Nilpotent variety of a reductive monoid

Mohan S. Putcha

Abstract In this paper we study the variety M_{nil} of nilpotent elements of a reductive monoid M. In general this variety has a completely different structure than the variety G_{uni} of unipotent elements of the unit group G of M. When M has a unique non-trivial minimal or maximal $G \times G$-orbit, we find a precise description of the irreducible components of M_{nil} via the combinatorics of the Renner monoid of M and the Weyl group of G. In particular for a semisimple monoid M, we find necessary and sufficient conditions for the variety M_{nil} to be irreducible.

1 Introduction

For a reductive group G, the variety G_{uni} of unipotent elements plays an important role, both in the study of structure of G and in the representation theory of the associated finite reductive groups, cf. [2]. In particular G_{uni} is an irreducible variety with finitely many conjugacy classes. We are interested in the variety M_{nil} of nilpotent elements of a reductive monoid M with unit group G. Nilpotent elements are a natural object of study since an analogue of the Fitting decomposition is valid for reductive monoids, cf. [7, Theorem 4.1]. When M is the multiplicative monoid of all $n \times n$ matrices, the two varieties M_{nil} and G_{uni} are isomorphic. But this is where the analogy ends. In general any connections between nilpotent and unipotent elements is either

M.S. Putcha (✉)
Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205, USA
e-mail: putcha@math.ncsu.edu

non-existent or subtle. The variety M_{nil} need not be irreducible and the number of conjugacy classes in M_{nil} is generally infinite. There is of course also the variety $\mathfrak{g}_{nil}$ of nilpotent elements of the Lie algebra $\mathfrak{g}$ of G. However $\mathfrak{g}_{nil}$ is much more closely related to G_{uni} than to M_{nil}.

We began the study of the variety M_{nil} in [10], in particular finding a close connection with the 'conjugacy decomposition' of M. In this paper we obtain an explicit description of the irreducible components of M_{nil} when M is $\mathcal{J}$-irreducible (has a unique minimal non-zero $G \times G$-orbit) or $\mathcal{J}$-coirreducible (has a unique maximal $G \times G$-orbit not equal to G). In the first case each component of M_{nil} intersects a unique maximal $\mathcal{J}$-class ($G \times G$-orbit) $J \neq G$. Moreover for a given J, at most three components of M_{nil} intersect J, and we give a precise description of each. For $\mathcal{J}$-coirreducible monoids, the irreducible components of M_{nil} are described in terms of the Coxeter elements of the Weyl group that are also minimal length coset representatives of an associated parabolic subgroup. In particular we find necessary and sufficient conditions for M_{nil} to be irreducible when M is semisimple (has a one dimensional center).

2 Preliminaries

Let κ be an algebraically closed field. By an *irreducible algebraic monoid*, we mean a monoid M with zero 0, whose underlying set is an irreducible affine variety and the product map is a morphism. We will say that M is a *reductive monoid* if the unit group G of M is reductive. Equivalently M is regular ($a \in aMa$ for all $a \in M$). We refer to [9, 16, 17] for the general theory of reductive monoids. For a reductive monoid M, the center $Z(G)$ of G has dimension at least one. M is said to be *semisimple* if the center $Z(G)$ of G is one dimensional. Equivalently 0 and 1 are the only central idempotents of M. The multiplicative monoid $M_n(\kappa)$ of all $n \times n$ matrices over κ is such an example. More generally if $G_0 \subseteq GL_n(\kappa)$ is a semisimple group then the lined closure $\overline{\kappa G_0}$ in $M_n(\kappa)$ is a semisimple monoid.

Let M be a reductive monoid with unit group G. The set $E(M)$ of idempotents of M is partially ordered in the usual way:

$$e \leq f \quad \text{if} \quad ef = e = fe$$

By [6, Section 6] (or see [9, Chapter 9]) there is a cross-section Λ of the $G \times G$-orbits ($\mathcal{J}$-classes) of M consisting of idempotents so that for $e, f \in \Lambda$,

$$e \leq f \Longleftrightarrow GeG \subseteq \overline{GfG} \Longleftrightarrow MeM \subseteq \overline{MfM}$$

Here the closure is with respect to the Zariski topology. Λ is a finite lattice and called the *cross-section lattice* of M. It is unique up to conjugacy by elements of G. We will let Λ_{min} denote the minimal elements of $\Lambda - \{0\}$ and Λ_{max} the maximal elements of $\Lambda - \{1\}$. All maximal chains in Λ have the same length. This yields *rank* and *corank* functions on Λ and hence on M (via the decomposition of M into $G \times G$-orbits):

$$\mathrm{rk} : M \longrightarrow \{0, \ldots, n\}, \quad \mathrm{crk} : M \longrightarrow \{0, \ldots, n\}$$

where n is the length of a maximal chain in Λ. Let

$$B = B(\Lambda) = \{x \in G \mid xe = exe \text{ for all } e \in \Lambda\}$$
$$B^- = B^-(\Lambda) = \{x \in G \mid ex = exe \text{ for all } e \in \Lambda\}$$

Then B, B^- are opposite Borel subgroups of G with maximal torus

$$T = C_G(\Lambda) = \{x \in G \mid xe = ex \text{ for all } e \in \Lambda\}$$

and dim T is equal to the length of any maximal chain in Λ. The set $E(\overline{T})$ of 'diagonal idempotents' of M is a finite lattice isomorphic to the face lattice of a polytope, with the set of vertices of the polytope corresponding to the set $E(\overline{T})_{min}$ of minimal idempotents of $E(\overline{T}) - \{0\}$.

Let $W = N_G(T)/T$ denote the *Weyl group* of G. Then

$$E(\overline{T}) = \bigcup_{x \in W} x \Lambda x^{-1}$$

The set

$$S = \{\alpha \in W \mid \alpha B \alpha \subseteq B \cup B \alpha B\}$$

of *simple reflections* of W generates W. We consider S as an unlabeled *Coxeter graph*. So for $\alpha, \beta \in S$, $\alpha \text{——} \beta$ means $\alpha\beta \neq \beta\alpha$. If S is connected, then we say that $\alpha \in S$ is an *endpoint* of S, if $S - \{\alpha\}$ is connected. If S is connected and $\alpha \in S$, then since W is finite, we see by the classification of Coxeter graphs that any component S' of $S - \{\alpha\}$ contains an endpoint of S and also contains a unique γ such that $\alpha \text{——} \gamma$. G admits a *Bruhat-decomposition* with respect to W:

$$G = \bigsqcup_{w \in W} BwB \tag{1}$$

The *Bruhat-Chevalley order* on W is given by [3]:

$$x \leq y \quad \text{if} \quad BxB \subseteq \overline{ByB}, \quad x, y \in W \tag{2}$$

Let $y \in W$. Then $y = \alpha_1 \cdots \alpha_m$ for some $\alpha_1, \ldots, \alpha_m \in S$. If m is minimal, then the *length* $l(y)$ of y is defined to be m and the *support*,

$$\text{supp } y = \{\alpha_1, \ldots, \alpha_m\}$$

If $\beta \in \text{supp } y$, we say that β *appears* in y. If $\beta \in S$ with $l(\beta y) < l(y)$, then by Matsumoto's exchange condition [4, 1.7],

$$\beta y = \alpha_1 \cdots \hat{\alpha_i} \cdots \alpha_m \quad \text{for some } i \tag{3}$$

For $x, y \in W$, let

$$x * y = \begin{cases} xy & \text{if } l(xy) = l(x) + l(y) \\ \text{undefined} & \text{otherwise} \end{cases}$$

If $\beta \in S$, $y \in W$, we will say that β *commutes with all of* y if $\alpha\beta = \beta\alpha$ for all $\alpha \in$ supp y. By [1, Lemma 1],

$$\beta \notin \text{supp } y, \ y\beta = \beta * y' \Longrightarrow \beta \text{ commutes with all of } y \tag{4}$$

If $x, y \in W$, $y = \alpha_1 \cdots \alpha_m$, $l(y) = m$, then it is known that the Bruhat-Chevalley order (2) has the following simple description:

$$x \leq y \Longleftrightarrow x = \alpha_{i_1} \cdots \alpha_{i_t}, \text{ for some } i_1 < \cdots < i_t \tag{5}$$

For $I \subseteq S$, $W_I = \langle I \rangle$ is called a *parabolic subgroup* of W. Let

$$D_I = \{x \in W \,|\, x\alpha = x * \alpha \forall \alpha \in I\},$$
$$D_I^{-1} = \{x \in W \,|\, \alpha x = \alpha * x \forall \alpha \in I\}.$$

Then

$$W = D_I W_I = W_I D_I^{-1} \tag{6}$$

We refer to [2, 4] for details. We are following here the notation in [2].

By [15] the Bruhat decomposition (1) of G extends to M:

$$M = \bigsqcup_{\sigma \in R} B\sigma B \tag{7}$$

where $R = \overline{N_G(T)}/T$ is the *Renner monoid* of M. R is a finite inverse monoid with unit group W and

$$R = \bigsqcup_{e \in \Lambda} WeW \tag{8}$$

Moreover WeW, $e \in \Lambda$, are the $\mathcal{J}$-classes ($W \times W$-orbits) of R. For $M_n(\kappa)$, R is the monoid of all partial permutation matrices and called the *rook monoid* in combinatorics and the *symmetric inverse semigroup* in semigroup theory. For $e \in \Lambda$, let

$$\lambda(e) = \{\alpha \in S \,|\, \alpha e = e\alpha\} \tag{9}$$

and

$$\lambda^*(e) = \bigcap_{f \geq e} \lambda(f), \quad \lambda_*(e) = \bigcap_{f \leq e} \lambda(f) \tag{10}$$

Then

$$W(e) = W_{\lambda(e)} = \{w \in W \,|\, we = ew\},$$

$$W^*(e) = W_{\lambda^*}(e),$$

$$W_*(e) = W_{\lambda_*(e)} = \{w \in W \,|\, we = e = ew\}$$

are parabolic subgroups of W with

$$W(e) = W^*(e) \times W_*(e)$$

$W(e)$ is the Weyl group of $C_G(e)$, $W^*(e)$ the Weyl group of the unit group $H(e)$ of eMe, and $W_*(e)$ the Weyl group of the identity component of

$$G_e = \{g \in G \mid ge = e = eg\}$$

Let

$$D(e) = D_{\lambda(e)} = \{y \in W \mid y\beta > y \text{ for all } \beta \in \lambda(e)\},$$

$$D(e)^{-1} = D_{\lambda(e)}^{-1} = \{y \in W \mid \beta y > y \text{ for all } \beta \in \lambda(e)\}$$

The Bruhat-Chevalley order (2) on W has a natural extension to R:

$$\sigma \leq \theta \text{ if } B\sigma B \subseteq \overline{B\theta B}, \ \sigma, \theta \in R \tag{11}$$

The order has been determined in [5] and more explicitly in [11, 12]. In particular for $e, e' \in \Lambda, x, y \in W$,

$$e \leq e' \Longrightarrow xey \leq xe'y \tag{12}$$

and for $y \in D(e)^{-1}, y' \in D(e')^{-1}$,

$$ey \leq e'y' \Longleftrightarrow y' \leq zy \text{ for some } z \in W(e) \tag{13}$$

In particular for $y, y' \in D(e)^{-1}$,

$$ey \leq ey' \Longleftrightarrow y' \leq y \tag{14}$$

In the study of reductive monoids, the cross-section lattice Λ along with the *type map*

$$\lambda : \Lambda \longrightarrow 2^S$$

takes the place of the Dynkin diagram for a reductive group. This data along with the Weyl group determines the Renner monoid R, and along with G determines the structure of the poset (or more precisely the biordered set) $E(M)$ of idempotents of M. In general the structure of Λ is very complicated and not well understood. The exceptions are when $|\Lambda_{min}| = 1$ ($\mathcal{J}$-irreducible monoids) and when $|\Lambda_{max}| = 1$ ($\mathcal{J}$-coirreducible monoids). We will discuss these monoids more in Sects. 5 and 6.

We will also need to consider another decomposition of M, related to conjugacy classes, studied by the author beginning in [8] and culminating in [10, 13]. First, following [15], let the set of *Gauss-Jordan elements* of R,

$$GJ = \{ey \mid e \in \Lambda, y \in D(e)^{-1}\}$$

For $M_n(\kappa)$, this consists of partial permutation matrices in row-echelon form. For $ey, e'y' \in GJ$, define $ey \sim e'y'$ if $weyw^{-1} = e'y'$ for some $w \in W$. This implies that $e = e'$. Let $\mathcal{C} = GJ/\sim$ and let $[ey]$ denote the $\sim$-class of ey. Then by [13], $\mathcal{C}$ is a poset if we define:

$$[ey] \leq [e'y'] \text{ if } weyw^{-1} \leq e'y' \text{ for some } w \in W \tag{15}$$

We call $\mathcal{C}$ the *conjugacy poset* of M, in that the associated decomposition (17) below leads to a description of the conjugacy classes of M. We note that in [10, 13, 16], $\mathcal{C}$ is denoted by R^*, $\tilde{R}$ and P, respectively. For $M_n(\kappa)$, $\mathcal{C}$ is the poset of all partitions of m, $m \le n$, with respect to a generalized dominance order. For $[ey] \in \mathcal{C}$, let

$$X(ey) = \bigcup_{g \in G} gBeyBg^{-1} \tag{16}$$

The *conjugacy decomposition* of M, obtained in [10, 13], is:

$$M = \bigsqcup_{[ey] \in \mathcal{C}} X(ey) \tag{17}$$

The conjugacy classes of M in $X(ey)$ are in a natural 1-1 correspondence with the twisted conjugacy classes of a finite collection of reductive groups with automorphisms. For $[ey], [e'y'] \in \mathcal{C}$,

$$X(ey) \subseteq \overline{X(e'y')} \Longleftrightarrow [ey] \le [e'y'] \tag{18}$$

and for $[ey] \in \mathcal{C}$,

$$\overline{X(ey)} = \bigsqcup_{[e'y'] \le [ey]} X(e'y') \tag{19}$$

Now let $e \in \Lambda_{max}$, $\lambda(e) = \lambda^*(e) = I$. Then for $y, y' \in D_I^{-1}$, we have by [13, Theorem 2.9] that

$$[ey] \le [ey'] \Longleftrightarrow wy' \le yw \text{ for some } w \in W_I \tag{20}$$

Then

$$l(w) + l(y') = l(wy') \le l(yw) \le l(w) + l(y)$$

So $l(y') \le l(y)$. If further $l(y) = l(y')$, then $l(wy') = l(yw)$. This implies that $w * y' = y * w$ and hence that $y = y'$. Thus for $e \in \Lambda_{max}$,

$$[ey] \le [ey'] \Longrightarrow l(y') \le l(y), \quad [ey] < [ey'] \Longrightarrow l(y') < l(y) \tag{21}$$

3 Linear and Coxeter elements

Let W be a finite Weyl group with generating set S of simple reflections. We will call $w \in W$ *linear* if $w = \alpha_1 \cdots \alpha_m$ with $\alpha_1, \dots, \alpha_m \in S$ distinct. If further $m = |S|$, then w is called a *Coxeter element*. Thus linear elements of W are Coxeter elements of parabolic subgroups of W. In this section we prove some technical results concerning linear and Coxeter elements that will be needed in our study of the nilpotent variety of a reductive monoid.

Lemma 3.1 *Let $y = \alpha_1 \cdots \alpha_m$ be linear, $\beta \in S$, $\beta y < y$. Then $\beta = \alpha_j$ for some j and $\beta \alpha_i = \alpha_i \beta$ for $1 \le i \le j$.*

Proof If $\beta \neq \alpha_j$ for all j, then clearly $\beta y > y$. So $\beta = \alpha_j$ for some j. By the exchange condition (3), there exists k such that

$$\beta\alpha_1 \cdots \alpha_m = \alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_m$$

So

$$\alpha_1 \cdots \alpha_k = \beta\alpha_1 \cdots \alpha_{k-1}$$

So $\alpha_k = \beta$ and $k = j$. Hence

$$\alpha_1 \cdots \alpha_j = \alpha_j\alpha_1 \cdots \alpha_{j-1}$$

By (4), $\alpha_i\alpha_j = \alpha_j\alpha_i$ for $1 \leq i \leq j$. □

Lemma 3.2 *Let S be connected, $\alpha \in S$. Then there is a unique Coxeter element $c_S(\alpha)$ of W in $D^{-1}_{S-\{\alpha\}}$.*

Proof We prove by induction on $|S|$. Let $S_1, \ldots, S_t$, $t \leq 3$, be the components of $S - \{\alpha\}$. Then there exist unique $\beta_i \in S_i$, $i = 1, \ldots, t$ such that $\alpha \text{———} \beta_i$ and $y = \alpha c_{S_1}(\beta_1) \cdots c_{S_t}(\beta_t)$ is a Coxeter element of W and is in $D^{-1}_{S-\{\alpha\}}$ by Lemma 3.1.

Conversely let y be a Coxeter element of W in $D^{-1}_{S-\{\alpha\}}$. Then $\alpha y < y$. Let $y = \alpha * y_0$. Then $y_0 = y_1 \cdots y_t$ where y_i is a Coxeter element of W_{S_i}. We claim that $y_i \in D^{-1}_{S_i-\{\beta_i\}}$. Otherwise $\gamma y_i < y_i$ for some $\gamma \in S_i, \gamma \neq \beta_i$. Then $\gamma\alpha = \alpha\gamma$ and γ commutes with all y_j, $j \neq i$. Hence

$$\gamma y = \gamma\alpha y_0 = \gamma\alpha y_1 \cdots y_t = \alpha y_1 \cdots \gamma y_i \cdots y_t < \alpha y_1 \cdots y_i \cdots y_t = \alpha y_0 = y,$$

a contradiction since $y \in D^{-1}_{S-\{\alpha\}}$. Hence $y_i \in D^{-1}_{S_i-\{\beta_i\}}$, $i = 1, \ldots, t$. By induction hypothesis, $y_i = c_{S_i}(\beta_i)$, $i = 1, \ldots, t$. Hence

$$y = \alpha y_0 = \alpha y_1 \cdots y_t = \alpha c_{S_1}(\beta_1) \cdots c_{S_t}(\beta_t)$$

This completes the proof. □

Lemma 3.3 *Let $I, K \subseteq S$ such that no component of K is contained in I. Then there is a Coxeter element z of W_K in D^{-1}_I.*

Proof Let $K_1, \ldots, K_t$ denote the components of K. Then there exists $\alpha_i \in K_i, \alpha_i \notin I$, $i = 1, \ldots, t$. Let $z_i = c_{K_i}(\alpha_i)$, $i = 1, \ldots, t$ as in Lemma 3.2. We claim that $z_i \in D^{-1}_I$. Otherwise $\beta z_i < z_i$ for some $\beta \in I$. So $\beta \in \text{supp } z_i \subseteq K_i$. Since $\beta \in I$, $\beta \neq \alpha_i$. So $z_i \notin D^{-1}_{K_i-\{\alpha_i\}}$, a contradiction. Thus $z_i \in D^{-1}_I$, $i = 1, \ldots, t$. Clearly $z = z_1 \cdots z_t$ is a Coxeter element of W_K. Also $z \in D^{-1}_I$ by Lemma 3.1. This completes the proof. □

Lemma 3.4 *Let $I \subseteq S$, $y \in D^{-1}_I$. Then there exists linear $y_0 \in D^{-1}_I$ such that $y_0 \leq y$ and $\text{supp } y_0 = \text{supp } y$.*

Proof Let $K = \operatorname{supp} y$. We prove by induction on $l(y)$. Let $y = x * \alpha_0$, $\alpha_0 \in K$. Then $x \in D_I^{-1}$. If α_0 appears in x, we are done by the induction hypothesis. So assume that α_0 does not appear in x. By the induction hypothesis, $\operatorname{supp} x = \{\alpha_1, \ldots, \alpha_m\}$, $z = \alpha_m \cdots \alpha_1 \in D_I^{-1}$. Then $y_0 = z\alpha_0 \leq y$. we claim that $z\alpha_0 \in D_I^{-1}$. Otherwise for some $\beta \in I$, $\beta z \alpha_0 < z\alpha_0$. Since $\beta z > z$, we see by Lemma 3.1 that $\beta = \alpha_0$ and that β commutes with all of z. So $y = x * \alpha_0 = \alpha_0 * x$ and $\beta y = \alpha_0 y = x < y$, a contradiction since $y \in D_I^{-1}$. Hence $y_0 = z\alpha_0 \in D_I^{-1}$. This completes the proof. $\square$

Lemma 3.5 *Let $I \subseteq S$, $\alpha_0 \in S - I$, I' a component of $S - \{\alpha_0\}$ contained in I. Let $\alpha_1, \ldots, \alpha_k \in I'$ be distinct such that $\alpha_0 \text{——} \alpha_1 \text{——} \cdots \text{——} \alpha_k$. Then*

(i) $y_0 = \alpha_0 \alpha_1 \cdots \alpha_k \in D_I^{-1}$.

(ii) *If $y \in D_I^{-1}$ is linear with $\alpha_0 y < y$ and if α_k appears in y, then $y_0 \leq y$.*

Proof (i) Let $\beta \in I$ such that $\beta y_0 < y_0$. Since $\alpha_0 \notin I$, we see by Lemma 3.1 that $\beta = \alpha_i$ for some $1 \leq i \leq k$ and that $\alpha_i \alpha_{i-1} = \alpha_{i-1} \alpha_i$, a contradiction. Hence $y_0 \in D_I^{-1}$.

(ii) We prove by induction on $l(y)$. Since $\alpha_0 y < y$, we see that $y = \alpha_0 * y_1$ and $\operatorname{supp} y_1 \subseteq S - \{\alpha_0\}$. So $y_1 = y_2 * y_3$ with $\alpha_k \in \operatorname{supp} y_2 \subseteq I'$. Hence $\alpha_0 y_2 \in D_I^{-1}$ and $\alpha_0 y_2 \leq y$. So by induction hypothesis we may assume that $\alpha_0 y_2 = y$. Hence $y_1 = y_2 \in W_{I'}$. Since W is finite, α_0 commutes with all elements of I' other than α_1. Since $y \in D_I^{-1}$, $y_1 = \alpha_1 * z$. Let $\gamma \in I'$, $\gamma \neq \alpha_1$, $\gamma y_1 < y_1$. Then

$$\gamma y = \gamma \alpha_0 y_1 = \alpha_0 \gamma y_1 < \alpha_0 y_1 = y$$

a contradiction since $y \in D_I^{-1}$. Hence $y_1 \in D_{I_1}^{-1}$, where $I_1 = I' - \{\alpha_1\}$. Since $y_1 \in W_{I'}$, the induction hypothesis implies that $\alpha_1 \cdots \alpha_k \leq y_1$. So $y_0 = \alpha_0 \alpha_1 \cdots \alpha_k \leq \alpha_0 y_1 = y$, completing the proof. $\square$

Lemma 3.6 *Let S be connected, $I \subsetneq S$, $y \in D_I^{-1}$, y linear. Suppose that for any $\alpha \in S$ with no component of $S - \{\alpha\}$ contained in I, α appears in y. Then y is a Coxeter element of W.*

Proof If $I = \emptyset$, the result is clear. So let $I \neq \emptyset$. We prove by induction on $|S - I|$. First suppose that $|S - I| = 1$ and $I = S - \{\alpha_0\}$. Let $\alpha \in I$, I' the component of α in I. Then

$$\alpha_0 \text{——} \alpha_1 \text{——} \cdots \text{——} \alpha \text{——} \cdots \text{——} \alpha_k$$

with α_k being an endpoint of S. Now $S - \{\alpha_k\}$ is connected and not contained in I. Hence α_k appears in y. Since $y \in D_I^{-1}$, $\alpha_0 y < y$. By Lemma 3.5, α_0, α appear in y. Hence y is a Coxeter element of W.

Next assume that $|S - I| > 1$. First suppose that $\beta y > y$ for some $\beta \in S - I$ and let $I_1 = I \cup \{\beta\}$. The $I_1 \neq S$ and $y \in D_{I_1}^{-1}$. Let $\alpha \in S$ such that no component of $S - \{\alpha\}$ is contained in I_1. Then no component of $S - \{\alpha\}$ is contained in I. Hence α appears in y. Since $|S - I_1| < |S - I|$, we see by the induction hypothesis that y is a Coxeter element of W. Hence assume that $\beta y < y$ for all $\beta \in S - I$. Then β appears

in y for all $\beta \in S - I$. Let $\alpha \in I$. If no component of $S - \{\alpha\}$ is contained in I, then α appears in y. So assume that some component I' of $S - \{\alpha\}$ is contained in I. Then I' contains an endpoint γ of S and

$$\alpha = \alpha_0 \text{———} \alpha_1 \text{———} \cdots \text{———} \alpha_k = \gamma, \quad \alpha_1, \ldots, \alpha_k \in I'$$

Now $S - \{\gamma\}$ is connected and not contained in I. So $\gamma = \alpha_k$ appears in y. Since $\alpha \in I$, there exists $\beta \in S - I$ such that

$$\beta = \beta_0 \text{———} \beta_1 \text{———} \cdots \text{———} \beta_m = \alpha \quad \text{with } \beta_1, \ldots, \beta_m \in I$$

Let I'' be the component of β in $S - \{\alpha\}$. Then since $I' \subseteq I$ and $\beta \notin I$, $I' \neq I''$. So $\beta_0, \ldots, \beta_{m-1} \in I''$ and

$$\beta = \beta_0 \text{———} \cdots \text{———} \beta_m \text{———} \alpha_1 \text{———} \cdots \text{———} \alpha_k$$

Since $\beta \notin I$, $\beta y < y$. Since also α_k appears in y, we see by Lemma 3.5 that $\alpha = \beta_m$ appears in y. Hence y is a Coxeter element of W, completing the proof. $\square$

4 Nilpotent variety

For a reductive monoid M, the *nilpotent variety*

$$M_{nil} = \{a \in M \mid a^k = 0 \text{ for some } k\}$$

This closed subset of M is in general not irreducible.

Example 4.1 Let

$$M = \{A \otimes B \mid A, B \in M_2(\kappa)\}$$

Then M_{nil} has two components:

$$\{A \otimes B \mid A^2 = 0\} \quad \text{and} \quad \{A \otimes B \mid B^2 = 0\}$$

The variety M_{nil} is best studied by analyzing the conjugacy poset $\mathcal{C}$. Let

$$\mathcal{C}_{nil} = \{[ey] \in \mathcal{C} \mid (ey)^k = 0 \text{ for some } k\}$$

The following result is proved in [10, Theorem 3.1(i)]. We include a proof here since the proof given in [10] needs minor elaboration.

Theorem 4.2 *Let* $0 \neq e \in \Lambda$, $y \in D(e)^{-1}$. *Then* $[ey] \in \mathcal{C}_{nil}$ *if and only if* supp $y \nsubseteq$ $\lambda(f)$ *for all* $f \in \Lambda_{min}$ *with* $f \leq e$.

Proof If $N = |W|$, then by [10, Equation (32)],

$$f_0 = (ey)^N = \prod_i y^i e y^{-i} \in \Lambda$$

Then $f_0 \leq e$ and

$$y \in W(f_0) = W^*(f_0)W_*(f_0) \subseteq W^*(e)W_*(f_0)$$

Since $y \in D(e)^{-1}$, $y \in W_*(f_0)$. So if ey is not nilpotent, then $f_0 \neq 0$ and $f_0 \geq f$ for some $f \in \Lambda_{min}$. So $e \geq f$ and

$$\text{supp } y \subseteq \lambda_*(f_0) \subseteq \lambda_*(f) = \lambda(f)$$

Conversely suppose that supp $y \subseteq \lambda(f)$ for some $f \in \Lambda_{min}$ with $e \geq f$. Then $y \in W(f) = W_*(f)$ and $yf = f$. Hence $f_0 f = (ey)^N f = f$ and $f_0 \geq f$. So $[ey] \notin \mathcal{C}_{nil}$. This completes the proof.   □

The following result is proved in [10, Theorem 3.1(ii),(iii)].

Theorem 4.3 *The variety M_{nil} decomposes as*:

$$M_{nil} = \bigsqcup_{[ey] \in \mathcal{C}_{nil}} X(ey)$$

The irreducible components of M_{nil} are $\overline{X(ey)}$, where $[ey]$ is a maximal element of $\mathcal{C}_{nil}$ with respect to the order (15).

The open problem then is to determine the maximal elements of $\mathcal{C}_{nil}$. We will solve this problem for $\mathcal{J}$-irreducible and $\mathcal{J}$-correducible monoids in Sects. 5 and 6, respectively. We end this section with the following auxiliary result for semisimple monoids.

Theorem 4.4 *Let M be a semisimple monoid. Then $M_{nil} \cap J \neq \emptyset$ for all $\mathcal{J}$-classes $J \neq G$ of M.*

Proof Let $e \in \Lambda$, $e \neq 0, 1$. If w_0 and v_0 are the respective maximum elements of W and $W(e)$, then $x = v_0 w_0$ is the maximum element of $D(e)^{-1}$. Let $f \in \Lambda_{min}$, $e \geq f$. Suppose $x \in W(f)$. Then for all $y \in D(e)^{-1}$, $y \leq x$ and hence $y \in W(f)$. So $D(e)^{-1} \subseteq W(f)$ and by (6),

$$W = W(e)D(e)^{-1} \subseteq W(e)W(f)$$

Let $w \in W$. Then $w = w_1 w_2$, $w_1 \in W(e)$, $w_2 \in W(f)$. So

$$w^{-1}ew = w_2^{-1}w_1^{-1}ew_1 w_2 = w_2^{-1}ew_2 \geq w_2^{-1}fw_2 = f$$

Hence

$$e' = \prod_{w \in W} w^{-1}ew \geq f$$

Clearly $we' = e'w$ for all $w \in W$. By [9, Theorem 6.30], $e' \in \overline{Z(G)}$. Since $e' \neq 1$ and M is semisimple, $e' = 0$. Hence $f = 0$, a contradiction. So $x \notin W(f)$. By Theorem 4.2, $[ex] \in \mathcal{C}_{nil}$. So $M_{nil} \cap GeG \neq \emptyset$, proving the theorem.   □

5 $\mathcal{J}$-irreducible monoids

If G_0 is a semisimple group and $\theta : G_0 \to GL_n(\kappa)$ an irreducible representation, then $\overline{\kappa\theta(G_0)}$ is a reductive monoid with a unique non-zero minimal $\mathcal{J}$-class. Call a reductive monoid M,$\mathcal{J}$-*irreducible*, if $|\Lambda_{min}| = 1$. Let $\Lambda_{min} = \{e_0\}$. Then $J_0 = Ge_0G$ is the unique non-zero minimal $\mathcal{J}$-class of M. Let $\lambda(e_0) = I$. We then say that M is a $\mathcal{J}$-irreducible monoid of *type I*. Now no component of S is contained in I. For suppose some component S' of S is contained in I. Then $W = W_{S'} \times W_{S''}$. Let $e \in E(\overline{T})_{min}$. Then $x^{-1}e_0x = e$ for some $x \in W$. Since $W_{S'} \subseteq W(e_0)$, $w^{-1}e_0w = e$ for some $w \in W_{S''}$. So for $u \in W_{S'}$,

$$u^{-1}eu = u^{-1}w^{-1}e_0wu = w^{-1}u^{-1}e_0uw = w^{-1}e_0w = e$$

If $f \in E(\overline{T})$, $f \neq 0$, then f is the join of some elements in $E(\overline{T})_{min}$. Hence $u^{-1}fu = f$. So by [9, Chapter 8], $u = 1$. Hence $S' = \emptyset$, a contradiction. Thus no component of S is contained in I. Let $S_1, \ldots, S_t$ be the components of S, $I_j = S_j \cap I$, $j = 1, \ldots, t$. Then $|I_j| < |S_j|$, $j = 1, \ldots, t$. We say that M is $\mathcal{J}$-irreducible of *maximal type* if $|I_j| = |S_j| - 1$, $j = 1, \ldots, t$. This concept is not used in this paper, but its dual concept is used in Sect. 6. Rather amazingly, the cross-section lattice Λ and the type map λ of a $\mathcal{J}$-irreducible monoid are completely determined its type I. The following result is proved in [14].

Theorem 5.1 *Let M be a $\mathcal{J}$-irreducible monoid of type I. Then*

  (i) *M is semisimple.*
 (ii) *If $e, e' \in \Lambda - \{0\}$, then $e \leq e'$ if and only if $\lambda^*(e) \subseteq \lambda^*(e')$.*
(iii) *If $e \in \Lambda - \{0\}$, then $\lambda_*(e) = \{\alpha \in I | \alpha\beta = \beta\alpha \text{ for all } \beta \in \lambda^*(e)\}$.*
 (iv) *If $K \subseteq S$, then $K = \lambda^*(e)$ for some $e \in \Lambda - \{0\}$ if and only if no component of K is contained in I.*
  (v) *If $e \in \Lambda - \{0\}$, then $|\lambda^*(e)| = rk\,(e) - 1 = |S| - crk\,(e)$. In particular if $e \in \Lambda_{max}$, then $\lambda(e) = \lambda^*(e) = S - \{\alpha\}$ for some $\alpha \in S$.*

We now proceed to determine the irreducible components of M_{nil} for a $\mathcal{J}$-irreducible monoid M of type I. We begin with:

Theorem 5.2 *Let M be a $\mathcal{J}$-irreducible monoid. Then every component of M_{nil} intersects a unique maximal $\mathcal{J}$-class $J \neq G$ of M.*

Proof We only need to prove existence of the maximal $\mathcal{J}$-class, since the uniqueness then follows from Theorem 4.3. We proceed by induction on $|S|$. Let $0 \neq [ey] \in \mathcal{C}_{nil}$. By Theorem 4.2, $y \notin W_I$. If $e \in \Lambda_{max}$, there is nothing to prove. So assume $e \notin \Lambda_{max}$. Since $y \notin W_I$, $y = y_1 * w$ with $w \in W_I$, $1 \neq y_1 \in D(e)^{-1} \cap D_I$. Then by (14), $ey \leq ey_1$ and by Theorem 4.2, $[ey_1] \in \mathcal{C}_{nil}$. Thus we may assume that $y = y_1 \in D(e)^{-1} \cap D_I$. Let $f \in \Lambda_{max}$, $e < f$. Suppose $y \notin W(f)$. Then $y = w * y'$ with $w \in W(f)$ and $1 \neq y' \in D(f)^{-1} \cap D_I$. So by (13), $ey \leq fy'$ and by Theorem 4.2, $fy' \in \mathcal{C}_{nil}$. So assume that $y \in W(f)$ for all $f \in \Lambda_{max}$ with $e < f$. Let $f \in \Lambda_{max}$ with $e < f$. Then applying induction hypothesis on the $\mathcal{J}$-irreducible monoid fMf,

we see that $[ey] \leq [e_1 z]$ with $[e_1 z] \in \mathcal{C}_{nil}$ and f covering e_1 in Λ. Thus we may assume that f covers e in Λ. Then by Theorem 5.1, $\lambda(f) = \lambda^*(f) = S - \{\alpha\}$, $\lambda^*(e) = S - \{\alpha, \beta\}$. If $S - \{\beta\}$ has no components contained in I, then by Theorem 5.1, there exists $f' \in \Lambda_{max}$ such that $\lambda(f') = \lambda^*(f') = S - \{\beta\}$. Then $e < f'$ and $ff' = e$. So

$$\text{supp } y \in \lambda(f) \cap \lambda(f') \subseteq \lambda(ff') = \lambda(e)$$

and $y \in W(e)$, a contradiction. So $S - \{\beta\}$ has a component contained in I. Since $S - \{\alpha, \beta\} = \lambda^*(e)$ has no components contained in I, it follows that $\{\alpha\}$ is a component of $S - \{\beta\}$ and contained in I. Thus $\lambda_*(e) = \{\alpha\}$ and $\lambda(e) = S - \{\beta\}$. So $\alpha\gamma = \gamma\alpha$ for all $\gamma \in S - \{\beta\}$. If also $\alpha\beta = \beta\alpha$, then $\{\alpha\}$ is a component of S contained in I, a contradiction. Hence $\alpha\beta \neq \beta\alpha$. Since $1 \neq y \in D(e)^{-1}$ and $\lambda(e) = S - \{\beta\}$, we see that $y = \beta * y_0$. Let

$$\alpha y = \alpha * y = w * x, w \in W(f), x \in D(f)^{-1} \quad (22)$$

Then by (13), $ey \leq fx$. So if $[fx] \in \mathcal{C}_{nil}$, we are done. So assume that fx is not nilpotent. Then by Theorem 4.2, $x \in W_I$. By (22), $\alpha(wx) = y < wx$. Also since $W(f) = S - \{\alpha\}$, α does not appear in w. So by the exchange condition (3),

$$y = \alpha wx = w * x', \ x' < x, \ l(x') = l(x) - 1$$

So $x' \in W_I$. Since $y \in D_I$, $x' = 1$. So $l(x) = 1$. Since α does not appear in w, we see by (22) that $x = \alpha$. Hence $w\alpha = \alpha y$ and by (4), α commutes with all of w. Since β appears in y, β appears in w. Hence $\alpha\beta = \beta\alpha$, a contradiction. This completes the proof.   □

For a maximal $\mathcal{J}$-class $J \neq G$ of M, we proceed to determine precisely the components of M_{nil} that intersect J. Let $\alpha \in S$, S_α the component of α in S. Then $S_\alpha \nsubseteq I = \lambda(e_0)$. Let $\mathcal{A}(I, \alpha)$ denote the set of all linear elements $y = \alpha_0 \cdots \alpha_k$ of S such that

$$\alpha = \alpha_0, \ \alpha_0 \text{------} \alpha_1 \text{------} \cdots \text{------} \alpha_k, \ \alpha_k \notin I, \ \alpha_i \in I \text{ for } i < k$$

Thus

$$\mathcal{A}(I, \alpha) = \{\alpha\} \Longleftrightarrow \alpha \notin I \quad (23)$$

Since $S_\alpha - \{\alpha\}$ has at most 3 components, we see by the classification of Coxeter graphs of finite Weyl groups that

$$1 \leq |\mathcal{A}(I, \alpha)| \leq 3 \quad (24)$$

Theorem 5.3 *Let M be a $\mathcal{J}$-irreducible monoid, $e \in \Lambda_{max}$, $J = GeG$, $\lambda(e) = S - \{\alpha\}$. Then $\mathcal{A}(I, \alpha) \subseteq D(e)^{-1}$ and the distinct irreducible components of M_{nil} that intersect J are $\overline{X(ey)}$, $y \in \mathcal{A}(I, \alpha)$. In particular, at most 3 components of M_{nil} intersect J.*

Proof If $y_0 \in \mathcal{A}(I, \alpha)$, then $y_0 \in D(e)^{-1}$ by Lemma 3.5(i), and $[ey_0] \in \mathcal{C}_{nil}$ by Theorem 4.2. Let $[ey] \in \mathcal{C}_{nil}$ be maximal with respect to the order (15). Then by (14), Lemma 3.4, and Theorem 4.2, y is linear. Let S_α denote the component of α in S. Since $y \in D(e)^{-1}$, $\text{supp } y \subseteq S_\alpha$. Let $\beta \in \text{supp } y$, $\beta \notin I$, $y = y_1 * \beta * y_2$. Then $y_1, y_1\beta \in D(e)^{-1}$. By (14) and the maximality of $[ey]$, $y = y_1\beta$ and $y_1 \in W_I$. Since $y \in D(e)^{-1}$, $\alpha y < y$. There exist distinct $\alpha = \alpha_0, \ldots, \alpha_k = \beta \in S$ such that $\alpha_0 \text{—} \alpha_1 \text{—} \cdots \text{—} \alpha_k$. By (14), the maximality of $[ey]$, and Lemma 3.5(ii), $y = \alpha_0\alpha_1 \cdots \alpha_k$. Hence $\alpha_j \in I$ for $j < k$. So $y \in \mathcal{A}(I, \alpha)$.

Let $y, y' \in \mathcal{A}(I, \alpha)$, $[ey'] < [ey]$. Then by (23), $\alpha \in I$. There exist distinct $\alpha_0, \ldots, \alpha_k \in S$ and distinct $\alpha'_0, \ldots, \alpha'_l \in S$ such that

$$\alpha = \alpha_0 = \alpha'_0, \quad \alpha_k, \alpha'_l \notin I, \quad \alpha_i, \alpha'_j \in I \text{ for } i < k, j < l$$

and

$$y = \alpha_0\alpha_1 \cdots \alpha_k, \quad \alpha_0 \text{—} \alpha_1 \text{—} \cdots \text{—} \alpha_k$$
$$y' = \alpha'_0\alpha'_1 \cdots \alpha'_l, \quad \alpha'_0 \text{—} \alpha'_1 \text{—} \cdots \text{—} \alpha'_l$$

We prove by induction on k that this is not possible. By (20), there exists $w \in W(e)$ such that $wy \leq y'w$. So α does not appear in w and

$$w * (\alpha\alpha_1 \cdots \alpha_k) \leq \alpha\alpha'_1 \cdots \alpha'_l w \tag{25}$$

Hence $w\alpha\alpha_1 \cdots \alpha_k \not\leq \alpha'_1 \cdots \alpha'_l w$. Thus

$$\alpha w\alpha\alpha_1 \cdots \alpha_k \leq \alpha'_1 \cdots \alpha'_l w$$

Hence α does not appear in $\alpha w\alpha\alpha_1 \cdots \alpha_k$. Thus

$$\alpha(w\alpha\alpha_1 \cdots \alpha_k) < w\alpha\alpha_1 \cdots \alpha_k$$

By the exchange condition (3), $\alpha w\alpha = w$. So $w * \alpha = \alpha * w$ and by (4), α commutes with all of w. Since $\alpha_1\alpha \neq \alpha\alpha_1$, α_1 does not appear in w. So by (25),

$$w * (\alpha_1 \cdots \alpha_k) \leq \alpha'_1 \cdots \alpha'_l w \tag{26}$$

Since α_1 does not appear in w, $\alpha_1 = \alpha'_j$ for some j. So $\alpha = \alpha'_0 \text{—} \alpha'_j$ and $j = 1$. By (26) and the induction hypothesis, $\alpha_1 \cdots \alpha_k = \alpha'_1 \cdots \alpha'_l$. Hence $y = y'$, a contradiction. Thus $[ey]$, $y \in \mathcal{A}(I, \alpha)$ are distinct and maximal in $\mathcal{C}_{nil}$. This completes the proof.   □
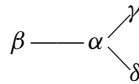
Example 5.4 Let

$$G_0 = \{A \otimes (A^{-1})^t | A \in SL_3(\kappa)\}$$

and let $M = \overline{\kappa G_0}$. See [9, Example 10.3]. Then $S = \{\alpha, \beta\}$ where $\alpha = (1\,2)$ and $\beta = (2\,3)$. M is a $\mathcal{J}$-irreducible monoid of type $\emptyset$ and $\Lambda_{max} = \{e_\alpha, e_\beta\}$ where $\lambda(e_\alpha) = S - \{\alpha\} = \{\beta\}$ and $\lambda(e_\beta) = S - \{\beta\} = \{\alpha\}$. The irreducible components of M_{nil} are:

$$\overline{X(e_\alpha\alpha)}, \ \overline{X(e_\beta\beta)}$$

corresponding to the elements $A \otimes B$ of M_{nil} with rk $B \leq 1$ and rk $A \leq 1$, respectively.

Example 5.5 Let $S = \{\alpha, \beta, \gamma, \delta\}$ be of type D_4 with β, γ, δ being the endpoints:

$$\beta \relbar\joinrel\relbar \alpha \Big\langle {}^{\displaystyle \gamma}_{\displaystyle \delta}$$

Let $I \subseteq S$ and consider a $\mathcal{J}$-irreducible monoid M of Type I. Then by Theorem 5.1, Λ_{max} consists of those $e_\pi, \pi \in S$, with $\lambda(e_\pi) = S - \{\pi\}$ having no components contained in I.

(i) If $I = \emptyset$, then $\Lambda_{max} = \{e_\alpha, e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 4 irreducible components:

$$\overline{X(e_\alpha \alpha)}, \ \overline{X(e_\beta \beta)}, \ \overline{X(e_\gamma \gamma)}, \ \overline{X(e_\delta \delta)}$$

(ii) If $I = \{\alpha\}$, then $\Lambda_{max} = \{e_\alpha, e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 6 irreducible components:

$$\overline{X(e_\alpha \alpha \beta)}, \ \overline{X(e_\alpha \alpha \gamma)}, \ \overline{X(e_\alpha \alpha \delta)}, \ \overline{X(e_\beta \beta)}, \ \overline{X(e_\gamma \gamma)}, \ \overline{X(e_\delta \delta)}$$

(iii) If $I = \{\beta\}$, then $\Lambda_{max} = \{e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 3 irreducible components:

$$\overline{X(e_\gamma \gamma)}, \ \overline{X(e_\delta \delta)}, \ \overline{X(e_\beta \beta \alpha)}$$

(iv) If $I = \{\alpha, \beta\}$, then $\Lambda_{max} = \{e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 4 irreducible components:

$$\overline{X(e_\gamma \gamma)}, \ \overline{X(e_\delta \delta)}, \ \overline{X(e_\beta \beta \alpha \gamma)}, \ \overline{X(e_\beta \beta \alpha \delta)}$$

(v) If $I = \{\beta, \gamma\}$, then $\Lambda_{max} = \{e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 3 irreducible components:

$$\overline{X(e_\beta \beta \alpha)}, \ \overline{X(e_\gamma \gamma \alpha)}, \ \overline{X(e_\delta \delta)}$$

(vi) If $I = \{\alpha, \beta, \gamma\}$, then $\Lambda_{max} = \{e_\beta, e_\gamma\}$ and M_{nil} has 2 irreducible components:

$$\overline{X(e_\beta \beta \alpha \delta)}, \ \overline{X(e_\gamma \gamma \alpha \delta)}$$

(vii) If $I = \{\beta, \gamma, \delta\}$, then $\Lambda_{max} = \{e_\beta, e_\gamma, e_\delta\}$ and M_{nil} has 3 irreducible components:

$$\overline{X(e_\beta \beta \alpha)}, \ \overline{X(e_\gamma \gamma \alpha)}, \ \overline{X(e_\delta \delta \alpha)}$$

Example 5.6 Let M be a $\mathcal{J}$-irreducible monoid of type $\emptyset$. Such a monoid is called a *canonical monoid* and is of considerable importance in embedding theory, cf. [16]. For $\alpha \in S$, let $e_\alpha \in \Lambda$ be such that $\lambda(e_\alpha) = S - \{\alpha\}$. Then the components of M_{nil} are $\overline{X(e_\alpha \alpha)}, \alpha \in S$.

6 $\mathcal{J}$-coirreducible monoids

Call a reductive monoid M, $\mathcal{J}$-*coirreducible*, if $|\Lambda_{max}| = 1$. Let $\Lambda_{max} = \{e_0\}$, $\lambda(e_0) = I$. We then say that M is $\mathcal{J}$-coirreducible of *type I*. Then by the dual of the argument given in the beginning of Section 5, no component of S is contained in I. Let $S_1, \ldots, S_t$ be the components of S, $I_j = S_j \cap I$, $j = 1, \ldots, t$. Then $|I_j| < |S_j|$, $j = 1, \ldots, t$. We say that M is $\mathcal{J}$-coirreducible of *maximal type* if $|I_j| = |S_j| - 1$, $j = 1, \ldots, t$. The proof of Theorem 5.1 given in [14] dualizes to yield the following result.

Theorem 6.1 *Let M be a $\mathcal{J}$-coirreducible monoid of type I. Then*

(i) *M is semisimple.*
(ii) *If $e, e' \in \Lambda - \{1\}$, then $e \le e'$ if and only if $\lambda_*(e') \subseteq \lambda_*(e)$.*
(iii) *If $e \in \Lambda - \{1\}$, then $\lambda^*(e) = \{\alpha \in I \,|\, \alpha\beta = \beta\alpha \,\forall \beta \in \lambda_*(e)\}$.*
(iv) *If $K \subseteq S$, then $K = \lambda_*(e)$ for some $e \in \Lambda - \{1\}$ if and only if no component of K is contained in I.*
(v) *If $e \in \Lambda - \{1\}$, then $\lambda_*(e) = crk\,(e) - 1 = |S| - rk\,(e)$. In particular if $e \in \Lambda_{min}$, then $\lambda(e) = \lambda_*(e) = S - \{\alpha\}$ for some $\alpha \in S$.*

We now describe the irreducible components of M_{nil} for a $\mathcal{J}$-coirreducible monoid.

Theorem 6.2 *Let M be a $\mathcal{J}$-coirreducible monoid of type I. Then the distinct irreducible components of M_{nil} are $\overline{X(e_0 x)}$ where x is a Coxeter element of W in D_I^{-1}.*

Proof Let $[ey] \in \mathcal{C}_{nil}$. We first show that $[ey] \le [e_0 x]$ for some $[e_0 x] \in \mathcal{C}_{nil}$. By (14), Lemma 3.4, and Theorem 4.2, we may assume that y is linear. Let $\lambda_*(e) = K$ and let

$$y = y_1 * w_1 * y_2 * w_2 * \cdots * y_m * w_m, \; w_i \in W_K, \; y_i \in W_{S-K}$$

Let $y' = y_1 y_2 \cdots y_m$. We claim that $y' \in D(e)^{-1}$. Otherwise there exists $\alpha \in \lambda(e)$ such that $\alpha y' < y'$. Then $\alpha \in supp\, y' \subseteq S - K$. So $\alpha \in \lambda(e) - \lambda_*(e) = \lambda^*(e)$. Since $\alpha y' < y'$ and y' is linear, we see by Lemma 3.1 that $\alpha y_i < y_i$ for some i and that α commutes with all of y_j for $j < i$. Since $\alpha \in \lambda^*(e), \alpha\beta = \beta\alpha$ for all $\beta \in \lambda_*(e) = K$. So α commutes with all of w_j, $j < i$. It follows that $\alpha y < y$, a contradiction since $y \in D(e)^{-1}$. Thus $y' \in D(e)^{-1}$.

By Theorem 6.1, no component of K is contained in I. So by Lemma 3.3, there is a Coxeter element z of W_K in D_I^{-1}. Let $x = z * y'$. Then x is linear. We claim that $x \in D_I^{-1}$. Let $\alpha \in I$ such that $\alpha x < x$. Since $z \in D_I^{-1}, \alpha z > z$. Since x is linear, we see by Lemma 3.1 that $\alpha y' < y'$ and that α commutes with all of z. Since z is a Coxeter element of W_K, $\alpha\beta = \beta\alpha$ for all $\beta \in K = \lambda_*(e)$. By Theorem 6.1(iii), $\alpha \in \lambda^*(e) \subseteq \lambda(e)$. This is a contradiction since $y' \in D(e)^{-1}$. Hence $x \in D_I^{-1}$. Next let $f \in \Lambda_{min}$. Then by Theorem 6.1(v), $\lambda(f) = \lambda_*(f) = S - \{\beta\}$ for some $\beta \in S$. First suppose $\beta \notin K = \lambda_*(e)$. Then $\lambda_*(e) \subseteq \lambda * (f)$ and by Theorem 6.1(ii), $f \le e$. Since $[ey] \in \mathcal{C}_{nil}$, we see by Theorem 4.2 that $\beta \in supp\, y$. Since $\beta \notin K$, $\beta \in supp\, y' \subseteq supp\, x$. Next

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let $\beta \in K$. Then since z is a Coxeter element of W_K, $\beta \in \text{supp } z \subseteq \text{supp } x$. Thus in all cases $x \notin W(f)$. So by Theorem 4.2, $[e_0 x] \in \mathcal{C}_{nil}$. Also by (13), (14),

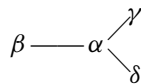$$ey \leq ey' \leq e_0 x$$

So $[ey] \leq [e_0 x]$.

Now let $[e_0 x]$ be a maximal element of $\mathcal{C}_{nil}$ with respect to the order (15). Then by (14), Lemma 3.4, and Theorem 4.2, x is linear. Let $\alpha \in S$ such that no component of $S - \{\alpha\}$ is contained in I. Then by Theorem 6.1, there exists $f \in \Lambda_{min}$ such that $\lambda(f) = S - \{\alpha\}$. By Theorem 4.2, α appears in x. By Lemma 3.6, x is a Coxeter element of W. Conversely if x is a Coxeter element of W in D_I^{-1}, then $[e_0 x] \in \mathcal{C}_{nil}$ by Theorem 4.2. Finally if x, x' are Coxeter elements of W in D_I^{-1} and if $[e_0 x] \leq [e_0 x']$, then $x = x'$ by (21). This completes the proof. □

Example 6.3 Let

$$G_0 = \{A \oplus (A^{-1})^t \mid A \in SL_3(\kappa)\}$$

and let $M = \overline{\kappa G_0}$. See [9, Example 10.4]. Then $S = \{\alpha, \beta\}$ where $\alpha = (1\ 2)$ and $\beta = (2\ 3)$. M is a $\mathcal{J}$-coirreducible monoid of type $\emptyset$. If $\Lambda_{max} = \{e_0\}$, then the irreducible components of M_{nil} are $\overline{X(e_0 \alpha \beta)}$ and $\overline{X(e_0 \beta \alpha)}$.

Example 6.4 Let $S = \{\alpha, \beta, \gamma, \delta\}$ be of type D_4 with β, γ, δ being the endpoints:

$$\beta \mathrel{\rule[0.5ex]{2em}{0.4pt}} \alpha \Big\langle \genfrac{}{}{0pt}{}{\gamma}{\delta}$$

Let $I \subseteq S$ and consider a $\mathcal{J}$-coirreducible M of type I. Let $\Lambda_{max} = \{e_0\}$.

(i) If $I = \emptyset$, then M_{nil} has 8 irreducible components:

$$\overline{X(e_0 \beta \alpha \gamma \delta)}, \ \overline{X(e_0 \gamma \alpha \beta \delta)}, \ \overline{X(e_0 \delta \alpha \beta \gamma)}, \ \overline{X(e_0 \beta \gamma \alpha \delta)},$$

$$\overline{X(e_0 \beta \delta \alpha \gamma)}, \ \overline{X(e_0 \gamma \delta \alpha \beta)}, \ \overline{X(e_0 \beta \gamma \delta \alpha)}, \ \overline{X(e_0 \alpha \beta \gamma \delta)}$$

(ii) If $I = \{\alpha\}$, then M_{nil} has 7 irreducible components:

$$\overline{X(e_0 \beta \alpha \gamma \delta)}, \ \overline{X(e_0 \gamma \alpha \beta \delta)}, \ \overline{X(e_0 \delta \alpha \beta \gamma)}, \ \overline{X(e_0 \beta \gamma \alpha \delta)},$$

$$\overline{X(e_0 \beta \delta \alpha \gamma)}, \ \overline{X(e_0 \gamma \delta \alpha \beta)}, \ \overline{X(e_0 \beta \gamma \delta \alpha)}$$

(iii) If $I = \{\beta\}$, then M_{nil} has 4 irreducible components:

$$\overline{X(e_0 \gamma \alpha \beta \delta)}, \ \overline{X(e_0 \delta \alpha \beta \gamma)}, \ \overline{X(e_0 \gamma \delta \alpha \beta)}, \ \overline{X(e_0 \alpha \beta \gamma \delta)}$$

(iv) If $I = \{\alpha, \beta\}$, then M_{nil} has 3 irreducible components:

$$\overline{X(e_0 \gamma \alpha \beta \delta)}, \ \overline{X(e_0 \delta \alpha \beta \gamma)}, \ \overline{X(e_0 \gamma \delta \alpha \beta)}$$

(v) If $I = \{\beta, \gamma\}$, then M_{nil} has 2 irreducible components:

$$\overline{X(e_0 \delta \alpha \beta \gamma)}, \ \overline{X(e_0 \alpha \beta \gamma \delta)}$$

  (vi) If $I = \{\alpha, \beta, \gamma\}$, then $M_{nil} = \overline{X(e_0\delta\alpha\beta\gamma)}$ is irreducible.
  (vii) If $I = \{\beta, \gamma, \delta\}$, then $M_{nil} = \overline{X(e_0\alpha\beta\gamma\delta)}$ is irreducible.

Example 6.5 Let M be a $\mathcal{J}$-coirreducible monoid of type $\emptyset$. Such a monoid is called a *dual canonical monoid*. Let $\Lambda_{max} = \{e_0\}$. Then the irreducible components of M_{nil} are $\overline{X(e_\alpha c)}$, where c is a Coxeter element of W.

    Finally we give necessary and sufficient conditions for M_{nil} to be irreducible when M is a semisimple monoid,

Theorem 6.6 *Let M be a semisimple monoid. Then M_{nil} is irreducible if and only if M is a $\mathcal{J}$-coirreducible monoid of maximal type.*

Proof By Theorems 4.3 and 4.4, we may assume that M is $\mathcal{J}$-coirreducible of type I. Let $\Lambda_{max} = \{e_0\}$. Let $S_1, \ldots, S_t$ be the irreducible components of S, $I_j = I \cap S_j$, $j = 1, \ldots, t$. Assume first that M is of maximal type and $I_j = S - \{\alpha_j\}$, $j = 1, \ldots, t$. Then by Lemma 3.2, $y = c_{S_1}(\alpha_1) \cdots c_{S_t}(\alpha_t)$ is the unique Coxeter element of W in D_I^{-1}. By Theorem 6.2, $M_{nil} = \overline{X(e_0y)}$ is irreducible.

    Assume conversely that M is not of maximal type. Let $\alpha_i \in S_i - I_i$, $i = 1, \ldots, t$. Then for some j, there exists $\alpha_j' \in S_j - I_j$, $\alpha_j \neq \alpha_j'$. By Lemma 3.2, $z_i = c_{S_i}(\alpha_i)$, $i = 1, \ldots, t$ and $z_j' = c_{S_j}(\alpha_j')$ are in D_I^{-1}. Moreover

$$\alpha_j z_j < z_j, \ \alpha_j z_j' > z_j'$$

Hence $z_j \neq z_j'$. Thus

$$z = z_1 \cdots z_j \cdots z_t, \ z' = z_1 \cdots z_j' \cdots z_t$$

are Coxeter elements of W in D_I^{-1}, $z \neq z'$. By Theorem 6.2, M_{nil} has at least two irreducible components, This completes the proof. $\square$

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