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Primitive elements in the matroid-minor Hopf algebra

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Abstract We introduce the *matroid-minor* coalgebra C, which has labeled matroids as distinguished basis and coproduct given by splitting a matroid into a submatroid and complementary contraction in all possible ways. We introduce two new bases for C; the first of these is related to the distinguished basis by Möbius inversion over the rank-preserving weak order on matroids, the second by Möbius inversion over the suborder excluding matroids that are irreducible with respect to the free product operation. We show that the subset of each of these bases corresponding to the set of irreducible matroids is a basis for the subspace of primitive elements of C. Projecting C onto the matroid-minor Hopf algebra H, we obtain bases for the subspace of primitive elements of H.

Keywords Matroid · Free product · Hopf algebra · Primitive elements

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Many of the Hopf algebras now of central importance in algebraic combinatorics share certain striking features, suggesting the existence of a natural, yet-to-be-identified, class of combinatorial Hopf algebras. These Hopf algebras are graded and cofree, each has a canonical basis consisting of, or indexed by, a family of (equivalence classes of) combinatorial objects that is equipped with a natural partial ordering, and in each case the algebraic structure is most clearly understood through the introduction of a second basis, related to the canonical one by Möbius inversion over the

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partial ordering. Indeed, in a number of key examples cofreeness becomes apparent once the coproduct is expressed in terms of the second basis, and this basis contains, as an easily recognizable subset, a basis for the subspace of primitive elements. The significance of primitive elements in this context was established by Loday and Ronco in [9], where they proved a Milnor–Moore type theorem characterizing cofree Hopf algebras in terms of their primitive elements.

Examples of such Hopf algebras include the algebra of quasisymmetric functions, introduced by Gessel in [6], the Hopf algebra structure of which was determined by Malvenuto in [10], the Malvenuto–Reutenauer Hopf algebra of permutations [10, 11], the Loday–Ronco Hopf algebra of planar binary trees [8], and, most recently, the Hopf algebra of uniform block partitions, due to Aguiar and Orellana [1]. Recent work by Taşkin [15] on various partial orderings of standard Young tableaux suggest that the Poirier–Reutenauer Hopf algebra of tableaux [12] also belongs to this class.

The canonical basis for quasisymmetric functions is indexed by compositions of nonnegative integers, whose natural partial ordering, given by refinement, is a disjoint union of Boolean algebras. In [2], Aguiar and Sottile use the weak order on the symmetric groups to elucidate the structure of the Malvenuto–Reutenauer Hopf algebra, and in [3] they use the Tamari order on planar binary trees in an analogous fashion to study the Loday–Ronco Hopf algebra. Furthermore, through the use of Galois connections between each pair of the aforementioned partial orders, they exhibit the myriad relationships among these Hopf algebras in a completely unified manner.

In this article, we use similar techniques to study a seemingly unrelated Hopf algebra, based on matroids. The *matroid-minor Hopf algebra*, introduced in [14], has as canonical basis the set of all isomorphism classes of matroids, with product induced by the direct sum operation, and coproduct of the isomorphism class e_M of a matroid $M = M(S)$ given by $\sum_{A\subseteq S} e_{M|A} \otimes e_{M/A}$, where $M|A$ is the submatroid obtained by restriction to A and M/A is the complementary contraction. The current authors showed in [5] that this Hopf algebra is cofree, and that its subspace of primitive elements has a basis indexed by those isomorphism classes of matroids that are irreducible with respect to the *free product* operation [4]. We approach the matroid-minor Hopf algebra here by first lifting to the matroid-minor *coalgebra*, which has as canonical basis the set of all *labeled* matroids whose underlying sets are subsets of some given infinite set. The coproduct of a matroid $M(S)$ in the matroid-minor coalgebra is given by $\sum_{A\subseteq S} M|A \otimes M/A$, so the natural projection, taking a matroid to its isomorphism class, is a coalgebra map. The set of labeled matroids is partially ordered by the (rank-preserving) *weak order*, under which $M \geq N$ means that M and N have the same underlying set, and each basis for N is also a basis for M. We introduce two new bases for the matroid-minor coalgebra, both related to the canonical basis by Möbius inversion; the first over the full weak order and the second over a suborder that excludes matroids which are irreducible with respect to free product. The subset of each of these new bases corresponding to the irreducible matroids is a basis for the subspace of primitive elements. Applying the projection map, we obtain bases for the subspace of primitive elements of the matroid-minor Hopf algebra, one of these is the basis previously identified in [5].

We note that it is not possible to extend the matroid-minor coalgebra to a Hopf algebra that maps onto the matroid-minor Hopf algebra without using equivalence classes of matroids, rather than labeled matroids, as basis elements. In order for the algebra to most directly reflect the underlying combinatorics, we work with labeled matroids, and thus only a coalgebra, rather than Hopf algebra, of matroids.

1 Posets and Möbius functions

In this section we gather together for later use some basic facts about partially ordered sets and their Möbius functions. Of the four results given here, two (Theorems 1.1 and 1.2) are classical, and stated without proof, one (Proposition 1.4) is apparently new, and one (Proposition 1.6) is trivial, but nonetheless useful to have on hand.

We assume that all partially ordered sets, or *posets*, for short, are locally finite, that is, given $x \le z$ in a poset P, the interval $[x,z] = \{y \in P : x \le y \le z\}$ is finite. The *Möbius function* $\mu = \mu_P$ of a poset P is the integer-valued function having the set of intervals in P as domain, defined by $\mu(x,x) = 1$, for all $x \in P$, and

$$\mu(x,z) = -\sum_{x \le y < z} \mu(x,y) = -\sum_{x < y \le z} \mu(y,z),$$

for all $x < z$ in P.

A *closure operator* on a poset P is an idempotent, order-preserving map $\varphi\colon P \to P$ such that $x \le \varphi(x)$, for all $x \in P$. Given a closure operator φ on P, we write P_φ for the subposet $\operatorname{im}\varphi = \{x \in P : x = \varphi(x)\}$ of *closed elements* of P. An essential ingredient in the proofs of our main results, Theorems 4.10 and 4.11, is the following well-known theorem, due to Rota [13], that expresses the Möbius function of P_φ in terms of that of P.

Theorem 1.1 *If φ is a closure operator on a poset P, then for all $a \le b$ in P,*

$$\sum_{x\colon \varphi(x)=b} \mu(a,x) = \begin{cases} \mu_\varphi(a,b) & \text{if } a,b \in P_\varphi, \\ 0 & \text{otherwise}, \end{cases}$$

where μ and μ_φ denote the Möbius functions of P and P_φ, respectively.

Given $x \le z$ in a poset P, a *chain* from x to z is a sequence $C = (x_0, \dots, x_k)$ of elements of P such that $x = x_0 < \cdots < x_k = z$. The *length* $\ell(C)$ of a chain C is $|C| - 1$. The following theorem, due to Philip Hall [7], provides an alternative definition of Möbius function that allows us to give a short proof of Proposition 1.4 below.

Theorem 1.2 *If $x \le z$ in a poset P, then $\mu_P(x,z) = \sum (-1)^{\ell(C)}$, where the sum is over all chains C from x to z in P.*

Suppose that P is a poset and $Q \subseteq P$. Given $x \le z$ in P, we denote by $[x,z]_Q$ the subposet $\{x,z\} \cup ([x,z] \cap Q)$ of $[x,z]$, and we write $[x,z)_Q$ for $[x,z]_Q \backslash \{z\}$. We extend the definition of the Möbius function of Q to all intervals in P by setting

$$\mu_Q(x,z) = \mu_{[x,z]_Q}(x,z), \tag{1.3}$$

for all $x < z$ in P.

Proposition 1.4 *Suppose that P is a poset, $Q \subseteq P$ and $R = P \backslash Q$. Then*

$$\mu_P(x,z) = \sum_{y \in [x,z)_Q} \mu_P(x,y)\mu_R(y,z), \tag{1.5}$$

for all $x < z$ in P.

Proof For each $y \in [x,z)_Q$, let $\mathcal{C}_y$ denote the set of all chains C from x to z in P such that $max(C \cap [x,z)_Q) = y$. From Theorem 1.2, we then have

$$\mu_P(x,y)\mu_R(y,z) = \sum_{C \in \mathcal{C}_y} (-1)^{\ell(C)},$$

for each such y. Since $\{\mathcal{C}_y : y \in [x,z)_Q\}$ is a partition of the set of all chains from x to z in P, Eq. 1.5 thus follows from Theorem 1.2. □

Proposition 1.6 *Suppose that P is a poset with minimum element x, and let $\widehat{P} = \{y \in P : \mu_P(x,y) \neq 0\}$. Then $\mu_{\widehat{P}}(x,z) = \mu_P(x,z)$, for all $z \in \widehat{P}$.*

Proof The proof is immediate from the recursive definition of Möbius function. □

2 The free product of matroids

We write $M(S)$ to indicate that M is a matroid with underlying set S. We denote by ρ_M the rank function of $M(S)$, and write $\rho(M)$ for the rank $\rho_M(S)$ of M. The *free product* of matroids $M(S)$ and $N(T)$ on disjoint sets S and T is the matroid $M \square N$ on the union $S \cup T$ whose bases are those sets $B \subseteq S \cup T$ of cardinality $\rho(M) + \rho(N)$ such that $B \cap S$ is independent in M and $B \cap T$ spans N.[1] The free product operation was introduced by the current authors in [4], where it was used to prove the conjecture of Welsh [16] that $f_{n+m} \geq f_n \cdot f_m$, where f_n is the number of distinct isomorphism classes of matroids on an n-element set. In [5], we studied the free product in detail; in particular we showed that this operation, which is noncommutative, is associative, and respects matroid duality in the sense that $(M \square N)^* = N^* \square M^*$, for all matroids M and N. We also characterized, in terms of cyclic flats, those matroids which are irreducible with respect to free product, and proved the following unique factorization theorem:

Theorem 2.1 *If $M_1 \square \cdots \square M_k \cong N_1 \square \cdots \square N_r$, where each M_i and N_j is irreducible with respect to free product, then $k = r$ and $M_i \cong N_i$, for $1 \leq i \leq k$.*

We gave in [5] a number of cryptomorphic definitions of free product, one of the most useful of which is the following proposition.

[1] This elegant characterization of free product in terms of bases is due to one of the referees of our paper [5].

Proposition 2.2 *The rank function of $M(S) \,\square\, N(T)$ is given by*

$$\rho_{M\square N}(A) = \rho_M(A \cap S) + \rho_N(A \cap T) + \min\{\rho_M(S) - \rho_M(A \cap S), \nu_N(A \cap T)\},$$

for all $A \subseteq S \cup T$.

It is worth contrasting the above formula with that for the rank function of the direct sum $M(S) \oplus N(T)$:

$$\rho_{M\oplus N}(A) = \rho_M(A \cap S) + \rho_N(A \cap T), \tag{2.3}$$

for all $A \subseteq S \cup T$.

We refer to a matroid as *irreducible* if it is irreducible with respect to free product, and *reducible*, otherwise.

Example 2.4 We denote by $Z(a)$ and $I(a)$, respectively, the matroids consisting of a single loop and single isthmus on $\{a\}$. For any set $S = \{a_1, \ldots, a_n\}$, and $0 \leq r \leq n$, the free product $Z(a_1) \,\square \cdots \square\, Z(a_{n-r}) \,\square\, I(a_{n-r+1}) \,\square \cdots \square\, I(a_n)$ is equal to $Z(a_1) \oplus \cdots \oplus Z(a_{n-r}) \oplus I(a_{n-r+1}) \oplus \cdots \oplus I(a_n)$, the direct sum of $n-r$ loops and r isthmi, while $I(a_1) \,\square \cdots \square\, I(a_r) \,\square\, Z(a_{r+1}) \,\square \cdots \square\, Z(a_n)$ is the uniform matroid $U_{r,n}(S)$ of rank r on S. The matroid $I(a) \,\square\, Z(b) \,\square\, I(c) \,\square\, Z(d)$ is a three-point line, with one point, ab, doubled.

Example 2.5 A matroid consisting of a single loop or isthmus is irreducible, and no matroid of size two or three is irreducible. Up to isomorphism, the unique irreducible matroid on four elements is the pair of double points $U_{1,2} \oplus U_{1,2}$, and on five elements the only irreducibles are $U_{1,3} \oplus U_{1,2}$ and its dual $U_{2,3} \oplus U_{1,2}$.

For any finite set S, we denote by $\mathcal{W}_S$ the collection of all matroids having S as ground set. The set $\mathcal{W}_S$ is partially ordered by the (*rank-preserving*) *weak order*, in which $M \leq N$ means that every basis for M is also a basis for N or, equivalently, M and N have the same rank and the identity map on S is a weak map from N to M. The second inequality of the following result was Proposition 4.2 in [5].

Proposition 2.6 *For all matroids $M(S)$, and all $U \subseteq S$, the relation*

$$M|U \oplus M/U \;\leq\; M \;\leq\; M|U \,\square\, M/U$$

holds in $\mathcal{W}_S$.

Proof Let $V = S\backslash U$. The bases of $M|U \oplus M/U$ are those subsets B of S such that $B \cap U$ is a basis for $M|U$ and $B \cap V$ is a basis for M/U, which is the case if and only if B is a basis for M such that $B \cap U$ spans $M|U$. Hence any basis for $M|U \oplus M/U$ is also a basis for M, and so $M|U \oplus M/U \leq M$.

By definition of free product, $B \subseteq S$ is a basis of $M|U \,\square\, M/U$ if and only if $|B| = \rho(M)$, the set $B \cap U$ is independent in M and $B \cap V$ spans M/U. Now $\rho_{M/U}(B \cap V) = \rho_M(B \cup U) - \rho_M(U)$ and $\rho(M/U) = \rho(M) - \rho_M(U)$, so $B \cap V$

spans M/U if and only $\rho_M(B \cup U) = \rho(M)$, that is, if and only if $B \cup U$ spans M. Hence any basis for M is also a basis for $M|U \,\square\, M/U$, and so $M \leq M|U \,\square\, M/U$. $\square$

The following result is one of the keys to understanding the coproduct, and thus the primitive elements, of the matroid-minor coalgebra.

Proposition 2.7 *For all matroids $M(S)$ and $N(T)$, with S and T disjoint, the set $\{L : L|S = M \text{ and } L/S = N\}$ is equal to the interval $[M \oplus N, M \,\square\, N]$ in $\mathcal{W}_{S \cup T}$.*

Proof If $L(S \cup T)$ is such that $L|S = M$ and $L/S = N$ then, by Proposition 2.6, $L \in [M \oplus N, M \,\square\, N]$. Conversely, suppose that $L \in [M \oplus N, M \,\square\, N]$, so that $\rho_{M \oplus N}(A) \leq \rho_L(A) \leq \rho_{M \square N}(A)$, for all $A \subseteq S \cup T$. If $A \subseteq S$ then, $\rho_N(A \cap T) = \nu_N(A \cap T) = 0$, and so by Proposition 2.2 and Eq. 2.3, we have $\rho_{M \oplus N}(A) = \rho_{M \square N}(A) = \rho_M(A)$. Hence $\rho_{L|S}(A) = \rho_L(A) = \rho_M(A)$, and therefore $L|S = M$.

Now if $A \subseteq T$, then $\rho_{M \oplus N}(A \cup S) = \rho_{M \square N}(A \cup S) = \rho_M(S) + \rho_N(A)$, and thus $\rho_L(A \cup S) = \rho_M(S) + \rho_N(A)$. Since $\rho_{L/S}(A) = \rho_L(A \cup S) - \rho_L(S)$, and $\rho_L(S) = \rho_M(S)$, it follows that $\rho_{L/S}(A) = \rho_N(A)$. Hence $L/S = N$. $\square$

3 The matroid-minor coalgebra

Let $\mathcal{U}$ be the set of all finite subsets of some fixed infinite set, and let K be a field. Denote by $\mathcal{W}$ the set of all matroids $M(S)$ whose ground set S belongs to $\mathcal{U}$, and denote by $\mathcal{W}_+$ the set of all nonempty $M \in \mathcal{W}$. We give $\mathcal{W}$ the (rank-preserving) weak order, in which $M \leq N$ means that M and N have the same ground set S and $M \leq N$ in $\mathcal{W}_S$; hence $\mathcal{W}$ is the disjoint union of the posets $\mathcal{W}_S$, for $S \in \mathcal{U}$. For sets S, U and V, we write $U + V = S$ to indicate that $U \cup V = S$ and $U \cap V = \emptyset$.

Let $K\{\mathcal{W}\}$ denote the free K-vector space having basis $\mathcal{W}$. Since $\{\mathcal{W}_S : S \in \mathcal{U}\}$ is a partition of $\mathcal{W}$, we have the direct sum decomposition

$$K\{\mathcal{W}\} = \bigoplus_{S \in \mathcal{U}} K\{\mathcal{W}_S\};$$

hence the vector space $\mathcal{W}$ is graded by the set $\mathcal{U}$, with each homogeneous component $K\{\mathcal{W}_S\}$ finite-dimensional.

We define a pairing $\langle \cdot, \cdot \rangle$ on $K\{\mathcal{W}\}$ by setting $\langle M, N \rangle$ equal to the Kronecker delta $\delta_{M,N}$, for all $M, N \in \mathcal{W}$, and thus identify $K\{\mathcal{W}\}$ with the graded dual space

$$K\{\mathcal{W}\}^* = \bigoplus_{S \in \mathcal{U}} K\{\mathcal{W}_S\}^*.$$

Let C be the K-coalgebra on $K\{\mathcal{W}\}$ with coproduct δ and counit ϵ determined by

$$\delta(M) = \sum_{A \subseteq S} M|A \otimes M/A \quad \text{and} \quad \epsilon(M) = \begin{cases} 1, & \text{if } S = \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for all $M(S) \in \mathcal{W}$. Let C^* be the K-algebra on $K\{\mathcal{W}\}$ dual to C; the product of C^* is thus determined by

$$\langle P \cdot Q, M\rangle = \langle \delta(M), P \otimes Q\rangle,$$

for all $M, P, Q \in \mathcal{W}$, and the unit element of C^* is ϵ. We remark that, even though the underlying vector space of C^* is the graded dual of that of C, we do not refer to C^* as the graded dual algebra of C, because C is not $\mathcal{U}$-graded as a coalgebra. In fact, since $\mathcal{U}$ has no given monoid structure, the concept of $\mathcal{U}$-graded coalgebra is meaningless.

Proposition 3.1 *In the algebra C^*, the product of matroids M and N on disjoint ground sets is given by*

$$M \cdot N \;=\; \sum_{M \oplus N \leq L \leq M \,\square\, N} L.$$

If M and N are not disjoint, then $M \cdot N = 0$ in C^*.

Proof First, observe that

$$\begin{aligned} M \cdot N &= \sum_{L \in \mathcal{W}} \langle M \cdot N, L\rangle\, L \\ &= \sum_{L \in \mathcal{W}} \langle M \otimes N,\, \delta(L)\rangle\, L. \end{aligned}$$

If M and N are not disjoint, the latter sum is empty, and hence $M \cdot N = 0$. If M and N are disjoint then, by Proposition 2.7, $\langle M \otimes N, \delta(L)\rangle$ is equal to one whenever $M \oplus N \leq L \leq M \,\square\, N$ in $\mathcal{W}$, and is zero otherwise. □

More generally, the product of matroids $M_1, \ldots, M_k$ in C^* is equal to

$$\sum_{M_1 \oplus \cdots \oplus M_k \leq L \leq M_1 \,\square \cdots \square\, M_k} L, \tag{3.2}$$

if the set $\{M_1, \ldots, M_k\}$ is pairwise disjoint and is zero otherwise. For all $S \in \mathcal{U}$, we let $\pi_S \colon K\{\mathcal{W}\} \to K\{\mathcal{W}_S\}$ denote the natural projection, and for any subset X of $K\{\mathcal{W}\}$, we write X_S for $\pi_S(X)$. In particular, we write C_S for $K\{\mathcal{W}_S\}$ when viewed as a subspace of the coalgebra C, and similarly for the algebra C^*.

Proposition 3.3 *The coproduct δ of C satisfies*

$$\delta(C_S) \subseteq \bigoplus_{U+V=S} C_U \otimes C_V,$$

for all $S \in \mathcal{U}$.

Proof The result follows immediately from the definition of δ. □

We remark that Proposition 3.3 may be stated alternatively as

$$\delta(\pi_S(x)) = \sum_{U+V=S} (\pi_U \otimes \pi_V)\delta(x) \tag{3.4}$$

for all $x \in C$ and $S \in \mathcal{U}$.

Proposition 3.3, together with the fact that $\epsilon(C_S) = 0$, for all $S \neq \emptyset$ says that C is something very much like a $\mathcal{U}$-graded coalgebra; the problem, as we indicated above, is that the disjoint union operation $+$ is only partially defined on $\mathcal{U}$ and so does not equip $\mathcal{U}$ with a monoid structure. To make precise the sense in which C is a "generalized graded" coalgebra, consider the *partial monoid algebra* $K\{\mathcal{U}\}$, with product determined by

$$ST = \begin{cases} S \cup T & \text{if } S \cap T = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and coproduct $\delta(S) = S \otimes S$, for all $S, T \in \mathcal{U}$. Then $K\{\mathcal{U}\}$ is a Hopf algebra, and the map $\psi\colon C \to K\{\mathcal{U}\} \otimes C$, determined by $x \mapsto S \otimes x$, for all $x \in C_S$, is a coaction, making C a $K\{\mathcal{U}\}$-comodule coalgebra. In the following situation: $\mathcal{U}$ a monoid, C_S a subspace of a coalgebra C, for all $S \in \mathcal{U}$ and $K\{\mathcal{U}\}$ the monoid (Hopf) algebra of $\mathcal{U}$, the above map ψ being well-defined and making C a $K\{\mathcal{U}\}$-comodule coalgebra is equivalent to C being $\mathcal{U}$-graded as a coalgebra, with homogeneous components C_S, for all $S \in \mathcal{U}$.

4 Primitive elements in the matroid-minor coalgebra

We denote by $\mathcal{I}$ and $\mathcal{R}$, respectively, the collections of all irreducible and all reducible matroids belonging to $\mathcal{W}$. For any $M \in \mathcal{W}$, we denote by $\mathcal{W}_M$ the order filter $\{N \in \mathcal{W}\colon N \geq M\}$ of $\mathcal{W}$ and define the following subposets of $\mathcal{W}_M$:

$$\mathcal{I}_M = (\mathcal{I} \cap \mathcal{W}_M) \cup \{M\} \quad \text{and} \quad \mathcal{R}_M = (\mathcal{R} \cap \mathcal{W}_M) \cup \{M\}.$$

For any $S \in \mathcal{U}$, we set

$$\mathcal{I}_S = \mathcal{I} \cap \mathcal{W}_S \quad \text{and} \quad \mathcal{R}_S = \mathcal{R} \cap \mathcal{W}_S.$$

Let $C_+ = \ker \epsilon$, and let $\bar{\delta}\colon C_+ \to C_+ \otimes C_+$ be the map determined by $\delta(x) = 1 \otimes x + x \otimes 1 + \bar{\delta}(x)$, for all $x \in C_+$, where 1 denotes the empty matroid. Then C_+ has basis $\mathcal{W}_+$, and $\bar{\delta}(x)$ satisfies

$$\bar{\delta}(M) = \sum_{\substack{A \subseteq S \\ A \neq \emptyset, S}} M|A \otimes M/A,$$

for all $M \in \mathcal{W}_+$. We write $P(C)$ for the subspace of primitive elements of C; hence $P(C) = \{x \in C_+\colon \bar{\delta}(x) = 0\}$.

Proposition 4.1 *The space of primitive elements $P(C)$ respects the grading of C by $\mathcal{U}$, that is,*

$$P(C)_S = P(C) \cap C_S,$$

for all $S \in \mathcal{U}$.

Proof For any $S \in \mathcal{U}$ we have $P(C) \cap C_S = \pi_S(P(C) \cap C_S) \subseteq P(C)_S$. On the other hand, if $x \in P(C)_S$, then $x = \pi_S(y)$ for some $y \in P(C)$ and thus, by Eq. 3.4,

$$\begin{aligned}\delta(x) &= \sum_{U+V=S} (\pi_U \otimes \pi_V)\delta(y) \\ &= \sum_{U+V=S} (\pi_U(y) \otimes \pi_V(1) + \pi_U(1) \otimes \pi_V(y)).\end{aligned}$$

Since

$$\pi_V(1) = \begin{cases} 1 & \text{if } V = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

we thus have

$$\begin{aligned}\delta(x) &= \pi_S(y) \otimes 1 + 1 \otimes \pi_S(y) \\ &= x \otimes 1 + 1 \otimes x.\end{aligned}$$

Hence $P(C)_S \subseteq P(C) \cap C_S$. □

An alternative way of stating Proposition 4.1 is the following:

$$P(C) = \bigoplus_{S \in \mathcal{U}} P(C) \cap C_S. \tag{4.2}$$

Denote by C_+^* the ideal in C^* consisting of all elements x such that $\langle x\,, 1\rangle = 0$. Note that C_+^* has basis $\mathcal{W}_+$, and that the ideal $(C_+^*)^2$ of C^* is spanned by the set of all products $P \cdot Q$ such that $P, Q \in \mathcal{W}_+$. For any subset X of $K\{\mathcal{W}\}$, we define $X^\perp = \{y \in K\{\mathcal{W}\} \colon \langle y, x\rangle = 0, \text{ for all } x \in X\}$. The following proposition is a standard result about connected coalgebras.

Proposition 4.3 *The subspace of primitive elements of C is given by $P(C) = C_+ \cap [(C_+^*)^2]^\perp$.*

Proof An element x of C belongs to $P(C)$ if and only if $\epsilon(x) = 0$ and $\bar{\delta}(x) = 0$, that is, if and only if $x \in C_+$ and $\langle P \otimes Q, \bar{\delta}(x)\rangle = 0$, for all $P, Q \in \mathcal{W}_+$. But, for nonempty P and Q, we have

$$\begin{aligned}\langle P \otimes Q,\, \bar{\delta}(x)\rangle &= \langle P \otimes Q,\, \delta(x)\rangle \\ &= \langle P \cdot Q,\, x\rangle.\end{aligned}$$

Hence, the condition that $\langle P \otimes Q, \bar{\delta}(x)\rangle = 0$, for all $P, Q \in \mathcal{W}_+$, is equivalent to x belonging to $[(C_+^*)^2]^\perp$. □

Corollary 4.4 *For all nonempty $S \in \mathcal{U}$ we have*

$$P(C)_S = ([(C^*_+)^2]_S)^\perp,$$

while $P(C)_\emptyset = \{0\}$.

Proof The result follows directly from Proposition 4.3 and the direct-sum decomposition (4.2). $\square$

Untangling the notation of Corollary 4.4, we see that $P(C)_S$ consists of all $x \in C_S$ such that $\langle x, P \cdot Q\rangle = 0$, for all matroids $P(U)$ and $Q(V)$ with U and V nonempty and $U + V = S$.

The following result streamlines the brute-force method of finding primitive elements in C.

Proposition 4.5 *The ideal $(C^*_+)^2$ of C^* is spanned by the set $\mathcal{P}$ of all products $P \cdot Q$ such that P is irreducible* (*with respect to free product*) *and Q is nonempty.*

Proof We prove the result by using (weak order) induction on P to show that if $P \cdot Q \in (C^*_+)^2$, then $P \cdot Q \in K\{\mathcal{P}\}$.

For the base case, suppose that $P \cdot Q \in (C^*_+)^2$ and that $P = P(S)$ is minimal in the weak order. Then P is the direct sum of k loops and r isthmi, where k is the nullity and r is the rank of P, and so, in the notation of Example 2.4, we have

$$\begin{aligned} P(S) &= Z(a_1) \oplus \cdots \oplus Z(a_k) \oplus I(a_{k+1}) \oplus \cdots \oplus I(a_{k+r}) \\ &= Z(a_1) \,\square \cdots \square\, Z(a_k) \,\square\, I(a_{k+1}) \,\square \cdots \square\, I(a_{k+r}), \end{aligned}$$

which, by (3.2), implies that P is equal to the product

$$Z(a_1) \cdots Z(a_k) \cdot I(a_{k+1}) \cdots I(a_{k+r}),$$

in C^*. Hence $P \cdot Q \in K\{\mathcal{P}\}$.

Now suppose that $P \cdot Q \in (C^*_+)^2$, and that $L \cdot Q \in K\{\mathcal{P}\}$, for all $L < P$. If P is irreducible, we are done; otherwise, write P as $M \,\square\, N$, where M is irreducible. By Proposition 3.1, we have

$$M \cdot N \cdot Q \;=\; \sum_{M \oplus N \leq L \leq P} L \cdot Q,$$

and hence

$$P \cdot Q = M \cdot N \cdot Q \;-\; \sum_{M \oplus N \leq L < P} L \cdot Q,$$

which, by induction, belongs to $K\{\mathcal{P}\}$. $\square$

Of course, the "right-handed" version of Proposition 4.5 also holds: the ideal $(C^*_+)^2$ of C^* is spanned by the set of all products $Q \cdot P$ such that P is irreducible and Q is nonempty.

Proposition 4.6 *The inequality* $\dim P(C)_S \le |\mathcal{I}_S|$ *holds for all* $S \in \mathcal{U}$.

Proof Define a map $\alpha\colon K\{\mathcal{R}_S\} \to (C^*_+)^2$ as follows: for each reducible $M(S)$, choose a sequence of irreducible matroids $M_1, \dots, M_k$ such that $M = M_1 \,\square \cdots \square\, M_k$ (recall that the sequence $M_1, \dots, M_k$ is uniquely determined by M only up to isomorphism), then set $\alpha(M)$ equal to the product $M_1 \cdots M_k$ in C^*. Clearly $\operatorname{im}\alpha \subseteq [(C^*_+)^2]_S$. By Proposition 3.1, we know that $\alpha(M)$ is equal to M plus matroids that are less than M in the weak order. It follows that α is injective, and so $|\mathcal{R}_S| \le \dim[(C^*_+)^2]_S$. Hence, by Corollary 4.4, we have

$$\begin{aligned}\dim P(C)_S &= \dim([(C^*_+)^2]_S)^\perp \\ &= |\mathcal{W}_S| - \dim[(C^*_+)^2]_S \\ &\le |\mathcal{W}_S| - |\mathcal{R}_S| \\ &= |\mathcal{I}_S|. \end{aligned}$$

□

We will see shortly, in Theorem 4.10, that the inequality in the above proposition is in fact an equality. Recall that a *free separator* of a matroid $M(S)$ is a subset U of S such that M factors as $M = M|U \,\square\, M/U$.

Proposition 4.7 *For all* $S \in \mathcal{U}$ *and* $U \subseteq S$, *the map* $\varphi_U : \mathcal{W}_S \to \mathcal{W}_S$ *given by*

$$\varphi_U(M) = M|U \,\square\, M/U,$$

for all $M \in \mathcal{W}_S$, *is a closure operator. A matroid* M *is* φ_U*-closed if and only if* U *is a free separator of* M.

Proof Let M be a matroid on S, and $U \subseteq S$. It is immediate from the definitions that U is a free separator of M if and only if $M = \varphi_U(M)$. To show that φ_U is a closure operator, we first note that, since U is a free separator of $\varphi_U(M)$, we have $\varphi_U(\varphi_U(M)) = \varphi_U(M)$; also, the inequality $M \le \varphi_U(M)$ follows from Proposition 2.6. It remains to show that, if $M \le N$ in $\mathcal{W}_S$, then $\varphi_U(M) \le \varphi_U(N)$. As we noted in the proof of Proposition 2.6, The bases of $M|U \,\square\, M/U$ are those subsets B of S with $|B| = \rho(M)$ such that $B \cap U$ is independent in M and $B \cup U$ spans M. Now if $N \ge M$ and $B \cap U$ is independent in M, then it is also independent in N; since $\rho(N) = \rho(M)$, it thus follows that any basis for $M|U \,\square\, M/U$ is also a basis for $N|U \,\square\, N/U$, that is, $\varphi_U(M) \le \varphi_U(N)$. □

Lemma 4.8 *For all matroids* $P(U)$, $Q(V)$ *and* $M(S)$, *with* $S = U + V$, *we have* $\varphi_U(M) = P \,\square\, Q$ *if and only if* $P \oplus Q \le M \le P \,\square\, Q$ *in* $\mathcal{W}_S$.

Proof Since $(P \,\square\, Q)|U = P$ and $(P \,\square\, Q)/U = Q$, for all matroids $P(U)$ and $Q(V)$, it follows that $P(U) \,\square\, Q(V) = \varphi_U(M) = M|U \,\square\, M/U$ if and only if $M|U = P$ and $M/U = Q$. By Proposition 2.7 this is the case if and only if $P \oplus Q \le M \le P \,\square\, Q$. □

We define bases $\{w_M : M \in \mathcal{W}\}$ and $\{r_M : M \in \mathcal{W}\}$ for the matroid-minor coalgebra C by setting

$$w_M = \sum_{N \in \mathcal{W}_M} \mu_{\mathcal{W}}(M, N)\, N \quad \text{and} \quad r_M = \sum_{N \in \mathcal{R}_M} \mu_{\mathcal{R}}(M, N)\, N,$$

for all $M \in \mathcal{W}$. We have written here $\mu_{\mathcal{R}}$ for the Möbius function of the poset $\mathcal{R}_M$; equivalently, we are using the convention (1.3) to extend the definition of the Möbius function of $\mathcal{R}$ to arbitrary intervals in $\mathcal{W}$. Note that $\{w_M : M \in \mathcal{W}\}$ and $\{r_M : M \in \mathcal{W}\}$ are indeed bases for C since, by Möbius inversion, we have

$$M = \sum_{N \in \mathcal{W}_M} w_N = \sum_{N \in \mathcal{R}_M} r_N,$$

for all $M \in \mathcal{W}$. The following proposition shows us how to change between the bases $\{w_M : M \in \mathcal{W}\}$ and $\{r_M : M \in \mathcal{W}\}$.

Proposition 4.9 *For all $M \in \mathcal{W}$*

$$w_M = \sum_{N \in \mathcal{I}_M} \mu_{\mathcal{W}}(M, N)\, r_N \quad \textit{and} \quad r_M = w_M - \sum_{N \in \mathcal{I}_M \setminus M} \mu_{\mathcal{R}}(M, N)\, w_N.$$

Proof By definition of r_N, w_M, and Proposition 1.4, we have

$$\begin{aligned}
\sum_{N \in \mathcal{I}_M} \mu_{\mathcal{W}}(M, N)\, r_N &= \sum_{N \in \mathcal{I}_M} \sum_{Q \in \mathcal{R}_N} \mu_{\mathcal{W}}(M, N)\mu_{\mathcal{R}}(N, Q)\, Q \\
&= M + \sum_{Q \in \mathcal{I}_M \setminus M} \mu_{\mathcal{W}}(M, Q) \\
&\quad + \sum_{Q \in \mathcal{R}_M \setminus M} \sum_{N \in [M, Q)_{\mathcal{I}}} \mu_{\mathcal{W}}(M, N)\mu_{\mathcal{R}}(N, Q)\, Q \\
&= \sum_{Q \in \mathcal{W}_M} \mu_{\mathcal{W}}(M, Q)\, Q \\
&= w_M,
\end{aligned}$$

thus establishing the first equality. For the second, we compute

$$\begin{aligned}
r_M &= \sum_{N\in\mathcal{R}_M} \mu_{\mathcal{R}}(M,N)\, N \\
&= \sum_{N\in\mathcal{R}_M} \sum_{Q\in\mathcal{W}_N} \mu_{\mathcal{R}}(M,N)\, w_Q \\
&= w_M + \sum_{Q\in\mathcal{R}_M\setminus M} \sum_{N\in[M,Q]_R} \mu_{\mathcal{R}}(M,N)\, w_Q + \sum_{Q\in\mathcal{I}_M\setminus M} \sum_{N\in[M,Q)_R} \mu_{\mathcal{R}}(M,N)\, w_Q.
\end{aligned}$$

By the recursive definition of Möbius function it follows that

$$\sum_{N\in[M,Q]_{\mathcal{R}}} \mu_{\mathcal{R}}(M,N) = 0 \quad \text{and} \quad \sum_{N\in[M,Q)_{\mathcal{R}}} \mu_{\mathcal{R}}(M,N) = -\mu_{\mathcal{R}}(M,Q),$$

for all $Q > M$ in $\mathcal{W}$; thus the result follows. □

We now come to the first of our main results.

Theorem 4.10 *The set $\{w_M : M \in \mathcal{I}\}$ is a homogeneous basis for $P(C)$.*

Proof It is clear from the definition that the w_M are homogeneous with respect to the $\mathcal{U}$-grading of C. Now, if M is irreducible, then it is nonempty and thus $\epsilon(w_M) = 0$. Suppose that $M(S)$ is irreducible and that $P(U)$ and $Q(V)$ are nonempty matroids. By Proposition 3.1, we have

$$\langle P \cdot Q, w_M\rangle = \sum_{\substack{N\in\mathcal{W}_M \\ P\oplus Q\le N\le P\,\square\, Q}} \mu_{\mathcal{W}}(M,N).$$

If it is not the case that $M \le P \,\square\, Q$, then the sum is empty; otherwise, according to Lemma 4.8, it is given by

$$\sum_{\substack{N\in\mathcal{W}_M \\ \varphi_U(N)=P\,\square\, Q}} \mu_{\mathcal{W}}(M,N).$$

Since U is a nonempty proper subset of S and M is irreducible, U is not a free separator of M, and so M is not φ_U-closed. Thus, by Theorem 1.1, the above sum is zero, and so it follows from Proposition 4.3 that w_M is primitive in C.

Since $\{w_M : M \in \mathcal{W}\}$ is a basis for C, the set $\{w_M : M \in \mathcal{I}_S\}$ is linearly independent in $P(C)_S$ and thus, by Proposition 4.6, is a basis for $P(C)_S$. It follows from the direct-sum decomposition (4.2) that $\{w_M : M \in \mathcal{I}\}$ is a basis for $P(C)$. □

Theorem 4.11 *The set $\{r_M : M \in \mathcal{I}\}$ is a homogeneous basis for $P(C)$. Furthermore, if an element x of $P(C)$ has the form $M + y$, where M is an irreducible matroid and y is a linear combination of reducible matroids, the $x = r_M$.*

Proof Observe that, for any $M(S)$, and $U \subseteq S$, the closure operator φ_U on $\mathcal{W}_M$ satisfies $\varphi_U(\mathcal{R}_M) \subseteq \mathcal{R}_M$, and thus restricts to a closure operator on $\mathcal{R}_M$. The proof that $\{r_M : M \in \mathcal{I}\}$ is a basis for $P(C)$ thus parallels that of Theorem 4.10, with the poset $\mathcal{R}_M$ used in place of $\mathcal{W}_M$. Uniqueness follows immediately from the fact that the set $\mathcal{I}$ of irreducible matroids is linearly independent in C. □

Example 4.12 Consider the irreducible matroid $D = U_{1,2}(a,b) \oplus U_{1,2}(c,d)$, consisting of two double points ab and cd. The poset $\mathcal{W}_D$ consists of D, the four-point line $Q = U_{2,4}(a,b,c,d)$, and the matroids $P_1 = I(a) \,\square\, Z(b) \,\square\, I(c) \,\square\, Z(d)$ and $P_2 = I(c) \,\square\, Z(d) \,\square\, I(a) \,\square\, Z(b)$ (see Example 2.4), with $D \le P_1, P_2 \le Q$, and P_1 and P_2 are incomparable. Since $\mathcal{W}_D$ contains no irreducible matroids other than D, we have $\mathcal{W}_D = \mathcal{R}_D$; hence, by Theorem 4.10,

$$w_D = r_D = D - P_1 - P_2 + Q$$

is primitive in C.

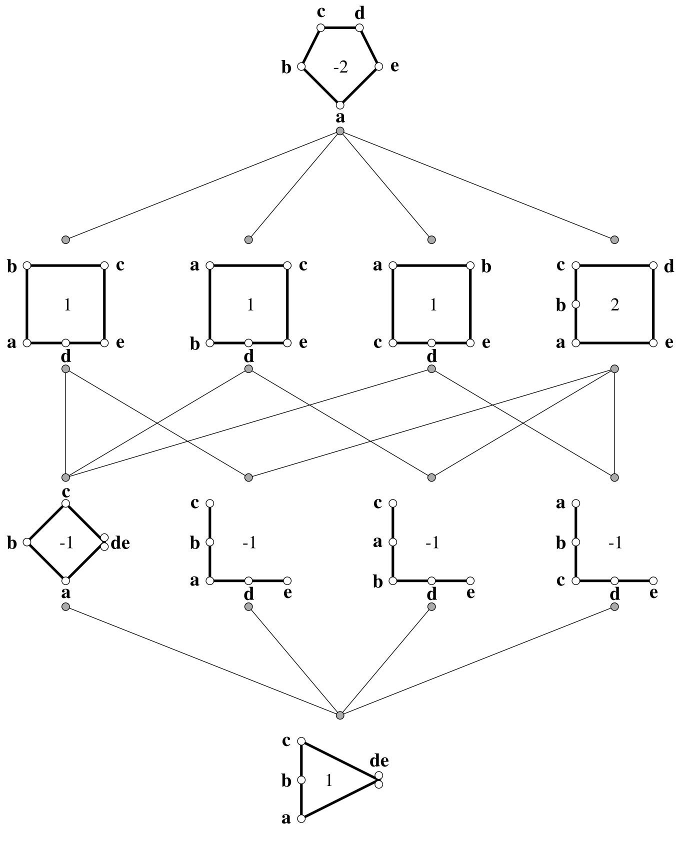
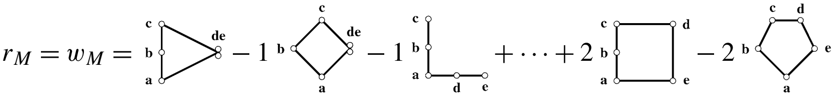


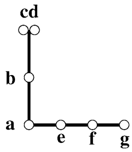
Fig. 1 The poset $\mathcal{W}_M = \mathcal{R}_M$, for $M = U_{2,3}(a,b,c) \oplus U_{1,2}(d,e)$

Example 4.13 The Hasse diagram of the poset $\mathcal{W}_M$, where M is the direct sum of the three-point line *abc* and the double point *de*, is shown in Fig. 1. Since $\mathcal{W}_M$ contains no irreducible matroids other than M, we have $\mathcal{W}_M = \mathcal{R}_M$. Each matroid N shown in the diagram is labeled by the Möbius function value $\mu_{\mathcal{W}}(M, N)$, so the primitive element

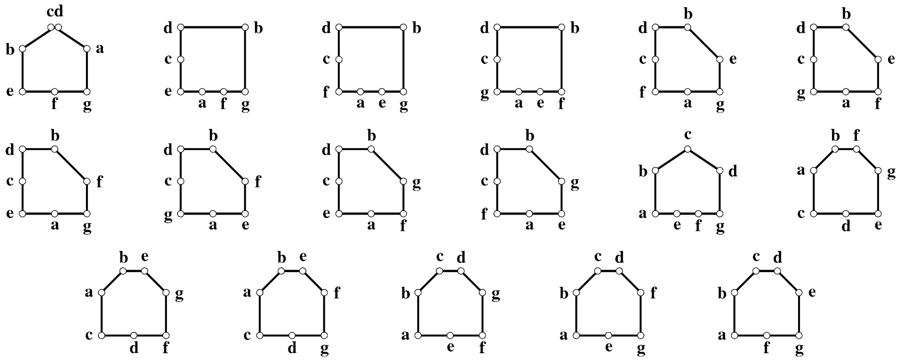


can be read from the diagram.

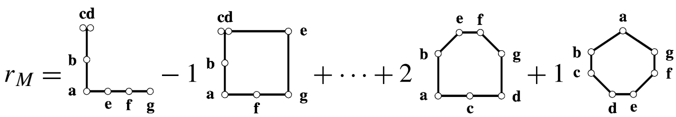
Example 4.14 Suppose that M is the matroid shown below.



The Hasse diagram of the poset $\widehat{\mathcal{R}}_M = \{N \in \mathcal{R}_M : \mu_{\mathcal{R}}(M, N) \neq 0\}$ is shown in Fig. 2. The matroids belonging to the set $\mathcal{R}_M \backslash \widehat{\mathcal{R}}_M = \{N \in \mathcal{R}_M : \mu_{\mathcal{R}}(M, N) = 0\}$ are shown below.



Each matroid N in the Hasse diagram is labeled by the Möbius function value $\mu_{\widehat{\mathcal{R}}}(M, N)$, which is equal to $\mu_{\mathcal{R}}(M, N)$, by Proposition 1.6. Hence the primitive element



can be read from the diagram. In contrast to the situation in Examples 4.12 and 4.13, we have $\mathcal{W}_M \neq \mathcal{R}_M$ here. The set $\mathcal{W}_M \backslash \mathcal{R}_M$ consists of 15 matroids, namely, the

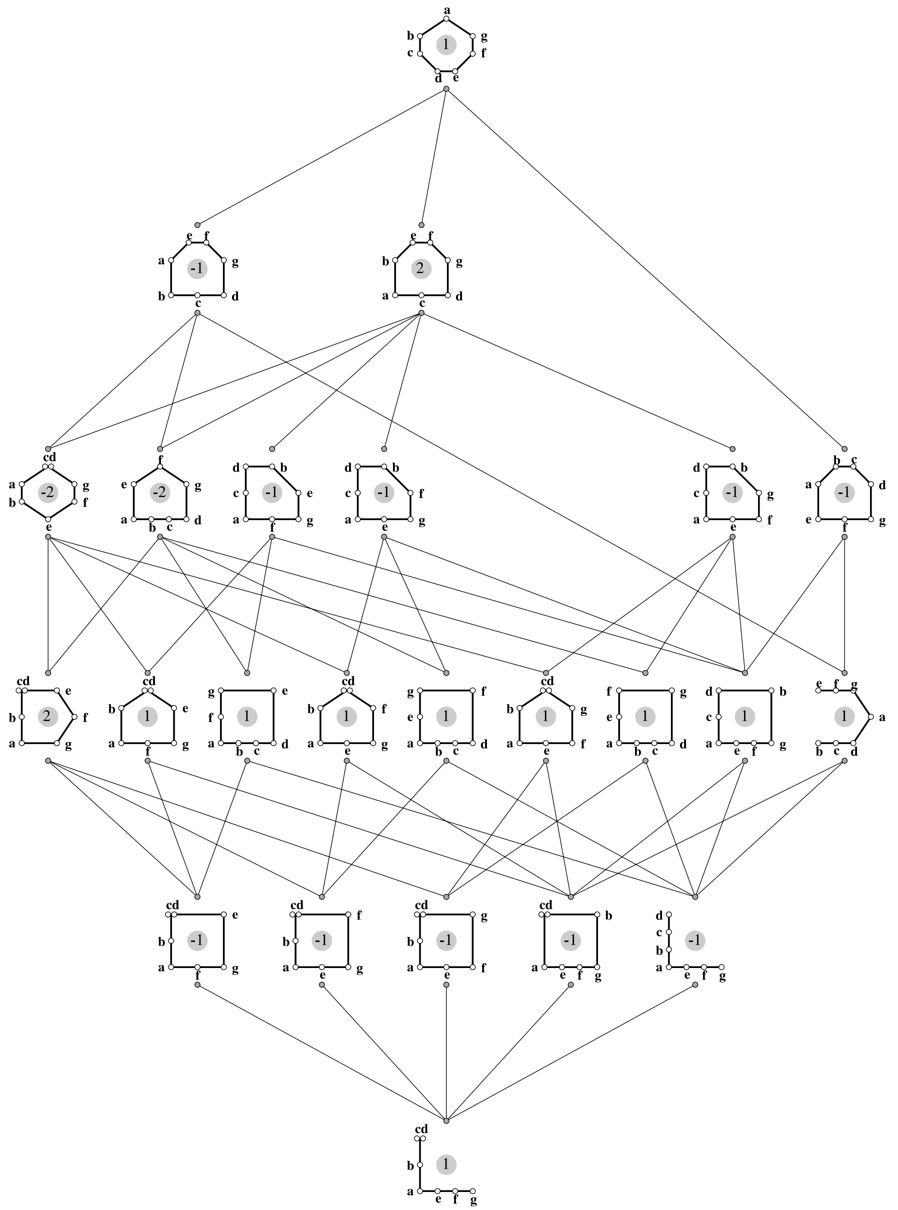
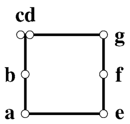
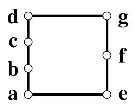
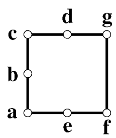


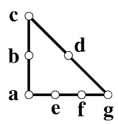
Fig. 2 The poset $\widehat{\mathcal{R}}_M$, for M the matroid of Example 4.14

following four









together with 11 others, one of which is isomorphic to the second of these, and five isomorphic to each of the last two.

5 The matroid-minor Hopf algebra

We now turn our attention to the matroid-minor Hopf algebra, which has distinguished basis consisting of isomorphism classes of matroids, rather than labeled matroids as we have been considering up to now. We denote by $\mathcal{M}$ and $\mathcal{M}_I$, respectively, the set of all isomorphism classes of matroids and the set of all isomorphism classes of irreducible matroids. (Recall that *irreducible* here means irreducible with respect to free product.) We write e_M for the isomorphism class of a matroid M, and note that $\mathcal{M} = \{e_M : M \in \mathcal{W}\}$. The (rank-preserving) *weak order* on $\mathcal{M}$ is the coarsest order relation such that the map $\mathcal{W} \to \mathcal{M}$, taking a matroid to its isomorphism class, is order preserving; thus $e_M \leq e_N$ in $\mathcal{M}$ if and only if there exist representatives $M' \in e_M$ and $N' \in e_N$ such that $M' \leq N'$ in $\mathcal{W}$.

The *matroid-minor Hopf algebra* H, first defined in [14], is the free vector space $K\{\mathcal{M}\}$, with product induced by direct sum, that is, $e_M e_N = e_{M \oplus N}$, for all matroids M and N, and with coproduct δ_H and counit ϵ determined by

$$\delta_H(e_M) = \sum_{A \subseteq S} e_{M|A} \otimes e_{M/A} \quad \text{and} \quad \epsilon(e_M) = \begin{cases} 1, & \text{if } S = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all M. Note that, as an algebra, H is free commutative, generated by the set $\mathcal{M}_c$ of isomorphism classes of connected matroids, that is, H is isomorphic to the polynomial algebra $K[\mathcal{M}_c]$.

As a means of determining the coalgebra structure of H, we first introduce a coalgebra, L, whose underlying vector space is also $K\{\mathcal{M}\}$, but with coproduct dual to the product on $K\{\mathcal{M}\}$ induced by the free product operation; that is, for all matroids M,

$$\delta_L(e_M) = \sum e_P \otimes e_Q,$$

where the sum is over all pairs $(e_P, e_Q) \in \mathcal{M} \times \mathcal{M}$ such that $P \,\square\, Q \cong M$. The counit of L is equal to that of H. Associativity of free product implies that δ_L is coassociative. By Theorem 2.1, for every $e_M \in \mathcal{M}$, there is a unique sequence $(e_{M_1}, \ldots, e_{M_k})$, for some $k \geq 0$, with each M_i irreducible, such that $M \cong M_1 \,\square \cdots \square\, M_k$. Denoting, for the moment, each element of $\mathcal{M}$ by its corresponding sequence, or *word*, we see that the coalgebra L has as basis all words on the set $\mathcal{M}_I$ of isomorphism classes of irreducible matroids, and its coproduct can be written

$$\delta_L(e_{M_1}, \ldots, e_{M_k}) = \sum_{i=0}^{k} (e_{M_1}, \ldots, e_{M_i}) \otimes (e_{M_{i+1}}, \ldots, e_{M_k}). \tag{5.1}$$

Hence L is cofree, and $\mathcal{M}_I$ is a basis for the space of primitive elements $P(L)$.

Our next result is a theorem from [5] which shows that H and L are isomorphic coalgebras, if the field K has characteristic zero. In the proof we give here, which is dual to the one given in that article, we use the following notation and terminology: If $M(S)$ is a matroid, and $A \subseteq B \subseteq S$, we denote by $M(A, B)$ the minor $(M|B)/A = (M/A)|(B\backslash A)$ of M determined by the interval $[A, B]$ in the Boolean algebra of subsets of S. For any set S, an S-*chain* is a sequence of sets $(S_0, \dots, S_k)$ such that S_{i-1} is strictly contained in S_i, for $1 \le i \le k$. Given S-chains $C = (S_0, \dots, S_k)$ and $D = (T_0, \dots, T_\ell)$ such that $S_k = T_0$, we write CD for the S-chain $(S_0, \dots, S_k = T_0, \dots, T_\ell)$. If S and T are disjoint sets, $C = (S_0, \dots, S_k)$ an S-chain and $D = (T_0, \dots, T_\ell)$ a T-chain with $T_0 = \emptyset$, we denote by $C \cdot D$ the $(S \cup T)$-chain $(S_0, \dots, S_k = T_0 \cup S_k, \dots, T_\ell \cup S_k)$. Given a matroid $M(S)$ and S-chain $C = (S_0, \dots, S_k)$, we write $M(C)$ for the free product $M(S_0, S_1) \,\Box \cdots \Box\, M(S_{k-1}, S_k)$. We refer to an S-chain C as M-*irreducible* if $S_0 = \emptyset$, $S_k = S$ and each of the minors $M(S_{i-1}, S_i)$ is irreducible with respect to free product. We write $\mathrm{IC}(M)$ for the set of all M-irreducible S-chains.

Theorem 5.2 [5] *If the field K has characteristic zero, then the map $\varphi\colon H \to L$, determined by*

$$\varphi(e_M) = \sum_{C \in \mathrm{IC}(M)} e_{M(C)},$$

for all M, is a coalgebra isomorphism.

Proof We compute, for $M = M(S)$,

$$\begin{aligned}\delta_L(\varphi(e_M)) &= \sum_{C \in \mathrm{IC}(M)} \delta_L(e_{M(C)}) \\ &= \sum_{C \in \mathrm{IC}(M)} \sum_{C_1 C_2 = C} e_{M(C_1)} \otimes e_{M(C_2)},\end{aligned}$$

and

$$\begin{aligned}(\varphi \otimes \varphi)\delta_H(e_M) &= \sum_{A \subseteq S} \varphi(M|A) \otimes \varphi(M/A) \\ &= \sum_{A \subseteq S} \sum_{\substack{D \in \mathrm{IC}(M|A) \\ E \in \mathrm{IC}(M/A)}} e_{M(D)} \otimes e_{M(E)}.\end{aligned}$$

It is readily verified that the map

$$\bigcup_{A \subseteq S} \mathrm{IC}(M|A) \times \mathrm{IC}(M/A) \to \{(C, A)\colon C \in \mathrm{IC}(M) \text{ and } A \in C\},$$

given by $(C_1, C_2) \mapsto (C_1 \cdot C_2, A)$, for all $A \subseteq S$ and $(C_1, C_2) \in \mathrm{IC}(M|A) \times \mathrm{IC}(M/A)$, is a bijection. It thus follows from the above computation that $\delta_L(\varphi(e_M)) = (\varphi \otimes \varphi)\delta_H(e_M)$, and so φ is a coalgebra map.

Proposition 2.6 implies that $e_M \le e_{M(C)}$ in $\mathcal{M}$, for all $C \in \mathrm{IC}(M)$; since K has characteristic zero, it thus follows that φ is a bijective and thus an isomorphism. □

For all $M \in \mathcal{M}$, we write p_M for $\varphi^{-1}(e_M) \in H$.

Theorem 5.3 [5] *If the field K has characteristic zero, then the matroid-minor Hopf algebra H is cofree, with $\{p_M : e_M \in \mathcal{M}_I\}$ a basis for the space $P(H)$ of primitive elements of H.*

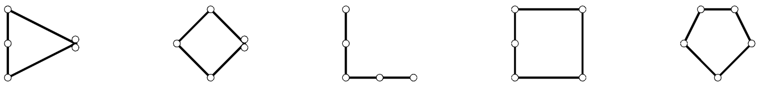
Proof The result follows immediately from Theorem 5.2 and the preceeding discussion of the coalgebra L. □

Observe that, since $e_M \leq e_{M(C)}$, for all matroids M and chains $C \in \mathrm{IC}(M)$, we may write

$$\varphi(e_M) = \sum_{e_N \geq e_M} c(e_M, e_N)\, e_N,$$

where $c(e_M, e_N)$ is the cardinality of the set $\{C \in \mathrm{IC}(M) : M(C) \cong N\}$. Hence we may compute $p_M = \varphi^{-1}(e_M)$ by computing the inverse of the matrix whose entries are the numbers $c(e_P, e_Q)$, for all $P \leq Q$ in $\mathcal{W}_M$.

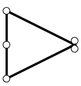
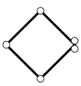
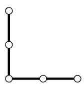
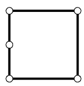
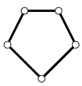
Example 5.4 Suppose that M is the 5-element matroid of Example 4.13. The isomorphism classes of matroids in $\mathcal{W}_M$ are the following



We refer to these isomorphism classes, in the order shown here, as $e_1, \ldots, e_5$. Note that we have the following factorizations into irreducibles: $e_1 = e_M$ is irreducible, $e_2 = IZIIZ$, $e_3 = ID$, $e_4 = IIZIZ$ and $e_5 = U_{3,5} = IIIZZ$, where we have written I and Z for the isomorphism classes of the single point and loop, and D for the irreducible isomorphism class $U_{1,2} \oplus U_{1,2}$ consisting of two double-points (see Example 4.12), and we have suppressed the symbol □ in writing free products of isomorphism classes. Denoting by Φ the matrix whose entry in position (i, j) is $c(e_i, e_j)$, for $1 \leq i \leq j$, we have

$$\Phi = \begin{pmatrix} 1 & 12 & 3 & 36 & 72 \\ 0 & 12 & 0 & 24 & 84 \\ 0 & 0 & 1 & 24 & 96 \\ 0 & 0 & 0 & 12 & 108 \\ 0 & 0 & 0 & 0 & 120 \end{pmatrix} \quad \text{and} \quad \Phi^{-1} = \begin{pmatrix} 1 & -1 & -3 & 5 & -2 \\ 0 & \frac{1}{12} & 0 & -\frac{1}{6} & \frac{11}{120} \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \frac{1}{12} & -\frac{3}{40} \\ 0 & 0 & 0 & 0 & \frac{1}{120} \end{pmatrix}.$$

It follows that the basis element $p_M = \varphi^{-1}(e_M)$ of $P(H)$ is given by

$p_M =$  $-\,1$  $-\,3$  $+\,5$  $-\,2$  (5.5)

The following result, which is analogous to Theorem 4.11, characterizes the basis elements p_M of $P(H)$.

Lemma 5.6 *Suppose that $x \in P(H)$ is of the form $x = e_M + \sum_N a_N e_N$, where M is irreducible and each N appearing in the sum is reducible. Then $x = p_M$.*

Proof For any matroid M, we have $\varphi(e_M) = \sum_{e_N \geq e_M} c(e_M, e_N)e_N$, where all isomorphism classes strictly greater than e_M appearing in the sum are reducible. If M is irreducible then $c(e_M, e_M) = 1$, and thus it follows that $p_M = \varphi^{-1}(e_M)$ is equal to e_M plus a linear combination of reducible isomorphism classes of matroids. Since the set $\mathcal{M}_I$ of irreducible isomorphism classes is linearly independent in H, and $\{p_M : M \in \mathcal{M}_I\}$ is a basis for H, the result follows. □

Theorem 5.7 *The basis elements p_M for $P(H)$ are given by*

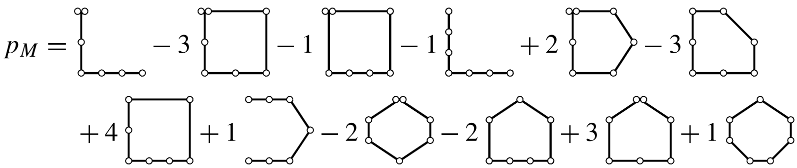
$$p_M = \sum_{N \in \mathcal{R}_M} \mu_{\mathcal{R}}(M, N)\, e_N,$$

for all irreducible matroids M.

Proof The natural surjection $\pi : C \to H$, from the matroid-minor coalgebra onto the matroid-minor Hopf algebra, taking a matroid M to its isomorphism class e_M, is clearly a coalgebra map, and so maps primitives to primitives. Thus, if M is irreducible, it follows from Theorem 4.11 that $\sum_{N \in \mathcal{R}_M} \mu_{\mathcal{R}}(M, N)\, e_N = \pi(r_M)$ belongs to $P(H)$, and hence, by Lemma 5.6, $\sum_{N \in \mathcal{R}_M} \mu_{\mathcal{R}}(M, N)\, e_N = p_M$. □

Example 5.8 Applying the projection π to the primitive element $r_M \in P(C)$ of Example 4.13, we obtain the primitive element $p_M \in P(H)$ given by Eq. 5.5.

Example 5.9 Applying π to the primitive element r_M of Example 4.14, we obtain the primitive element



of H. This example exhibits the following curious phenomenon: the isomorphism class of the three coatoms of the poset $\widehat{\mathcal{R}}_M$ does not appear in p_M, since these matroids are the only elements of their isomorphism class belonging to $\widehat{\mathcal{R}}_M$ and the sum of their Möbius function values is zero. We thus have a situation in which certain matroids are needed in order to provide all the requisite cancellations so that r_M is primitive in the matroid-minor coalgebra, while the isomorphism class of these matroids is not required in order to make $p_M = \pi(r_M)$ primitive in the matroid-minor Hopf algebra.

6 Further work

One line of inquiry clearly suggested by the results of the last two sections is that of developing techniques for computing the Möbius functions of the posets

$\mathcal{W}_M$ and $\mathcal{R}_M$. Due to the complexity of these posets, and the fact that so little is known about them, this is likely to be very difficult. A more modest, but still worthwhile, goal would be to characterize the matroids belonging to $\widehat{\mathcal{R}}_M = \{N \in \mathcal{R}_M : \mu_{\mathcal{R}}(M, N) \neq 0\}$. For M irreducible this amounts to identifying precisely those matroids appearing with nonzero coefficient in the expression for the primitive element r_M in terms of the basis $\mathcal{W}$ of matroids.

A related problem, suggested by phenomenon of "disappearing" matroids observed in Example 4.14, is to characterize those matroids appearing with nonzero coefficient in r_M whose isomorphism classes have zero coefficient in $\pi(r_M)$, where $\pi : C \to H$ is the natural projection mapping a matroid to its isomorphism class. In other words, we wish to find those matroids N in $\widehat{\mathcal{R}}_M$ such that $\sum \mu_{\mathcal{R}}(M, N') = 0$, where the sum is over all $N' \in \widehat{\mathcal{R}}_M$ such that $N' \cong N$. Once we have a characterization of the matroids belonging to $\widehat{\mathcal{R}}_M$, a solution to this problem would provide a characterization of the isomorphism classes that appear with nonzero coefficient in p_M.

We showed in Sect. 4 that r_M and w_M are primitive elements of C whenever M is irreducible (and the results of that section imply that the converse is also true; if M is reducible, then r_M and w_M are not primitive). The more general problem of expressing the coproduct of C in terms of the bases $\{r_M : M \in \mathcal{W}\}$ and $\{w_M : M \in \mathcal{W}\}$ remains open. Examples we have looked at so far suggest that such expressions will not be straightforward formulas along the lines of Eq. 5.1 but, rather, will involve some form of Möbius inversion and rely on the fairly subtle interplay between the free product operation and the weak order on matroids. We also note that C is not cofree (since it follows from Proposition 3.1 that the dual algebra C^* is not free), and thus the coproduct of C cannot take the precise form of Eq. 5.1.

According to Theorem 5.2 and Eq. 5.1, the coproduct of H, expressed in terms of the basis $\{p_M : M \in \mathcal{M}\}$ has the form

$$\delta(p_M) = \sum_{i=0}^{k} p_{M_1 \,\Box\, \cdots \,\Box\, M_i} \otimes p_{M_{i+1} \,\Box\, \cdots \,\Box\, M_k},$$

whenever $M = M_1 \,\Box\, \cdots \,\Box\, M_k$, with all M_i irreducible. Thus, in terms of this basis (and keeping in mind unique factorization, Theorem 2.1), the cofreeness of H becomes apparent. We showed in Theorem 5.7 that the basis element r_M of C maps to p_M under the projection $\pi : C \to H$, for irreducible M, thus giving, in this case, a combinatorial interpretation for the coefficients of p_M, as sums of Möbius function values. The result does not hold for general M, however; indeed, the coefficients of isomorphism classes appearing in p_M are not even necessarily integers when M is reducible (see Example 5.4). It would be of interest to find a generalization of Theorem 5.7 that holds for all matroids. This would amount to determining elements of C, expressed in terms of the basis $\{r_N : N \in \mathcal{W}\}$ (or perhaps in terms of the basis $\{w_N : N \in \mathcal{W}\}$) that project to p_M, for all matroids M. A combinatorial description of such elements would provide, in turn, a combinatorial interpretation of all coefficients of the basis elements p_M.

Our proof of Theorem 5.7 is quite indirect, relying on the uniqueness assertions of Theorem 4.11 and Lemma 5.6, then using only the fact that r_M and p_M are primitive, for M irreducible. A direct proof of this result might reveal useful information about the relationship between the posets $\mathcal{R}_M$ and $\{e_N : e_N \geq e_M \text{ in } \mathcal{M}\}$.

We mention finally the project of determining the precise manner in which the matroid-minor Hopf algebra and coalgebra fit into the framework developed by Aguiar and Sottile in [3] and [2]. A first step is to look for natural mappings between H, and/or C, and the Hopf algebras of symmetric functions, permutations, and planar binary trees.

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