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Nested set complexes of Dowling lattices and complexes of Dowling trees

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Abstract Given a finite group G and a natural number n, we study the structure of the complex of nested sets of the associated Dowling lattice $\mathcal{Q}_n(G)$ (Proc. Internat. Sympos., 1971, pp. 101–115) and of its subposet of the G-symmetric partitions $\mathcal{Q}_n^0(G)$ which was recently introduced by Hultman (http://www.math.kth.se/~hultman/, 2006), together with the complex of G-symmetric phylogenetic trees $\mathcal{T}_n^G$. Hultman shows that the complexes $\mathcal{T}_n^G$ and $\widetilde{\Delta}(\mathcal{Q}_n^0(G))$ are homotopy equivalent and Cohen–Macaulay, and determines the rank of their top homology.

An application of the theory of building sets and nested set complexes by Feichtner and Kozlov (*Selecta Math. (N.S.)* **10**, 37–60, 2004) shows that in fact $\mathcal{T}_n^G$ is subdivided by the order complex of $\mathcal{Q}_n^0(G)$. We introduce the complex of Dowling trees $\mathcal{T}_n(G)$ and prove that it is subdivided by the order complex of $\mathcal{Q}_n(G)$. Application of a theorem of Feichtner and Sturmfels (*Port. Math. (N.S.)* **62**, 437–468, 2005) shows that, as a simplicial complex, $\mathcal{T}_n(G)$ is in fact isomorphic to the Bergman complex of the associated Dowling geometry.

Topologically, we prove that $\mathcal{T}_n(G)$ is obtained from $\mathcal{T}_n^G$ by successive coning over certain subcomplexes. It is well known that $\mathcal{Q}_n(G)$ is shellable, and of the same dimension as $\mathcal{T}_n^G$. We explicitly and independently calculate how many homology spheres are added in passing from $\mathcal{T}_n^G$ to $\mathcal{T}_n(G)$. Comparison with work of Gottlieb and Wachs (*Adv. Appl. Math.* **24**(4), 301–336, 2000) shows that $\mathcal{T}_n(G)$ is intimely related to the representation theory of the top homology of $\mathcal{Q}_n(G)$.

Keywords Posets · Lattices · Combinatorial blowups · Building sets · Nested sets · Dowling lattices · Complexes of trees · Phylogenetic trees

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Introduction

Dowling lattices are named after T.A. Dowling, who first studied a particular class of arrangements of hyperplanes whose intersection lattices can be obtained by enriching the partition lattice with sets of elements of the cyclic groups $\mathbb{Z}_n$ [8]. In a separate work [9], Dowling took a step to a general combinatorial point of view, and thoroughly studied what he called 'posets based on finite groups.' We wish to point to [9] as a very readable and comprehensive introduction to the subject.

Both approaches to these structures were followed since the work of Dowling. Let us mention, as two examples, the work of Ehrenborg and Readdy [10, 11], who introduced the notion of *Dowling transform* of an arrangement of hyperplanes and showed that this transformation preserves supersolvability, and the study of a combinatorial generalization of Dowling lattices by Hanlon [20].

The theory of building sets and nested set complexes was initiated and developed by Feichtner and Kozlov in [14] as the combinatorial framework of the De Concini–Procesi models for hyperplane arrangements. A complex of *nested sets* is associated to any meet-semilattice and any of its *building sets*—i.e., subsets of the semilattice satisfying some conditions that are inspired by the special properties of the set of irreducible elements in a geometric lattice.

The study of such structures was carried out in different contexts (see [6, 7, 12, 13, 15–17]), leading to new results or sharpening the understanding of previously studied objects. In particular, this theory has proved to be a very useful tool for the study of *complexes of trees* of various kind (see [7, 12]).

The study of abstract simplicial complexes whose cells are indexed by combinatorial types of rooted trees on a fixed number of leaves was recently brought to a broad attention by the work of Billera, Holmes and Vogtmann [3]. They considered the space of all possible phylogenetic trees of a certain set of biological species, with lengths on edges representing the genetic distance of two mutations. This space is a cone with apex the unique tree with all edge lengths equal to zero. The base of this cone (the link of the apex) is obtained by considering the trees with unit edge lengths (i.e., cutting the cone by a transversal hyperplane). This space has a natural stratification, with each cell corresponding to a combinatorial type of trees—this is the *complex of phylogenetic trees.*

Complexes of trees were studied for different purposes, before and after [3], see [5, 19, 21, 23, 25, 26]. In particular, in [23] Robinson and Whitehouse determined the homotopy type of the complex of phylogenetic trees on n leaves to be a wedge of $(n-1)!$ spheres of dimension $n-3$. Later on, shellability of these complexes was proved by Trappmann and Ziegler [25] and, independently, by Wachs (unpublished, acknowledged in [12]). Ardila and Klivans [2] proved that the complex of trees can be subdivided by the order complex of the partition lattice. One of the recent applications of the theory of nested set complexes sharpened this last result: in [12], Feichtner showed that the order complex of the partition lattice is obtained from the complex of trees by a sequence of stellar subdivisions. This property was proven to hold even in the more general class of k-trees (see [7]).

This strong topological and combinatorial connection between the order complex of different kinds of partition posets and the corresponding complexes of trees sharpens the homotopy equivalences proved in [19, 21, 23].

In all these cases, the motivation for introducing and studying variations of the complex of trees came from the connection of their top cohomology groups with the homogeneous component of certain associated free Lie algebras, both viewed as modules over the symmetric group (see [21, 23]).

Our purpose is to study the nested set complexes of Dowling lattices and look for a corresponding complex of trees (in combinatorial, topological and algebraic sense). The initial input came from the work of Hultman [22], who defined the *complex of G-symmetric phylogenetic trees* and showed that it has the same homotopy type as a particular subposet of the Dowling lattice. We sharpen this homotopy equivalence by showing that it is in fact a homeomorphism: the two complexes are related by a sequence of stellar subdivisions (Theorem 3.4). Moreover, we put these objects into the classical theory of Dowling lattices by introducing a combinatorially and topologically suitable notion of Dowling trees (Definition 4.3, Corollary 4.5) and by studying their topological relationship with Hultman's complexes (Theorem 4.11, Remark 4.12).

After a first version of this paper was circulating, Federico Ardila [1] pointed out that Gottlieb and Wachs defined a certain complex of trees associated to Dowling lattices [18]. Their aim was to encode the generators of the top homology group of the Dowling lattice, thereby relating this group to certain Lie superalgebras in a way that generalize the results of [21, 23] on philogenetic trees and k-trees (for the precise statement see Remark 4.7 or [18]). It turns out that our complex of Dowling trees is isomorphic to Gottlieb and Wachs' complex as a simplicial complex (see Remark 4.7). Thus, our Corollary 4.5 sharpens the homotopy equivalence of [18] in the same way as [12] and [7] do for [23] and [21].

Moreover, Ardila [1] pointed out that, in unpublished joint work with Caroline Klivans, he proved that the complex of trees of [18] equals the Bergman complex of the associated Dowling geometry (i.e., the matroid whose lattice of flats is the corresponding Dowling lattice). Using our description in terms of nested sets we can use a theorem by Feichtner and Sturmfels [16] to prove that the Bergman complex of a Dowling geometry equals the nested set complex of the minimal building set of the associated Dowling lattice. This implies Ardila and Klivans' result and shows that the order complex of the Dowling lattice can be obtained from the corresponding Bergman complex by a sequence of stellar subdivisions.

This paper is organized as follows: in Sects. 1 and 2 we give a detailed picture of Dowling lattices and G-symmetric phylogenetic trees, reviewing the basics and developing a notation that will prove to be appropriate for a direct application of the theory of nested set complexes. This theory will enter the picture in Sect. 3, where the result of Hultman is sharpened by showing that the complex of G-symmetric phylogenetic trees actually is subdivided by the associated subposet of the Dowling lattice. In Sect. 4 the notion of a *Dowling tree* is introduced as naturally associated to the nested set complex of the full Dowling lattice, thus being probably the appropriate Dowling generalization of the complex of phylogenetic trees. We support our definition by describing the connection with Gottlieb and Wachs' complex of trees and the relation with the Bergman complex of the corresponding Dowling geometry. The complex of Dowling trees contains the complex of G-symmetric phylogenetic trees as a simplicial subcomplex. In the last section we describe explicitly how the bigger complex can be obtained from the smaller one by successively coning over certain subcomplexes. By keeping under control the topology of those subcomplexes, we calculate how many homology spheres arise at each step, thus explicitly relating the "numerologically" suggestive expressions of the top homology ranks of the two complexes.

1 Posets based on finite groups

Before starting out, let us fix some general notations. In this paper we will deal with finite partially ordered sets, briefly called *posets*. The main topological structure associated to a poset P is its *order complex* $\Delta(P)$, the (abstract) simplicial complex of the totally ordered subsets of P. We will almost never distinguish between an abstract simplicial complex and its geometric realization, thus speaking of 'topological properties' of an abstract simplicial complex. It is easily seen that if P possesses a maximal element (that is customarily denoted $\hat{1}$), then $\Delta(P)$ is a cone over $\Delta(P \setminus \{\hat{1}\})$. An analogous statement holds of course if P has a minimal element (usually denoted $\hat{0}$). To capture the 'essential' topological information we define the *reduced order complex* $\widetilde{\Delta}(P)$ as the order complex of the poset obtained from P by removing the maximal and the minimal element, if P has any.

In considering partitions, we will switch between the set-theoretic notation $\sigma = S_0 \amalg S_1 \amalg \cdots \amalg S_k$ and the (more customary) 'block notation' $\sigma = S_0|\cdots|S_k$. Sets of partitions can be ordered *by refinement*, i.e., setting $\sigma' < \sigma$ if every block of σ' is contained in a block of σ.

1.1 Dowling lattices

Definition 1.1 Let G be a finite group and n a natural number. Consider the action of G on the set $\{0\} \cup ([n] \times G)$ defined by $g((i,h)) = (i, gh)$ and $g(0) = 0$. The partitions of $\{0\} \cup ([n] \times G)$ such that this action induces an action on the blocks are called *G-symmetric partitions*. A block of a G-symmetric partition is called simple if its orbit under this action has length $|G|$.

We call $\mathcal{Q}_n(G)$ the set of G-symmetric partitions such that the only non-simple block is the block containing 0. The ordering by refinement turns it into a lattice, called the *Dowling lattice*.

The 'forgetful' map $\{0\} \cup ([n] \times G) \to \{0\} \cup [n]$ defined by $(i, g) \mapsto i$ and $0 \mapsto 0$ induces a mapping of $\mathcal{Q}_n(G)$ onto the poset $\Pi_{n,0}$ of partitions of the set $\{0\} \cup [n]$. This map sends $\omega \in \mathcal{Q}_n(G)$ to its *associated partition* $\underline{\omega} \in \Pi_{n,0} \cong \Pi_{n+1}$. An element of $\Pi_{n,0}$ will be written as $\alpha := A_0|A_1|\dots|A_k$, where we agree to choose the indexing such that 0 is always contained in the block indexed by 0. Of course, $\alpha = \underline{\omega}$ has one block A_i for every orbit of blocks in ω, and the A_i with $i > 1$ correspond to the simple blocks.

We now see that we can encode in a unique way any $\omega \in \mathcal{Q}_n(G)$ in the following data:

- A partition $\alpha \in \Pi_{n,0}$, called the *associated partition* of ω.
- For every block $A_j = \{i_1, \dots, i_k\}$ of α with $j > 0$ an $|A_j|$-tuple $A_j^G := (id, g_{i_2}^{\alpha}, g_{i_3}^{\alpha}, \dots, g_{i_k}^{\alpha}) \subset G^k$.

This gives a good encoding of G-symmetric partitions if we agree to always order the elements in the blocks A_j in increasing order: $i_1 < i_2 < \cdots < i_k$.

The order relation of the poset translates to the following: $\omega_1 < \omega_2$ if and only if

- the associated partitions satisfy $\underline{\omega}_1 < \underline{\omega}_2$
- for every block B of $\underline{\omega}_2$, if $C_1, \ldots, C_k$ are the blocks of $\underline{\omega}_1$ that subdivide it, there are elements $h_1, \ldots, h_k$ in G such that for every $j \in B \cap C_i$ we have $g_j^{\omega_2} = h_i g_j^{\omega_1}$.

Dowling gave a complete and very readable survey on these objects in his seminal paper [9], to which we point as a general reference. Although the definition we gave is inspired by the setting of [22], it can be easily seen to be equivalent to the definition of Dowling, e.g. by thinking in terms of the encoding we presented above.

The main statement on the topology of these lattices that can be deduced from [9] is summarized in the following proposition.

Proposition 1.1 *The reduced order complex* $\widetilde{\Delta}(\mathcal{Q}_n(G))$ *is homotopy equivalent to a wedge of* $(|G|+1)(2|G|+1)\cdots((n-1)|G|+1)$ *spheres of dimension* $(n-2)$.

Proof The proof is a concatenation of arguments from [9] (where $\mathcal{Q}_n(G)$ is shown to be supersolvable), [4] (where it is proved that supersolvable lattices are shellable), and [24] (for the enumeration of the number of homology spheres). Details are left to the reader. □

1.2 The subposet $\mathcal{Q}_n^0(G)$

In [22], the author introduces a subposet of $\mathcal{Q}_n(G)$ that is of particular interest in connection with G-symmetric phylogenetic trees.

Definition 1.2 We define $\mathcal{Q}_n^0(G)$ to be the subposet of $\mathcal{Q}_n(G)$ consisting of the partitions with trivial zero block.

Remark 1.3 Our definition is slightly different from the one given in [22], because we allow the minimal element of $\mathcal{Q}_n(G)$ to be in $\mathcal{Q}_n^0(G)$.

In particular, given $\sigma \in \mathcal{Q}_n^0(G)$, every element of $[n] \times G$ is contained in a simple block of σ.

In general, $\mathcal{Q}_n^0(G)$ is not a lattice. Indeed, consider the following two elements of $\mathcal{Q}_3^0(\mathbb{Z}_2)$:

$$\sigma_1 := \{0\}\{(1,0),(2,0)\}\{(1,1)(2,1)\}\{(3,0)\}\{(3,1)\},$$
$$\sigma_2 := \{0\}\{(1,0),(2,1)\}\{(1,1)(2,0)\}\{(3,0)\}\{(3,1)\}.$$

Their join is not contained in $\mathcal{Q}_3^{\mathbb{Z}_2}$, where there is no element that is bigger than both σ_1 and σ_2.

Nevertheless, $\mathcal{Q}_n^0(G)$ is a meet-semilattice. We prove the following easy lemma.

Lemma 1.4 *For any* $\sigma \in \mathcal{Q}_n^0(G)$, *we have an isomorphism*

$$(\mathcal{Q}_n^0(G))_{\leq\sigma} \simeq (\Pi_n)_{\leq\underline{\sigma}}.$$

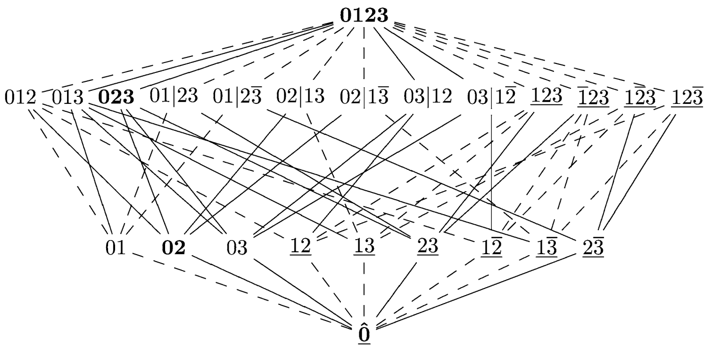

Fig. 1 The lattice $\mathcal{Q}_3(\mathbb{Z}_2)$ and its subposet $\mathcal{Q}_3^0(\mathbb{Z}_2)$ (underlined). The bold elements give a modular chain, the dashed chains are the 15 homology chains of the corresponding shelling

Proof The map defined by $\sigma \mapsto \underline{\sigma}$ is clearly a poset morphism, and obviously surjective. For injectivity note that, given $\underline{\sigma}' \leq \underline{\sigma}$, any set of representatives of the orbit blocks for σ forces the choice of the ℓ-tuples associated to the blocks to $\underline{\sigma}'$. Indeed, if B is a block of $\underline{\sigma}$ that is associated to $(id, g_2^\sigma, \dots, g_\ell^\sigma)$, then if $C = \{i_1, \dots, i_k\} \subseteq B$ is a block of $\underline{\sigma}'$, the only possibility for a partition associated to $\underline{\sigma}'$ to be below σ is to associate to C the k-tuple $(id, g_{i_2}^\sigma (g_{i_1}^\sigma)^{-1}, \dots, g_{i_k}^\sigma (g_{i_1}^\sigma)^{-1})$. $\square$

This makes the work of Sect. 3 possible, where the theory of building sets and nested set complexes will be applied to the posets $\mathcal{Q}_n^0(G)$.

As an example, we depict in Fig. 1 the Dowling lattice $\mathcal{Q}_3(\mathbb{Z}_2)$, where we write only one representative for every nonsingleton block. The numbers refer to the associated partitions, and an overline over an element indicates that this number is associated with the nonidentity element of $\mathbb{Z}_2$. Thus we will write $01|2\overline{3}$ instead of $\{0, (1,0), (1,1)\}\{(2,0), (3,1)\}\{(2,1), (3,0)\}$. The elements of $\mathcal{Q}_3^0(\mathbb{Z}_2)$ are underlined.

The homotopy type of $\widetilde{\Delta}(\mathcal{Q}_n^0(G))$ was determined in [22] by comparison with the complex of G-symmetric phylogenetic trees (see Sect. 2), as an application of discrete Morse theory.

2 G-Symmetric phylogenetic trees

Recall that we fixed once and for all a natural number n. In this context, given a finite group G, a G-tree is a rooted tree whose leaves are in bijection with the set $[n] \times G$. The group G acts on the set of leaves by means of the 'standard' action described in the previous section.

We now specify a class of G-trees that will be the object of our study.

Definition 2.1 A *G-symmetric phylogenetic tree* is a G-tree satisfying the following additional conditions:

(1) Every internal vertex (except the root) has degree at least 3.
(2) The tree is invariant under the G-action.

(3) For any two different elements $g, h \in G$ and any $i \in [n]$, the (unique) shortest path connecting the leaves labelled (i, g) and (i, h) passes through the root.

The set of G-symmetric phylogenetic trees is denoted by $\mathcal{T}_n^G$.

This is Definition 3.1 of [22], where some properties of those trees are listed. Here we need only recall that every internal edge t of a tree $T \in \mathcal{T}_n^G$ generates an orbit $\mathfrak{o}(t)$ (called *inner orbit*) of cardinality $|G|$ under the action of G. Following [22], one can associate to $\mathfrak{o}(t)$ the partition $\pi(\mathfrak{o}(t))$ of $[n] \times G$ obtained by putting in the same block all labels of leaves that are in the same connected component of T after removing all the edges in $\mathfrak{o}(t)$, and adding 0 to the block corresponding to the component containing the root. We want to slightly modify this definition.

2.1 Some notation

Note that in all partitions of $\{0\} \cup [n]$ that are associated to some $\sigma \in \mathcal{Q}_n^0(G)$ the element 0 is alone in its block. Therefore we may reconstruct every such partition from the corresponding partition of $[n]$ by just adding the block $\{0\}$. We then agree that, for $\sigma \in \mathcal{Q}_n^0(G)$, in this section we will let $\underline{\sigma} \in \Pi_n$.

Now consider a tree T satisfying Definition 2.1. For any vertex v of T let λ_v denote the set of leaf labels such that the path connecting them to the root traverses v. Let

$$\lambda_{Gv} := \bigcup_{g \in G} \lambda_{gv}$$

denote the set of labels of leaves that are separated from the root by a vertex of the form gv for some $g \in G$.

Then, given an inner edge t of T, let $\lambda_t := \lambda_v$ where v is the vertex of t that is further from the root. For every internal edge t we can then define a partition $\sigma(t) \in \mathcal{Q}_n^0(G)$ as

$$\coprod_{(i,g) \notin \lambda_{Gt}} \{(i, g)\} \coprod_{g \in G} \lambda_{gt}.$$

It is clear that $\lambda_{gt} = \{(i, gh) | (i, h) \in \lambda_t\} = g\lambda_t$, and therefore we see that $\sigma(t)$ has only one nonsingleton orbit. Thus the associated partition $\underline{\sigma}(t) \in \Pi_n$ has only one nonsingleton block.

The next two subsections present some material of [22] in a language and from a viewpoint that are well-suited to our methods.

2.2 Inner orbit contraction

The contraction of all edges in an inner orbit $\mathfrak{o}(t)$ turns a G-symmetric tree T into another tree $T' \in \mathcal{T}_n^G$. We can then define the following partial order on $\mathcal{T}_n^G$.

Definition 2.2 Given $T, T' \in \mathcal{T}_n^G$, define $T \le T'$ if and only if T' can be obtained from T by a sequence of inner orbit contractions.

The importance of this ordering is shown in the following proposition.

Proposition 2.1 [22, Corollary 3.5] *With the partial ordering of Definition 2.2, $\mathcal{T}_n^G$ is the face poset of a pure simplicial complex of dimension $n-2$.*

Before turning our attention to the operation of inner orbit *extension*, which is inverse to the contraction defined above, let us see what kind of relation one can draw between partitions associated to different contractions on the same tree.

So suppose again a tree $T \in \mathcal{T}_n^G$ be given, and consider two inner edges t, t' of T that are not in the same G-orbit. We have seen that the associated partition $\underline{\sigma}(t)$ (respectively $\underline{\sigma}(t')$) has exactly one nonsingleton block, say B (resp. B'), associated to the unique nonsingleton orbit of $\sigma(t)$ ($\sigma(t')$), of which we consider a representative block S (S'). Suppose we first contract the edges in the orbit $\mathfrak{o}(t)$.

If $S' \subseteq hS$ for some $h \in G$, then by definition $gS' \subseteq g(hS)$ for all $g \in G$. We conclude that in this case $\sigma(t) > \sigma(t')$. Therefore, the unique nonsingleton block B' of the partition $\underline{\sigma}(t')$ is contained in B, so $\underline{\sigma}(t') < \underline{\sigma}(t)$.

If for some $h \in G$ the reverse inclusion $S' \supset hS$ holds, then of course the conclusion above holds with t and t' switched.

The fact that T is a tree excludes the possibility that, if neither of the previous cases enters, $S' \cap hS \neq \emptyset$ for some $h \in G$.

We summarize the conclusion for later reference.

Remark 2.3 If t, t' are two inner edges of a G-symmetric tree T such that $\mathfrak{o}(t) \neq \mathfrak{o}(t')$ and if $\sigma(t)$ and $\sigma(t')$ are incomparable, then the sets λ_{Gt} and $\lambda_{Gt'}$ are disjoint. In particular the associated nonsingleton blocks B, B' of $\underline{\sigma}(t)$ and $\underline{\sigma}(t')$ are either contained in one another, or are disjoint.

In particular, we can associate to every tree $T \in \mathcal{T}_n^G$ a subset $N(T) \subset \mathcal{Q}_n^0(G)$ defined as

$$N(T) := \big\{\sigma(t) \mid t \text{ is (a representative of the orbit of) an inner edge of } T\big\}.$$

$N(T)$ has the property that the unique nonsingleton block orbits of any incomparable $\sigma, \sigma' \in N(T)$ are disjoint.

2.3 Inner orbit extension

We now discuss the inverse of the above operation: *inner orbit extension*.

For this, we suppose a tree $T \in \mathcal{T}_n^G$ to be given together with a partition $\sigma \in \mathcal{Q}_n^0(G)$ that has exactly one nonsingleton orbit $\mathfrak{o}$ (of which we consider a representative block S). The preceding observations suggest to require the following condition to be satisfied by σ:

(∗) *For any inner edge t of T, if neither $S \subset \lambda_t$ nor $S \supset \lambda_t$ then $S \cap \lambda_t = \emptyset$.*

In the following we will show how these data give rise to a tree $T' \in \mathcal{T}_n^G$ such that T is obtained from T' by an inner orbit contraction that is represented by σ.

First of all it is clear that there is a unique vertex v of T such that the component that is separated from the root by removing v is minimal with the property of containing all leaves labelled by elements of S. The family of sets $\{\lambda_{gv} \setminus gS | g \in G\}$ can be obtained from the representative $\lambda_v \setminus S$ by the action of G.

Because of property $(*)$, we may partition the edges f incident to v and with $\lambda_f \subset \lambda_v$ into the two following classes:

$$F := \{f|\lambda_f \subset S\}, \qquad E := \{f|\lambda_f \cap S = \emptyset\}.$$

All is now prepared for the extension. We first delete all edges $f \in E \cup F$, and for each of them we get a connected component T_f not containing the root.

We then grow an edge t below v appending all edges in E (and their whole components) to v, and the edges of F (with their connected components) to the other vertex of t.

In the tree that we have now constructed we clearly have $\lambda_t = S$. Of course we may repeat the whole process inserting edges gt below gv for any $g \in G$, eventually reaching a tree T' satisfying the requirements.

The tree T could have been reached by a sequence of inner orbit extensions. If we consider the set N of all partitions that correspond to an edge orbit of T we indeed may reconstruct T starting from the unique tree without inner edges by recursively performing the above process on all elements of N. We have seen that the unique condition enabling to perform such an extension is given at each step by $(*)$. We summarize our considerations with the following two statements.

Condition N: *Given a subset $X \subset \mathcal{Q}_n^0(G)$ of partitions that have only one orbit consisting of non-singleton blocks, we say that X* satisfies condition N *if for any two incomparable $\sigma, \sigma' \in X$ the only non-singleton blocks of $\underline{\sigma}, \underline{\sigma'}$ are disjoint.*

Remark 2.4 To any set N of one-nonsingleton-orbit partitions from $\mathcal{Q}_n^0(G)$ satisfying condition N we can naturally associate a tree T, and this is such that, with the notation of Remark 2.3, $N = N(T)$.

3 Homeomorphism through subdivisions

The reader familiar with the subject will have already noticed that in the previous section all the material has been prepared for a direct application of the theory of building sets and nested set complexes. This theory was first developed by Feichtner and Kozlov in [14] as the combinatorial framework of the De Concini–Procesi models for hyperplane arrangements. We refer to that paper for a thorough introduction to this subject. Here we recall only the main definitions.

Definition 3.1 Let $\mathcal{L}$ be a meet-semilattice. A *building set* of $\mathcal{L}$ is a subset $\mathcal{G} \subseteq \mathcal{L} \setminus \{\hat{0}\}$ such that for any $x \in \mathcal{L} \setminus \hat{0}$ there is an isomorphism

$$\varphi_x : \prod_{z \in \max \mathcal{G}_{\leq x}} [\hat{0}, z] \to [\hat{0}, x]$$

with $\varphi_x(0, \ldots, 0, z, 0, \ldots, 0) = z$ for $z \in \max \mathcal{G}_{\leq x}$.

We call a set $N \subseteq \mathcal{G}$ *nested* ($\mathcal{G}$-nested, if specification is needed) if, for any set $\{x_1, \ldots, x_\ell\} \subseteq N$ ($\ell \geq 2$) of incomparable elements, the join $x_1 \vee \ldots \vee x_\ell$ exists and is not an element of $\mathcal{G}$. The *nested set complex* of $\mathcal{L}$ with respect to $\mathcal{G}$, denoted $\mathcal{N}(\mathcal{L}, \mathcal{G})$, is the abstract simplicial complex of all nonempty $\mathcal{G}$-nested sets. If $\mathcal{L}$ has a maximal element $\hat{1}$ and $\mathcal{G}$ contains it, then the nested set complex is a cone with apex $\{\hat{1}\}$. The base of this cone is the *reduced nested set complex* $\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{G})$.

One of the main topological features of this theory is the following theorem which first appeared in [15] in a version for atomic lattices. It was then extended to its full generality in [6, 7], and to these papers we refer for a careful topological treatment of the concept of *stellar subdivision* of an abstract simplicial complex. Here we only mention that the geometric realizations of two abstract simplicial complexes that are related by subdivisions are homeomorphic (see [7, Definition 2.1]).

Theorem 3.2 *Consider two building sets $\mathcal{G}_1, \mathcal{G}_2$ in a meet-semilattice $\mathcal{L}$. If $\mathcal{G}_1 \subset \mathcal{G}_2$, then the simplicial complex $\mathcal{N}(\mathcal{L},\mathcal{G}_2)$ can be obtained from $\mathcal{N}(\mathcal{L},\mathcal{G}_1)$ by a sequence of stellar subdivisions.*

Note that, for any semilattice, there is a unique minimal building set. It is given by the set of all elements x such that the interval $[\hat{0}, x]$ cannot be decomposed in a product of smaller principal order ideals. For example, in the partition lattice Π_n those elements are the partitions with only one nonsingleton block. The minimal building set of Π_n will be denoted by $\mathcal{I}$.

On the other hand, for any meet-semilattice the maximal building set is the whole poset, and the associated reduced nested set complex is then $\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{L}) = \widetilde{\Delta}(\mathcal{L})$. This proves the following corollary.

Corollary 3.3 *Let $\mathcal{L}$ be a meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. Then $\widetilde{\Delta}(\mathcal{L})$ can be obtained from $\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{G})$ by a sequence of stellar subdivisions.*

In analogy with the partition poset let us define a subset $\mathcal{I}^G \subset \mathcal{Q}_n^0(G)$ as follows:

$$\mathcal{I}^G := \left\{\sigma \in \mathcal{Q}_n^0(G) \mid \underline{\sigma} \in \mathcal{I}\right\}.$$

The following proposition shows that this is indeed "the right definition".

Proposition 3.1 *$\mathcal{I}^G$ is the minimal building set of $\mathcal{Q}_n^0(G)$.*

Proof With Lemma 1.4 the claim follows immediately by comparison with Π_n. □

Proposition 3.2 *The complexes $\mathcal{T}_n^G$ and $\mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^0(G))$ are isomorphic.*

Proof The condition of being nested in $\mathcal{I}^G$ is equivalent to condition N of the previous section. □

We are ready to state the main result of this section, which is now an easy application of Corollary 3.3.

Theorem 3.4 *The order complex $\widetilde{\Delta}(\mathcal{Q}_n^0(G))$ is obtained from the complex of G-symmetric trees $\mathcal{T}_n^G$ by a sequence of stellar subdivisions.*

Hultman calculated the homotopy type of $\mathcal{T}_n^G$ in [22]. We include this result in the following corollary, that is intended to summarize our topological knowledge about G-symmetric partitions and G-symmetric phylogenetic trees.

Corollary 3.5 *The simplicial complexes $\widetilde{\Delta}(\mathcal{Q}_n^0(G))$ and $\mathcal{T}_n^G$ are PL-homeomorphic. They are homotopy equivalent to a wedge of*

$$(|G|-1)(2|G|-1)\cdots((n-1)|G|-1)$$

spheres of dimension $(n-2)$.

4 Dowling trees

The natural task at this point is to study the nested set complexes of the full Dowling lattice $\mathcal{Q}_n(G)$.

4.1 Nested set complexes in $\mathcal{Q}_n(G)$

We want to determine the minimal building set $\mathcal{J}^G$ of $\mathcal{Q}_n(G)$. By an easy check (or by comparing to Theorem 2 of [9]) one sees that, given any $\omega \in \mathcal{Q}_n(G)$ with zero block S_0 and fixed chosen orbit representatives S_i, $i = 1, \dots, k$, there is a natural isomorphism

$$(\mathcal{Q}_n(G))_{\le\omega} \xrightarrow{\sim} \Pi^G_{m(\omega)} \times \Pi_{|S_1|} \times \cdots \times \Pi_{|S_k|}$$

where $m(\omega) := \frac{(|S_0|-1)}{|G|}$.

With this decomposition, we see that any element of $\mathcal{J}^G$ having a nonzero block that is not a singleton must have $S_0 = \{0\}$, thus be an element of $\mathcal{I}^G$. Moreover, $\mathcal{J}^G \setminus \mathcal{I}^G$ consists of the partitions where all simple blocks are singletons.

We will distinguish these two types of elements in $\mathcal{J}^G$ by calling any $x \in \mathcal{J}^G \cap \mathcal{I}^G$ *of type* 1, while we will refer to the elements in $\mathcal{J}^G \setminus \mathcal{I}^G$ as to those *of type* 0.

Remark 4.1 A subset X of $\mathcal{J}^G$ is nested if and only if, for any $\omega, \omega' \in X$, the only nonsingleton blocks of the associated partitions $\underline{\omega}, \underline{\omega}'$ of $\{0\} \cup [n]$ are either disjoint or contained in one another.

The following facts are now at hand, and we collect them for later reference.

Lemma 4.2 *Let $\mathcal{J}^G$ denote the minimal building set of $\mathcal{Q}_n(G)$.*

(1) $\mathcal{J}^G \cap \mathcal{Q}_n^0(G) = \mathcal{I}^G$.
(2) $\mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^0(G)) \subseteq \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$.
(3) *For any $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$, $X \cap \mathcal{I}^G \in \mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^0(G))$.*
(4) *If $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$, the subset $X \setminus \mathcal{I}^G$ given by the elements of type 0 is linearly ordered.*

4.2 Dowling trees

The last question we want to address is whether the nested set complex of the full Dowling lattice has an interpretation in terms of trees. The answer is positive, and leads to the definition of what we would like to call *the complex of Dowling trees*.

Definition 4.3 Given a natural number n and a finite group G, a Dowling tree is a G-tree T with some distinguished vertices, called *zero vertices*, satisfying the following conditions:

(0) The root is a zero vertex.
(1) Every internal vertex (except the root) has degree at least 3.
(2) The tree is invariant under the G-action, and the zero vertices are fixed by this action.
(3) For any two different elements $g, h \in G$ and any $i \in [n]$, the (unique) shortest path connecting the leaves labelled (i, g) and (i, h) passes through exactly one zero vertex.
(4) The zero vertices are the vertices of a path beginning at the root.

On Dowling trees the operation of inner orbit contraction and extension are defined analogously as in $\mathcal{T}_n^G$ with the only difference that for every edge t that connects two zero vertices we have $\mathfrak{o}(t) = \{t\}$. Therefore the Dowling trees form an abstract simplicial complex that we will denote by $\mathcal{T}_n(G)$.

We state the theorem relating Dowling trees and Nested set complexes of Dowling lattices. The way of encoding trees with nested sets is the same as in [12].

Theorem 4.4 *$\mathcal{T}_n(G)$ is isomorphic to $\widetilde{\mathcal{N}}(\mathcal{J}^G, \mathcal{Q}_n(G))$ as an abstract simplicial complex.*

Proof Consider a simplex $X \in \widetilde{\mathcal{N}}(\mathcal{J}^G, \mathcal{Q}_n(G))$. Since $\mathcal{Q}_n^0(G)$ is an order ideal in $\mathcal{Q}_n(G)$, we may choose a linear extension of the ordering in X such that all elements of type 1 come before all those of type 0. We will perform our inner orbit extensions according to the chosen linear order of X. After having exhausted all elements of type 1 we are clearly left with a tree $T \in \mathcal{T}_n^G$, that can be turned into a good Dowling

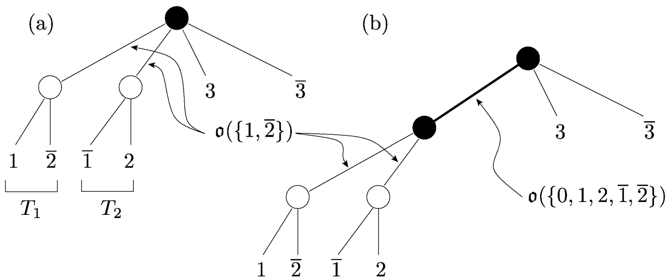

Fig. 2 **a** The tree corresponding to $\{1, \overline{2}\}$. **b** The Dowling tree constructed from the nested set $\{1\overline{2}, 012\}$. The zero vertices are *black*. As above, only a representative of every nonsingleton orbit is indicated. See Fig. 1

tree just by declaring the root as the only zero vertex. On this tree we now have to perform the 'type 0'-orbit extensions.

So let ω be a type-0 partition, with zero block S_0. As above, there is a vertex v of T such that the union of the leaves in the connected components $T_1, \dots, T_s$ of T not containing the root that arise by deleting v is minimal with the property of containing the set $S_0 \backslash \{0\}$. In particular, this v is fixed by the action of G and therefore is a zero vertex; by 4.2(4) we know that the type 0 elements of X that correspond to already performed extensions lie on a chain below ω, so that their zero blocks are all contained in S_0. Thus, v can be only the root.

Let $\tau_1, \dots, \tau_k$ denote the elements of X that are maximal among those below ω. By construction, to every τ_i corresponds an inner orbit of edges that are incident to the root. Again by construction, all elements in $S_0 \backslash \{0\}$ that are not contained in a nonsingleton block of some τ_i are directly appended to the root. We may then renumber the T_i's in such a way that the union of the labels of the leaves of the first s' trees is exactly $S_0 \setminus \{0\}$. Note that $s' < s$ because $\hat{1} \notin X$ (in Fig. 2 we have $s' = 2$ and the corresponding trees T_1, T_2 are indicated).

Then we build a tree starting with an edge t that joins the root to a new vertex w (which we declare to be a zero vertex). Below w we grow s' edges $e_1, \dots e_{s'}$, and append to those the trees $T_1, \dots T_{s'}$. The trees $T_{s'+1}, \dots, T_s$ will be appended directly to the root via edges $e_{s'+1}, \dots, e_s$.

Now check that this is again a Dowling tree: we only have to worry about the zero vertices. Both sets $\bigcup_{i \le s'} \lambda_{e_i}$ and $\bigcup_{i > s'} \lambda_{e_i}$ contain the full orbit of each of their elements, and therefore properties (1), (2) and (3) follow immediately. For property (4) recall Lemma 4.2 to see that all zero vertices (except the root) are in some T_i with $i \le s'$.

We have thus constructed a unique Dowling tree $T(X)$ from a nested set $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$. The inverse operation is now easy: given a Dowling tree T identify the orbits of all inner edges under the action of G, and note that the set of corresponding one-block-orbit elements of $\mathcal{Q}_n(G)$ is nested.

It is clear that the bijection $T : \widetilde{\mathcal{N}}(\mathcal{J}^G, \mathcal{Q}_n(G)) \to \mathcal{T}_n(G)$ extends to an isomorphism of simplicial complexes, if we take the operation of orbit contraction as boundary operator in $\mathcal{T}_n(G)$. □

Summarizing, we can formulate the following corollary, that is a suggestive counterpart of Corollary 3.5.

Corollary 4.5 *The complex of Dowling trees $\mathcal{T}_n(G)$ is a pure simplicial complex of dimension $(n-2)$. It is subdivided by the reduced order complex $\widetilde{\Delta}(\mathcal{Q}_n(G))$ of the Dowling lattice. Their realizations are therefore PL-homeomorphic. They are homotopy equivalent to a wedge of*

$$(|G|+1)(2|G|+1)\cdots((n-1)|G|+1)$$

spheres.

Example 4.6 For the examples considered above, where $n = 3$ and $G = \mathbb{Z}_2$, we have that the complexes $\widetilde{\Delta}(\mathcal{Q}_3^0(\mathbb{Z}_2))$ and $\mathcal{T}_3^{\mathbb{Z}_2}$ are each homotopy equivalent of a wedge of

$(2-1)(4-1)=3$ circles, while $\widetilde{\Delta}(\mathcal{Q}_3(\mathbb{Z}_2))$ and $\mathcal{T}_3(\mathbb{Z}_2)$ have the homotopy type of a wedge of $(2+1)(2\cdot 2+1)=15$ circles.

Remark 4.7 After the last version of this paper was posted on the ArXiv, Federico Ardila pointed out to me that a complex of trees associated with Dowling lattices was defined by Gottlieb and Wachs in [18] and that, together with Caroline Klivans, he could directly prove that it equals the Bergman complex of the Dowling geometry (i.e., the matroid whose lattice of flats is $\mathcal{Q}_n(G)$) [1], thus being subdivided by the order complex of the Dowling lattice (see [2]).

In fact, the complex defined in [18] can be seen to be isomorphic to our complex of Dowling trees by simply changing the way of drawing trees: Gottlieb and Wachs append the leaves corresponding to elements of the zero block to one of the deepest internal nodes, while we append them directly to the root. The goal of [18] was to study the representation of the wreath product $S_n \wr G$ on the cohomology of $\mathcal{Q}_n(G)$. More precisely, it was proved that, as a $S_n \wr G$-module, the only nonvanishing cohomology group $H^{n-3}(\mathcal{Q}_n(G))$ is isomorphic to the multilinear component of the enveloping algebra of the fixed point subalgebra of the free Lie (super)algebra on $[n]\times G$. This result was achieved by explicitly describing the generators of $H^{n-3}(\mathcal{Q}_n(G))$ in terms of certain binary trees. The involved $S_n \wr G$-equivariant isomorphism between $H^{n-3}(\mathcal{Q}_n(G))$ and $H^{n-3}(\mathcal{T}_n(G))$ follows from the homeomorphism of simplicial complexes of our Corollary 4.5.

Motivated by Ardila's observation, we prove the following fact.

Corollary 4.8 *The complex of Dowling Trees $\mathcal{T}_n(G)$ equals the Bergman complex of the associated Dowling geometry* (*i.e.*, *the matroid having $\mathcal{Q}_n(G)$ as its lattice of flats*). *Thus*, *Corollary 4.5 implies in particular that the Bergman complex of a Dowling geometry is subdivided by the order complex of the corresponding Dowling lattice*.

Proof The Dowling geometries satisfy the necessary and sufficient condition given in [16, Theorem 5.3] for the complex of nested sets of a geometric lattice $\mathcal{L}$ to be equal to the Bergman complex of the corresponding matroid $\mathcal{M}$. Indeed, the condition is that the matroid obtained from $\mathcal{M}$ by restricting to any connected flat G and then contracting any flat $F\subset G$ is connected. Translated in the language of geometric lattices, this means that any interval $[F,G]$ in $\mathcal{L}$ is indecomposable if $\mathcal{L}_{\leq G}$ is indecomposable (see e.g. [27, Theorem 5.3.2]). Since a poset is indecomposable if and only if it is not isomorphic to a nontrivial product of posets, we conclude that $\mathcal{Q}_n(G)_{\leq\omega}$ is indecomposable if and only if the associate partition $\underline{\omega}\in\Pi_{n+1}$ has only one nonsingleton block, and then we know $\mathcal{Q}_n(G)_{\leq\omega}\cong(\Pi_{n+1})_{\leq\underline{\omega}}$. So, any interval $[\omega_1,\omega_2]$ in $\mathcal{Q}_n(G)$ with ω_2 indecomposable is isomorphic to the interval $[\underline{\omega}_1,\underline{\omega}_2]$ in Π_{n+1} with $\underline{\omega}_2$ indecomposable. Since Π_{n+1} is the lattice of flats of the graphic matroid associated to the complete graph on $n+1$ vertices, indecomposability of intervals $[\underline{\omega}_1,\underline{\omega}_2]$ as above follows from [12, Remark 3.4.(2)]. □

4.3 From $\mathcal{T}_n^G$ to $\mathcal{T}_n(G)$

The description in terms of nested set complexes allows us to explicitly reconstruct $\mathcal{T}_n(G)$ from $\mathcal{T}_n^G$ by successively coning over subcomplexes having the homotopy type of wedges of $(n-3)$-spheres. This gives another proof of the fact that $\mathcal{T}_n^G$ is Cohen–Macaulay and allows to explicitly calculate the difference between the numbers of spheres in the homotopy type of $\mathcal{T}_n^G$ and $\mathcal{T}_n(G)$.

First of all we want to distinguish three types of simplices in $\mathcal{T}_n(G)$. We call simplices of *type 0*, respectively of *type 1*, those simplices consisting only of elements of type 0, respectively of type 1. The nested sets containing elements of both types will be called simplices *of mixed type*. We remark that the subcomplex given by the simplices of type 1 is exactly $\mathcal{T}_n^G$, and that any simplex X of mixed type is contained in the star of a unique maximal simplex X_0 of type 0, namely $X_0 = X \setminus \mathcal{I}^G$.

The idea is therefore to start with $\mathcal{T}_n^G$ and glue successively the stars of all simplices of type 0. Topologically, this means coning over the link of those simplices: to keep track of the change of topology, we need some definitions and a lemma.

Definition 4.9 Let $\mathcal{T}_J$ denote the subcomplex of $\mathcal{T}_n(G)$ consisting of all simplices of type 0, i.e.

$$\mathcal{T}_J := \{X \in \mathcal{T}_n(G) \mid X \subset \mathcal{J}^G \setminus \mathcal{I}^G\}.$$

From the above considerations we know that any $X \in \mathcal{T}_J$ is a chain $\omega = \omega_1 < \omega_2 < \cdots < \omega_\ell$ of elements of type 0. The length of the chain is the number of its elements and will be denoted $\ell(\omega) = \ell(X)$. The associated partitions $\underline{\omega_i} \in \Pi_{n,0}$ have only one nonsingleton block, namely the one containing 0, which we call w_i. Setting $w_0 := \{0\}$ and $w_{\ell+1} := \{0, 1, 2, \ldots, n\}$, we define numbers $p_0(\omega), \ldots, p_\ell(\omega) \in \mathbb{N}$ as

$$p_i(\omega) := |w_{i+1} \setminus w_i|.$$

If the chain is understood, we will simply write p_i. For $m = 1, \ldots, n-1$ we define the subcomplex of $\mathcal{T}_n(G)$ consisting of $\mathcal{T}_n^G$ and the stars of all simplices $X \in \mathcal{T}_J$ with $\ell(X) \le m$:

$$\mathcal{K}_m := \mathcal{T}_n^G \cup \{X \in \mathcal{T}_n(G) \mid |X \cap \mathcal{J}^G| \le m\}.$$

Lemma 4.10 *The link of any $X \in \mathcal{T}_J$ with $\ell(X) = m$ in $\mathcal{K}_m$ is*

$$lk_{\mathcal{K}_m}(X) \simeq \widetilde{\Delta}(B_m) * \widetilde{\Delta}(\mathcal{Q}_{p_0}^G) * \cdots * \widetilde{\Delta}(\mathcal{Q}_{p_m}^G),$$

where B_m denotes the boolean lattice on m elements.

Proof Any simplex Y in the link can be written as

$$Y = Y' \amalg Y_0 \amalg \cdots \amalg Y_m,$$

where Y' is a (proper!) subset of X, and Y_i is a nested subset of $\mathcal{I}^G$ such that the only nonsingleton block of the associated partitions in $\Pi_{n,0}$ contains only elements from $w_{i+1} \setminus w_i$. The subcomplex of such Y_i can of course be identified with $\widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_i}^G)$, whereas the possible choices of Y' give a subcomplex with a face lattice that can be identified with the proper part of B_m, the boolean lattice on m elements. Note that any choice of $Y' \in B_m \setminus \{\hat{1}\}$ and $Y_i \in \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}^G_{p_i})$ gives a simplex in the link.

Since all complexes $\widetilde{\Delta}(B_m)$ and $\widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}^G_{p_i})$ are flag complexes, we have that the link of X is a simplicial complex that is isomorphic to the join

$$\widetilde{\Delta}(B_m) * \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}^G_{p_0}) * \cdots * \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}^G_{p_m}).$$

With Corollary 3.3 the claim follows. □

In order to simplify notation, let us define numbers q_i^ω associated to any chain ω that gives rise to a simplex in $\mathcal{T}_J$. Recall Definition 4.9 and let

$$q_i^\omega := \prod_{j=1}^{p_i(\omega)-1} (j|G| - 1).$$

The numbers $Q(\omega)$ are then defined for any chain ω as

$$Q(\omega) := q_0^\omega q_1^\omega \dots q_{\ell(\omega)}^\omega.$$

Now we can state a theorem which follows easily from our previous work.

Theorem 4.11 *The link of any $X \in \mathcal{T}_J$ with $\ell(X) = m$ in $\mathcal{K}_m$ is homotopy equivalent to a wedge of $Q(\omega)$ spheres of dimension $(n-3)$, where ω is the chain obtained by ordering the elements of X. Each of those spheres bounds in $\mathcal{K}_m$.*

Proof After Hultman [22] we know that, for any $p_i(\omega)$, $\widetilde{\Delta}(\mathcal{Q}^G_{p_i(\omega)})$ is homotopy equivalent to a wedge of q_i^ω spheres of dimension $(p_i(\omega) - 2)$. It is a standard fact that $\widetilde{\Delta}(B_m) \simeq S^{(m-1)}$. We have then to compute the homotopy type of

$$S^{(m-1)} * \bigvee_{q_0^\omega} S^{(p_0-2)} * \cdots * \bigvee_{q_m^\omega} S^{(p_m-2)},$$

where the index under the wedges indicates how many copies of the corresponding sphere come into play. By basic topological facts we may rewrite this as:

$$S^{(m-2)} * \bigvee_{q_0^\omega \dots q_m^\omega} S^{\sum_{i=0}^m (p_i-2)+m} = \bigvee_{Q(\omega)} S^{m-2+n-2(m+1)+m+1} = \bigvee_{Q(\omega)} S^{(m-3)},$$

where in the second equality we used that $p_0 + p_1 + \cdots + p_m = n$. This proves the first part of the corollary.

The last assertion is proved by induction on m, after remarking that actually the link of X in $\mathcal{K}_m$ is contained in $\mathcal{K}_{m-1}$ (we define $\mathcal{K}_0 = \mathcal{T}_n^G$). For $m = 1$ the assertion holds because $\mathcal{T}_n^G$ is CM of dimension $(n-2)$, thus each $(n-3)$-sphere bounds. Let the claim hold for $m \geq 1$. Then in particular $\mathcal{K}_m$ was obtained by repeatedly coning over spheres that were already boundaries—therefore $\mathcal{K}_m$ is also CM of dimension $(n-2)$, and any of its $(n-3)$-cycles bounds. □

Note Since $p_i(\omega) < n$, we need the result of [22] only in dimension strictly smaller than the one in which the conclusion of the corollary holds. Therefore we may in principle omit the use of [22], thus reproving fully independently the result, by induction on n. We preferred not to do this in order to keep the argument simpler and more readable.

Remark 4.12 We have proved that any chain $\omega \in \widetilde{\Delta}(B_n)$ indexes a simplex of $\mathcal{T}_J$ that contributes $Q(\omega)$ times to the difference between the numbers of spheres in the homotopy types of $\mathcal{T}_n^G$ and $\mathcal{T}_n(G)$.

Example 4.13 For our favourite example $\mathcal{Q}_3(\mathbb{Z}_2)$, we have 12 chains in $\widetilde{\Delta}(B_3)$, each with $Q(\omega) = 1$, therefore $\sum_{\omega\in\widetilde{\Delta}(B_3)} Q(\omega) = 12$, which in fact gives $12 + 3 = 15$.

We may even combine the results of Dowling about $\widetilde{\Delta}(\mathcal{Q}_n(G))$, of Hultman about $\mathcal{T}_n^G$ and our above considerations to state the following arithmetic equality:

Corollary 4.14 *Let integers $k \geq 1$ and $n \geq 2$ be given, and for $\pi \in \Pi_n$ let $h(\pi, j)$ denote the height of the j-th column of the Young tableau of π. Then*

$$\prod_{j=1}^{n}(jk+1) - \prod_{j=1}^{n}(jk-1) = \sum_{\sigma\in\Pi_n}\prod_{j=1}^{n}(jk-1)^{h(\sigma,j)}.$$

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