

The discrete fundamental group of the order complex of B_n

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Abstract A few years ago Kramer and Laubenbacher introduced a discrete notion of homotopy for simplicial complexes. In this paper, we compute the discrete fundamental group of the order complex of the Boolean lattice. As it turns out, it is equivalent to computing the discrete homotopy group of the 1-skeleton of the permutahedron. To compute this group we introduce combinatorial techniques that we believe will be helpful in computing discrete fundamental groups of other polytopes. More precisely, we use the language of words, over the alphabet of simple transpositions, to obtain conditions that are necessary and sufficient to characterize the equivalence classes of cycles. The proof requires only simple combinatorial arguments. As a corollary, we also obtain a combinatorial proof of the fact that the first Betti number of the complement of the 3-equal arrangement is equal to $2^{n-3}(n^2 - 5n + 8) - 1$. This formula was originally obtained by Björner and Welker in 1995.

Keywords Permutahedron · Boolean lattice · Homotopy groups · A -theory · Subspace arrangements · Symmetric group

1 Introduction

In this paper we give an application of a discrete notion of homotopy theory (A -theory hereafter) to subspace arrangements, which yields an unexpected connection between

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word problems on the symmetric group and the computation of the first Betti number of the k -equal arrangement in \mathbb{R}^n . More importantly, in order to compute those Betti numbers we were led to introduce new combinatorial techniques for evaluating the rank of the discrete fundamental group of the 1-skeleton of the permutahedron. It is those techniques that we believe will be useful for computing the discrete fundamental group of the 1-skeleton of other polytopes.

Since the early 1980's, subspace arrangements have been extensively studied both by topologists and combinatorialists. One of the reasons for such activities is the fact that certain complexity theory problems arising in computer science have a reformulation in terms of subspace arrangements, thus yielding a beautiful interaction between combinatorics, topology, geometry, and complexity theory. Björner gives a nice account of this interaction in [6] and [7].

On the other hand, A -theory is a recently introduced notion of discrete homotopy theory for graphs and simplicial complexes that was originally developed, in the mid 90's, in the context of social and technological networks. See [3] for a survey of this topic, and [4, 12–15], for applications. More recently, Dochtermann introduced a different notion of homotopy of graph maps. His notion is based on the categorical product of graphs rather than on the Cartesian product. For a nice account of the relation between these two notions of homotopy see [10] and [11].

In 2001, Babson [3] (and independently Björner [5]) proved that the discrete fundamental A -group, $A_1^{n-3}(\Delta(B_n))$, of the order complex of the boolean lattice is isomorphic to the fundamental group, $\pi_1(M_{n,3})$, of the complement, $M_{n,3}$, of the real 3-equal arrangement. Namely,

$$A_1^{n-3}(\Delta(B_n)) \simeq \pi_1(M_{n,3}), \quad (1)$$

where $M_{n,3} = \mathbb{R}^n - V_{n,3}$ with

$$V_{n,3} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_{i_1} = x_{i_2} = x_{i_3}, \text{ for some } 1 \leq i_1 < i_2 < i_3 \leq n\}.$$

In [9], Björner and Welker showed that in fact the cohomology groups, $H^i(M_{n,k})$, of the complement of the k -equal arrangements are free. Furthermore, they give formulae for some of the non-vanishing rank of the cohomology groups. From these formulae one can deduce that the first Betti number for the (real) complement of the 3-equal arrangement is equal to

$$\text{rank } H^1(M_{n,3}) = 2^{n-3}(n^2 - 5n + 8) - 1.$$

Their proof is quite intricate and relies on the Goresky-MacPherson and the Ziegler-Živaljević formulae and on some combinatorial methods for computing the homotopy type of partially ordered sets. A different approach, based on non-pure shellability of the lattice $\pi_{n,k}$ (the lattice of intersections of the subspaces associated with the k -equal arrangement) can be found in [8].

Because of the above isomorphism (1), one realizes that computing the rank of the abelianization of $A_1^{n-3}(\Delta(B_n))$ will also yield the first Betti number of $M_{n,3}$. We show here how to compute this rank using only combinatorial arguments on $\Gamma(B_n)$, the 1-skeleton of the permutahedron, a graph that A -theory associates with $\Delta(B_n)$.

As it turns out, every discrete homotopy argument can be reformulated in terms of word problems for the symmetric group S_n , making the arguments simple and, we hope, shedding new light on the theory of subspace arrangements.

In Section 2, we review the fundamental notions of A -theory that are needed throughout this article. In a nutshell, the discrete fundamental group, $A_1^q(\Delta)$ of a simplicial complex Δ is a certain quotient of its (classical) fundamental group, $\pi_1(\Delta)$. In order to compute the rank of this group a graph Γ_Δ is constructed. Next, viewing Γ_Δ as a 1-dimensional simplicial complex, one attaches 2-cells to all of its 3 and 4-cycles, after which it suffices to compute the fundamental group of this 2-cell complex. In practice, one is left to count the number of equivalence classes (under an appropriate equivalence relation) of cycles of Γ_Δ of length at least 5, which is equal to rank $A_1^q(\Delta)^{ab}$ when the group is free.

In the case of the order complex of the boolean lattice, $\Delta(B_n)$, the associated graph is $\Gamma(B_n)$. The $n!$ vertices of $\Gamma(B_n)$ can be identified with the permutations of the symmetric group S_n , and there is an edge between two permutations σ and τ if there exists a simple transposition, $s_i = (i \ i + 1)$, such that $\sigma = \tau s_i$. Note, we multiply permutations from right to left. Moreover, in order to compute the rank of $A_1^{n-3}(\Delta(B_n))^{ab}$ it will suffice to find the number of equivalence classes of 6-cycles in $\Gamma(B_n)$. Note that in $\Gamma(B_n)$ every *primitive* 6-cycle (one that is not the concatenation of two 4-cycles) can be associated with a pair of consecutive simple transpositions s_i and s_{i+1} . While the permutahedron is a well studied polytope (see for e.g. [18]), some of the properties needed to compute the above mentioned rank come to light when we realize that $\Gamma(B_n)$ can be obtained by taking the Cartesian product of smaller graphs and then removing some of the edges. The crux of the argument relies on the fact that the maximal chains of the direct product of two graded posets L_1, L_2 can be expressed as a shuffle of maximal chains from L_1 and L_2 . The various constructions involved are described in Sections 3 and 4. For greater details the interested reader may wish to consult the second author’s Ph.D. thesis [16].

Section 5 contains the main theorem of this paper. Namely, (and somewhat informally) two 6-cycles, C_1 and C_2 of $\Gamma(B_n)$ associated with the simple transpositions s_i and s_{i+1} are in the same equivalence class if and only if there exists an integer $k \geq 1$ such that

$$C_2 = C_1 \tau_1 \dots \tau_k$$

where the τ_j are transpositions disjoint from s_i and s_{i+1} . From this result it is relatively easy to compute the total number of equivalence classes of 6-cycles, that is

$$\text{rank } A_1^{n-3}(\Delta(B_n))^{ab} = \text{rank } H^1(M_{n,3}) = 2^{n-3}(n^2 - 5n + 8) - 1.$$

In order to prove the main theorem we translate the notion of G -homotopic loops in $\Gamma(B_n)$ to an equivalent notion on a set of words (on the alphabet \mathbf{S} of all simple transpositions of S_n) that are naturally associated to the loops in $\Gamma(B_n)$. Two words are equivalent if one can be obtained from the other by a series of transformations involving only operations of the form $s_j^2 = 1$ and $s_k s_j = s_j s_k$ for $|k - j| \geq 2$. This translation facilitates the burden of the proof of the main result. In this paper, all transpositions are simple, and thus we shall write transposition in lieu of simple transposition.

2 Discrete homotopy theory for graphs and simplicial complexes

In this section we briefly review some of the basic concepts of discrete homotopy that will be needed throughout the rest of this paper. All details can be found in [2] and [1].

Definition 2.1 Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be *simple* graphs, with no loops or parallel edges.

- (1) A *graph map* $f : \Gamma \rightarrow \Gamma'$ is a set map $V \rightarrow V'$ that preserves adjacency, that is, if $vw \in E$, then either $f(v)$ is adjacent to $f(w)$ in Γ' , denoted by $f(v) \sim_{\Gamma'} f(w)$, or $f(v) = f(w)$.
- (2) Let $v \in V$ and $v' \in V'$ be distinguished vertices. A *based graph map* is a graph map $f : (\Gamma, v) \rightarrow (\Gamma', v')$ such that $f(v) = v'$.

We note that if Γ is connected, then the image of f is a connected subgraph of Γ' .

Definition 2.2 The *Cartesian product* $\Gamma \square \Gamma'$ of two graphs, Γ and Γ' , is the graph with vertex set $V \times V'$ and an edge between (v, v') and (w, w') if either

- (1) $v = w$ and $v' \sim_{\Gamma'} w'$, or
- (2) $v \sim_{\Gamma} w$ and $v' = w'$.

Let I_n be the path on $n + 1$ vertices, with vertices labeled from 0 to n , and let I be the infinite path with vertices labeled $0, 1, 2, \dots$. Two based graph maps $f, g : \Gamma \rightarrow \Gamma'$ of simple graphs are G -homotopic (written $f \simeq_G g$) relative to v'_0 and v'_1 , if there is an integer n and a graph map $F : \Gamma \square I_n \rightarrow \Gamma'$ such that

- (1) $F(v, 0) = f(v)$ and $F(v, n) = g(v)$ for all $v \in V$
- (2) $F(v_0, j) = v'_0$ and $F(v_1, j) = v'_1$ for $0 \leq j \leq n$.

Given a simple graph Γ with distinguished vertex v_0 , let $A_1^G(\Gamma, v_0)$ be the set of all equivalence classes of based graph maps $f : I \rightarrow \Gamma$ such that $f(0) = v_0$ and $f(m) = v_0$ for all $m \geq N_f$, where N_f is a positive integer that depends on f . Concatenation of loops is well-defined on this set and it is easy to show that $A_1^G(\Gamma, v_0)$ is indeed a group. As in classical topology, if Γ is connected, the discrete fundamental group $A_1^G(\Gamma, v_0)$ is independent of the choice of base vertex. In this case, we refer to $A_1^G(\Gamma)$ simply as the A_1 -group of Γ .

Figure 1 shows an example of two G -homotopic graph maps, $f, g : I \rightarrow \Gamma$ where Γ is a cycle of length 4. Graph map f corresponds to going around the 4-cycle once, while g is the constant map v_0 . The vertices of the graph (grid) $I \square I_2$ are labeled with the image of a G -homotopy from f to g . The G -homotopy F is itself a graph map, and as such must preserve all adjacencies. Thus for each horizontal or vertical edge (v_i, v_j) in the grid, we must have either (v_i, v_j) is an edge of Γ , or $v_i = v_j$.

Furthermore, it is straightforward to show that any based graph map from I to the 4-cycle is G -homotopic to the constant map g , so the A_1 -group of the 4-cycle, and similarly of the 3-cycle, is trivial. However, a graph map that maps onto a cycle of length ≥ 5 is not G -homotopic to the constant map. In fact, it can be shown that if Γ is a cycle of length at least 5, then $A_1^G(\Gamma) \simeq \mathbb{Z}$.

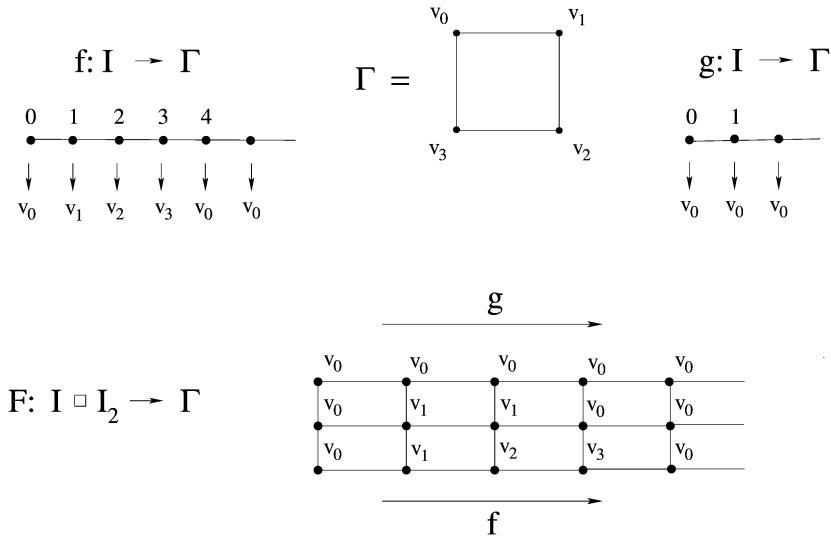


Fig. 1 A G -homotopy from f to g

In [2], Barcelo et al. also show that $A_1^G(\Gamma, v) \simeq \pi_1(\Gamma, v)/N$, where $\pi_1(\Gamma, v)$ is the classical fundamental group of Γ when viewed as a 1-dimensional simplicial complex and N is the normal subgroup generated by 3- and 4-cycles. Thus, computing the A_1^G -group of a graph is equivalent to attaching 2-cells to the 3- and 4-cycles of the graph and computing the classical fundamental group of the resulting 2-cell complex.

There is an equivalent definition of discrete homotopy for simplicial complexes which includes a graded version of the discrete fundamental group, related to the dimension of the intersection of simplices. Let Δ be a simplicial complex of dimension d , let $0 \leq q \leq d$ be a fixed integer, and let $\sigma_0 \in \Delta$ be a maximal simplex (with respect to inclusion) of dimension at least q . A q -chain in Δ is a sequence of simplices (not necessarily distinct),

$$\sigma, \sigma_1, \sigma_2, \dots, \sigma_n, \tau,$$

such that any two consecutive simplices share a q -face. A q -loop in Δ based at σ_0 is a q -chain beginning and ending at σ_0 . Two such loops are equivalent if they can be deformed into each other without breaking any q -dimensional connections. More precisely, the equivalence relation, \simeq_A , on the collection of q -loops in Δ , based at σ_0 , is generated by the following two conditions.

(1) The q -loop

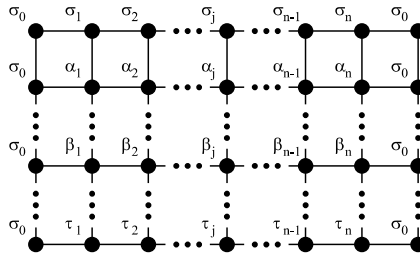
$$(\sigma) = (\sigma_0, \dots, \sigma_i, \sigma_{i+1}, \dots, \sigma_n, \sigma_0)$$

is equivalent to the q -loop

$$(\sigma') = (\sigma_0, \dots, \sigma_i, \sigma_i, \sigma_{i+1}, \dots, \sigma_n, \sigma_0).$$

That is, we can “stretch” loops by repeating a simplex without changing its equivalence class.

Fig. 2 Equivalent q -loops



(2) Suppose that (σ) and (τ) have the same length. They are equivalent if there is a diagram as in Fig. 2. The diagram is to be interpreted as follows. A horizontal or vertical edge between two simplices indicates that they share a q -face. Each horizontal row in the diagram is a q -loop based at σ_0 . Thus, (σ) is equivalent to (τ) if there is a sequence of q -loops based at σ_0 connecting them.

This equivalence relation is called *A-homotopy*, and its ensuing set of equivalence classes is denoted by $A_1^q(\Delta, \sigma_0)$. Concatenation of q -loops yields a group structure on $A_1^q(\Delta, \sigma_0)$. In [2], it was shown that in fact $A_1^q(\Delta, \sigma_0) \cong \pi_1(\Gamma_\Delta^q, v_0)/N$, where Γ_Δ^q is the graph whose vertices correspond to all maximal simplices of Δ of dimension at least q . Two vertices v_σ and v_τ in Γ_Δ^q are adjacent if and only if the corresponding simplices σ and τ share (at least) a q -face, and v_0 is the distinguished vertex of Γ_Δ^q corresponding to σ_0 . One sees that there is a close relation between the A_1^G -groups defined for graphs, and the A_1^q -groups defined for simplicial complexes. Indeed the relation is given by

$$A_1^q(\Delta, \sigma_0) \cong A_1^G(\Gamma_\Delta^q, v_0),$$

thus *G*-homotopy and *A*-homotopy are equivalent concepts.

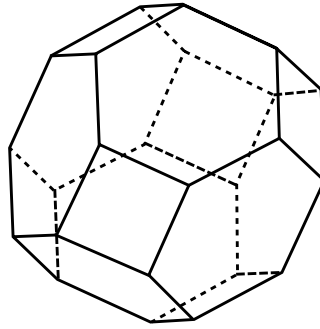
3 The product of lattices

Recall that one of our goals is to use the techniques of *A*-theory to compute the first Betti number of $M_{n,3}$, the complement of the 3-equal arrangement. As mentioned in the introduction, in [3] it was shown that

$$A_1^{n-3}(\Delta(B_n)) \simeq \pi_1(M_{n,3}),$$

where $\Delta(B_n)$ is the order complex of the boolean lattice, $B_n - \{\hat{0}, \hat{1}\}$. Thus to find rank $H^1(M_{n,3})$ we will need to count the number of distinct equivalence classes of cycles, $[C]$, in the graph $\Gamma_{\Delta(B_n)}^{n-3}$. From here on, for any poset L of rank r , we will only be interested in its top $A_1^q(\Delta(L))$, which is $A_1^{r-3}(\Delta(L))$, and thus we shall write $\Gamma(L)$ in lieu of $\Gamma_{\Delta(L)}^q$. The vertices of $\Gamma(B_n)$ correspond to the maximal faces of $\Delta(B_n)$, which are the maximal chains in $B_n - \{\hat{0}, \hat{1}\}$, which further correspond to the permutations of S_n . Two vertices in $\Gamma(B_n)$ are adjacent if the two maximal chains C_1 and C_2 differ in precisely one element, in which case we also say that

Fig. 3 The 1-skeleton of the permutahedron P_3



the chains are adjacent, $C_1 \sim C_2$. Equivalently, the associated permutations differ by multiplication on the right by a (simple) transposition s_i for some $1 \leq i \leq n - 1$.

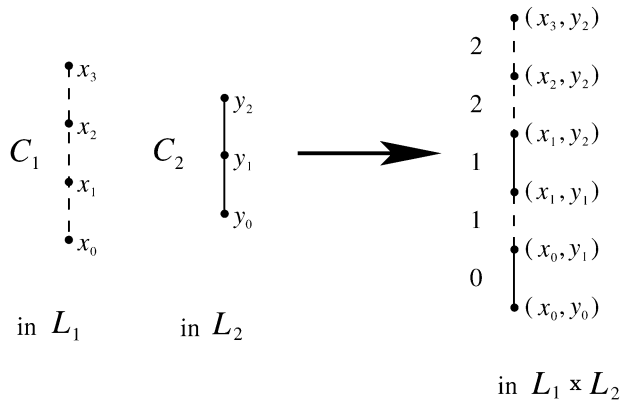
We note that $\Gamma(B_n)$ is the 1-skeleton of the permutahedron P_{n-1} [18]; Fig. 3 represents $\Gamma(B_4)$. Any path in $\Gamma(B_n)$ corresponds to a product of transpositions, $s_{i_1}, \dots, s_{i_p} = w$. We view w both as a word in the alphabet \mathbf{S} , as well as an element of the symmetric group S_n . Thus any cycle is a representation of the identity as a product of transpositions. We can see that if we attach 2-cells to the 4-cycles in $\Gamma(B_4)$, we are left with eight 6-cycles. In general, in order to compute the rank of $A_1^G(\Gamma(B_n))^{ab}$, we will need to count the distinct equivalence classes of cycles, with primitive 6-cycles as representatives. Moreover, $A_1^G(\Gamma(B_4))^{ab}$ is a free group (see [3]) on seven generators, not eight, so unfortunately simply counting 6-cycles will not suffice.

The breakthrough that allows us to understand the G -homotopy relation on $\Gamma(B_n)$ is the simple observation that B_n can be viewed as the direct product of smaller boolean lattices; in fact, $B_n \simeq B_1^n$. It is useful to express this isomorphism as $B_n \simeq B_{n-1} \times B_1$, because the graph $\Gamma(B_1)$ is a single vertex corresponding to the empty chain. However, clearly $\Gamma(B_n) \not\cong \Gamma(B_{n-1}) \square \Gamma(B_1)$, since $\Gamma(B_n)$ has $n!$ vertices, compared to $(n - 1)!$ vertices in $\Gamma(B_{n-1}) \square \Gamma(B_1)$. In general, $\Gamma(L_1 \times L_2) \not\cong \Gamma(L_1) \square \Gamma(L_2)$ for nontrivial posets L_1 and L_2 , nevertheless, there is a relationship between the graphs. We now introduce a method to obtain $\Gamma(L_1 \times L_2)$ from $\Gamma(L_1)$ and $\Gamma(L_2)$.

Each maximal chain in $L_1 \times L_2$, may be viewed as a combination of one maximal chain from each of L_1 and L_2 ; however, those two chains can be combined in more than one way (for more details on product of posets, see [17]). Thus, $\Gamma(L_1) \square \Gamma(L_2)$ is a subgraph of $\Gamma(L_1 \times L_2)$, but we must also find a way to reflect the various combinations of chains that are possible.

To remedy this problem, we first introduce a new graph, the *shuffle graph*, with vertices corresponding to each possible shuffle of a pair of maximal chains from L_1 and L_2 . Next, we construct the Cartesian product of the shuffle graph with $\Gamma(L_1) \square \Gamma(L_2)$. This solves the problem of too few vertices, but replaces it with a new obstacle of too many edges. Finally, we determine which edges are superfluous and remove them so that the resulting graph is the desired $\Gamma(L_1 \times L_2)$. In the following section, we then apply this construction to $B_{n-1} \times B_1$ to create a representation of the permutahedron, $\Gamma(B_n)$, where we can better understand its structure.

Fig. 4 The shuffle of C_1 and C_2 associated with the 3-sequence (1, 2, 2) and 2-sequence (0, 1)



Step 1: The shuffle graph, $\Gamma_{shuffle}^{k,l}$

To construct $\Gamma(L_1 \times L_2)$, we begin by considering the ways in which the edges of chains from L_1 and L_2 may be combined to create a new chain. Let C_1 and C_2 be maximal chains in two graded posets L_1 and L_2 of rank k and l , respectively. A shuffle of the edges of C_1 and C_2 creates a maximal chain C in $L_1 \times L_2$. In C , label each edge from C_1 with the number of edges from C_2 below it in the shuffle. The ordered, weakly increasing collection of labels, $\kappa = (a_1, a_2, \dots, a_k)$, is the k -sequence associated with that shuffle. Similarly, label each edge from C_2 with the number of edges from C_1 below it in the shuffle and the ordered collection of labels is an l -sequence, λ .

We now introduce the shuffle graph, $\Gamma_{shuffle}^{k,l}$, from which we will construct $\Gamma(L_1 \times L_2)$. The vertices of $\Gamma_{shuffle}^{k,l}$ correspond to the $\binom{k+l}{k}$ shuffles of chains of L_1 and L_2 of length k and l respectively. Label each vertex with the pair (κ, λ) that corresponds to each shuffle. A shuffle is uniquely determined by either its k -sequence or l -sequence, but both will be useful later in the construction of $\Gamma(L_1 \times L_2)$.

Definition 3.1 Two k -sequences, $\kappa = (a_1, \dots, a_k)$ and $\kappa' = (a'_1, \dots, a'_k)$ are adjacent, $\kappa \sim \kappa'$, if and only if

- (1) $a_i = a'_i \pm 1$ for some $1 \leq i \leq k$, and
- (2) $a_j = a'_j \forall j \neq i$.

Two shuffles of $\Gamma_{shuffle}^{k,l}$ are said to be adjacent if their k -sequences are adjacent, $\kappa \sim \kappa'$.

We note that if two k -sequences are adjacent then the associated l -sequences are also adjacent, so we only need to refer to one of the sequences when determining if two shuffles are adjacent. A pair of chains in $L_1 \times L_2$ resulting from the use of adjacent shuffles differ by a diamond (formed where the order of the pair of edges is reversed), as shown in Fig. 5. Figure 6 shows $\Gamma_{shuffle}^{3,2}$ with sequences for the ten possible shuffles of C_1 and C_2 from Fig. 5.

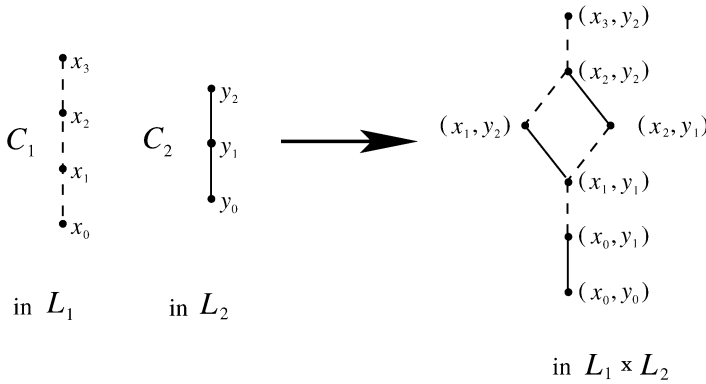
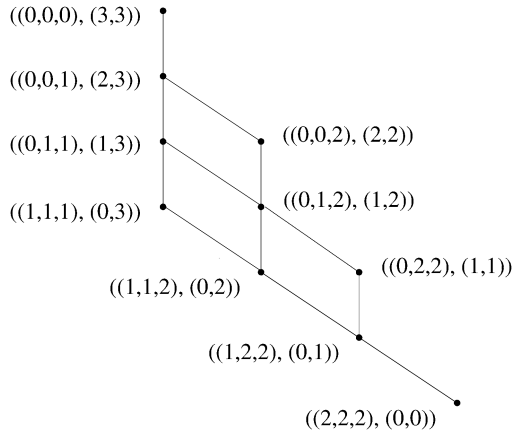


Fig. 5 Chains C_1 and C_2 combined using adjacent shuffles with 3-sequences $(1, 2, 2)$ and $(1, 1, 2)$

Fig. 6 $\Gamma_{shuffle}^{3,2}$ labeled with 3-sequences and 2-sequences



Step 2: The intermediate graph $\tilde{\Gamma}(L_1 \times L_2)$

Let $\tilde{\Gamma}(L_1 \times L_2) = \Gamma(L_1) \square \Gamma(L_2) \square \Gamma_{shuffle}^{k,l}$. Label each vertex of $\tilde{\Gamma}(L_1 \times L_2)$ with the ordered triple $(C_1, C_2, (\kappa, \lambda))$, for all maximal chains $C_i \in L_i, i = 1, 2$ and for all possible shuffles (κ, λ) . The set of vertices of $\tilde{\Gamma}(L_1 \times L_2)$ corresponds to all possible shuffles of pairs of maximal chains from L_1 and L_2 , thus there is a one-to-one correspondence between the vertices of $\tilde{\Gamma}(L_1 \times L_2)$ and the maximal chains of $L_1 \times L_2$. From the definition of a Cartesian product of graphs, two vertices in $\tilde{\Gamma}(L_1 \times L_2)$, $(C_1, C_2, (\kappa, \lambda))$ and $(C'_1, C'_2, (\kappa', \lambda'))$, are adjacent if they satisfy precisely one of the following conditions:

- (1) $C_1 = C'_1, C_2 = C'_2$, and $\kappa \sim \kappa'$
- (2) $C_1 = C'_1, C_2 \sim C'_2$ in L_2 , and $\kappa = \kappa'$
- (3) $C_1 \sim C'_1$ in $L_1, C_2 = C'_2$, and $\kappa = \kappa'$.

While $\tilde{\Gamma}(L_1 \times L_2)$ has the right number of vertices, it has too many edges. For example, Fig. 7 shows the possible result of shuffling C_1 with adjacent chains C_2

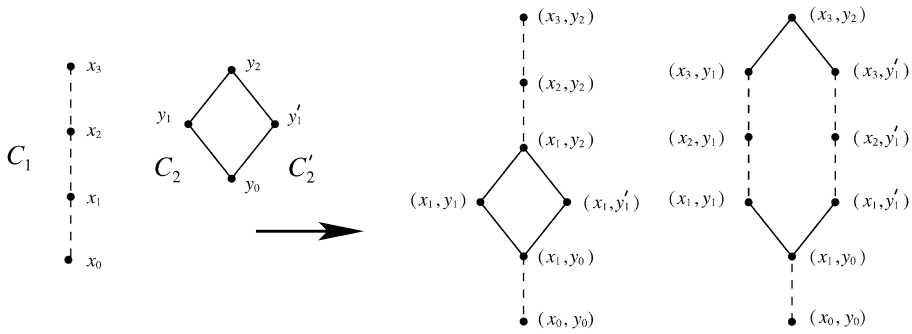


Fig. 7 Combining C_1 with C_2 and C'_2 using two different shuffles

and C'_2 . In one case, the shuffle results in a pair of adjacent chains in $L_1 \times L_2$. However, using another shuffle results in chains that differ at three ranks in $L_1 \times L_2$. This difficulty is easily remedied in Step 3 by removing a well-defined set of edges, determined by the rank where a pair of adjacent chains in one poset differs, along with the k - and l -sequences of the shuffle used.

Step 3: Removing edges from $\tilde{\Gamma}(L_1 \times L_2)$

Each edge in $\tilde{\Gamma}(L_1 \times L_2)$ can be classified as Type 1, 2, or 3, according to which of the above conditions is satisfied by $(C_1, C_2, (\kappa, \lambda))$ and $(C'_1, C'_2, (\kappa', \lambda'))$. The next step in the process of constructing $\Gamma(L_1 \times L_2)$ is to examine each type of edge in $\tilde{\Gamma}(L_1 \times L_2)$, to determine which ones are between a pair of adjacent chains in $L_1 \times L_2$ and which are not. Once edges corresponding to pairs of non-adjacent chains have been removed from the graph, the result will be the desired final graph $\Gamma(L_1 \times L_2)$.

Type 1 edges. $C_1 = C'_1, C_2 = C'_2$, and $\kappa \sim \kappa'$. It is easy to see that none of these edges need to be removed. See Fig. 5 for an example of this type.

Type 2 edges. $C_1 = C'_1, C_2 \sim C'_2$ in L_2 , and $\kappa = \kappa'$. The diagram of C_2 and C'_2 contains a diamond in L_2 at some rank i where the two chains differ. When we shuffle C_2 and C'_2 with C_1 , this diamond may be stretched by the insertion of edges from C_1 , depending on which shuffle is used. If $i \notin \kappa$ then the resulting chains are adjacent; but if $i \in \kappa$ then the resulting chains are not adjacent and the edge must be removed. Figure 7 shows the result of combining C_1 with both C_2 and C'_2 using the shuffles associated with 3-sequences $(0, 2, 2)$ and $(0, 1, 1)$. Chains C_2 and C'_2 differ at rank 1, but $(0, 2, 2)$ does not contain an element 1, so the shuffle does not stretch the diamond and the resulting chains are adjacent in $L_1 \times L_2$. However, the shuffle associated with $(0, 1, 1)$ stretches the diamond by inserting two edges from C_1 , so this Type 2 edge must be removed from $\tilde{\Gamma}(L_1 \times L_2)$.

Type 3 edges. $C_1 \sim C'_1$ in $L_1, C_2 = C'_2$, and $\kappa = \kappa'$. As in the previous case, we must first identify the rank i where C_1 and C'_1 differ. If $i \in \lambda$ then the chains are not adjacent in $L_1 \times L_2$ and the edge is removed from $\tilde{\Gamma}(L_1 \times L_2)$.

This completes the determination of which edges to remove from $\tilde{\Gamma}(L_1 \times L_2)$, resulting in the final graph $\Gamma(L_1 \times L_2)$.

4 The Boolean lattice

While in [2], it was shown that for any connected graphs Γ_1 and Γ_2 , we have that $A_1^G(\Gamma_1 \square \Gamma_2) \cong A_1^G(\Gamma_1) \times A_1^G(\Gamma_2)$, it is not immediately clear from the construction of $\Gamma(L_1 \times L_2)$ if there is an easily defined relationship between the groups $A_1^G(\Gamma(L_1))$, $A_1^G(\Gamma(L_2))$, and $A_1^G(\Gamma(L_1 \times L_2))$. However, applying the construction defined in the previous section to B_n leads to a better understanding of $\Gamma(B_n)$ that allows us to use combinatorial methods to compute the abelianization of its A_1 -group. We now reconstruct $\Gamma(B_n)$ and characterize all of its primitive 6-cycles.

In Fig. 8, the vertices of $\Gamma(B_3)$ and $\tilde{\Gamma}(B_4)$ are labeled with permutations written in one line notation. The graph $\Gamma(B_1)$ consists of a single vertex, thus there are no Type 2 edges in $\tilde{\Gamma}(B_n)$ and we can simply consider the 1-sequences of $\Gamma_{shuffle}^{3,1}$. Labeling $\Gamma(B_1)$ with the element 4 enables us to see the relationship between the 1-sequences labeling the shuffle graph and the position of 4 in the resulting permutations in S_4 .

The graph in Fig. 9 is another representation of the permutahedron we saw in Fig. 3. The vertices are labeled with permutations of S_4 , written in one line notation, and each edge corresponds to a (simple) transposition. $\Gamma(B_4)$ illustrates the following (most of them well-known) properties of $\Gamma(B_n)$. All properties easily follow from the definition of the graph.

Fig. 8 The intermediate graph $\tilde{\Gamma}(B_4)$

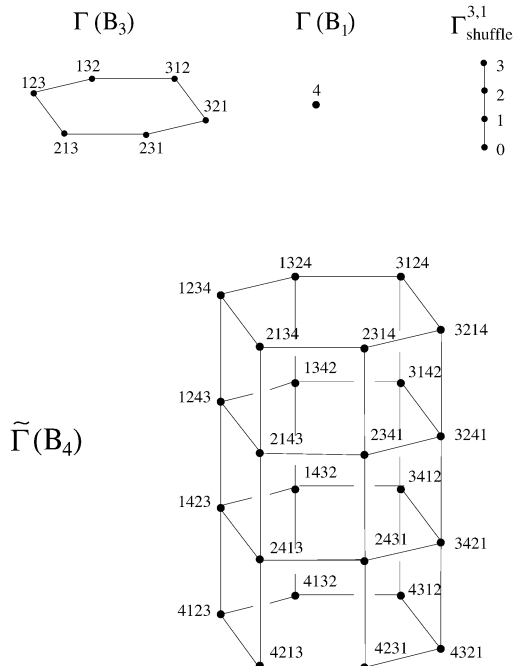
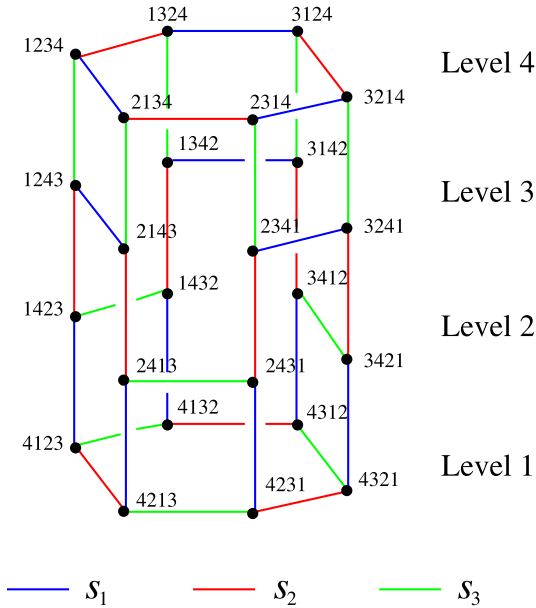


Fig. 9 The final graph $\Gamma(B_4)$, which corresponds to the 1-skeleton of the permutahedron P_3



Properties of $\Gamma(B_n)$

- (1) The graph $\Gamma(B_n)$ is $(n - 1)$ -regular, with each vertex incident to precisely one edge for each of the $n - 1$ transpositions $s_i \in \mathbf{S}$, $1 \leq i \leq n - 1$. Label each edge with its associated transposition.
- (2) Let *level* i be the set of all permutations $\sigma \in S_n$ such that $\sigma^{-1}(n) = i$. The graph has n levels, and each level was initially a copy of $\Gamma(B_{n-1})$ before we removed edges from $\Gamma(B_n)$.
- (3) The graph $\Gamma(B_n)$ is bipartite, with vertices partitioned into even and odd permutations, and all cycles in the graph are of even length.
- (4) The sequence of transpositions labeling the edges of a cycle in $\Gamma(B_n)$ form a representation of the identity in S_n . Each 4-cycle in the graph corresponds to $(s_k s_j)^2$ for some $1 \leq k, j \leq n - 1$, where $|j - k| \geq 2$. Each primitive 6-cycle corresponds to $(s_j s_{j+1})^3$ for some $1 \leq j \leq n - 2$.

Due to the structure of S_n , all other cycles of length ≥ 8 can be expressed as the concatenation of 4- and 6-cycles; we can therefore limit our investigation to primitive 6-cycles. We want to count the G -homotopy equivalence classes of 6-cycles in B_n , which yields the rank of $A_1^G(\Gamma(B_n))^{ab}$. The following definition gives us an additional description of edges and cycles in $\Gamma(B_n)$ that will help us determine equivalence classes of 6-cycles.

Definition 4.1

- (1) An edge in $\Gamma(B_n)$ between σ and τ is *horizontal* if $\sigma^{-1}(n) = \tau^{-1}(n)$, or *vertical* if $\sigma^{-1}(n) = \tau^{-1}(n) \pm 1$.
- (2) All vertices in a *horizontal* 6-cycle are at the same level. A *vertical* 6-cycle contains two vertices in each of three consecutive levels.

We note that all vertical edges between levels i and $i + 1$ are labeled with s_i . We identify each vertical 6-cycle with the middle of its three levels. For example, 1243-2143-2413-4213-4123-1423 is a vertical 6-cycle at level 2 in $\Gamma(B_4)$, and its edges are labeled with s_1 and s_2 .

Let C be a cycle in $\Gamma(B_{n-1})$. There are n copies of C in $\tilde{\Gamma}(B_n)$, one in each level of the graph. Each copy of C in $\tilde{\Gamma}(B_n)$ is a horizontal cycle that is connected to the copies in neighboring levels by vertical edges, forming a net of vertical 4-cycles connecting all of the copies. For example, we see in Fig. 8 that the 6-cycles forming levels 1 and 4 are connected by such a net. Removing vertical edges from $\tilde{\Gamma}(B_n)$ to form $\Gamma(B_n)$ may remove edges from this net, however, it is easy to see that the copies of C remain connected by a net of vertical 4- and 6-cycles, as seen in Fig. 9.

5 Equivalence classes

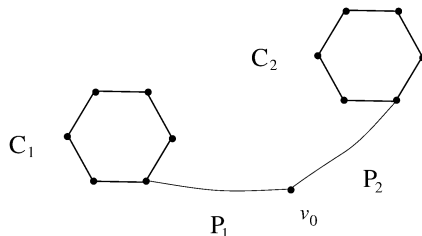
In this section, we describe how to distinguish between different G -homotopy equivalence classes in $\Gamma(B_n)$ so that we may count them. Specifically, we consider graph maps whose images in $\Gamma(B_n)$ are primitive 6-cycles connected to the base vertex by a path, and we determine when they are G -homotopic to one another. We denote this relationship by $C_1 \simeq_G C_2$, referring to the 6-cycles in the images rather than the graph maps themselves.

First, we show that if two 6-cycles are in the same equivalence class then they are associated with the same pair of transpositions, s_i and s_{i+1} , for some i , $1 \leq i \leq n - 1$. We then prove a stronger theorem: 6-cycles C_1 and C_2 are in the same equivalence class if and only if they differ by a sequence of transpositions disjoint from s_i and s_{i+1} . This theorem, when combined with our new understanding of the structure of $\Gamma(B_n)$, gives us the means to describe the equivalence classes, and to find a formula for the rank of $A_1^G(\Gamma(B_n))^{ab}$.

Let C_1 and C_2 be distinct primitive 6-cycles in $\Gamma(B_n)$. Let σ_0 be the base permutation and let P_1 and P_2 be paths from σ_0 to C_1 and C_2 , respectively. Then by definition $C_1 \simeq_G C_2$ if and only if there exists a G -homotopy grid such that the first row is $P_1C_1P_1^{-1}$ and the last row is $P_2C_2P_2^{-1}$.

Suppose that $C_1 \simeq_G C_2$ and we have a G -homotopy grid from $P_1C_1P_1^{-1}$ to $P_2C_2P_2^{-1}$. We label each vertex in the grid with the corresponding permutation in S_n . Recall that a G -homotopy is itself a graph map that preserves adjacency, thus two vertices, σ_1 and σ_2 , are adjacent in the grid if and only if $\sigma_1 = \sigma_2$, or $\sigma_2 = \sigma_1s_i$ for some s_i , $1 \leq i \leq n - 1$. If $\sigma_2 = \sigma_1s_i$, then we label the edge with s_i ; if $\sigma_1 = \sigma_2$ the

Fig. 10 Two 6-cycles based at v_0



edge remains unlabeled. Each row in the grid is a loop based at σ_0 , to which we associate a word in S_n formed by the sequence of transpositions labeling the edges from left to right in the row. Let

$$W_1 = s_{i_1}s_{i_2} \dots s_{i_m} (s_i s_{i+1})^3 s_{i_m} \dots s_{i_2}s_{i_1} \quad \text{and}$$

$$W_2 = s'_{i_1}s'_{i_2} \dots s'_{i_n} (s_j s_{j+1})^3 s'_{i_n} \dots s'_{i_2}s'_{i_1}$$

be the words associated with $P_1C_1P_1^{-1}$ and $P_2C_2P_2^{-1}$, respectively. We note that each word is a non-reduced representation of the identity and σ_0W_1 and σ_0W_2 are *based words*. Thus a G -homotopy between primitive 6-cycles C_1 and C_2 corresponds to a sequence of operations that transform W_1 into W_2 .

We now consider the possible changes that we may make from one row to the next which preserve the G -homotopy relation, and describe each change in terms of operations on the associated words. Two graph maps are G -homotopic if they differ by 3- and 4-cycles, however, $\Gamma(B_n)$ contains no 3-cycles. Furthermore, we saw in Fig. 1 that it is not possible to contract a 4-cycle in a single step, thus the changes described below are the only permissible changes.

- (T1) **Repeating vertices.** The G -homotopy relation is preserved by the repetition of a vertex in the image of a row, and the corresponding word does not change.
- (T2) **Traversing an edge in both directions.** Traversing an edge once in each direction also preserves the G -homotopy relation and is equivalent to inserting s_i^2 into the associated word. Similarly, we may reverse this by removing such an edge, and deleting s_i^2 from the word.

Fig. 11 Repeating a vertex

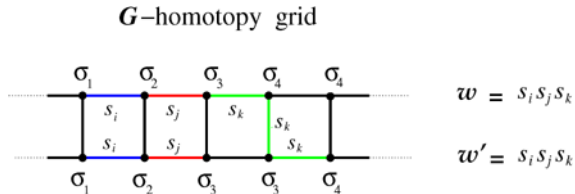


Fig. 12 Traversing an edge in both directions

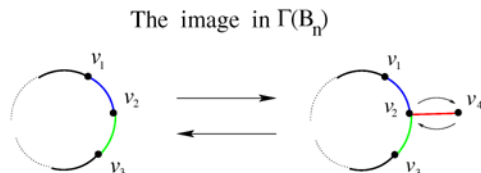
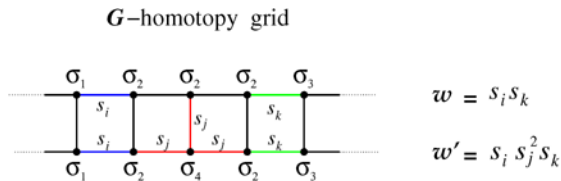
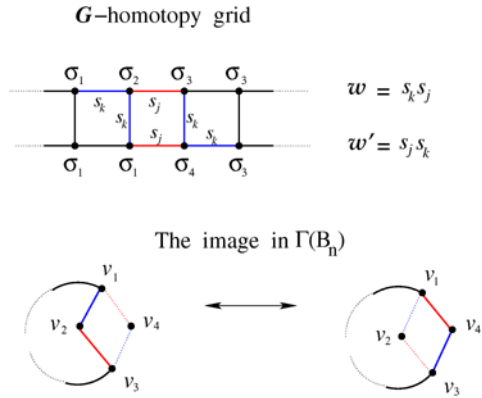


Fig. 13 Exchanging pairs of edges



(T3) Exchanging pairs of edges of a 4-cycle. Since 4-cycles are G -homotopic to the identity, we can replace two consecutive edges of a 4-cycle with the other two edges in the cycle. Recall that a 4-cycle is associated with s_j and s_k , with $|j - k| \geq 2$. We see that the effect of this change is to commute s_j and s_k in the word. See Fig. 13.

A key feature of each of the changes described above is that they preserve the parity of the number of transpositions s_i (for each $1 \leq i \leq n - 1$) in the associated words. We also note that these changes do not include the use of the relation $(s_j s_{j+1})^3 = 1$ because that would involve exchanging consecutive edges of a primitive 6-cycle with the remaining edges. This would obviously not preserve G -homotopy because any cycle of length ≥ 5 is not contractible to a single vertex.

Note that the above two types of operations, **(T2)** and **(T3)** generate an equivalence relation on the set $\sigma_0 \mathbf{W}$ of (based) loop-words, where

$$\mathbf{W} = \{W = s_{j_1} s_{j_2} \dots s_{j_{2k}}, \forall k \geq 1 \mid s_{j_i} \in \mathbf{S}, W = 1\}.$$

Two based words $\sigma_0 W$ and $\sigma_0 W'$ are equivalent, $\sigma_0 W \sim \sigma_0 W'$, if one can be obtained from the other by a sequence of **(T2)** and **(T3)** operations. Furthermore, two 6-cycles based at σ_0 , C_1 and C_2 , are G -homotopic if and only if their corresponding words, $\sigma_0 W_1$ and $\sigma_0 W_2$ are equivalent. Therefore we can continue our investigation of equivalence classes of 6-cycles using the language of graphs or words interchangeably.

Proposition 5.1 *Let C_1 and C_2 be two G -homotopic primitive 6-cycles in $\Gamma(B_n)$. If $C_1 \simeq_G C_2$, then $C_1 = C_2 = (s_i s_{i+1})^3$, for some $i, 1 \leq i \leq n - 2$.*

Proof For $i = 1, 2$, let P_i be a path from σ_0 to C_i . By assumption $P_1 C_1 P_1^{-1} \simeq_G P_2 C_2 P_2^{-1}$, and the corresponding words $w_1 (s_i s_{i+1})^3 w_1^{-1}$, $w_2 (s_j s_{j+1})^3 w_2^{-1}$ are equivalent. This means that we must be able to transform the first word into the second using only **(T2)** and **(T3)** operations. A simple parity argument (s_i appears an odd number of times in $w_1 (s_i s_{i+1})^3 w_1^{-1}$) shows that this is possible only if $i = j$. \square

Association with the same pair of transpositions is a necessary condition for G -homotopy of 6-cycles, but it turns out not to be sufficient. For $1 \leq i \leq 6$, let σ_i and

γ_i be the vertices of the cycles C_1 and C_2 respectively. Since $\Gamma(B_n)$ is connected there exists a path from C_1 to C_2 . Let $\tau_i \in \mathbf{S}$ and let $\tau_1, \tau_2, \dots, \tau_k$ be a shortest path from C_1 to C_2 . Moreover, assume that $\gamma_1 = \sigma_1 \tau_1 \tau_2 \dots \tau_k$, and that we go around the cycles C_1 and C_2 in the s_i direction. That is, $\sigma_2 = \sigma_1 s_i$, $\sigma_3 = \sigma_1 s_i s_{i+1}$ and so on. Similarly $\gamma_2 = \gamma_1 s_i$, $\gamma_3 = \gamma_1 s_i s_{i+1}$, etc. With this notation in mind we have the following theorem.

Theorem 5.2 *Let C_1 and C_2 be two distinct primitive 6-cycles in $\Gamma(B_n)$. Then $C_1 \simeq_G C_2$ iff there exists $k \geq 1$, and transpositions $\tau_1 \tau_2 \dots \tau_k$ disjoint from s_i and s_{i+1} , such that $\gamma_i = \sigma_i \tau_1 \dots \tau_k$, for all $i = 1, \dots, 6$.*

Proof The first part of the proof is constructive: assuming $C_2 = C_1 \tau_1 \dots \tau_k$, we are able to construct a G -homotopy that connects C_1 to C_2 by a net of 4-cycles of type $(s_i \tau_j)^2$ or $(s_{i+1} \tau_j)^2$. Note that if τ_j is not disjoint from s_i and s_{i+1} , we do not get 4-cycles. Figure 14 is the image of one such G -homotopy from C_1 to $C_2 = C_1 \tau_1 \tau_2 \tau_3$.

For the second part of the proof assume that C_1 and C_2 are G -homotopic primitive 6-cycles, thus $C_1 = C_2 = (s_i s_{i+1})^3$, for some $1 \leq i \leq n - 1$. Let P be a shortest path from C_1 to C_2 , with the vertices of both cycles as described above the theorem. See Fig. 15.

While neither C_1 nor C_2 can be contracted, the loop $C_1 P C_2^{-1} P^{-1}$ is contractible to a single vertex. Let $w = \tau_1 \tau_2 \dots \tau_k$ be the word corresponding to P , thus

$$W = (s_i s_{i+1})^3 w (s_{i+1} s_i)^3 w^{-1}$$

corresponds to $C_1 P C_2^{-1} P^{-1}$. Since as a permutation $W = 1$ we must be able to reduce the word W to the empty word using only the relations $s_j^2 = 1$ and $s_j s_k = s_k s_j$ if $|j - k| \geq 2$.

Our goal is to show that w consists solely of transpositions disjoint from s_i and s_{i+1} , and consequently that $\gamma_i = \sigma_i \tau_1 \tau_2 \dots \tau_k$, for all $1 \leq i \leq 6$. This is proven by first showing that we must be able to reduce W to the empty word without having to insert s_j , for any $1 \leq j \leq n - 1$. Next, we consider the types of transpositions that

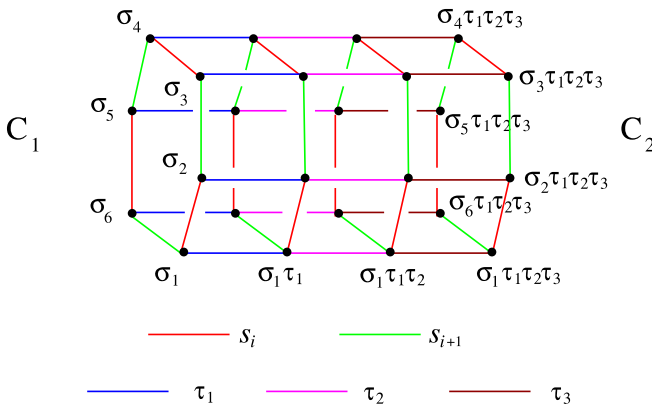


Fig. 14 A G -homotopy from C_1 to C_2

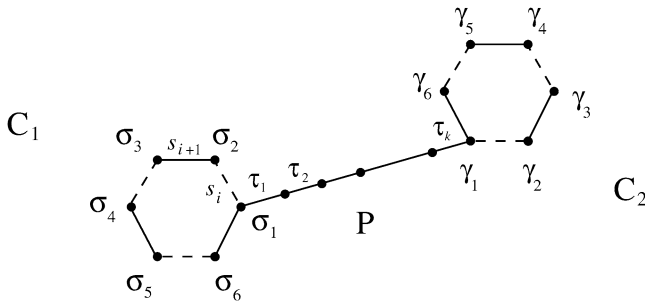


Fig. 15 The loop $C_1PC_2^{-1}P^{-1}$ can be contracted to a single vertex

may occur in w , and show that it may only contain those that are disjoint from s_i and s_{i+1} . This part of the proof requires checking many cases. Once we have shown that $w = \tau_1\tau_2 \dots \tau_k$, where each of the τ_j are disjoint from s_i and s_{i+1} , then we can see that

$$\begin{aligned} \gamma_1 &= \sigma_1\tau_1\tau_2 \dots \tau_k \quad \text{and} \\ \gamma_2 &= \gamma_1s_i \\ &= \sigma_1\tau_1\tau_2 \dots \tau_k s_i \\ &= \sigma_1s_i\tau_1\tau_2 \dots \tau_k \\ &= \sigma_2\tau_1\tau_2 \dots \tau_k. \end{aligned}$$

By a similar argument, $\gamma_j = \sigma_j\tau_1\tau_2 \dots \tau_k$ for $3 \leq j \leq 6$.

Insertion of s_j^2 is not needed.

Suppose we insert s_j^2 , at some step in the process. By the end of the process, each of these two s_j will have been removed. If those two s_j were removed together as a single pair, then they did not assist us in removing other occurrences of s_j , so it was not necessary to insert them at all.

If the s_j were removed separately, each paired with another occurrence of s_j , then whatever commuting that was done to put them into position next to their new partners could have been done in the opposite direction by the partner transpositions, which we could then remove. Thus we must be able to reduce W to the empty word without inserting transpositions.

The word w does not contain s_{i-1} or s_{i+2} .

Without loss of generality, suppose there is at least one s_{i-1} in w . Since w is a shortest path the only way to cancel that s_{i-1} is by pairing it with the first s_{i-1} in w^{-1} . But there is an odd number of transpositions s_i between the last occurrence of s_{i-1} in w and its first occurrence in w^{-1} ; three from $(s_{i+1}s_1)^3$ and an even number, if any, from w and w^{-1} . Thus, even if there were to be some cancellation there will be at least one occurrence of s_i left between the s_{i-1} , preventing their pairing.

The word w does not contain s_i or s_{i+1} .

Assume now that there are some s_i or s_{i+1} in w . Consider the subsequence of all the transpositions s_i and s_{i+1} in w . We show that if this subsequence is not empty then P is not a shortest path.

First, we show that a subsequence cannot have consecutive occurrences of either s_i or s_{i+1} . Next, we show that the subsequences must be of length shorter than six. We then show that transforming W to the empty word is not possible if the alternating subsequence is of odd length. Finally, we deal with the subsequences of even length.

Subsequence of s_i, s_{i+1} alternates. Suppose that the subsequence in w consisting of all transpositions s_i and s_{i+1} contains a consecutive pair of the same transposition, say s_i . Since s_i commutes with all transpositions in w different than s_{i+1} , we can commute those two s_i within w until they are adjacent at which point they can be removed. But this yields a shorter word w , and thus a shorter path P .

Length of subsequence is less than six. Next we consider which lengths of alternating subsequences of s_i and s_{i+1} are possible in w . If we have an alternating subsequence of length six or longer, then we can commute s_i and s_{i+1} with the other transpositions in w until we have the subword $(s_i s_{i+1})^3$. The new word corresponds to a path from C_1 to C_2 that contains a 6-cycle. The same path without the 6-cycle would be shorter.

Odd subsequences. For subsequences of odd length, Fig. 16 illustrates the possible subsequences consisting of s_i and s_{i+1} in W . This includes the six transpositions from each of C_1 and C_2 , as well as those from w and w^{-1} . In each case, we see that we are not able to remove all transpositions in the entire subsequence, contradicting the assumption that C_1 and C_2 are G -homotopic.

Even subsequences. Assume that the alternating subsequence in P is of length two or four. To illustrate this, consider the case where

$$w = \tau_1 \tau_2 s_i \tau_3 s_{i+1} \tau_4.$$

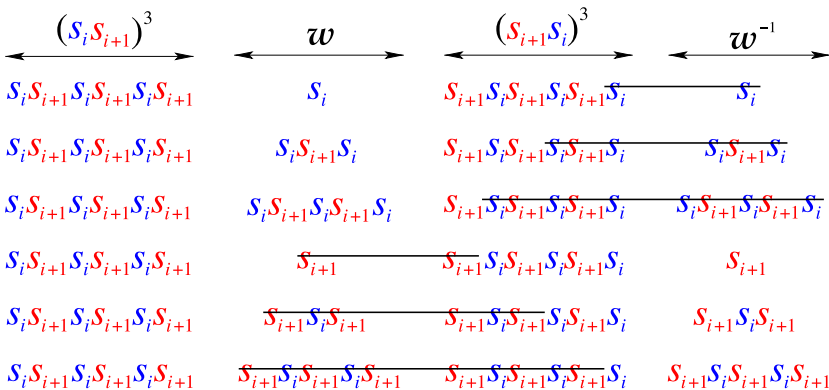
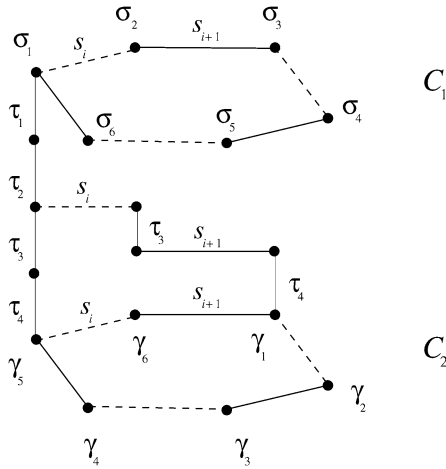


Fig. 16 We cannot reduce $W = (s_i s_{i+1})^3 w (s_{i+1} s_i)^3 w^{-1}$ to the empty word when subsequences are of odd length

Fig. 17 A shorter path without s_i and s_{i+1}



We can commute these transpositions within w to obtain

$$w' = s_i s_{i+1} \tau_1 \tau_2 \tau_3 \tau_4,$$

with the transpositions τ_j in the same relative order as they were in w . Note that as permutations in S_n , $w = w'$.

Recall that σ_j and γ_j , $1 \leq j \leq 6$, are the permutations associated with the vertices in C_1 and C_2 , respectively, with σ_1 the first vertex in P , and γ_1 the last. Our reordered word w' also corresponds to a path from σ_1 to γ_1 in $\Gamma(B_n)$, thus we may write

$$\begin{aligned} \gamma_1 &= \sigma_1 s_i s_{i+1} \tau_1 \tau_2 \tau_3 \tau_4 \\ &= \sigma_3 \tau_1 \tau_2 \tau_3 \tau_4. \end{aligned}$$

and similarly we have

$$\begin{aligned} \gamma_2 &= \sigma_4 \tau_1 \tau_2 \tau_3 \tau_4, \\ \gamma_3 &= \sigma_5 \tau_1 \tau_2 \tau_3 \tau_4, \\ \gamma_4 &= \sigma_6 \tau_1 \tau_2 \tau_3 \tau_4, \\ \gamma_5 &= \sigma_1 \tau_1 \tau_2 \tau_3 \tau_4, \\ \gamma_6 &= \sigma_2 \tau_1 \tau_2 \tau_3 \tau_4. \end{aligned}$$

We see in Fig. 17 that the word $\tau_1 \tau_2 \tau_3 \tau_4$ corresponds to a shorter path from σ_1 to C_2 . This new path now meets C_2 at γ_5 , but it still satisfies the condition that the first edge traversed in C_2 is s_i , thus contradicting the assumption that P is a shortest path. A similar argument holds true for each of the remaining three cases. \square

The proof of Theorem 5.2 relies on the definition of a G -homotopy from C_1 to C_2 , and the limitations on the types of changes we are able to make to the words associated with each row of the G -homotopy grid. We can combine this theorem with the

properties of $\Gamma(B_n)$ to obtain the following corollaries about equivalence classes of primitive 6-cycles.

Corollary 5.3 *Horizontal and vertical 6-cycles are in different equivalence classes.*

Proof Suppose C_1 is a horizontal 6-cycle at level i . As observed earlier, for all vertices σ in C_1 , $\sigma^{-1}(n) = i$. Furthermore, if $C_2 = C_1\tau_1 \dots \tau_k$, then $(\sigma\tau_1 \dots \tau_k)^{-1}(n) = \tau_k \dots \tau_1(i)$ and consequently C_2 is also a horizontal 6-cycle. Similarly, if C_1 is a vertical 6-cycle, then C_2 is a vertical 6-cycle. Therefore we may count horizontal and vertical equivalence classes separately. \square

Corollary 5.4 *Vertical 6-cycles at different levels of $\Gamma(B_n)$ are in different equivalence classes.*

Proof This is a direct consequence of Proposition 5.1 and the observation we made in Section 4 that a vertical 6-cycle at level i , $2 \leq i \leq n - 1$ is associated with transpositions s_{i-1} and s_i . \square

Corollary 5.5 *For all $2 \leq i \leq n - 1$, the number of vertical equivalence classes at level i in $\Gamma(B_n)$ is given by*

$$\frac{(n - 1)!}{2(i - 2)!(n - i - 1)!} = \binom{n - 1}{i} \binom{i}{2}.$$

Proof There are $(n - 1)!$ vertices in each level of $\Gamma(B_n)$, and each vertical 6-cycle at level i contains 2 vertices in that level, thus there are $\frac{(n-1)!}{2}$ vertical 6-cycles at level i . The number of 6-cycles in each vertical equivalence class at level i is equal to the order of the subgroup of S_n generated by transpositions disjoint from s_{i-1} and s_i , which is $(i - 2)!(n - i - 1)!$, thus the number of equivalence classes at level i is $\frac{(n-1)!}{2(i-2)!(n-i-1)!}$. \square

Consequently, the total number of vertical equivalence classes is

$$\begin{aligned} \sum_{i=2}^{n-1} \binom{n - 1}{i} \binom{i}{2} &= \sum_{i=2}^{n-1} \frac{(n - 1)!}{2(i - 2)!(n - i - 1)!} \\ &= \sum_{i=0}^{n-3} \frac{(n - 1)!}{2(i)!(n - i - 3)!} \\ &= \left(\sum_{i=0}^{n-3} \binom{n - 3}{i} \right) \frac{(n - 1)(n - 2)}{2} \\ &= 2^{n-3} \binom{n - 1}{2}. \end{aligned}$$

Corollary 5.6 *The number of horizontal equivalence classes in $A_1^G(\Gamma(B_n))^{ab}$ is equal to the rank of $A_1^G(\Gamma(B_{n-1}))^{ab}$.*

Proof We count the horizontal equivalence classes in level n , which remains a copy of $\Gamma(B_{n-1})$ even after we removed edges from $\tilde{\Gamma}(B_n)$ in the construction of $\Gamma(B_n)$. Let C be a horizontal 6-cycle in the graph that is not at level n . Recall that there is a copy, C' , of C at level n of $\Gamma(B_n)$, and there is a net of vertical 4- and 6-cycles connecting C to C' . While C may not be in the same equivalence class as C' , its equivalence class can be expressed in terms of those of C' and the vertical 6-cycles in the net. □

We are now able to see why $A_1^G(\Gamma(B_4))^{ab}$ has seven generators rather than eight. In $\Gamma(B_4)$, the horizontal 6-cycle at level 4 and the six vertical 6-cycles are each in distinct equivalence classes; the remaining 6-cycle equivalence class at level 1 is clearly expressible in terms of the previous classes.

Theorem 5.7 *For all $n \geq 1$,*

$$\text{rank } A_1^G(\Gamma(B_n))^{ab} = 2^{n-3}(n^2 - 5n + 8) - 1.$$

Proof From Theorem 5.2 and all of its corollaries we have that there are $\sum_{k=1}^n 2^{k-3} \binom{k-1}{2}$ equivalence classes of 6-cycles in $\Gamma(B_n)$. Any other cycle in $\Gamma(B_n)$ of length ≥ 8 can be expressed as the concatenation of 4-cycles and 6-cycles, so those 6-cycles classes generate the free group $A_1^{n-3}(\Delta(B_n))^{ab}$.

Let $f(n) = \sum_{k=1}^n 2^{k-3} \binom{k-1}{2} = \text{rank } A_1^G(\Gamma(B_n))^{ab}$. Then

$$\begin{aligned} \sum_{n \geq 0} f(n)x^n &= \sum_{n \geq 0} \sum_{k=0}^n 2^{n-k-3} \binom{n-k-1}{2} x^n \\ &= \sum_{k \geq 0} 2^{-2} x^{k+1} \sum_n 2^{n-k-1} \binom{n-k-1}{2} x^{n-k-1} \\ &= \sum_{k \geq 0} \frac{1}{4} x^{k+1} \sum_r \binom{r}{2} (2x)^r \\ &= \sum_{k \geq 0} \frac{1}{4} x^{k+1} \frac{(2x)^2}{(1-2x)^3} \\ &= \frac{x^3}{(1-2x)^3} \sum_{k \geq 0} x^k \\ &= \frac{x^3}{(1-2x)^3(1-x)}. \end{aligned}$$

Using partial fraction decomposition, we see that

$$f(n) = 2^{n-3}(n^2 - 5n + 8) - 1. \quad \square$$

In 2001, Babson observed that attaching 2-cells to the 4-cycles in $\Gamma(B_n)$ results in a topological space that is homotopy equivalent to the complement (in \mathbb{R}^n) of the 3-equal hyperplane arrangement. But this result holds true for all k -equal arrangements, $k \geq 3$ (see [3]). That is

$$A_1^{n-k}(\Delta(B_n)) \simeq \pi_1(M_{n,k}).$$

So, if we want to compute the first Betti number, $H^1(M_{n,k})$ of the complement (in \mathbb{R}^n) of the k -equal arrangement, it will suffice to compute the rank $A_1^{n-k}(\Delta(B_n))^{ab}$, which means counting the number of equivalence classes of 6-cycles in $\Gamma_{\Delta(B_n)}^{n-k}$. The vertices of this graph correspond to the maximal chains of $\Delta(B_n)$ and there is an edge between two such maximal chains if they share (when viewed as simplices of $\Delta(B_n)$) at least an $(n - k)$ -face. Thus $\Gamma_{\Delta(B_n)}^{n-3} (= \Gamma(B_n))$ is a subgraph of $\Gamma_{\Delta(B_n)}^{n-k}$ for all $k \geq 4$, with both graphs having the same set of vertices. In fact,

$$\Gamma_{\Delta(B_n)}^{n-3} \subset \Gamma_{\Delta(B_n)}^{n-4} \subset \Gamma_{\Delta(B_n)}^{n-5} \subset \dots,$$

and the only difference between any two of these graphs is that $\Gamma_{\Delta(B_n)}^{n-k}$ has more edges than $\Gamma_{\Delta(B_n)}^{n-j}$ for all $k > j$. Already when $k = 4$ it is easy to see that the graph $\Gamma_{\Delta(B_n)}^{n-4}$ no longer has primitive 6-cycles. Indeed, if σ is a permutation of a 6-cycle, and m is the corresponding maximal chain of B_n , then the maximal chain m' corresponding to the permutation $\sigma s_i s_{i+1} s_i$ differs from m in exactly two levels. Hence, the permutations σ and $\sigma s_i s_{i+1} s_i$ are adjacent in $\Gamma_{\Delta(B_n)}^{n-4}$ as their corresponding simplices share an $(n - 4)$ -face. Thus, in $\Gamma_{\Delta(B_n)}^{n-4}$ all 6-cycles are concatenation of at least two 4-cycles, which means that every cycle is contractible and $A_1^{n-4}(\Delta(B_n))$ is the trivial group. Moreover, given the fact that $\Gamma_{\Delta(B_n)}^{n-k}$ contains $\Gamma_{\Delta(B_n)}^{n-4}$ for all $k \geq 4$ none of these graphs have primitive cycles of length ≥ 4 , and we have the following theorem.

Theorem 5.8 For all $k \geq 4$, $A_1^{n-k}(\Delta(B_n)) = 1$.

Thus, we can conclude that for all $k \geq 4$, the first Betti number, $H^1(M_{n,k})$, of the complement of the k -equal arrangement is zero, a fact that can also be recovered from Theorem 1.1(b) of [9].

Conclusion. As we mentioned in the introduction, once all is said and done we believe that the crux of the paper lies in the combinatorial techniques that were introduced. In particular, realizing that the Γ -graph of the order complex of a product of two posets is obtained by taking the box product of three graphs, one of them being the new *shuffle* graph, and removing some (easily identifiable) edges, should prove useful in computing the discrete fundamental group of other posets. Moreover, the translation of the grid-homotopy equivalence relations in the language of equivalence of words (over the symmetric group), is also promising. Indeed, all Coxeter arrangements will have such algebraic structure that could prove fundamental in computing the A_1^q -group of the appropriate complexes.

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