forests are discussed. Finally, we introduce one-parameter families interpolating between different structures constructed on the same combinatorial objects.

Keywords Hopf algebras · Quasi-symmetric functions · Parking functions · Trees · Graphs

1 Introduction

Many examples of Hopf algebras based on combinatorial structures are known. Among these, certain algebras based on permutations and planar binary trees play a prominent role and arise in seemingly unrelated contexts [4, 7, 17, 19]. As Hopf algebras, both are noncommutative and noncocommutative and in fact self-dual.

More recently, cocommutative Hopf algebras of binary trees and permutations have been constructed [1, 21]. In [21], binary trees arise as sums over rearrangements

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classes in an algebra of parking functions, while in [1], cocommutative Hopf algebras are obtained as the graded coalgebras associated with coradical filtrations.

In [23], a general method for constructing commutative Hopf algebras based on various kinds of graphs has been presented. The aim of this note is to investigate Hopf algebras based on permutations, trees, and various other combinatorial structures and constructed by the method developed in [23]. These commutative algebras are, by definition, realized in terms of polynomials in an infinite set of doubly indexed indeterminates. The dual Hopf algebras are then realized by means of noncommutative polynomials in variables a_{ij}. We show that the algebras based on permutations and planar binary trees are isomorphic (in a nontrivial way) to those of [1] and study some generalizations such as endofunctions, parking functions, set partitions, trees, forests, and so on.

The possibility to obtain in an almost systematic way commutative and in general noncocommutative versions of the usual combinatorial Hopf algebras leads us to the conjecture that these standard versions should be considered as some kind of quantum groups, i.e., can be incorporated into one-parameter families containing an enveloping algebra and its dual for special values of the parameter. A few results supporting this point of view are presented in the final section.

Throughout, $\mathbb{K}$ is a field of characteristic zero. We adopt the notation from [8, 23]. This paper is an expanded and updated version of the preprint [12].

2 A commutative Hopf algebra of endofunctions

Permutations can be regarded in an obvious way as labeled and oriented graphs whose connected components are cycles. Actually, arbitrary *endofunctions* (functions from $[n] := \{1, \dots, n\}$ to itself) can be regarded as labeled graphs connecting i with $f(i)$ for all i so as to fit in the framework of [23], where a general process for building Hopf algebras of graphs is described.

In the sequel, we identify an endofunction f of $[n]$ with the word

$$w_f = f(1)f(2)\cdots f(n) \in [n]^n. \tag{1}$$

Let $\{x_{i\,j} \mid i, j \geq 1\}$ be an infinite set of commuting indeterminates, and let $\mathcal{J}$ be the ideal of $R = \mathbb{K}[x_{i\,j} \mid i, j \geq 1]$ generated by the relations

$$x_{i\,j}x_{i\,k} = 0 \quad \text{for all } i, j, k. \tag{2}$$

For an endofunction $f : [n] \to [n]$, define

$$M_f := \sum_{i_1 < \cdots < i_n} x_{i_1\, i_{f(1)}} \cdots x_{i_n\, i_{f(n)}} \tag{3}$$

in $R/\mathcal{J}$.

From [23, Sect. 4], it follows:

Theorem 2.1 *The M_f span a subalgebra EQSym of the commutative algebra $R/\mathcal{J}$. More precisely, there exist nonnegative integers $C^h_{f,g}$ such that*

$$M_f M_g = \sum_h C^h_{f,g} M_h. \tag{4}$$

Examples 2.2

$$M_1 M_{22} = M_{133} + M_{323} + M_{223}. \tag{5}$$

$$M_1 M_{331} = M_{1442} + M_{4241} + M_{4431} + M_{3314}. \tag{6}$$

$$M_{12} M_{21} = M_{1243} + M_{1432} + M_{4231} + M_{1324} + M_{3214} + M_{2134}. \tag{7}$$

$$M_{12} M_{22} = M_{1244} + M_{1434} + M_{4234} + M_{1334} + M_{3234} + M_{2234}. \tag{8}$$

$$M_{12} M_{133} = 3M_{12355} + 2M_{12445} + 2M_{12545} + M_{13345} + M_{14345} + M_{15345}. \tag{9}$$

The *shifted concatenation* of two endofunctions $f : [n] \to [n]$ and $g : [m] \to [m]$ is the endofunction $h := f \bullet g$ of $[n+m]$ such that $w_h := w_f \bullet w_g$, that is,

$$\begin{cases} h(i) = f(i) & \text{if } i \le n, \\ h(i) = n + g(i-n) & \text{if } i > n. \end{cases} \tag{10}$$

We can now give a combinatorial interpretation of the coefficient $C^h_{f,g}$: if $f : [n] \to [n]$ and $g : [m] \to [m]$, this coefficient is the number of permutations τ in the shuffle product $(1 \dots n) \sqcup\!\sqcup (n+1 \dots n+m)$ such that

$$h = \tau^{-1} \circ (f \bullet g) \circ \tau. \tag{11}$$

For example, with $f = 12$ and $g = 22$, one finds the set (see (8))

$$\{1244, 1434, 4234, 1334, 3234, 2234\}. \tag{12}$$

Now, still following [23], define the coproduct by

$$\Delta M_h := \sum_{(f,g);\, f \bullet g = h} M_f \otimes M_g. \tag{13}$$

This endows *EQSym* with a (commutative, noncocommutative) Hopf algebra structure.

Examples 2.3

$$\Delta M_{626124} = M_{626124} \otimes 1 + 1 \otimes M_{626124}. \tag{14}$$

$$\Delta M_{4232277} = M_{4232277} \otimes 1 + M_{42322} \otimes M_{22} + 1 \otimes M_{4232277}. \tag{15}$$

Define a *connected* endofunction as a function that cannot be obtained by nontrivial shifted concatenation. For example, the connected endofunctions for $n = 1,\ 2,\ 3$ are

$$\begin{aligned} &1, \qquad 11,\ 21,\ 22,\\ &111,\ 112,\ 121,\ 131,\ 211,\ 212,\ 221,\ 222,\ 231,\ 232,\\ &233,\ 311,\ 312,\ 313,\ 321,\ 322,\ 323,\ 331,\ 332,\ 333, \end{aligned} \tag{16}$$

and the generating series of their numbers begins with

$$t + 3t^2 + 20t^3 + 197t^4 + 2511t^5 + 38924t^6 + 708105t^7 + 14769175t^8 + \cdots. \tag{17}$$

Then, the definition of the coproduct of the M_f implies the following:

Proposition 2.4 *If* $(\mathbf{S}^f)$ *denotes the dual basis of* (M_f), *the graded dual* **ESym** := *EQSym** *is free over the set*

$$\{\mathbf{S}^f \mid f \text{ connected}\}. \tag{18}$$

Indeed, (13) is equivalent to

$$\mathbf{S}^f \mathbf{S}^g = \mathbf{S}^{f \bullet g}. \tag{19}$$

Now, **ESym** being a graded connected cocommutative Hopf algebra, from the Cartier–Milnor–Moore theorem it follows that

$$\mathbf{ESym} = U(L) \tag{20}$$

where L is the Lie algebra of its primitive elements. Let us now prove the following:

Theorem 2.5 *As a graded Lie algebra, the primitive Lie algebra* L *of* **ESym** *is free over a set indexed by connected endofunctions.*

Proof Assume it is the case. By standard arguments on generating series, one finds that the number of generators of L in degree n is equal to the number of algebraic generators of **ESym** in degree n, parametrized for example by connected endofunctions. We will now show that L has at least this number of generators and that those generators are algebraically independent, determining completely the dimensions of the homogeneous components L_n of L whose generating series begins with

$$t + 3t^2 + 23t^3 + 223t^4 + 2800t^5 + 42576t^6 + 763220t^7 + 15734388t^8 + \cdots. \tag{21}$$

Following Reutenauer [27, p. 58], denote by π_1 the Eulerian idempotent, that is, the endomorphism of **ESym** defined by $\pi_1 = \log^*(Id)$. It is obvious, thanks to the definition of $\mathbf{S}^f$, that

$$\pi_1(\mathbf{S}^f) = \mathbf{S}^f + \cdots, \tag{22}$$

where the dots stand for terms $\mathbf{S}^g$ such that g is not connected. Since the $\mathbf{S}^f$ associated with connected endofunctions are independent, the dimension of L_n is at least equal to the number of connected endofunctions of size n. So L is free over a set of primitive elements parametrized by connected endofunctions. □

There are many Hopf subalgebras of *EQSym* which can be defined by imposing natural restrictions to maps: being bijective (see Sect. 3), idempotent ($f^2 = f$), involutive ($f^2 = id$), or more generally the Burnside classes ($f^p = f^q$), and so on. We shall start with the Hopf algebra of permutations.

3 A commutative Hopf algebra of permutations

We will use two different notations for permutations depending on whether they are considered as bijections from $[1, n]$ onto itself or as products of cycles. In the first case, we will write $\sigma = 31542$ for the bijection σ where $\sigma(i) = \sigma_i$. In the second case, the same permutation will be written $\sigma = (1352)(4)$, since 31542 is composed of two cycles: the cycle (1352) sending each element to the next one (circularly) in the sequence and the cycle (4) composed of only one element.

3.1 The Hopf algebra of bijective endofunctions

Let us define $\mathfrak{S}$*QSym* as the subalgebra of *EQSym* spanned by the

$$M_\sigma = \sum_{i_1 < \cdots < i_n} x_{i_1\, i_{\sigma(1)}} \cdots x_{i_n\, i_{\sigma(n)}} \tag{23}$$

where σ runs over bijective endofunctions, i.e., permutations. Note that $\mathfrak{S}$*QSym* is also isomorphic to the image of *EQSym* in the quotient of $R/\mathcal{J}$ by the relations

$$x_{i\,k} x_{j\,k} = 0 \quad \text{for all } i, j, k. \tag{24}$$

By the usual argument, it follows that:

Proposition 3.1 *The M_σ span a Hopf subalgebra $\mathfrak{S}$QSym of the commutative Hopf algebra EQSym.*

As already mentioned, there exist nonnegative integers $C_{\alpha,\beta}^{\gamma}$ such that

$$M_\alpha M_\beta = \sum_\gamma C_{\alpha,\beta}^{\gamma} M_\gamma. \tag{25}$$

The combinatorial interpretation of the coefficients $C_{f,g}^{h}$ seen in Sect. 2 can be reformulated in the special case of permutations. Write α and β as a union of disjoint cycles. Split the set $[n+m]$ into a set A of n elements and its complement B in all possible ways. For each splitting, apply to α (resp. β) in A (resp. B) the unique increasing morphism of alphabets from $[n]$ to A (resp. from $[m]$ to B) and return the list of permutations with the resulting cycles. If, for example, $\alpha = (1)(2) = 12$ and $\beta = (13)(2) = 321$, this yields

$$\begin{aligned}&(1)(2)(53)(4),\ (1)(3)(52)(4),\ (1)(4)(52)(3),\ (1)(5)(42)(3),\ (2)(3)(51)(4),\\&(2)(4)(51)(3),\ (2)(5)(41)(3),\ (3)(4)(51)(2),\ (3)(5)(41)(2),\ (4)(5)(31)(2).\end{aligned} \tag{26}$$

This set corresponds to the permutations and multiplicities of (31).

The third interpretation of this product comes from the dual coproduct point of view: $C^{\gamma}_{\alpha,\beta}$ is the number of ways of getting (α, β) as the standardized words of pairs (a, b) of two complementary subsets of cycles of γ. For example, with $\alpha = 12$, $\beta = 321$, and $\gamma = 52341$, one has three solutions for the pair (a, b), namely,

$$\big((2)(3), (4)(51)\big), \quad \big((2)(4), (3)(51)\big), \quad \big((3)(4), (2)(51)\big), \tag{27}$$

which is coherent with (26) and (31).

Examples 3.2

$$M_{12\ldots n} M_{12\cdots p} = \binom{n+p}{n} M_{12\ldots(n+p)}. \tag{28}$$

$$M_1 M_{21} = M_{132} + M_{213} + M_{321}. \tag{29}$$

$$M_{12} M_{21} = M_{1243} + M_{1324} + M_{1432} + M_{2134} + M_{3214} + M_{4231}. \tag{30}$$

$$M_{12} M_{321} = M_{12543} + M_{14325} + 2M_{15342} + M_{32145} + 2M_{42315} + 3M_{52341}. \tag{31}$$

$$\begin{aligned}M_{21} M_{123} = {}& M_{12354} + M_{12435} + M_{12543} + M_{13245} + M_{14325}\\&+ M_{15342} + M_{21345} + M_{32145} + M_{42315} + M_{52341}.\end{aligned} \tag{32}$$

$$\begin{aligned}M_{21} M_{231} = {}& M_{21453} + M_{23154} + M_{24513} + M_{25431} + M_{34152}\\&+ M_{34521} + M_{35412} + M_{43251} + M_{43512} + M_{53421}.\end{aligned} \tag{33}$$

3.2 Duality

Recall that the coproduct is given by

$$\Delta M_\sigma := \sum_{(\alpha,\beta);\alpha\bullet\beta=\sigma} M_\alpha \otimes M_\beta. \tag{34}$$

As in Sect. 2, this implies:

Proposition 3.3 *If* $(\mathbf{S}^\sigma)$ *denotes the dual basis of* (M_σ), *the graded dual* $\mathfrak{S}\mathbf{Sym} := \mathfrak{S}QSym^*$ *is free over the set*

$$\{\mathbf{S}^\alpha \mid \alpha \text{ connected}\}. \tag{35}$$

Indeed, (34) is equivalent to

$$\mathbf{S}^\alpha \mathbf{S}^\beta = \mathbf{S}^{\alpha\bullet\beta}. \tag{36}$$

Note that $\mathfrak{S}$**Sym** is both a subalgebra and a quotient of **ESym**, since $\mathfrak{S}QSym$ is both a quotient and a subalgebra of $EQSym$.

Now, as before, $\mathfrak{S}$**Sym** being a graded connected cocommutative Hopf algebra, from the Cartier–Milnor–Moore theorem it follows that

$$\mathfrak{S}\mathbf{Sym} = U(L), \tag{37}$$

where L is the Lie algebra of its primitive elements.

The same argument as in Sect. 2 proves the following:

Theorem 3.4 *The graded Lie algebra L of primitive elements of $\mathfrak{S}$**Sym** is free over a set indexed by connected permutations.*

Corollary 3.5 $\mathfrak{S}$**Sym** *is isomorphic to H_O, the Grossman–Larson Hopf algebra of heap-ordered trees* [9], *and to the cocommutative Hopf algebra of permutations of Patras–Reutenauer* [25].

According to [1], $\mathfrak{S}QSym$ ($= \mathfrak{S}\mathbf{Sym}^*$) is therefore also isomorphic to the quotient of Free quasi-symmetric functions [7] (or Malvenuto–Reutenauer algebra of permutations [19]) by its coradical filtration.

3.3 Cyclic tensors and $\mathfrak{S}QSym$

For a vector space V, let $\Gamma^n V$ be the subspace of $V^{\otimes n}$ spanned by *cyclic tensors*, i.e., sums of the form

$$\sum_{k=0}^{n-1} (v_1 \otimes \cdots \otimes v_n)\gamma^k \tag{38}$$

where γ is the cycle $(1\,2\ldots n)$, the right action of permutations on tensors being as usual

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{39}$$

Clearly, $\Gamma^n V$ is stable under the action of $GL(V)$, and its character is the symmetric function "cyclic character" [15, 30]:

$$l_n^{(0)} = \frac{1}{n}\sum_{d|n} \phi(d) p_d^{n/d} \tag{40}$$

where ϕ is Euler's function.

Let $L_n^{(0)}$ be the subspace of $\mathbb{C}\mathfrak{S}_n$ spanned by cyclic permutations. This is a submodule of $\mathbb{C}\mathfrak{S}_n$ for the conjugation action $\rho_\tau(\sigma) = \tau\sigma\tau^{-1}$ with Frobenius characteristic $l_n^{(0)}$. Then one can define the analytic functor Γ (see [13, 18]):

$$\Gamma(V) = \sum_{n\geq 0} V^{\otimes n} \otimes_{\mathbb{C}\mathfrak{S}_n} L_n^{(0)}. \tag{41}$$

Let $\overline{\Gamma}(V)=\bigoplus_{n\geq 1}\Gamma^n(V)$. Its symmetric algebra $H(V)=S(\overline{\Gamma}(V))$ can be endowed with a Hopf algebra structure by declaring the elements of $\overline{\Gamma}(V)$ primitive.

As an analytic functor, $V\mapsto H(V)$ corresponds to the sequence of $\mathfrak{S}_n$-modules $M_n=\mathbb{C}\mathfrak{S}_n$ endowed with the conjugation action, that is,

$$H(V)=\bigoplus_{n\geq 0}V^{\otimes n}\otimes_{\mathbb{C}\mathfrak{S}_n}M_n, \tag{42}$$

so that basis elements of $H_n(V)$ can be identified with symbols $\left[\begin{smallmatrix}w\\ \sigma\end{smallmatrix}\right]$ with $w\in V^{\otimes n}$ and $\sigma\in\mathfrak{S}_n$ subject to the equivalences

$$\begin{bmatrix}w\tau^{-1}\\ \tau\sigma\tau^{-1}\end{bmatrix}\equiv\begin{bmatrix}w\\ \sigma\end{bmatrix}. \tag{43}$$

Let $A:=\{a_n\mid n\geq 1\}$ be an infinite linearly ordered alphabet, and let $V=\mathbb{C}A$. We identify a tensor product of letters $a_{i_1}\otimes\cdots\otimes a_{i_n}$ with the corresponding word $w=a_{i_1}\dots a_{i_n}$ and denote by $(w)\in\Gamma^nV$ the circular class of w. A basis of $H_n(V)$ is then given by the commutative products

$$\underline{m}=(w_1)\cdots(w_p)\in S(\overline{\Gamma}V) \tag{44}$$

of circular words with $|w_1|+\cdots+|w_p|=n$ and $|w_i|\geq 1$ for all i.

With such a basis element, we can associate a permutation by the following standardization process. Fix a total order on circular words, for example, the lexicographic order on minimal representatives. Write $\underline{m}$ as a nondecreasing product

$$\underline{m}=(w_1)\cdots(w_p)\quad\text{with } (w_1)\leq(w_2)\leq\cdots\leq(w_p), \tag{45}$$

where the w_i are minimal representatives of the circular classes, and compute the ordinary standardization σ' of the word $w=w_1\cdots w_p$. Then σ is the permutation obtained by parenthesing the word σ' like $\underline{m}$ and interpreting the factors as cycles. For example, if

$$\begin{aligned}
\underline{m}&=(cba)(aba)(ac)(ba)=(aab)(ab)(ac)(acb),\\
w&=aababacacb,\\
\sigma'&=1\,2\,6\,3\,7\,4\,9\,5\,10\,8,\\
\sigma&=(126)(37)(49)(5\,10\,8),\\
\sigma&=(2,6,7,9,10,1,3,5,4,8).
\end{aligned} \tag{46}$$

We set $\sigma=\operatorname{cstd}(\underline{m})$ and call it the *circular standardization* of $\underline{m}$.

Let $H_\sigma(V)$ be the subspace of $H_n(V)$ spanned by those $\underline{m}$ such that $\operatorname{cstd}(\underline{m})=\sigma$, and let $\pi_\sigma:H(V)\to H_\sigma(V)$ be the projector associated with the direct sum decomposition

$$H(V)=\bigoplus_{n\geq 0}\bigoplus_{\sigma\in\mathfrak{S}_n}H_\sigma(V). \tag{47}$$

Computing the convolution of such projectors then yields the following:

Theorem 3.6 *The π_σ span a subalgebra of the convolution algebra* $\mathrm{End}^{\mathrm{gr}} H(V)$, *isomorphic to $\mathfrak{S}QSym$ via $\pi_\sigma \mapsto M_\sigma$.*

Proof First, note that $\pi_\alpha * \pi_\beta$ is a sum of π_γ: indeed, regarding $\mathrm{cstd}(\underline{m})$ as an element of $H(V)$,

$$\mathrm{cstd}\big(\pi_\alpha * \pi_\beta(\underline{m})\big) = \pi_\alpha * \pi_\beta\big(\mathrm{cstd}(\underline{m})\big), \tag{48}$$

since the circular standardization of a subword of $\underline{m}$ is equal to the circular standardization of the same subword of $\mathrm{cstd}(\underline{m})$. So

$$\pi_\alpha * \pi_\beta = \sum_\sigma D^\sigma_{\alpha,\beta}\pi_\sigma. \tag{49}$$

Now, by the definition of π_α and by the third interpretation of the product of the M_σ, one concludes that $D^\sigma_{\alpha,\beta}$ is equal to the $C^\sigma_{\alpha,\beta}$ of (25). □

3.4 Interpretation of $H(V)$

The Hopf algebra $H(V)$ can be interpreted as an algebra of functions as follows. Assume that V has finite dimension d, and let $a_1, \ldots, a_d$ be a basis of V.

To the generator $(a_{i_1} \cdots a_{i_n})$ of $H(V)$, we associate the function of d square matrices of arbitrary size N

$$f_{(i_1,\ldots,i_n)}(A_1, \ldots, A_d) = \mathrm{tr}(A_{i_1} \cdots A_{i_n}). \tag{50}$$

These functions are clearly invariant under simultaneous conjugation $A_i \mapsto MA_iM^{-1}$ and it is easy to prove that they generate the ring of invariants of $GL(N,\mathbb{C})$ in the symmetric algebra

$$M_N(\mathbb{C})^{\oplus d} \simeq M_N(\mathbb{C}) \otimes V. \tag{51}$$

Indeed, let us set $U = \mathbb{C}^N$ and identify $M_N(\mathbb{C})$ with $U \otimes U^*$. Then, using the notation of [18] for symmetric functions, the character of $GL(U)$ in $S^n(U \otimes U^* \otimes V)$ is $h_n(XX^\vee N)$, where $X = \sum x_i$, $X^\vee = \sum x_i^{-1}$.

By the Cauchy formula,

$$h_n(XX^\vee N) = \sum_{\lambda \vdash n} s_\lambda(NX)s_\lambda(X^\vee), \tag{52}$$

and the dimension of the invariant subspace is

$$\begin{aligned} \dim S^n(U \otimes U^* \otimes V)^{GL(U)} &= \langle h_n(XX^\vee N), 1\rangle_{GL(U)} \\ &= \sum_{\lambda \vdash n} \langle s_\lambda(NX), s_\lambda(X)\rangle \quad \text{(by (52))} \\ &= \sum_{\lambda,\mu \vdash n} s_\lambda * s_\mu(N)\langle s_\lambda, s_\mu\rangle \\ &= \sum_{\lambda \vdash n} (s_\lambda * s_\lambda)(N) \end{aligned} \tag{53}$$

where $f(N)$ means $f(1,1,\dots,1)$, N times. The characteristic of the conjugation action of $\mathfrak{S}_n$ on $\mathbb{C}\mathfrak{S}_n$ is precisely $\sum_{\lambda\vdash n}(s_\lambda * s_\lambda)$, so this is the dimension of $H_n(V)$. We have therefore established:

Theorem 3.7 *Let $F_N^{(d)}$ be the algebra of $GL(N,\mathbb{C})$-invariant polynomial functions on $M_N(\mathbb{C})^{\oplus d} \simeq M_N(\mathbb{C}) \otimes V$ endowed with the comultiplication*

$$\Delta f\big(A'_1,\dots,A'_d;A''_1,\dots,A''_d\big) := f\big(A'_1\oplus A''_1,\dots,A'_d\oplus A''_d\big), \tag{54}$$

where in the r.h.s., f is interpreted as an element of $F_{2N}^{(d)}$ via the obvious embedding.

Then the map $(a_{i_1}\cdots a_{i_k}) \mapsto f_{(i_1,\dots,i_k)}$ is an epimorphism of bialgebras $H(V)\to F_N^{(d)}$. In the limit $N\to\infty$, this map is an isomorphism.

3.5 Subalgebras of $\mathfrak{S}QSym$

3.5.1 *Symmetric functions in noncommuting variables (dual)*

For a permutation $\sigma\in\mathfrak{S}_n$, let $\mathrm{supp}(\sigma)$ be the partition π of the set $[n]$ whose blocks are the supports of the cycles of σ. Define the *standardized partition* of a partition of a finite subset of the nonnegative integers as the unique set partition on the first nonnegative integers preserving the relative order between the elements, so that the standardized partition of $\{\{3,6\},\{7\},\{2,8\}\}$ is $\{\{2,3\},\{4\},\{1,5\}\}$. The sums

$$U_\pi := \sum_{\mathrm{supp}(\sigma)=\pi} M_\sigma \tag{55}$$

span a Hopf subalgebra *ΠQSym* of $\mathfrak{S}QSym$, which is isomorphic to the graded dual of the Hopf algebra of symmetric functions in noncommuting variables (such as in [3, 29], not to be confused with **Sym**), which we denote here by **WSym**(A), for *Word symmetric functions*. Indeed, from the product rule of the M_σ given in (25) one easily finds

$$U_{\pi'}U_{\pi''} := \sum C^{\pi}_{\pi',\pi''}U_\pi \tag{56}$$

where $C^{\pi}_{\pi',\pi''}$ is the number of ways of splitting the parts of π into two subpartitions whose standardized partitions are π' and π''. For example,

$$\begin{aligned} U_{\{\{1,2,4\},\{3\}\}}U_{\{\{1\}\}} &= U_{\{\{1,2,4\},\{3\},\{5\}\}} + 2U_{\{\{1,2,5\},\{3\},\{4\}\}} \\ &\quad + U_{\{\{1,3,5\},\{4\},\{2\}\}} + U_{\{\{2,3,5\},\{4\},\{1\}\}}. \end{aligned} \tag{57}$$

The dual **WSym**(A) of *ΠQSym* is the subspace of $\mathbb{K}\langle A\rangle$ spanned by the orbits of $\mathfrak{S}(A)$ on A^*. These orbits are naturally labeled by set partitions of $[n]$, the orbit corresponding to a partition π being constituted of the words

$$w = a_1\dots a_n \tag{58}$$

such that $a_i = a_j$ iff i and j are in the same block of π. The sum of these words will be denoted by M_π.

For example,

$$M_{\{\{1,3,6\},\{2\},\{4,5\}\}} := \sum_{a\neq b;b\neq c;a\neq c} abacca. \tag{59}$$

It is known that the natural coproduct of **WSym** (given as usual by the ordered sum of alphabets) is cocommutative [3] and that **WSym** is free over connected set partitions. The same argument as in Theorem 2.5 shows that $\Pi QSym^*$ is free over the same graded set, hence that $\Pi QSym$ is indeed isomorphic to $\mathbf{WSym}^*$.

3.5.2 Quasi-symmetric functions

One can also embed $QSym$ into $\Pi QSym$: take as total ordering on finite sets of integers $\{i_1 < \cdots < i_r\}$ the lexicographic order on the words $(i_1, \ldots, i_r)$. Then, any set partition π of $[n]$ has a canonical representative B as a nondecreasing sequence of blocks $(B_1 \leq B_2 \leq \cdots \leq B_r)$. Let $I = K(\pi)$ be the composition $(|B_1|, \ldots, |B_r|)$ of n. The sums

$$U_I := \sum_{K(\pi)=I} U_\pi = \sum_{K(\sigma)=I} M_\sigma, \tag{60}$$

where $K(\sigma)$ denotes the ordered cycle type of σ, span a Hopf subalgebra of $\Pi QSym$ and $\mathfrak{S}QSym$, which is isomorphic to $QSym$. Indeed, from the product rule of the M_σ given in (25) one easily finds

$$U_{I'}U_{I''} := \sum_I C^I_{I',I''} U_I \tag{61}$$

where $C^I_{I',I''}$ is the coefficient of I in $I' \sqcup\!\sqcup I''$. For example,

$$\begin{aligned} U_{(1,3,1)}U_{(1,2)} = 2U_{(1,1,2,3,1)} + 2U_{(1,1,3,1,2)} + 2U_{(1,1,3,2,1)} \\ + U_{(1,2,1,3,1)} + 2U_{(1,3,1,1,2)} + U_{(1,3,1,2,1)}. \end{aligned} \tag{62}$$

3.5.3 Symmetric functions

Furthermore, if we denote by $\Lambda(I)$ the partition associated with a composition I by sorting I and by $\Lambda(\pi)$ the partition λ whose parts are the sizes of the blocks of π, the sums

$$u_\lambda := \sum_{\Lambda(I)=\lambda} U_I = \sum_{\Lambda(\pi)=\lambda} U_\pi = \sum_{Z(\sigma)=\lambda} M_\sigma \tag{63}$$

where $Z(\sigma)$ denotes the cycle type of σ, span a Hopf subalgebra of $QSym$, $\Pi QSym$, and $\mathfrak{S}QSym$, which is isomorphic to Sym (ordinary symmetric functions). As an example of the product, one has

$$u_{(3,3,2,1)}u_{(3,1,1)} = 9u_{(3,3,3,2,1,1,1)}. \tag{64}$$

Indeed, from (61) it follows that an explicit Hopf embedding of Sym into $\mathfrak{S}QSym$ is given by

$$j : p^*_\lambda \to u_\lambda \tag{65}$$

where $p_\lambda^* = \frac{p_\lambda}{z_\lambda}$ is the adjoint basis of products of power sums. The images of the usual generators of Sym under this embedding have simple expressions in terms of the infinite matrix $X = (x_{ij})_{i,j\le 1}$:

$$j(p_n) = \mathrm{tr}\bigl(X^n\bigr), \tag{66}$$

which implies that $j(e_n)$ is the sum of the diagonal minors of order n of X:

$$j(e_n) = \sum_{i_1<\cdots<i_n} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) x_{i_1 i_{\sigma(1)}} \dots x_{i_n i_{\sigma(n)}}, \tag{67}$$

and $j(h_n)$ is the sum of the same minors of the permanent:

$$j(h_n) = \sum_{i_1<\cdots<i_n} \sum_{\sigma\in\mathfrak{S}_n} x_{i_1 i_{\sigma(1)}} \dots x_{i_n i_{\sigma(n)}}. \tag{68}$$

More generally, the sum of the diagonal immanants of type λ gives

$$j(s_\lambda) = \sum_{i_1<\cdots<i_n} \sum_{\sigma\in\mathfrak{S}_n} \chi^\lambda(\sigma) x_{i_1 i_{\sigma(1)}} \dots x_{i_n i_{\sigma(n)}}. \tag{69}$$

3.5.4 Involutions

Finally, one can check that the M_σ with σ involutive span a Hopf subalgebra of $\mathfrak{S}QSym$. Since the number of involutions of $\mathfrak{S}_n$ is equal to the number of standard Young tableaux of size n, this algebra can be regarded as a commutative version of the Poirier–Reutenauer algebra [26] denoted by **FSym** and realized in [7]. Note that this version is also isomorphic to the image of $\mathfrak{S}QSym$ in the quotient of $R/\mathcal{J}$ by the relations

$$x_{ij}x_{jk} = 0 \quad \text{for all } i \neq k. \tag{70}$$

This construction generalizes to the algebras built on permutations of arbitrary given order.

3.6 Quotients of **WSym**

Since we have built a subalgebra of $\Pi QSym$ isomorphic to $QSym$, we can define the embedding

$$i : QSym \hookrightarrow \Pi QSym = \mathbf{WSym}^*, \tag{71}$$

so that, dually, there is a Hopf epimorphism $i^* : \mathbf{WSym} \twoheadrightarrow \mathbf{Sym}$.

The dual basis V^I of the U_I defined in (60) can be identified with equivalence classes of $\mathbf{S}^\sigma$ under the relation

$$\mathbf{S}^\sigma \simeq \mathbf{S}^{\sigma'} \quad \text{if and only if} \quad K(\sigma) = K(\sigma'). \tag{72}$$

The $\mathbf{S}^\sigma$ with σ a full cycle are primitive, so that we can take for V_n any sequence of primitive generators of **Sym**.

It turns out that there is another natural epimorphism from **WSym** to **Sym**. Using the canonical ordering of set partitions introduced in Sect. 3.5, that is, the lexicographic ordering on the nondecreasing representatives of the blocks, we can as above associate a composition $K(\pi)$ with π and define the equivalence relation

$$\pi \sim \pi' \quad \text{if and only if} \quad K(\pi) = K(\pi'). \tag{73}$$

Then, the ideal $\mathfrak{J}$ of **WSym** generated by the differences

$$M_\pi - M_{\pi'}, \quad \pi \sim \pi', \tag{74}$$

is a Hopf ideal, and the quotient

$$\mathbf{WSym}/\mathfrak{J} \tag{75}$$

is isomorphic to **Sym**. The images V_I of the $\mathbf{M}_\pi$ under the canonical projection are analogs of the monomial symmetric functions in **Sym**. Indeed, the commutative image v_λ of $\mathbf{M}_\pi$ is proportional to a monomial function:

$$v_\lambda = \left(\prod_i m_i(\lambda)!\right) m_\lambda, \tag{76}$$

where $m_i(\lambda)$ is the number of occurrences of i in λ. It is worth noticing that, if we define coefficients c_λ by

$$v_1^n = \sum_{\lambda \vdash n} c_\lambda v_\lambda, \tag{77}$$

then the multivariate polynomials

$$B_n(x_1, \ldots, x_n) = \sum_{\lambda \vdash n} c_\lambda x_\lambda, \tag{78}$$

where $x_\lambda := x_{\lambda_1} \cdots x_{\lambda_n}$, are the exponential Bell polynomials defined by

$$\sum_{n \geq 0} \frac{t^n}{n!} B_n(x_1, \ldots, x_n) = e^{\sum_{n\geq 1} \frac{x_n}{n!} t^n}. \tag{79}$$

3.7 The stalactic monoid

The constructions of Sect. 3.6 can be interpreted in terms of a kind of Robinson–Schensted correspondence and of a plactic-like monoid. The *stalactic congruence* is the congruence $\equiv$ on A^* generated by the relations

$$a\,w\,a \equiv a\,a\,w \tag{80}$$

for all $a \in A$ and $w \in A^*$.

Each stalactic class has a unique representative, its *canonical representative*, of the form

$$a_1^{m_1} a_2^{m_2} \ldots a_r^{m_r} \tag{81}$$

with $a_i \neq a_j$ for $i \neq j$.

We can represent such a canonical word by a tableau-like planar diagram, e.g.,

$$c^3ad^3b^2 \longleftrightarrow \begin{array}{|c|c|c|c|} c & a & d & b \\ c & & d & b \\ c & & d & \end{array}. \tag{82}$$

Now, there is the obvious algorithm which consists in scanning a word from left to right and arranging its identical letters in columns, creating a new column to the right when one scans a letter for the first time. Let us call $P(w)$ the resulting canonical word, or, equivalently, its planar representation. We can compute $P(w)$ along with a Q-symbol recording the intermediate shapes of the P-symbol. For example, to insert $w = cabccdbdd$, we have the steps

$$\emptyset, \emptyset \xrightarrow{c} \begin{array}{c} c \end{array}, \begin{array}{c} 1 \end{array} \xrightarrow{a} \begin{array}{cc} c & a \end{array}, \begin{array}{cc} 1 & 2 \end{array} \xrightarrow{b} \begin{array}{ccc} c & a & b \end{array}, \begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\xrightarrow{c} \begin{array}{ccc} c & a & b \\ c & & \end{array}, \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \end{array} \xrightarrow{c} \begin{array}{ccc} c & a & b \\ c & & \\ c & & \end{array}, \begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ 5 & & \end{array} \xrightarrow{d} \begin{array}{cccc} c & a & b & d \\ c & & & \\ c & & & \end{array}, \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & & & \\ 5 & & & \end{array}$$

$$\xrightarrow{b} \begin{array}{cccc} c & a & b & d \\ c & & b & \\ c & & & \end{array}, \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & & 7 & \\ 5 & & & \end{array} \xrightarrow{d} \begin{array}{cccc} c & a & b & d \\ c & & b & d \\ c & & & \end{array}, \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & & 7 & 8 \\ 5 & & & \end{array} \xrightarrow{d} \begin{array}{cccc} c & a & b & d \\ c & & b & d \\ c & & & d \end{array}, \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & & 7 & 8 \\ 5 & & & 9 \end{array}$$

Clearly, the Q-symbol can be interpreted as a set partition of $[n]$, whose blocks are columns. In our example,

$$Q(cabccdbdd) = \big\{\{1, 4, 5\}, \{2\}, \{3, 7\}, \{6, 8, 9\}\big\}. \tag{83}$$

We now see that the natural basis $\mathbf{M}_\pi$ of **WSym** can be characterized as

$$\mathbf{M}_\pi = \sum_{Q(w)=\pi} w. \tag{84}$$

This is completely similar to the definition of the bases R_I of **Sym** via the hypoplactic congruence, of $\mathbf{S}_t$ of **FSym** via the plactic congruence, and of $\mathbf{P}_T$ of **PBT** via the Sylvester congruence [7, 11, 14].

A similar (but different) construction, due to M. Rey, based on the patience sorting algorithm, leads to a self-dual Hopf algebra structure based on set partitions [28].

Note that the defining relation of the stalactic monoid can be presented in a plactic-like way as

$$a\,u\,ba \equiv a\,u\,ab \tag{85}$$

for all $a, b \in A$ and $u \in A^*$.

3.8 Other Hopf algebras derived from the stalactic congruence

It is always interesting to investigate the behavior of certain special classes of words under analogs of the Robinson–Schensted correspondence. This section presents a few examples leading to interesting combinatorial sequences.

3.8.1 Parking functions

Recall that a word on $A=\{1,\dots\}$ is a parking function if its nondecreasing reordering $u_1\dots u_k$ satisfies $u_i\le i$. The number of stalactic classes of parking functions of size n can be combinatorially determined as follows.

Since the congruence does not change the evaluation of a word and since if a word is a parking function, then so are its permutations, one can restrict to a rearrangement class containing a unique nondecreasing parking function. Those are known to be counted by Catalan numbers. Now, the rearrangement class of a nondecreasing parking function p has exactly $l!$ congruence classes if l is the number of different letters of p.

The counting of nondecreasing parking functions p by their number of different letters obviously is the same as the counting of Dyck paths by their number of peaks given by the Narayana triangle (sequence A001263 in [31]). To get the number of stalactic classes of parking functions, one has to multiply the ith column by $i!$. This is the definition of the unsigned Lah numbers (sequence A089231 in [31]), which count with the additional parameter "number of lists," the number of sets of lists (sequence A000262 in [31]). Equivalently, these are set partitions of $[n]$ with an ordering inside each block but no order among the blocks. The first numbers of stalactic classes of parking functions are:

$$1,\ 3,\ 13,\ 73,\ 501,\ 4051,\ 37633,\ 394353,\ 4596553,\dots, \tag{86}$$

whereas the first rows of the Narayana and unsigned Lah triangles are given in Fig. 1.

The number of stalactic classes of parking functions of size n occurs as the sum of the nth row of the unsigned Lah triangle. One can also derive this last result from a character calculation. The Frobenius characteristic of the representation of $\mathfrak{S}_n$ on PF_n is

$$ch(\mathrm{PF}_n)=\frac{1}{n+1}h_n\big((n+1)X\big)=\frac{1}{n+1}\sum_{\mu\vdash n}m_\mu(n+1)h_\mu(X). \tag{87}$$

In this expression, each $h_\mu(X)$ is the characteristic of the permutation representation on a rearrangement class of words, with μ_1 occurrences of some letter i_1, μ_2 of some other letter i_2, and so on. Hence, the number of stalactic classes in each such rearrangement class is $l(\mu)!$, and the total number of stalactic classes of parking functions is

$$\alpha_n=\frac{1}{n+1}\sum_{\mu\vdash n}m_\mu(n+1)l(\mu)!. \tag{88}$$

Fig. 1 The Narayana and unsigned Lah triangles

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

1					
1	2				
1	6	6			
1	12	36	24		
1	20	120	240	120	
1	30	300	1200	1800	720

Since $g(z)=\sum_{n\geq 0} z^n \operatorname{ch}(\mathrm{PF}_n)$ solves the functional equation [24]

$$g(z)=\sum_{n\geq 0} z^n h_n(X) g(z)^n, \tag{89}$$

we see that the exponential generating function is

$$A(z)=\sum_{n\geq 0}\alpha_n \frac{z^n}{n!}=\int_0^\infty \mathrm{e}^{-x} g_x(z)\,\mathrm{d}x, \tag{90}$$

where $g_x(z)$ is the specialization of g, where $h_n = x$ for all $n\geq 1$. In this case,

$$g_x(z)=1+x\frac{zg_x(z)}{1-zg_x(z)}, \tag{91}$$

and (90) yields

$$A(z)=\exp\left(\frac{z}{1-z}\right). \tag{92}$$

Hence, α_n is the number of 'sets of lists,' giving back sequence A000262 of [31]. It would be interesting to find a natural bijection between stalactic classes of parking functions and sets of lists compatible with the algebraic structures.

Now, recall that the Hopf algebra of parking functions $\mathbf{PQSym}^*$ is spanned by the polynomials

$$\mathbf{G_a}(A) := \sum_{\mathrm{Park}(w)=\mathbf{a}} w. \tag{93}$$

As usual, it is easy to show that the equivalence defined by

$$\mathbf{G_{a'}}(A)\equiv \mathbf{G_{a''}}(A) \tag{94}$$

iff $\mathbf{a}'$ and $\mathbf{a}''$ are stalactically congruent is such that the quotient $\mathbf{PQSym}^*/\equiv$ is a Hopf algebra.

3.8.2 Endofunctions

The same methods allow one to see that the number β_n of stalactic classes of endofunctions on n letters is given by

$$\beta_n=\sum_{k=0}^{n}\binom{n-1}{k-1}\binom{n}{k} k!. \tag{95}$$

It is sequence A052852 of [31] whose first terms are

$$1, 4, 21, 136, 1045, 9276, 93289, 1047376, 12975561, 175721140, \ldots \tag{96}$$

and whose exponential generating series is

$$\frac{z}{1-z}\exp\left(\frac{z}{1-z}\right). \tag{97}$$

Fig. 2 The *Tw* and *Endt* triangles

1					
2	1				
3	6	1			
4	18	12	1		
5	40	60	20	1	
6	75	200	150	30	1

1					
2	2				
3	12	6			
4	36	72	24		
5	80	360	380	120	
6	150	1200	3600	3600	720

As before, the enumeration of the classes can be refined by introducing the additional parameter given by the number of different letters of their representatives, so that one gets a new triangle, the *Endt* triangle. Moreover, this triangle is obtained from the triangle referenced as sequence A103371 of [31] counting the integer compositions of $2n$ in n parts with a given number of ones, by multiplying column i by $i!$. Let us call this one the *Tw* triangle. The first rows of both triangles are shown in Fig. 2.

From the algebraic point of view, the Frobenius characteristic of $[n]^n$ is

$$\mathrm{ch}([n]^n) = h_n(nX) = \sum_{\mu\vdash n} m_\mu(n) h_\mu(X), \tag{98}$$

so that $\beta_n = \sum_{\mu\vdash n} m_\mu(n) l(\mu)!$, and the same method directly gives the exponential generating series of β_n.

As above, the quotient of **ESym** by $\mathbf{S}^{f'} \equiv \mathbf{S}^{f''}$ iff f' and f'' are stalactically congruent is a Hopf algebra.

3.8.3 Initial words

Recall that initial words are words on the alphabet of integers so that, if n appears in w, then $n-1$ also appears. The previous method allows one to see that the number γ_n of stalactic classes of initial words on n letters is

$$\gamma_n = \sum_{k=0}^{n} \binom{n-1}{k-1} k!. \tag{99}$$

It is sequence A001339 of [31] whose first terms are

$$1, 3, 11, 49, 261, 1631, 11743, 95901, 876809, 8877691, 98641011, \ldots \tag{100}$$

and whose exponential generating series is

$$\frac{\mathrm{e}^z}{(1-z)^2}. \tag{101}$$

As before, the classes counted by these numbers can be counted with the additional parameter given by the number of different letters of their representatives, so that one gets a new triangle, the *Arr* triangle. Once more, this triangle is obtained by multiplying column i by $i!$ in a simpler triangle, here the Pascal triangle. Here are the first rows of both triangles given in Fig. 3.

Again, the stalactic quotient of **WQSym** [10, 20] is a Hopf algebra.

Fig. 3 The Pascal and *Arr* triangles

1						1					
1	1					1	2				
1	2	1				1	4	6			
1	3	3	1			1	6	18	24		
1	4	6	4	1		1	8	36	96	120	
1	5	10	10	5	1	1	10	60	240	600	720

3.8.4 Generic case

Let A be an alphabet, and let $V=\mathbb{C}A$. The generic symmetric function

$$f_n=\sum_{\mu\vdash n} l(\mu)!m_\mu(X) \tag{102}$$

is the character of $GL(V)$ in the image of $V^{\otimes n}$ in the quotient of $T(V)\sim\mathbb{K}\langle A\rangle$ by the stalactic congruence. It is Schur-positive and can be explicitly expanded on the Schur basis. Indeed, one has

$$f_n=\sum_{k=0}^{n} c_k s_{(n-k,k)}. \tag{103}$$

The coefficients c_k are given by sequence A000255 of [31] whose first terms are

$$1, 1, 3, 11, 53, 309, 2119, 16687, 148329, 1468457, 16019531, \dots. \tag{104}$$

To see this, write again

$$l(\mu)!=\int_0^\infty t^{l(\mu)}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{105}$$

Then

$$f=\sum_{n\ge 0} f_n=\int_0^\infty \mathrm{e}^{-t}\sum_\mu t^{l(\mu)}m_\mu\,\mathrm{d}t \tag{106}$$

and

$$\begin{aligned}\sum_\mu t^{l(\mu)}m_\mu&=\prod_{i\ge 1}\left(1+\frac{tx_i}{1-x_i}\right)=\prod_{i\ge 1}\frac{1-(1-t)x_i}{1-x_i}\\&=\left(\sum_{k\ge 0}(t-1)^k e_k(X)\right)\left(\sum_{l\ge 0}h_l(X)\right).\end{aligned} \tag{107}$$

Since

$$\int_0^\infty \mathrm{e}^{-t}(t-1)^k\,\mathrm{d}t=d_k, \tag{108}$$

the number of derangements in $\mathfrak{S}_n$, we finally have

$$f=\sum_{n\ge 0}\sum_{k=0}^{n} d_k e_k(X)h_{n-k}(X). \tag{109}$$

Expanding

$$e_k h_{n-k} = s_{(n-k,1^k)} + s_{n-k+1,1^{k-1}}, \tag{110}$$

we get (103). Alternatively, we can express c_k as

$$c_k = \int_0^\infty \mathrm{e}^{-t} t(t-1)^k \, \mathrm{d}t, \tag{111}$$

since the term of degree n in (107) is

$$t \sum_{k=0}^{n} (t-1)^k s_{(n-k,1^k)}. \tag{112}$$

The exponential generating series of these numbers is given by

$$\frac{\mathrm{e}^{-z}}{(1-z)^2}. \tag{113}$$

4 Structure of $\mathfrak{S}$Sym

4.1 A realization of $\mathfrak{S}$Sym

In the previous section, we have built a commutative algebra of permutations $\mathfrak{S}QSym$ from explicit polynomials on a set of auxiliary variables x_{ij}. One may ask whether its noncommutative dual admits a similar realization in terms of noncommuting variables a_{ij}.

We shall find such a realization, in a somewhat indirect way, by first building from scratch a Hopf algebra of permutations **ΦSym** $\subset \mathbb{K}\langle a_{ij} \mid i, j \geq 1 \rangle$, whose operations can be described in terms of the cycle structure of permutations. Its coproduct turns out to be cocommutative, and the isomorphism with $\mathfrak{S}$**Sym** follows as above from the Cartier–Milnor–Moore theorem.

Let $\{a_{ij},\ i, j \geq 1\}$ be an infinite set of noncommuting indeterminates. We use the biword notation

$$a_{ij} \equiv \begin{bmatrix} i \\ j \end{bmatrix}, \quad \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \quad \dots \quad \begin{bmatrix} i_n \\ j_n \end{bmatrix} \equiv \begin{bmatrix} i_1 \dots i_n \\ j_1 \dots j_n \end{bmatrix}. \tag{114}$$

Let $\sigma \in \mathfrak{S}_n$, and let $(c_1, \dots, c_k)$ be a decomposition of σ into disjoint cycles. With any cycle $c = (i_1 \dots i_r)$, one associates its *cycle words* $w = i_k \dots i_r i_1 \dots i_{k-1}$ for any k, i.e., the words such that $c = (w)$. For example, the cycle words associated with the cycle (1362) are $1362, 2136, 3621, 6213$. For a word w, let $C(w)$ be the cycle of which $\mathrm{Std}(w)^{-1}$ is a cycle word. For example, $C(6213) = (3241)$.

Let $B = \left[\begin{smallmatrix} x \\ a \end{smallmatrix}\right]$ be a biword. Let $alph(x)$ be the letters appearing in x. For any i in $alph(x)$, let

$$\begin{bmatrix} u \\ a' \end{bmatrix} = \begin{bmatrix} ii \dots i \\ a_{j_1} \dots a_{j_r} \end{bmatrix}$$

be the sub-biword of B obtained by erasing the biletters $\left[{k \atop l}\right]$ such that $k \neq i$. Then define $\tau_i = \mathrm{Std}(a')^{-1}$ and $w_i = j_{\tau(1)} \dots j_{\tau(r)}$. Finally, let $Cd(B)$ be the permutation σ whose cycles are the (w_i). Such biwords will be said to have σ as *cycle decomposition*.

For example, let

$$B = \begin{bmatrix} 112121 \\ 421151 \end{bmatrix}.$$

Then $\tau_1 = 3421$, $w_1 = 4621$ and $\tau_2 = 12$, $w_2 = 35$, so that $Cd(B) = (4621)(35) = 415632$.

We now define

$$\phi_\sigma := \sum_{Cd(B)=\sigma} B. \tag{115}$$

Note that any biword appears in the expansion of exactly one ϕ_σ with coefficient 1.

Examples 4.1

$$\phi_{12} = \sum_{x \neq y} \begin{bmatrix} x\ y \\ a\ b \end{bmatrix}. \tag{116}$$

$$\phi_{41352} = \sum_{x \neq y;\, \mathrm{Std}(abde)^{-1} = 1342,\, 2134,\, 3421, \text{ or } 4213} \begin{bmatrix} x\ x\ y\ x\ x \\ a\ b\ c\ d\ e \end{bmatrix}. \tag{117}$$

Theorem 4.2 *The ϕ_σ span a subalgebra* $\mathbf{\Phi Sym}$ *of* $\mathbb{K}\langle a_{ij} \mid i, j \geq 1 \rangle$. *More precisely,*

$$\phi_\alpha \phi_\beta = \sum g^\sigma_{\alpha,\beta} \phi_\sigma \tag{118}$$

where $g^\sigma_{\alpha,\beta} \in \{0, 1\}$. *Moreover,* $\mathbf{\Phi Sym}$ *is free over the set*

$$\{\phi_\alpha \mid \alpha \text{ connected}\}. \tag{119}$$

Proof Let σ' and σ'' be two permutations, and let w be a biword appearing in $\phi_{\sigma'}\phi_{\sigma''}$. The multiplicity of w is 1. To get the first part of the theorem, we only need to prove that all words w' appearing in the same ϕ_σ as w also appear in this product. Given a letter x appearing in the first row of w at some positions, the subwords a of the elements of ϕ_σ taken from the *second* row at those positions have the same image by C as the corresponding element of w.

Thus, we only have to prove that all words w having the same image by C satisfy that all their prefixes (and suffixes) of a given length have also the same image by C. It is sufficient to prove the result on permutations. Now, given two permutations σ and τ, σ^{-1} and τ^{-1} are cycle words of the same cycle iff, for some k, $\tau = \gamma_n^k \sigma$, where γ_n is the cyclic permutation $(12\dots n)$. If $u = \sigma_1 \dots \sigma_r$ and $v = \tau_1 \dots \tau_r$ are the prefixes of length r of σ and τ, then it is obvious that $\alpha = \mathrm{Std}(u)^{-1}$ and $\beta = \mathrm{Std}(v)^{-1}$ are cycles words of the same cycle, since there exists an integer l such that $\beta^{-1} = \gamma_r^l \alpha^{-1}$. The same argument works also for the suffixes, and so the property holds.

Moreover, any biword can be uniquely written as a concatenation of a maximal number of biwords such that no letter appears in the first row of two different biwords and that the letters of the second row of a biword are all smaller than the letters of the second row of the next one. This proves that the ϕ_α where α is connected are free. The usual generating series argument then proves that those elements generate **ΦSym**. □

To give the precise expression of the product $\phi_\alpha\phi_\beta$, we first need to define two operations on cycles.

The first operation is just the circular shuffle on disjoint cycles: if c'_1 and c''_1 are two disjoint cycles, their *cyclic shuffle* $c'_1 \sqcup\!\sqcup c''_1$ is the set of cycles c_1 such that their cycle words are obtained by applying the usual shuffle on the cycle words of c'_1 and c''_1. This definition makes sense because a shuffle of cycle words associated with two words on disjoint alphabets splits as a union of cyclic classes.

For example, the cyclic shuffle $(132) \sqcup\!\sqcup (45)$ gives the set of cycles

$$\begin{aligned}\{ &(13245), (13425), (13452), (14325), (14352), (14532),\\ &(13254), (13524), (13542), (15324), (15342), (15432)\}.\end{aligned} \tag{120}$$

These cycles correspond to the following list of permutations which are those appearing in (127), except for the first one which will be found later:

$$\begin{aligned}\{ &34251, 35421, 31452, 45231, 41532, 41253,\\ &35214, 34512, 31524, 54213, 51423, 51234\}.\end{aligned} \tag{121}$$

Let us now define an operation on two sets C_1 and C_2 of cycles on mutually disjoint alphabets. We call *matching* an unordered list of all those cycles, some of the cycles being paired, always one of C_1 with one of C_2. The cycles remaining alone are considered to be associated with the empty cycle. We associate to such a matching the set of sets of cycles obtained by the cyclic product $\sqcup\!\sqcup$ of any pair of cycles. The union of those sets of cycles is denoted by $C_1 \smile C_2$.

For example, the matchings corresponding to $C_1 = \{(1), (2)\}$ and $C_2 = \{(3), (4)\}$ are:

$$\begin{aligned}&\{(1)\}\{(2)\}\{(3)\}\{(4)\},\ \{(1)\}\{(2),(3)\}\{(4)\},\ \{(1)\}\{(2),(4)\}\{(3)\},\\ &\{(1),(3)\}\{(2)\}\{(4)\},\ \{(1),(3)\}\{(2),(4)\},\ \{(1),(4)\}\{(2)\}\{(3)\},\\ &\{(1),(4)\}\{(2),(3)\},\end{aligned} \tag{122}$$

and the corresponding products $C_1 \smile C_2$ are:

$$\begin{aligned}&\{(1),(2),(3),(4)\},\ \{(1),(23),(4)\},\ \{(1),(24),(3)\},\\ &\{(13),(2),(4)\},\ \{(13),(24)\},\ \{(14),(2),(3)\},\ \{(14),(23)\}.\end{aligned} \tag{123}$$

Note that this calculation is identical with the Wick formula in quantum field theory (see [5] for an explanation of this coincidence).

We are now in a position to describe the product $\phi_\sigma\phi_\tau$:

Proposition 4.3 *Let C_1 be the cycle decomposition of σ, and C_2 be the cycle decomposition of τ shifted by the size of σ. Then the permutations indexing the elements appearing in the product $\phi_\sigma\phi_\tau$ are the permutations whose cycle decompositions belong to $C_1 \smile C_2$.*

Proof Recall that the product $\phi_\sigma\phi_\tau$ is a sum of biwords with multiplicity one, since any biword appears in exactly one ϕ_σ. So we only have to prove that the biwords appearing in $\phi_\sigma\phi_\tau$ are the same as those biwords whose cycle decompositions are contained in $C_1 \smile C_2$. First, by the definition of the cyclic shuffle and of the operation $\smile$, if a biword has its cycle decomposition in $C_1 \smile C_2$, its prefix of size n has cycle decomposition C_1 whereas its suffix of size p has cycle decomposition C_2, where n (resp. p) is the size of σ (resp. τ).

Conversely, let w_1 (resp. w_2) be a biword with cycle decomposition C_1 (resp. C_2), and let us consider $w = w_1 \cdot w_2$. For all letters in the first row of w, either it only appears in w_1, or only in w_2, or in both w_1 and w_2. In the first two cases, we obtain the corresponding cycle of C_1 (or C_2, shifted). In the remaining case, the cycle decomposition of the word of the second row corresponding to this letter belongs to the cyclic shuffle of the corresponding cycles of C_1 and C_2 (hence matching those two cycles). Indeed, by the definition of the cyclic shuffle, if $\mathrm{Std}^{-1}(w_1)$ is a cycle word of c_1 and $\mathrm{Std}^{-1}(w_2)$ is a cycle word of c_2, then $\mathrm{Std}^{-1}(w)$ is a cycle word of an element of $c_1 \sqcup\!\sqcup c_2$. □

For example, with $\sigma = \tau = 12$, one finds that $C_1 = \{(1), (2)\}$ and $C_2 = \{(3), (4)\}$. It is then easy to check that one goes from (123) to (125) by computing the corresponding permutations.

Examples 4.4

$$\phi_{12}\phi_{21} = \phi_{1243} + \phi_{1342} + \phi_{1423} + \phi_{3241} + \phi_{4213}. \tag{124}$$

$$\phi_{12}\phi_{12} = \phi_{1234} + \phi_{1324} + \phi_{1432} + \phi_{3214} + \phi_{3412} + \phi_{4231} + \phi_{4321}. \tag{125}$$

$$\phi_{1}\phi_{4312} = \phi_{15423} + \phi_{25413} + \phi_{35421} + \phi_{45123} + \phi_{51423}. \tag{126}$$

$$\begin{aligned}\phi_{312}\phi_{21} = {} & \phi_{31254} + \phi_{31452} + \phi_{31524} + \phi_{34251} + \phi_{34512} + \phi_{35214} + \phi_{35421} \\ & + \phi_{41253} + \phi_{41532} + \phi_{45231} + \phi_{51234} + \phi_{51423} + \phi_{54213}.\end{aligned} \tag{127}$$

Let us recall a general recipe to obtain the coproduct of a combinatorial Hopf algebra from a realization in terms of words on an ordered alphabet X. Assume that X is the ordered sum of two mutually commuting alphabets X' and X''. Then define the coproduct as $\Delta(F) = F(X' \dot{+} X'')$, identifying $F' \otimes F''$ with $F'(X')F''(X'')$ [7, 22].

There are many different ways to define a coproduct on **ΦSym** compatible with the realization, since there are many ways to order an alphabet of biletters: order the letters of the first alphabet, order the letters of the second alphabet, or order lexicographically with respect to one alphabet and then to the second.

In the sequel, we only consider the coproduct obtained by ordering the biletters with respect to the first alphabet so that ϕ_σ is primitive if σ consists of only one cycle. More precisely, thanks to the definition of the elements ϕ_σ, it is easy to see that it corresponds to the unshuffling of the cycles of a permutation:

$$\Delta\phi_\sigma := \sum_{(\alpha,\beta)} \phi_\alpha \otimes \phi_\beta \tag{128}$$

where the sum is taken over all pairs of permutations (α, β) such that the cycle decomposition of α is obtained by renumbering the elements of a subset of cycles of σ (preserving the relative order of values), and β by doing the same on the complementary subset of cycles. For example, if $\sigma = (1592)(36)(4)(78)$, the subset $(1592)(4)$ gives $\alpha = (1452)(3)$ and $\beta = (12)(34)$.

Examples 4.5

$$\Delta\phi_{12} = \phi_{12} \otimes 1 + 2\phi_1 \otimes \phi_1 + 1 \otimes \phi_{12}. \tag{129}$$

$$\Delta\phi_{312} = \phi_{312} \otimes 1 + 1 \otimes \phi_{312}. \tag{130}$$

$$\Delta\phi_{4231} = \phi_{4231} \otimes 1 + 2\phi_{321} \otimes \phi_1 + \phi_{21} \otimes \phi_{12} + \phi_{12} \otimes \phi_{21} + 2\phi_1 \otimes \phi_{321} + 1 \otimes \phi_{4231}. \tag{131}$$

The following theorem is a direct consequence of the definition of the coproduct on the realization.

Theorem 4.6 *Δ is an algebra morphism, so that* **ΦSym** *is a graded bialgebra (for the grading* $\deg \phi_\sigma = n$ *if* $\sigma \in \mathfrak{S}_n$*). Moreover, Δ is cocommutative.*

The same reasoning as in Sect. 3 shows that:

Theorem 4.7 **𝔖Sym** *and* **ΦSym** *are isomorphic as Hopf algebras.*

To describe such an isomorphism in the pair of bases (ϕ_σ), $(\mathbf{S}_\tau)$, let us first recall that a *connected permutation* is a permutation σ such that $\sigma([1, k]) \neq [1, k]$ for any $k \in [1, n-1]$. Any permutation σ has a unique maximal factorization $\sigma = \sigma_1 \bullet \cdots \bullet \sigma_r$ into connected permutations. We then define

$$\mathbf{S}'_\sigma := \phi_{\sigma_1} \dots \phi_{\sigma_r}. \tag{132}$$

Then

$$\mathbf{S}'_\sigma = \phi_{\sigma_1 \bullet \cdots \bullet \sigma_r} + \sum_\mu \phi_\mu \tag{133}$$

where the second sum ranges over permutations μ with strictly less than r cycles in their cycle decomposition. So the $\mathbf{S}'$ form a basis of **ΦSym**. Moreover, they are a multiplicative basis with product given by shifted concatenation of permutations, so that they multiply as the $\mathbf{S}$ do. Moreover, the coproduct of $\mathbf{S}'_\sigma$ is the same as for ϕ_σ, so the same as for $\mathbf{S}^\sigma$. So both bases $\mathbf{S}$ and $\mathbf{S}'$ have the same product and the same coproduct. This proves the following:

Proposition 4.8 *The linear map* $\mathbf{S}^\sigma \mapsto \mathbf{S}'_\sigma$ *realizes the Hopf isomorphism between* $\mathfrak{S}$**Sym** *and* $\mathbf{\Phi}$**Sym**.

There is another natural isomorphism: define

$$\mathbf{S}''_\sigma := \sum_{x,a} \begin{bmatrix} x \\ a \end{bmatrix} \tag{134}$$

where the sum ranges over all words x, a such that $x_i = x_j$ if (but *not* only if) i and j belong to the same cycle of σ and such that the standardized word of the subword of a consisting of the indices of cycle c_l is equal to the inverse of the standardized word of a cycle word of c_l.

The basis elements $\mathbf{S}'_\sigma$ and $\mathbf{S}''_\sigma$ satisfy the same product and coproduct rules because, if $(c_1)\dots(c_p)$ is the cycle decomposition of σ, then

$$\mathbf{S}''_\sigma = \sum_{(c)\in(c_1)\smile(c_2)\smile\cdots\smile(c_p)} \phi_{(c)}. \tag{135}$$

For example,

$$\begin{aligned} \mathbf{S}''_{2431} = \mathbf{S}''_{(124)(3)} &= \phi_{(124)(3)} + \phi_{(1423)} + \phi_{(1234)} + \phi_{(1324)} \\ &= \phi_{2431} + \phi_{4312} + \phi_{2341} + \phi_{3421}. \end{aligned} \tag{136}$$

4.2 Quotients of $\mathbf{\Phi}$**Sym**

Let I be the ideal of $\mathbf{\Phi}$**Sym** generated by the differences

$$\phi_\sigma - \phi_\tau \tag{137}$$

where σ and τ have the same cycle type.

The definitions of its product and coproduct directly imply that I is a Hopf ideal. Since the cycle types are parametrized by integer partitions, the quotient $\mathbf{\Phi}$**Sym**$/I$ has a basis Y_λ corresponding to the class of ϕ_σ, where σ has λ as cycle type.

From (124–127) one finds:

Examples 4.9

$$Y_{11}Y_2 = Y_{211} + 4Y_{31}, \qquad Y_{11}^2 = Y_{1111} + 2Y_{22} + 4Y_{211}. \tag{138}$$

$$Y_1Y_4 = Y_{41} + 4Y_5, \qquad Y_3Y_2 = Y_{32} + 12Y_5. \tag{139}$$

Theorem 4.10 $\mathbf{\Phi}$**Sym**$/I$ *is isomorphic to Sym, the Hopf algebra of ordinary symmetric functions.*

If one writes $\lambda = (\lambda_1, \ldots, \lambda_p) = (1^{m_1}, \ldots, k^{m_k})$, *an explicit isomorphism is given by*

$$Y_\lambda \mapsto \frac{\prod_{i=1}^{k} m_i!}{\prod_{j=1}^{p} (\lambda_j - 1)!} m_\lambda. \tag{140}$$

Proof This follows from the description of $\phi_\sigma \phi_\tau$ given in Proposition 4.3. □

5 Parking functions and trees

5.1 A commutative algebra of parking functions

It is also possible to build a commutative version of the Hopf algebra of parking functions introduced in [21]: let PF_n be the set of parking functions of length n. For $\mathbf{a} \in \mathrm{PF}_n$, set, as before,

$$M_{\mathbf{a}} := \sum_{i_1 < \cdots < i_n} x_{i_1\, i_{\mathbf{a}(1)}} \cdots x_{i_n\, i_{\mathbf{a}(n)}}. \tag{141}$$

Then, using once more the same arguments as in Sect. 2, we conclude that the $M_{\mathbf{a}}$ form a linear basis of a $\mathbb{Z}$-subalgebra *PQSym* of *EQSym*, which is also a subcoalgebra if one defines the coproduct in the usual way.

Examples 5.1

$$M_1 M_{11} = M_{122} + M_{121} + M_{113}. \tag{142}$$

$$M_1 M_{221} = M_{1332} + M_{3231} + M_{2231} + M_{2214}. \tag{143}$$

$$M_{12} M_{21} = M_{1243} + M_{1432} + M_{4231} + M_{1324} + M_{3214} + M_{2134}. \tag{144}$$

$$\Delta M_{525124} = M_{525124} \otimes 1 + 1 \otimes M_{525124}. \tag{145}$$

$$\Delta M_{4131166} = M_{4131166} \otimes 1 + M_{41311} \otimes M_{11} + 1 \otimes M_{4131166}. \tag{146}$$

The main interest of the noncommutative and noncocommutative Hopf algebra of parking functions defined in [21] is that it leads to two algebras of trees. The authors obtain a cocommutative Hopf algebra of planar binary trees by summing over the distinct permutations of parking functions and an algebra of planar trees by summing over hypoplactic classes.

We shall now investigate whether similar constructions can be found for the commutative version *PQSym*.

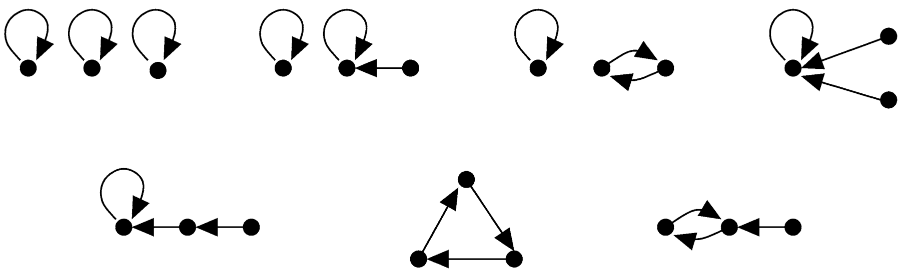
5.2 From labeled to unlabeled parking graphs

A first construction, which applies to all Hopf algebras of labeled graphs is to build a subalgebra by summing over labelings. Notice that this subalgebra is the same as the subalgebra obtained by summing endofunction graphs over their labelings. Those graphs are also known as endofunctions (hence considered there as unlabeled graphs) in [2].

The dimension of this subalgebra in degree n is equal to the number of unlabeled parking graphs

$$1, 1, 3, 7, 19, 47, 130, 343, 951, 2615, 7318, 20491, 57903, 163898, \ldots. \tag{147}$$

This is sequence A001372 in [31]. For example, here are the 7 unlabeled parking graphs of size 3 (to be compared with the 16 parking functions):



The product of two such unlabeled graphs is the concatenation of graphs and the coproduct of an unlabeled graph is the unshuffle of its connected subgraphs. So this algebra is isomorphic to the polynomial algebra on generators indexed by connected parking graphs.

5.3 Binary trees and nondecreasing parking functions

One can easily check that, in *PQSym*, summing over parking functions having the same reordering does not lead to a subalgebra. However, if we denote by I the subspace of *PQSym* spanned by the $M_{\mathbf{a}}$ where $\mathbf{a}$ is not nondecreasing, I is an ideal and a coideal, and $CQSym := PQSym/I$ is therefore a commutative Hopf algebra with basis given by the classes $M_\pi := \overline{M_{\mathbf{a}}}$ labeled by nondecreasing parking functions.

Note that *CQSym* is also isomorphic to the image of *PQSym* in the quotient of $R/\mathcal{J}$ by the relations

$$x_{ij}x_{kl} = 0 \quad \text{for all } i < k \text{ and } j > l. \tag{148}$$

The dual basis of M_π is

$$S^\pi := \sum_{\mathbf{a}} S^{\mathbf{a}}, \tag{149}$$

where the sum is taken over all permutations of π.

The dual $CQSym^*$ is free over the set S^π, where π runs over connected nondecreasing parking functions. So if one denotes by **CQSym** the Catalan algebra defined in [21], the usual Cartier–Milnor–Moore argument then shows that

$$\mathbf{CQSym} \sim CQSym^*, \qquad CQSym \sim \mathbf{CQSym}^*. \tag{150}$$

5.4 From nondecreasing parking functions to rooted forests

Nondecreasing parking functions correspond to parking graphs of a particular type, namely, rooted forests with a particular labeling (it corresponds to nondecreasing maps), the root being given by the loops in each connected component.

Taking sums over such admissible labelings of a given rooted forest, we get that the elements

$$M_F := \sum_{\mathrm{supp}(\pi)=F} M_\pi \tag{151}$$

span a commutative Hopf algebra of rooted forests, which can be regarded as a commutative and cocommutative version of the Connes–Kreimer algebra [6].

6 Quantum versions

6.1 Quantum quasi-symmetric functions

When several Hopf algebra structures can be defined on the same class of combinatorial objects, it is tempting to try to interpolate between them. This can be done, for example, with compositions: the algebra of quantum quasi-symmetric functions $QSym_q$ [7, 32] interpolates between quasi-symmetric functions and noncommutative symmetric functions.

However, the natural structure on $QSym_q$ is not exactly that of a Hopf algebra but rather of a *twisted Hopf algebra* [16].

Recall that the coproduct of $QSym(X)$ is obtained by replacing X by the ordered sum $X' \dot{+} X''$ of two isomorphic and mutually commuting alphabets. On the other hand, $QSym_q$ can be realized by means of an alphabet of q-commuting letters

$$x_j x_i = q x_i x_j \quad \text{for } j > i. \tag{152}$$

Hence, if we define the coproduct on $QSym_q$ by

$$\Delta_q f(X) = f\big(X' \dot{+} X''\big), \tag{153}$$

with X' and X'' q-commuting with each other, it will be an algebra morphism

$$QSym_q \to QSym_q\big(X' \dot{+} X''\big) \simeq QSym_q \otimes_\chi QSym_q \tag{154}$$

for the *twisted product of tensors*

$$(a \otimes b) \cdot (a' \otimes b') = \chi(b, a')(aa' \otimes bb') \tag{155}$$

where

$$\chi(b, a') = q^{\deg(b) \cdot \deg(a')} \tag{156}$$

for homogeneous elements b and a'.

It is easily checked that Δ_q is actually given by the same formula as the usual coproduct of $QSym$, that is,

$$\Delta_q M_I = \sum_{H \cdot K = I} M_H \otimes M_K. \tag{157}$$

The dual twisted Hopf algebra, denoted by $\mathbf{Sym}_q$, is isomorphic to $\mathbf{Sym}$ as an algebra. If we denote by S^I the dual basis of M_I, $S^I S^J = S^{I \cdot J}$, and $\mathbf{Sym}_q$ is freely generated by the $S^{(n)} = S_n$, whose coproduct is

$$\Delta_q S_n = \sum_{i+j=n} q^{ij} S_i \otimes S_j. \tag{158}$$

As above, Δ_q is an algebra morphism

$$\mathbf{Sym}_q \to \mathbf{Sym}_q \otimes_\chi \mathbf{Sym}_q \tag{159}$$

where χ is again defined by (156).

6.2 Quantum free quasi-symmetric functions

The previous constructions can be lifted to **FQSym**. Recall from [7] that $\phi(\mathbf{F}_\sigma) = q^{l(\sigma)} F_{c(\sigma)}$ is an algebra homomorphism $\mathbf{FQSym} \to QSym_q$, which is in fact induced by the specialization $\phi(a_i) = x_i$ of the underlying free variables a_i to q-commuting variables x_i.

The coproduct of **FQSym** is also defined by

$$\Delta \mathbf{F}(A) = \mathbf{F}\big(A' \dot{+} A''\big), \tag{160}$$

where A' and A'' are two mutually commuting copies of A [7]. If, instead, one sets $a''a' = qa'a''$, one obtains again a twisted Hopf algebra structure $\mathbf{FQSym}_q$ on **FQSym**, for which ϕ is a homomorphism. With these definitions at hand, one can see that the arguments given in [32] to establish the results recalled in Sect. 6.1 prove in fact the following more general result:

Theorem 6.1 *Let A' and A'' be q-commuting copies of the ordered alphabet A*, i.e., *$a''a' = qa'a''$ for $a' \in A'$ and $a'' \in A''$. Then, the coproduct*

$$\Delta_q f = f\big(A' \dot{+} A''\big) \tag{161}$$

defines a twisted Hopf algebra structure. It is explicitly given in the basis $\mathbf{F}_\sigma$ by

$$\Delta_q \mathbf{F}_\sigma = \sum_{\alpha \cdot \beta = \sigma} q^{\mathrm{inv}(\alpha,\beta)} \mathbf{F}_{\mathrm{Std}(\alpha)} \otimes \mathbf{F}_{\mathrm{Std}(\beta)} \tag{162}$$

where $\mathrm{inv}(\alpha, \beta)$ *is the number of inversions of* σ *with one element in* α *and the other in* β.

More precisely, Δ_q *is an algebra morphism with values in the twisted tensor product of graded algebras* $\mathbf{FQSym} \otimes_\chi \mathbf{FQSym}$, *where* $(a \otimes b)(a' \otimes b') = \chi(b, a')(aa' \otimes bb')$ *and* $\chi(b, a') = q^{\deg(b).\deg(a')}$ *for homogeneous elements* b, a'.

The map $\phi : \mathbf{FQSym}_q \to QSym_q$ *defined by*

$$\phi(\mathbf{F}_\sigma) = q^{l(\sigma)} F_{c(\sigma)} \tag{163}$$

is a morphism of twisted Hopf algebras.

Examples 6.2

$$\Delta_q \mathbf{F}_{2431} = \mathbf{F}_{2431} \otimes 1 + q^3 \mathbf{F}_{132} \otimes \mathbf{F}_1 + q^3 \mathbf{F}_{12} \otimes \mathbf{F}_{21} + q\mathbf{F}_1 \otimes \mathbf{F}_{321} + 1 \otimes \mathbf{F}_{2431}. \tag{164}$$

$$\Delta_q \mathbf{F}_{3421} = \mathbf{F}_{3421} \otimes 1 + q^3 \mathbf{F}_{231} \otimes \mathbf{F}_1 + q^4 \mathbf{F}_{12} \otimes \mathbf{F}_{21} + q^2 \mathbf{F}_1 \otimes \mathbf{F}_{321} + 1 \otimes \mathbf{F}_{3421}. \tag{165}$$

$$\Delta_q \mathbf{F}_{21} = \mathbf{F}_{21} \otimes 1 + q\mathbf{F}_1 \otimes \mathbf{F}_1 + 1 \otimes \mathbf{F}_{21}. \tag{166}$$

$$\begin{aligned}(\Delta_q \mathbf{F}_{21})(\Delta_q \mathbf{F}_1) = {} & (\mathbf{F}_{213} + \mathbf{F}_{231} + \mathbf{F}_{321}) \otimes 1 + (\mathbf{F}_{21} + q^2(\mathbf{F}_{12} + \mathbf{F}_{21})) \otimes \mathbf{F}_1 \\ & + \mathbf{F}_1 \otimes (q^2 \mathbf{F}_{21} + q(\mathbf{F}_{12} + \mathbf{F}_{21})) + 1 \otimes (\mathbf{F}_{213} + \mathbf{F}_{231} + \mathbf{F}_{321}).\end{aligned} \tag{167}$$

$$\Delta_q \mathbf{F}_{213} = \mathbf{F}_{213} \otimes 1 + \mathbf{F}_{21} \otimes \mathbf{F}_1 + q\mathbf{F}_1 \otimes \mathbf{F}_{12} + 1 \otimes \mathbf{F}_{213}. \tag{168}$$

$$\Delta_q \mathbf{F}_{231} = \mathbf{F}_{231} \otimes 1 + q^2 \mathbf{F}_{12} \otimes \mathbf{F}_1 + q\mathbf{F}_1 \otimes \mathbf{F}_{21} + 1 \otimes \mathbf{F}_{231}. \tag{169}$$

$$\Delta_q \mathbf{F}_{321} = \mathbf{F}_{321} \otimes 1 + q^2 \mathbf{F}_{21} \otimes \mathbf{F}_1 + q^2 \mathbf{F}_1 \otimes \mathbf{F}_{21} + 1 \otimes \mathbf{F}_{321}. \tag{170}$$

Finally, one can also define a one-parameter family of ordinary Hopf algebra structures on **FQSym** by restricting formula (162) for Δ_q to connected permutations σ and requiring that Δ_q be an algebra homomorphism. Then, for $q = 0$, Δ_q becomes cocommutative, and it is easily shown that the resulting Hopf algebra is isomorphic to $\mathfrak{S}$**Sym**.

However, from [7] it follows that, for generic q, the Hopf algebras defined in this way are all isomorphic to **FQSym**. This suggests to interpret **FQSym** as a kind of quantum group: it would be the generic element of a quantum deformation of the enveloping algebra $\mathfrak{S}\mathbf{Sym} = U(L)$. Similar considerations apply to various examples, in particular, to the Loday–Ronco algebra **PBT**, whose commutative version obtained in [21] can be quantized in the same way as *QSym* by means of q-commuting variables [22].

There is another way to obtain *QSym* from **FQSym**: it is known [7] that *QSym* is isomorphic to the image of $\mathbf{FQSym}(A)$ in the hypoplactic algebra $\mathbb{K}[A^*/\equiv_H]$. One may then ask whether there exists a q-analogue of the hypoplactic congruence leading directly to $QSym_q$.

Recall that the hypoplactic congruence can be presented as the bi-Sylvester congruence:

$$\begin{aligned} bvca &\equiv bvac \quad \text{with } a < b \le c, \\ cavb &\equiv acvb \quad \text{with } a \le b < c, \end{aligned} \tag{171}$$

and $v \in A^*$.

A natural q-analogue compatible with the above q-commutation is

$$\begin{aligned} bvca &\equiv_{qH} q\, bvac \quad \text{with } a < b \le c, \\ cavb &\equiv_{qH} q\, acvb \quad \text{with } a \le b < c, \end{aligned} \tag{172}$$

and $v \in A^*$. Then we have the following:

Theorem 6.3 *The image of* **FQSym**(A) *under the natural projection* $\mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle/\equiv_{qH}$ *is isomorphic to* $QSym_q$ *as an algebra, and also as a twisted Hopf algebra for the coproduct* $A \to A' \dot{+} A''$, A' *and* A'' *being* q*-commuting alphabets.*

Moreover, it is known that if one only considers the Sylvester congruence

$$cavb \equiv_S acvb, \tag{173}$$

the quotient **FQSym**(A) under the natural projection $\mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle/\equiv_S$ is isomorphic to the Hopf algebra of planar binary trees of Loday and Ronco [11, 17]. The previous construction provides natural twisted q-analogs of this Hopf algebra. Indeed, let the q-Sylvester congruence $\equiv_{qS}$ be

$$cavb \equiv_{qS} q\, acvb \quad \text{with } a \le b < c. \tag{174}$$

Then, since this congruence is compatible with the q-commutation, we have the following:

Theorem 6.4 *The image of* **FQSym**(A) *under the natural projection* $\mathbb{K}\langle A\rangle \to \mathbb{K}\langle A\rangle/\equiv_{qS}$ *is a twisted Hopf algebra with basis indexed by planar binary trees.*

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